Center of mass and momentum

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Physics Enhancement Programme for Gifted Students

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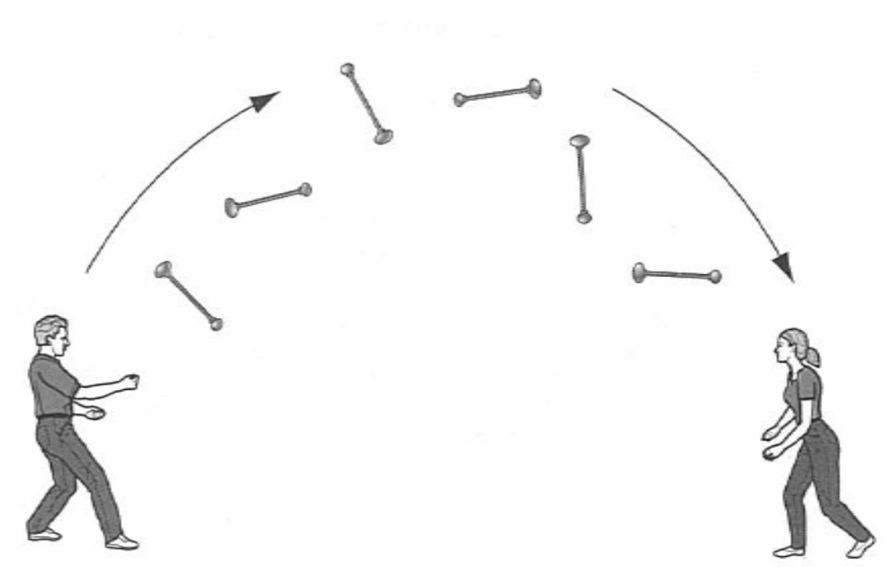
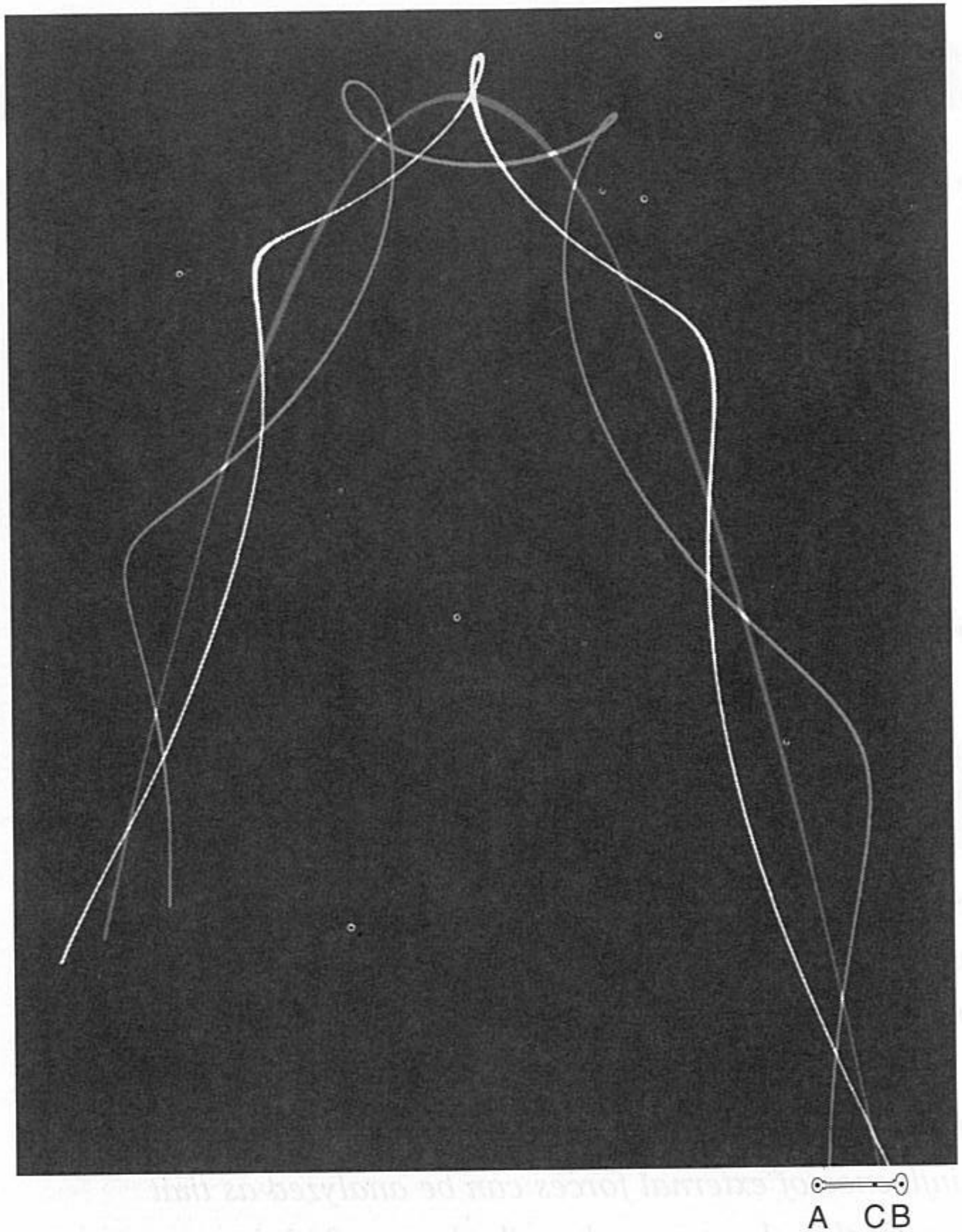
Contents

- Center of mass of a object/system
- Collisions between objects
- Linear momentum, impulse
- Conservation of momentum

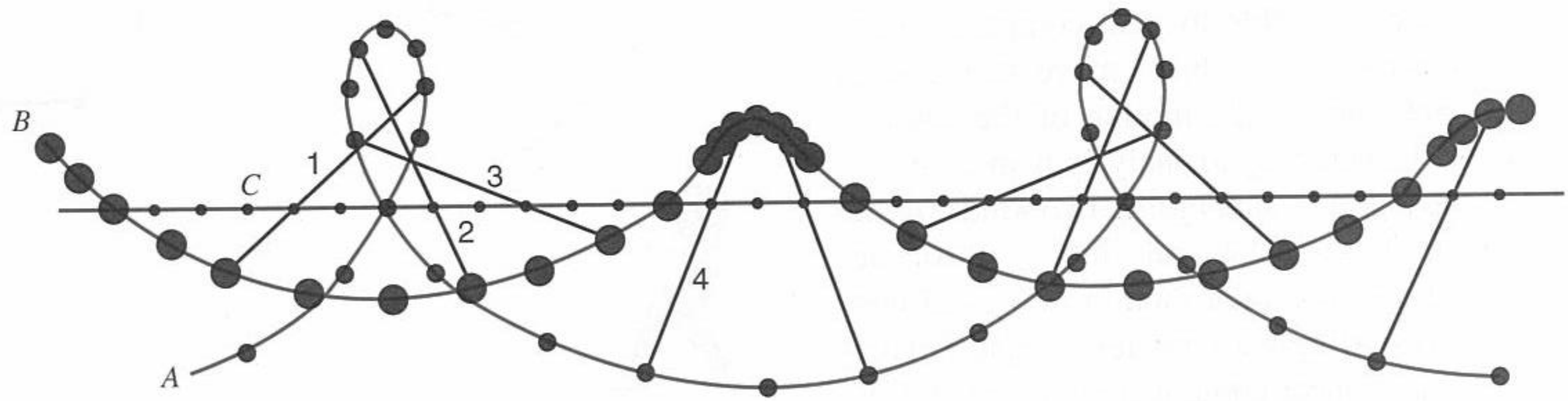
Systems of particles

- We will learn how to find the center of mass of an object.
- Newton's law can be used to describe the motion of the center of mass of a complex system.





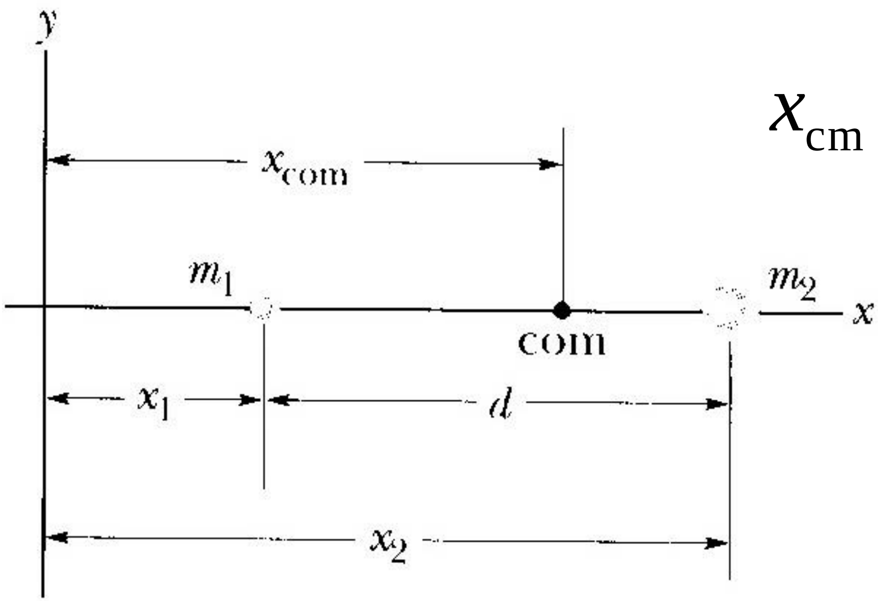
Time exposure photograph of tossed baton with the motion of three points (A, B, C) indicated by lights. Point C (the center of mass) follows a simple parabolic path.



The motion of two particles attached to a connecting rod. The dots represent "snapshots" showing the locations of point A, B, C at successive intervals of time. Point C on the rod follows a straight-line path and its successive positions are equally spaced.

- The center of mass of a body or a system of bodies is the point that moves as though all of the mass were concentrated there and all external forces were applied there.

For example, a 2 particles system in one dimension



$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m_1 x_1 + m_2 x_2}{M}$$

For n particles,

$$x_{\mathrm{cm}} = \frac{m_1 x_1 + \cdots + m_n x_n}{M}$$

In general, n particles in three dimension space, the coordinates of center of mass (x_{cm}, y_{cm}, z_{cm}) are

$$x_{\mathrm{cm}} = \frac{1}{M} \sum_{i=1}^{n} m_i x_i ,$$

$$y_{\mathrm{cm}} = \frac{1}{M} \sum_{i=1}^{n} m_i y_i ,$$

$$z_{\mathrm{cm}} = \frac{1}{M} \sum_{i=1}^{n} m_i z_i .$$

If we express it in vector form,

$$\vec{r}_{\mathrm{cm}} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i .$$

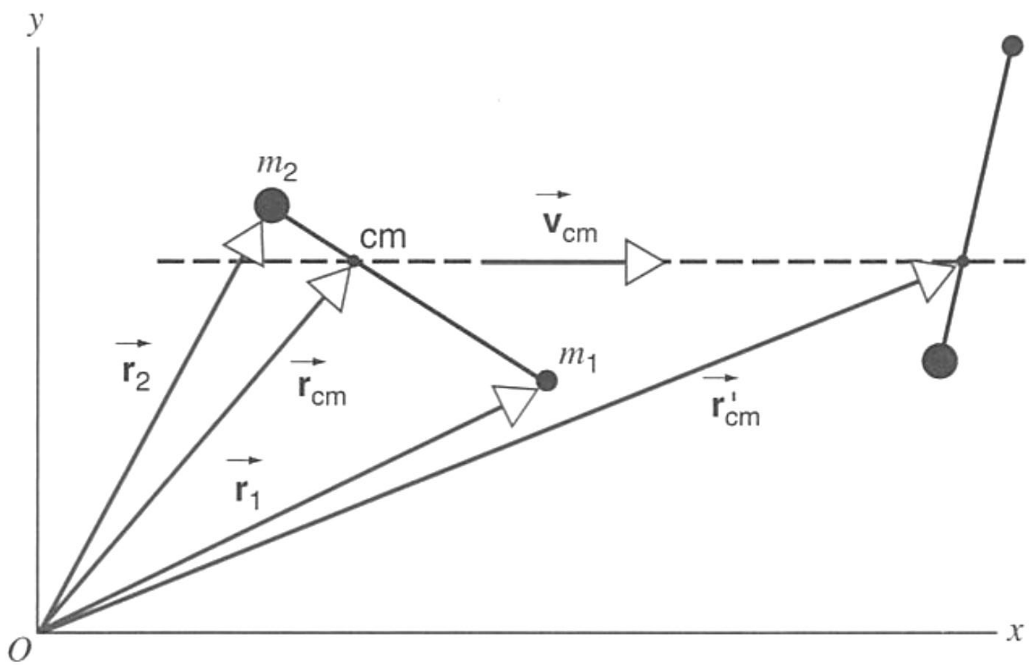


FIGURE 7-5. A coordinate system for locating the center of mass of our system at a particular time. At a later time, the center of mass is at $\vec{\mathbf{r}}\,'_{\mathrm{cm}}$.

$$\vec{\mathbf{r}}_{\mathrm{cm}} = \frac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2}{m_1 + m_2}$$

$$\vec{\mathbf{v}}_{\text{cm}} = \frac{d\,\vec{\mathbf{r}}_{\text{cm}}}{dt} = \frac{m_1 \dfrac{d\,\vec{\mathbf{r}}_1}{dt} + m_2 \dfrac{d\,\vec{\mathbf{r}}_2}{dt}}{m_1 + m_2}$$

or

$$\vec{\mathbf{v}}_{\text{cm}} = \frac{m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2}{m_1 + m_2},$$

and

$$\vec{\mathbf{a}}_{\text{cm}} = \frac{d\,\vec{\mathbf{v}}_{\text{cm}}}{dt} = \frac{m_1 \dfrac{d\,\vec{\mathbf{v}}_1}{dt} + m_2 \dfrac{d\,\vec{\mathbf{v}}_2}{dt}}{m_1 + m_2}$$

or

$$\vec{\mathbf{a}}_{\text{cm}} = \frac{m_1\vec{\mathbf{a}}_1 + m_2\vec{\mathbf{a}}_2}{m_1 + m_2}.$$

For solid bodies

$$x_{\mathrm{cm}} = \frac{1}{M} \lim_{\delta m \to 0} \sum x_n \, \delta m_n = \frac{1}{M} \int x \, dm,$$

$$y_{\mathrm{cm}} = \frac{1}{M} \lim_{\delta m \to 0} \sum y_n \, \delta m_n = \frac{1}{M} \int y \, dm,$$

$$z_{\mathrm{cm}} = \frac{1}{M} \lim_{\delta m \to 0} \sum z_n \, \delta m_n = \frac{1}{M} \int z \, dm.$$

If the object has uniform density,

Density $\rho = \dfrac{dm}{dV} = \dfrac{M}{V}.$

Rewriting $dm = \rho dV$ and $m = \rho V$, we obtain

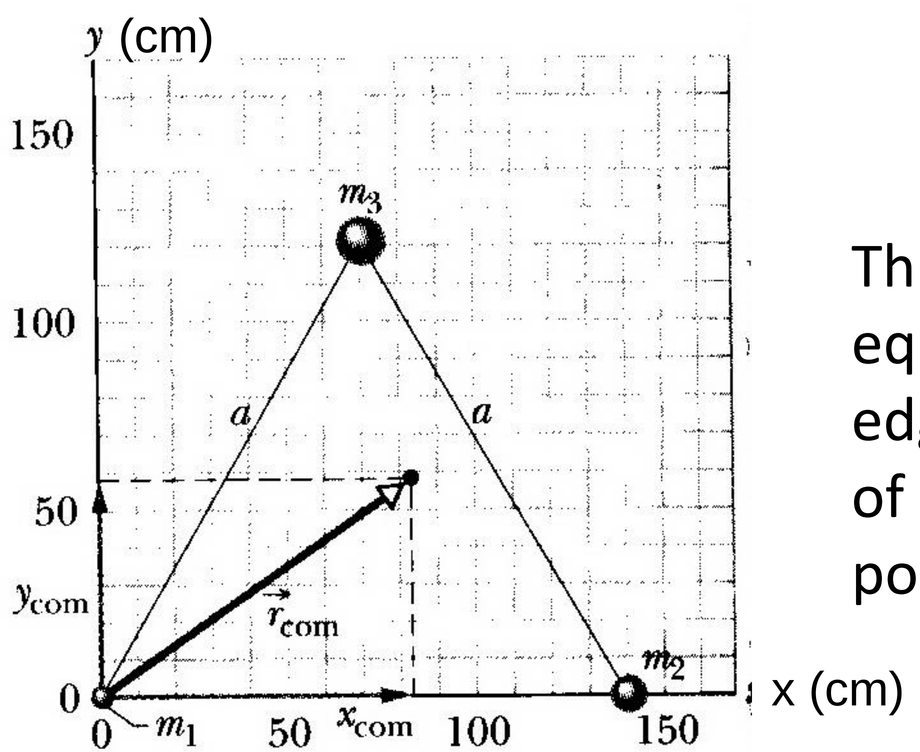
$$x_{\mathrm{cm}} = \frac{1}{V} \int x dV,$$

$$y_{\mathrm{cm}} = \frac{1}{V} \int y dV,$$

$$z_{\mathrm{cm}} = \frac{1}{V} \int z dV.$$

Example

Three particles of masses m_1 = 1.2 kg, m_2 = 2.5 kg, m_3 = 3.4 kg are located at the corners of an equilateral triangle of edge a = 140 cm. Where is the center of mass?



Three particles form an equilateral triangle of edge length a. The center of mass is located by the position vector $\boldsymbol{r}_{com}$

Mass (kg)	*x (cm)*	*y (cm)*
$m_1 = 1.2$	$x_1 = 0$	$y_1 = 0$
$m_2 = 2.5$	$x_2 = 140$	$y_2 = 0$
$m_3 = 3.4$	$x_3 = 140\cos 60^\circ$	$y_3 = 140\sin 60^\circ$

$$x_{\mathrm{cm}} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{M}$$

$$= \frac{(1.2)(0) + (2.5)(140) + (3.4)(70)}{7.1}$$

= 83 cm

$$y_{\mathrm{cm}} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{M}$$

$$= \frac{(1.2)(0) + (2.5)(0) + (3.4)(121)}{7.1}$$

= 58 cm

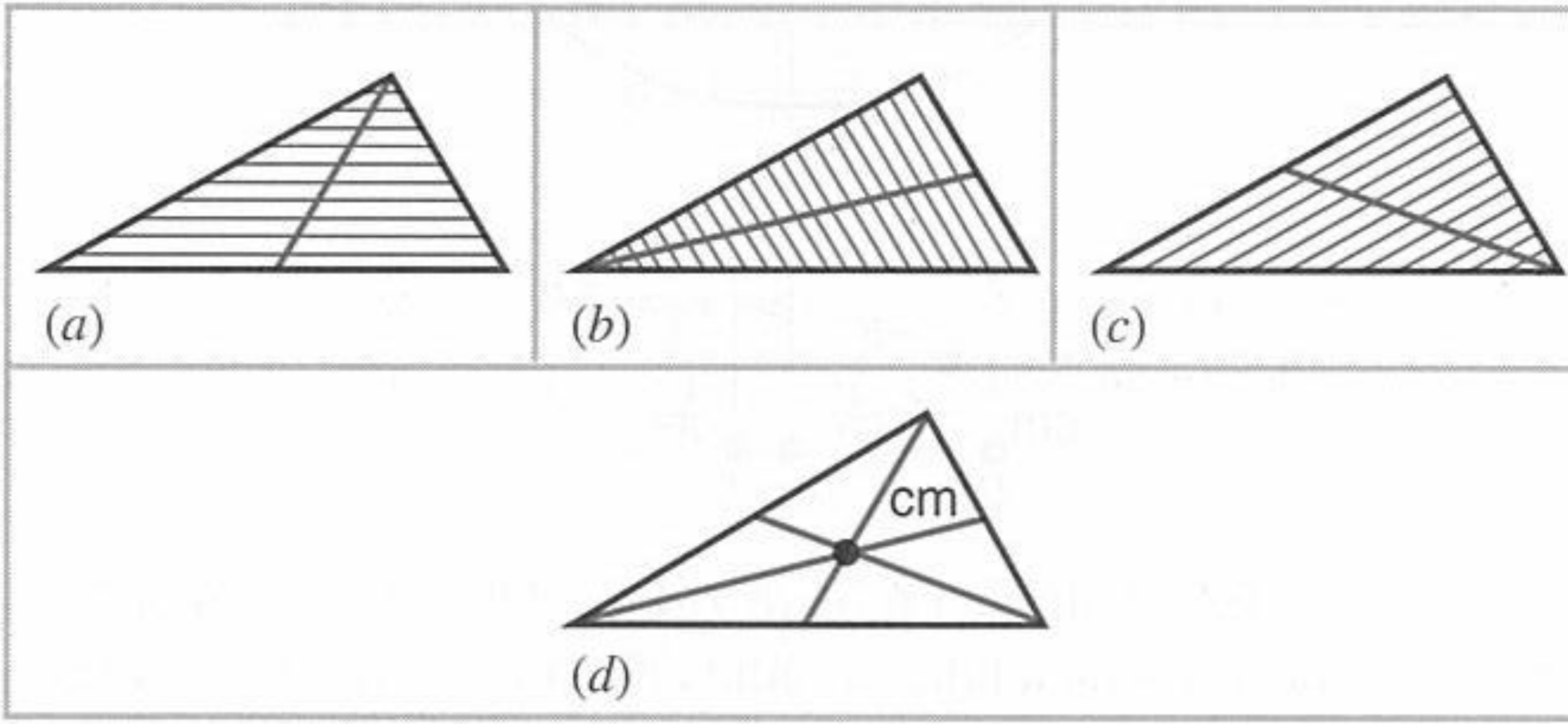
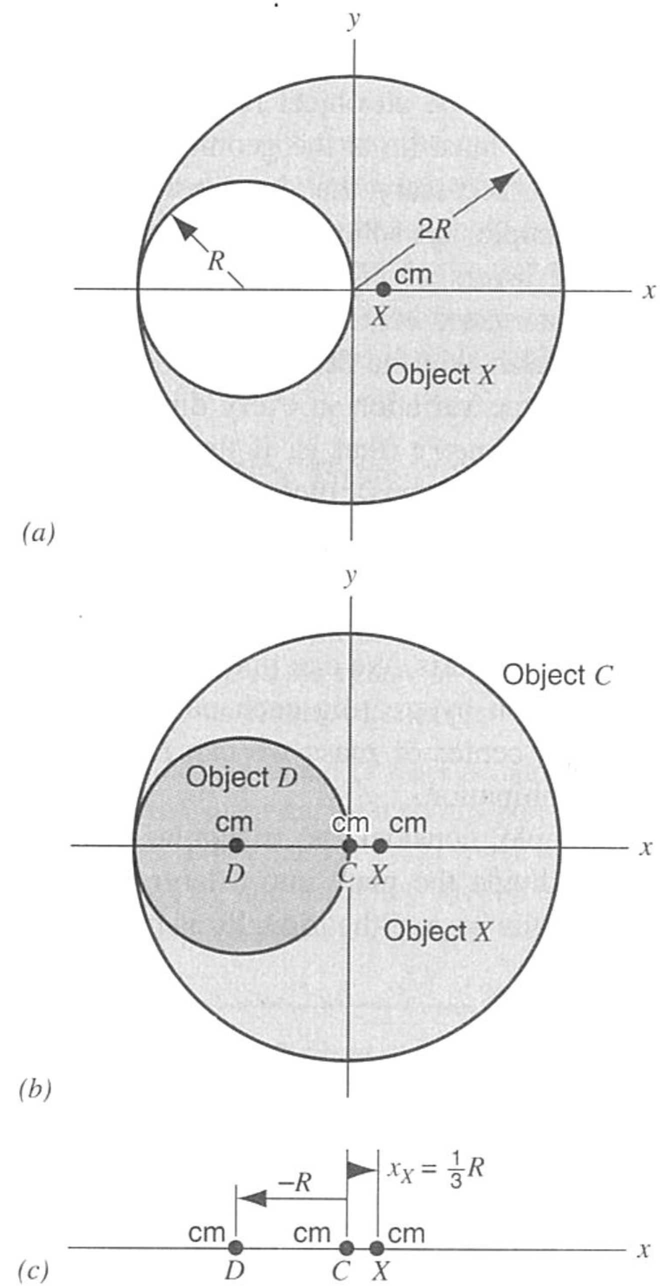


FIGURE 7-12. In (*a*), (*b*), and (*c*), the triangle is divided into thin strips, parallel to each of the three sides. The center of mass must lie along the symmetrical dividing lines shown. (*d*) The dot, the only point common to all three lines, is the position of the center of mass.

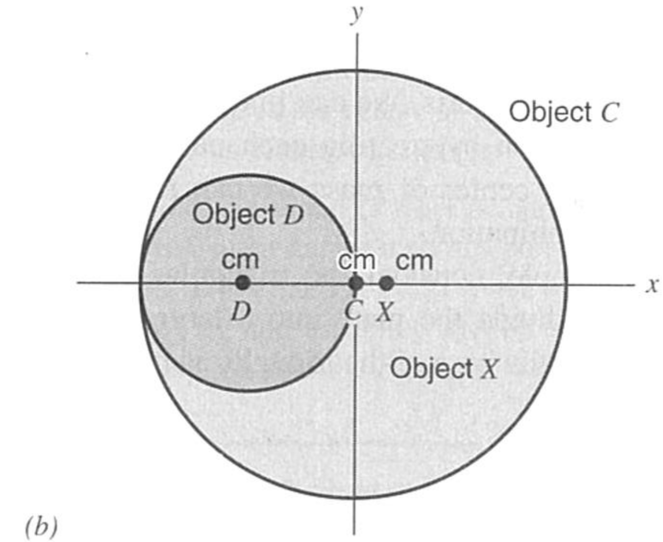
Example

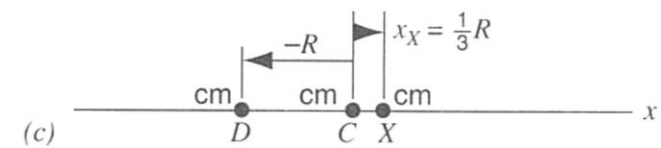
A uniform circular metal plate (Object X) of radius 2*R* has a disk (Object D) of radius *R* removed from it. Locate its centre of mass lying on the x axis



$$x_C = \frac{m_D x_D + m_X x_X}{m_D + m_X},$$

$$x_X = -\left(\frac{m_D}{m_X}\right) x_D.$$





$$\frac{m_D}{m_X} = \frac{\text{area of } D}{\text{area of } X} = \frac{\text{area of } D}{\text{area of } C - \text{area of } D}$$

$$= \frac{\pi R^2}{\pi(2R)^2 - \pi R^2} = \frac{1}{3}.$$

With $x_D = -R$, we obtain

$$x_X = \tfrac{1}{3}R.$$

Linear Momentum

For a single particle, the linear momentum is equal to its mass multiply by its velocity

$$\underbrace{\vec{p}}_{\text{momentum}} = \underbrace{m}_{\text{mass}}\underbrace{\vec{v}}_{\text{velocity}}.$$

By the Newton's law:

$$\vec{F}_{\text{net}} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d\vec{p}}{dt}.$$

For a system of particles, the total linear momentum is

$$\vec{P} = \vec{p}_1 + \cdots + \vec{p}_n = m_1\vec{v}_1 + \cdots + m_n\vec{v}_n.$$

Differentiating the position of the center of mass,

$$M\vec{r}_{\text{com}} = m_1\vec{r}_1 + \cdots + m_n\vec{r}_n.$$

$$M\vec{v}_{\text{com}} = m_1\vec{v}_1 + \cdots + m_n\vec{v}_n.$$

$$\vec{P} = M\vec{v}_{\text{com}}.$$

Thus the linear momentum of a system of particles is equal to the product of the total mass *M* of the system and the velocity of the center of mass of the system.

Newton's law for a system of particles:

From $\vec{P} = M\vec{v}_{\mathrm{com}}$.

$$\frac{d\vec{P}}{dt} = M\,\frac{d\vec{v}_{\mathrm{cm}}}{dt} = M\vec{a}_{\mathrm{cm}}.$$

Hence

$$\vec{F}_{\mathrm{net}} = \frac{d\vec{P}}{dt}.$$

Newton's Second Law for a System of Particles

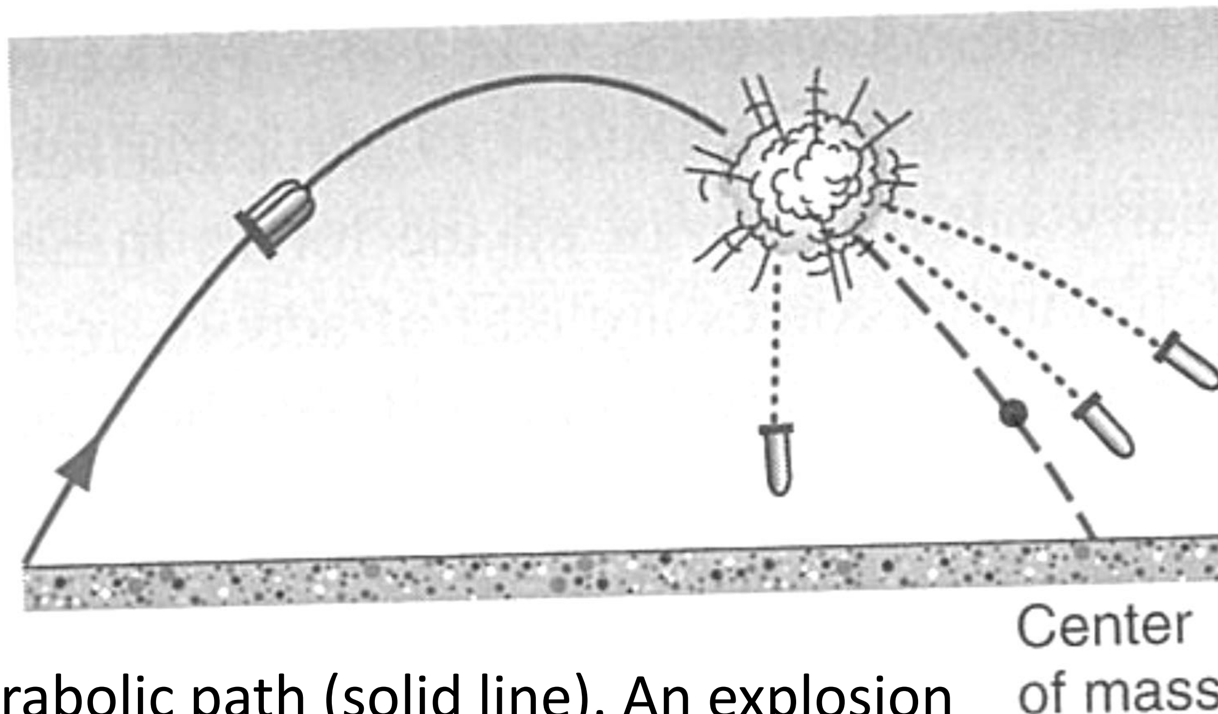
$$\vec{F}_{\text{net}} = M\vec{a}_{\text{cm}}.$$

In terms of components,

$$F_{\text{net},x} = Ma_{\text{cm},x},$$

$$F_{\text{net},y} = Ma_{\text{cm},y},$$

$$F_{\text{net},z} = Ma_{\text{cm},z}.$$



A projectile follows a parabolic path (solid line). An explosion breaks the projectile into three fragments, which travel so that their center of mass follows the original parabolic path.

Collision and Impulse

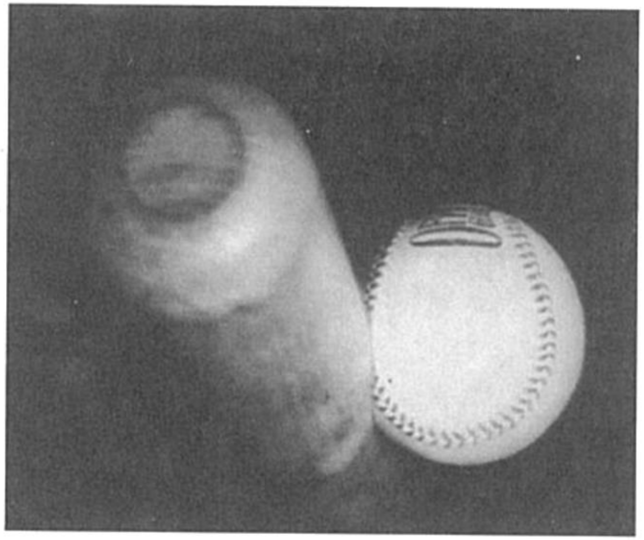


FIGURE 6-1. A high-speed photograph of a bat striking a baseball. Note the deformation of the ball, indicating the large impulsive force exerted by the bat.

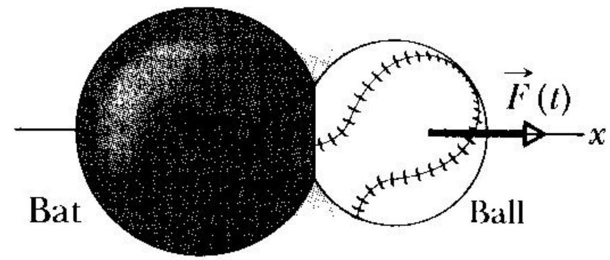


FIG. 9-8 Force $\vec{F}(t)$ acts on a ball as the ball and a bat collide.

Both the ball and the bat are deformed during the collision. Forces that act for a time that is short compared with the time of observation of the system are called impulsive forces.

Collision and Impulse

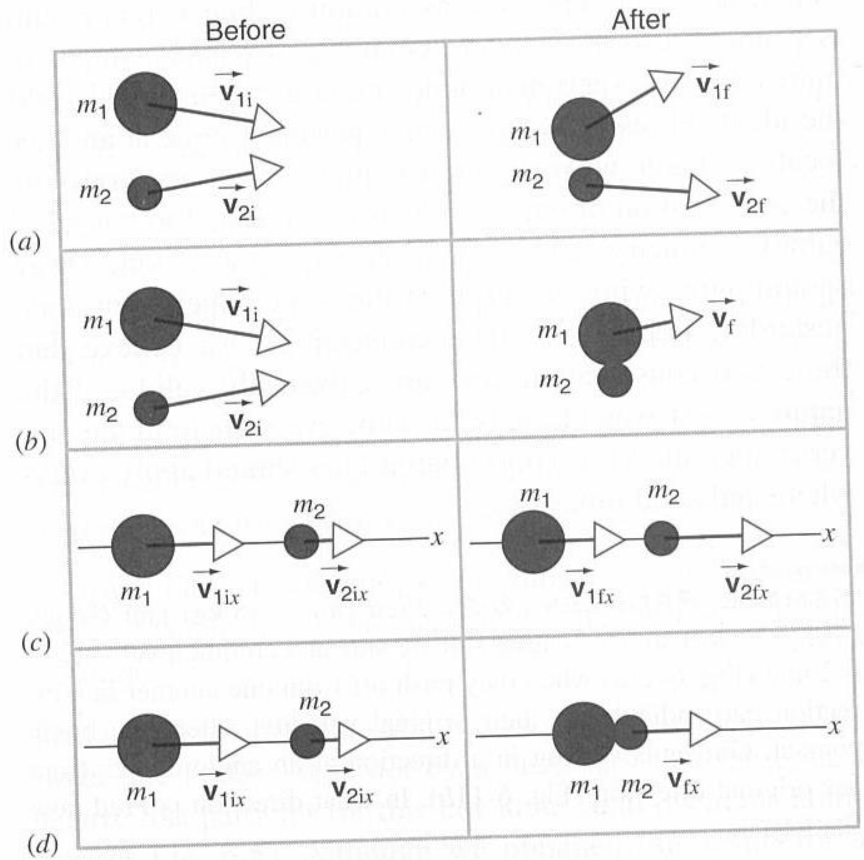


FIGURE 6-13. The initial and final velocities in various two-body collisions.

We will learn what is the relationship between the momentum of the system before and after the collision

By Newton's law:

$$\vec{F} = \frac{d\vec{p}}{dt} \Rightarrow d\vec{p} = \vec{F}(t)dt.$$

Integrating from just before the collision to just afterwards,

$$\int_{\vec{p}_i}^{\vec{p}_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F}(t)dt.$$

Change in linear momentum during the collision:

$$\vec{p}_f - \vec{p}_i = \Delta\vec{p} = \int_{t_i}^{t_f} \vec{F}(t)dt.$$

impulse $\vec{J}$

Change in momentum

The impulse of the net force acting on a particle during a given time interval is equal to the change in momentum of the particle during that interval

The impulse depends on the strength of the force and on its duration. The impulse is a vector and it has the same units and dimensions as momentum

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)dt.$$

Impulse-linear momentum theorem:

$$\vec{p}_f - \vec{p}_i = \Delta\vec{p} = \vec{J}.$$

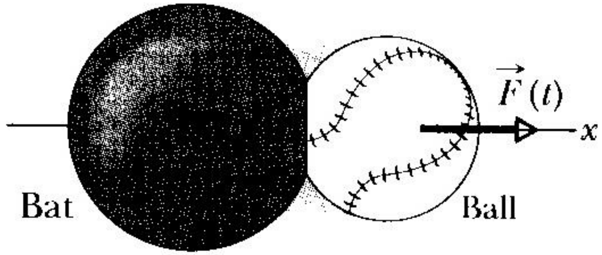
In terms of components,

$$p_{fx} - p_{ix} = \Delta p_x = J_x,$$
$$p_{fy} - p_{iy} = \Delta p_y = J_y,$$
$$p_{fz} - p_{iz} = \Delta p_z = J_z.$$

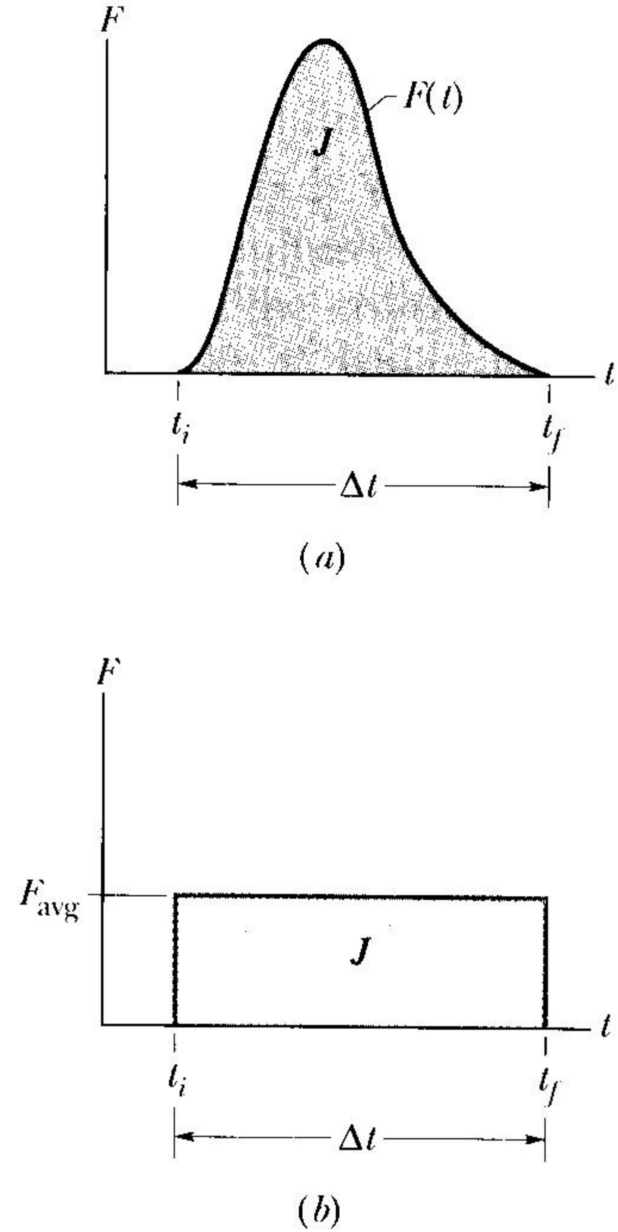
Average force:

$$J = \overline{F}\Delta t,$$

where Δt is the duration of the collision.



(a) The curve shows the magnitude of the time-varying force F(t) that acts on the ball by the bat in the collision. The area under the curve is equal to the magnitude of the impulse on the ball in the collision. (b) The height of the rectangle represents the average force F_{avg} acting on the ball over the time interval Δt. The area within the rectangle is equal to the area under the curve in (a).



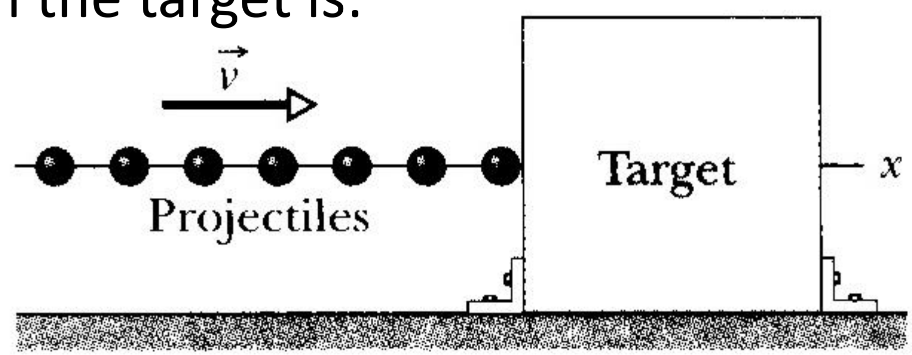
Series of Collisions

n particles collide with target in time interval Δt.
Total change in momentum = $n\Delta p$.
According to Newton's law,
average force acting on the particles = rate of change of momentum

$$= \frac{n}{\Delta t}\Delta p = \frac{n}{\Delta t} m\Delta v$$

Hence the average force acting on the target is:

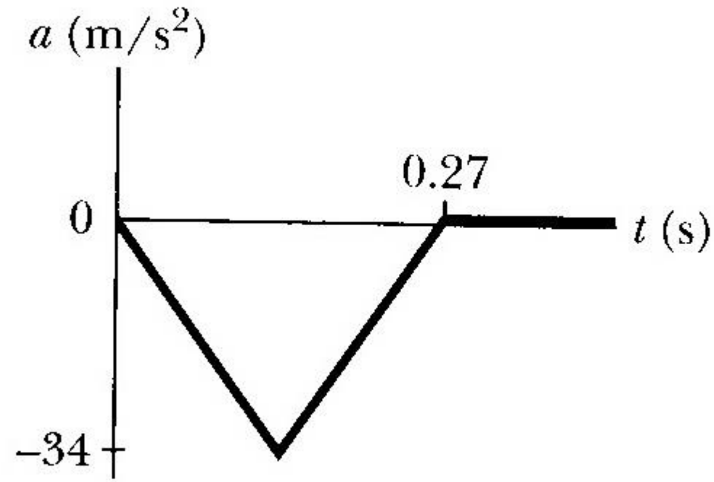
$$\overline{F} = -\frac{n}{\Delta t} m\Delta v$$



If the colliding particles stop upon impact, $\Delta v = -v$.
If the colliding particles bounds elastically upon impact, $\Delta v = -2v$.

Examples

shows the typical acceleration of a male bighorn sheep when he runs head-first into another male. Assume that the sheep's mass is 90.0 kg. What are the magnitudes of the impulse and average force due to the collision?



Since

$$a = \frac{dv}{dt}$$

$$\Delta v(t) = \int_0^t a(t')dt'$$

Hence the change in velocity is given by the area enclosed by the *a*-*t* curve

$$\Delta v = -\frac{1}{2}(0.27)(34) = -4.59\ \mathrm{ms}^{-1}$$

Using the impulse-momentum theorem,

$$J = \Delta p = m\Delta v = (90)(-4.59) = -413\ \mathrm{kg\ ms}^{-1}$$

The magnitude of the impulse is 413 kg ms^{-1}

The magnitude of the average force is

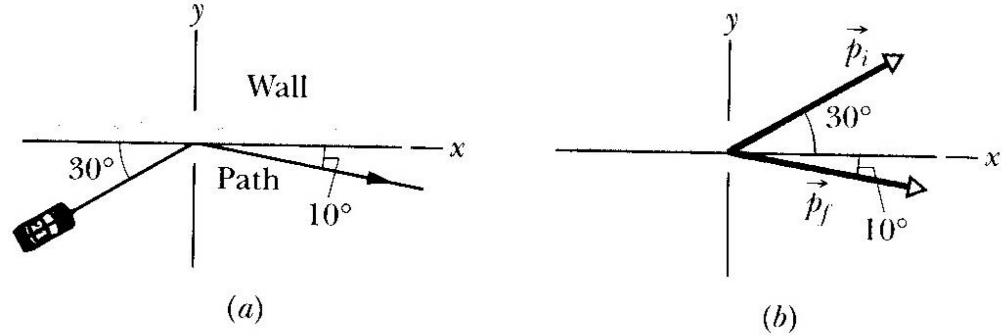
$$F_{\text{av}} = \frac{|J|}{\Delta t} = \frac{413}{0.27} = 1530 \text{ N}$$

Remark: The collision time is prolonged by the flexibility of the horns. If the sheep were to hit skull-to-skull or skull-to-horn, the collision duration would be 1/10 of what we used, and the average force would be 10 times of what we calculated!

Example

Race-car wall collision. A race car collides with a race track wall at speeds v_i = 70 ms^{-1} and v_f = 50 ms^{-1} before and after collision respectively (Fig. 9-12a). His mass m is 80 kg.

(a) What is the impulse on the driver due to the collision?
(b) The collision lasts for 14 ms. What is the magnitude of the average force on the driver during the collision?



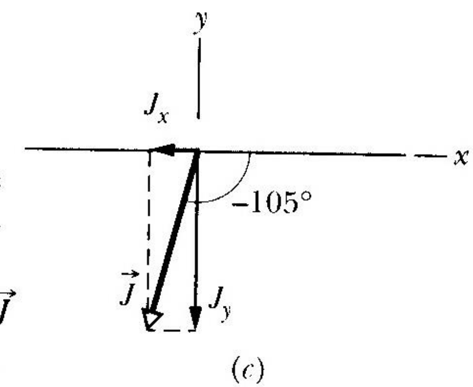


FIG. 9-12 (*a*) Overhead view of the path taken by a race car and its driver as the car slams into the racetrack wall. (*b*) The initial momentum $\vec{p}_i$ and final momentum $\vec{p}_f$ of the driver. (*c*) The impulse $\vec{J}$ on the driver during the collision.

(a) Using the impulse-momentum theorem,

$$\vec{J} = \vec{p}_f - \vec{p}_i = m\vec{v}_f - m\vec{v}_i$$
$$= m(\vec{v}_f - \vec{v}_i)$$

x component:

$$J_x = m(v_{fx} - v_{ix})$$
$$= (80)(50\cos 10^{\circ} - 70\cos 30^{\circ})$$
$$= -910 \text{ kg ms}^{-1}$$

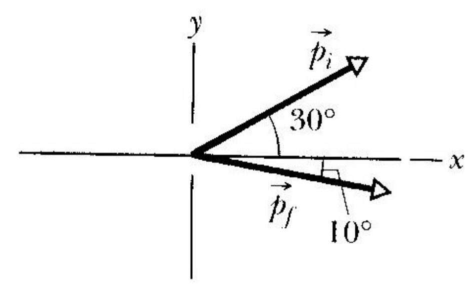
y component:

$$J_y = m(v_{fy} - v_{iy})$$
$$= (80)(-50\sin 10^{\circ} - 70\sin 30^{\circ})$$
$$= -3495 \text{ kg ms}^{-1}$$

The impulse is then

$$\vec{J} = (-910\hat{i} - 3500\hat{j}) \text{ kg ms}^{-1}$$

$$J = \sqrt{J_x^2 + J_y^2} = 3616 \text{ kg ms}^{-1} \approx 3600 \text{ kg ms}^{-1}$$

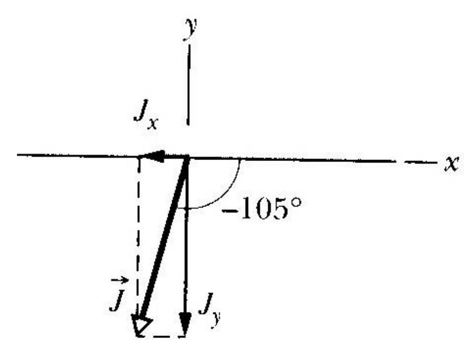


(b)

Direction:

$$\theta = \tan^{-1}\frac{J_y}{J_x} = 75.4^{\circ} - 180^{\circ} = -105^{\circ}$$

(b) $$F_{\text{av}} = \frac{J}{\Delta t} = \frac{3616}{0.014} = 2.583 \times 10^5 \text{ N} \approx 2.6 \times 10^5 \text{ N}$$



(c)

Conservation of Linear Momentum

If the system of particles is isolated (i.e. there are no external forces) and closed (i.e. no particles leave or enter the system), then

$$\vec{P} = \text{constant}$$

Law of conservation of linear momentum:

$$\vec{P}_i = \vec{P}_f$$

Initial state momentum is equal to final state momentum

Examples

Imagine a spaceship and cargo module, of total mass M, travelling in deep space with velocity v_i = 2100 km/h relative to the Sun. With a small explosion, the ship ejects the cargo module, of mass $0.20M$. The ship then travels 500 km/h faster than the module; that is, the relative speed v_{rel} between the module and the ship is 500 km/h. What then is the velocity v_f of the ship relative to the Sun?

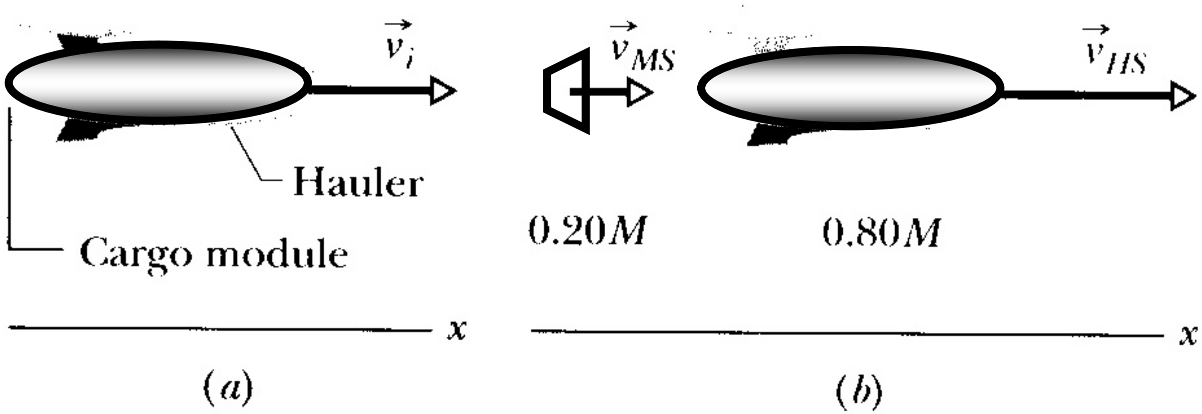


FIG. 9-13 (a) A space hauler, with a cargo module, moving at initial velocity $\vec{v}_i$. (b) The hauler has ejected the cargo module. Now the velocities relative to the Sun are $\vec{v}_{MS}$ for the module and $\vec{v}_{HS}$ for the hauler.

Using the conservation of linear momentum,

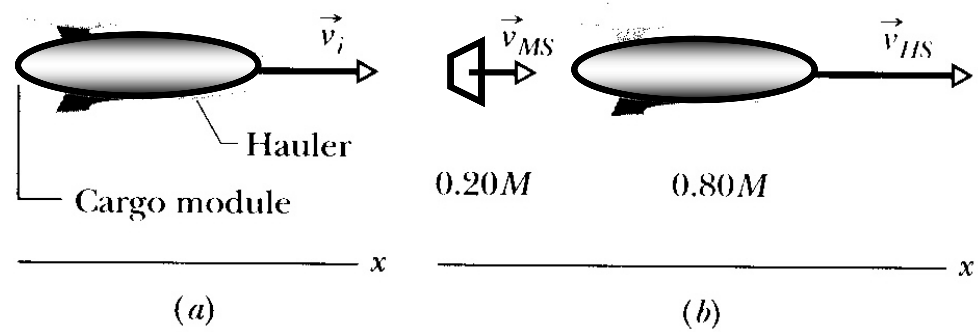
$$P_i = P_f$$

$$Mv_i = 0.2M(v_f - v_{\text{rel}}) + 0.8Mv_f$$

$$v_i = v_f - 0.2v_{\text{rel}}$$

$$v_f = v_i + 0.2v_{\text{rel}}$$

= 2100 + (0.2)(500)
= 2200 km/h



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Example

A firecracker placed inside a coconut of mass M, initially at rest on a frictionless floor, blows the fruit into three pieces and sends them sliding across the floor. An overhead view is shown in the figure. Piece C, with mass $0.3M$, has final speed v_{fc}=5.0ms^{-1}.

(a) What is the speed of piece B, with mass $0.2M$?

(b) What is the speed of piece A, *with mass 0.5M*?

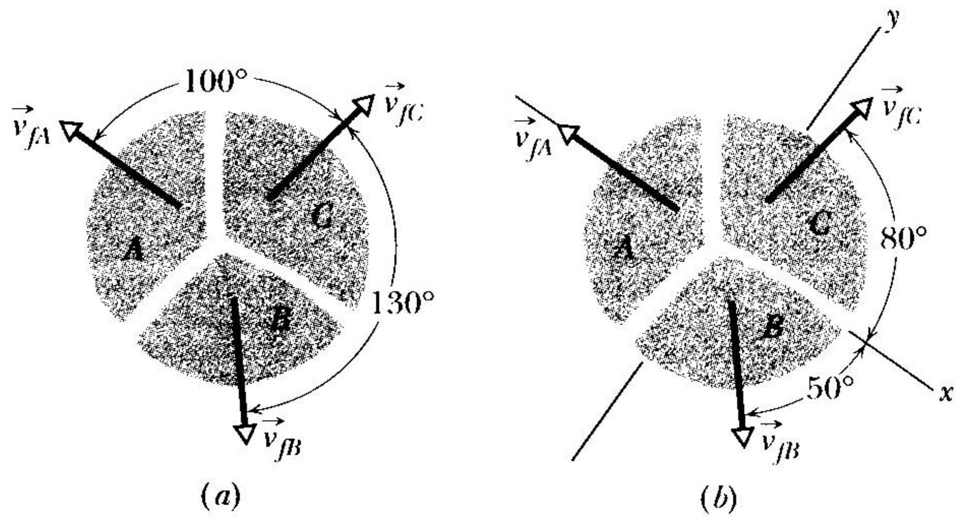


FIG. 9-14 Three pieces of an exploded coconut move off in three directions along a frictionless floor. (*a*) An overhead view of the event. (*b*) The same with a two-dimensional axis system imposed.

(a) Using the conservation of linear momentum,

$$P_{ix} = P_{fx} \qquad P_{iy} = P_{fy}$$

$$m_C v_{fC} \cos 80^\circ + m_B v_{fB} \cos 50^\circ - m_A v_{fA} = 0$$

$$m_C v_{fC} \sin 80^\circ - m_B v_{fB} \sin 50^\circ = 0$$

$$m_A = 0.5M,\ m_B = 0.2M,\ m_C = 0.3M$$

$$0.3M v_{fC} \sin 80^\circ - 0.2M v_{fB} \sin 50^\circ = 0$$

$$v_{fB} = \left(\frac{0.3\sin 80^\circ}{0.2\sin 50^\circ}\right)5 = 9.64\ \text{ms}^{-1} \approx 9.6\ \text{ms}^{-1}$$

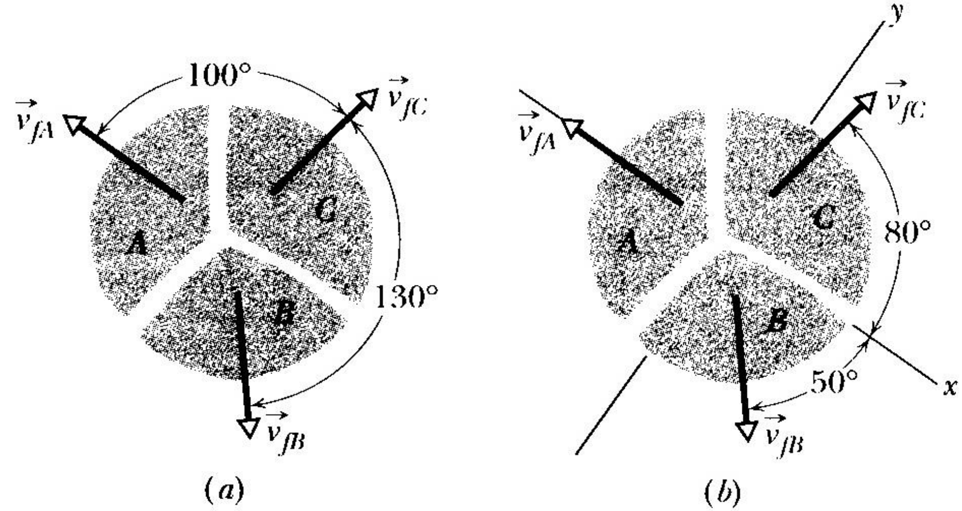


FIG. 9-14 Three pieces of an exploded coconut move off in three directions along a frictionless floor. (*a*) An overhead view of the event. (*b*) The same with a two-dimensional axis system imposed.

(b)

$$0.3M v_{fC} \cos 80^\circ + 0.2M v_{fB} \cos 50^\circ = 0.5M v_{fA}$$

$$v_{fA} = \frac{(0.3)(5)\cos 80^\circ + (0.2)(9.64)\cos 50^\circ}{0.5} = 3.0\ \text{ms}^{-1}$$

Inelastic collisions in one Dimension

In an inelastic collision, the kinetic energy of the system of colliding bodies is not conserved.
In a completely inelastic collision, the colliding bodies stick together after the collision.
However, the conservation of linear momentum still holds.

$$m_1 v = (m_1 + m_2)V,$$

or

$$V = \frac{m_1}{m_1 + m_2} v.$$

object 1 and object 2 stick together in completely inelastic collision

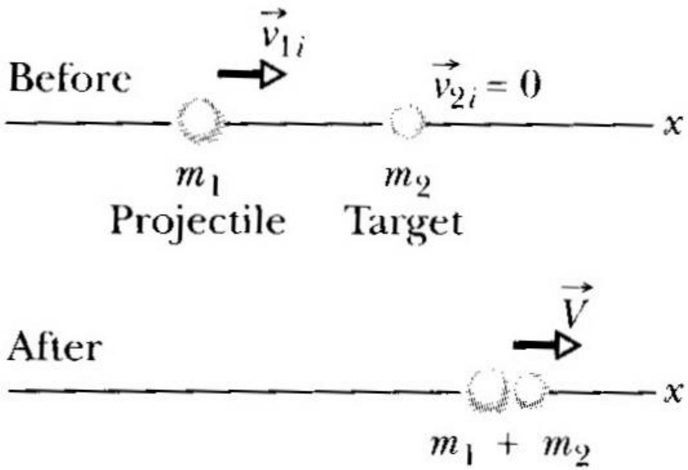


FIG. 9-16 A completely inelastic collision between two bodies. Before the collision, the body with mass m_2 is at rest and the body with mass m_1 moves directly toward it. After the collision, the stuck-together bodies move with the same velocity $\vec{V}$.

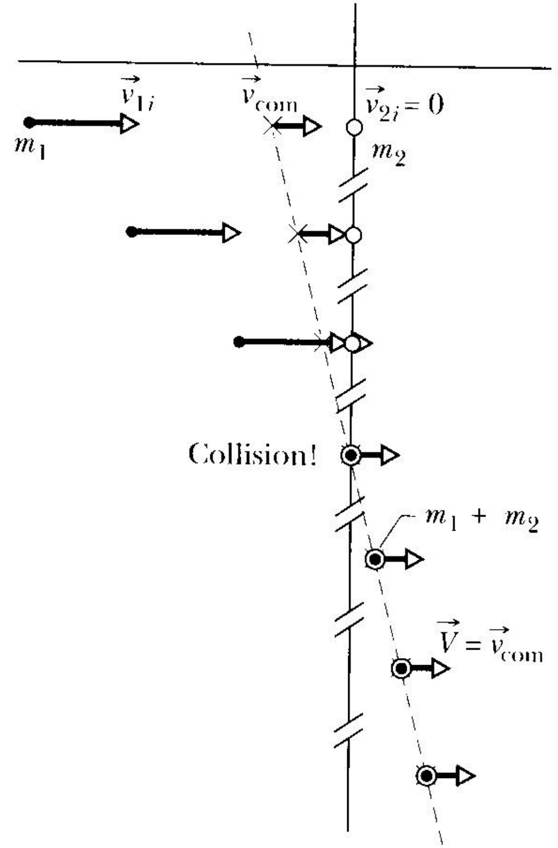


FIG. 9-17 Some freeze-frames of the two-body system in Fig. 9-16, which undergoes a completely inelastic collision. The system's center of mass is shown in each freeze-frame. The velocity $\vec{v}_{com}$ of the center of mass is unaffected by the collision. Because the bodies stick together after the collision, their common velocity $\vec{V}$ must be equal to $\vec{v}_{com}$.

Example

The ballistic pendulum was used to measure the speeds of bullets before electronic timing devices were developed. Here it consists of a large block of wood of mass M = 5.4 kg, hanging from two long cords. A bullet of mass m = 9.5 g is fired into the block, coming quickly to rest. The *block + bullet* then swing upward, their center of mass rising a vertical distance h = 6.3 cm before the pendulum comes momentarily to rest at the end of its arc. What was the speed v of the bullet just prior to the collision?

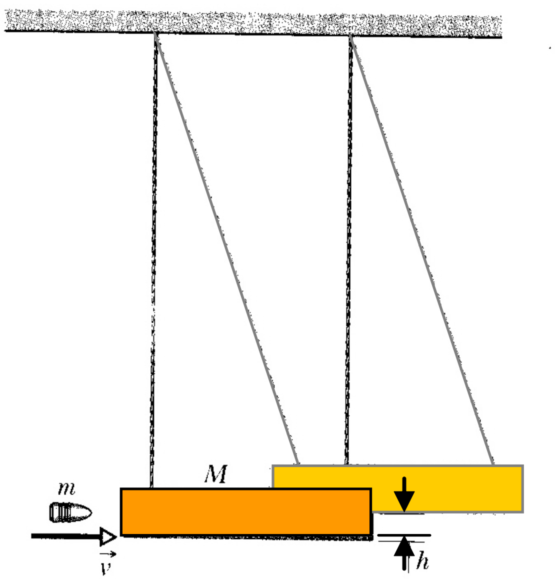


FIG. 9-18 A ballistic pendulum, used to measure the speeds of bullets.

Using the conservation of momentum during collision,

$$mv = (M + m)V$$

Using the conservation of energy after collision,

$$\frac{1}{2}(M + m)V^2 = (M + m)gh$$

Kinetic energy Potential energy

$$V = \sqrt{2gh}$$

$$v = \frac{M + m}{m}V = \frac{M + m}{m}\sqrt{2gh}$$

$$= \frac{5.4 + 0.0095}{0.0095}\sqrt{(2)(9.8)(0.063)} = 630\ \mathrm{ms}^{-1}$$

Example

Consider the collision of cars 1 and 2 with initial velocities v_{1i} = +25 ms^{-1} and v_{2i} = –25 ms^{-1} respectively. Let each car carry one driver. The total mass of cars 1 and 2 are m_1 = 1400 kg and m_2 = 1400 kg respectively.

(a) What are the changes Δv_1 and Δv_2 during their head-on and completely inelastic collision?

(b) Repeat the calculation with an 80 kg passenger in car 1.

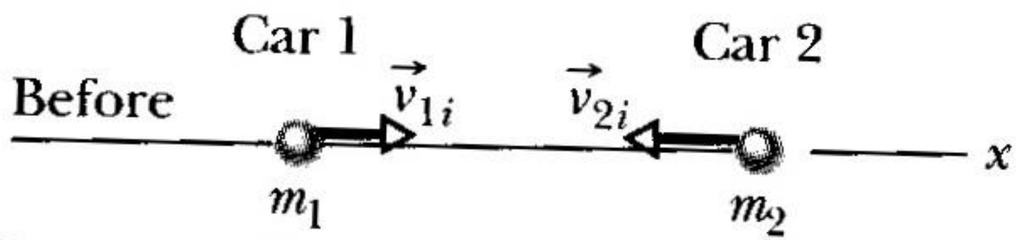


FIG. 9-19 Two cars about to collide head-on.

(a) Using the conservation of momentum,

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2)V$$

Since the collision is completely inelastic, $v_{1f} = v_{2f} = V$.

$$V = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2}$$

$$= \frac{(1400)(+25) + (1400)(-25)}{1400 + 1400} = 0$$

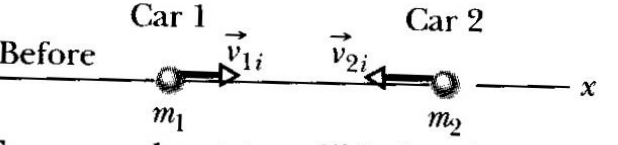


FIG. 9-19 Two cars about to collide head-on.

$$\Delta v_1 = v_{1f} - v_{1i} = 0 - (+25) = -25\,\text{ms}^{-1}$$

$$\Delta v_2 = v_{2f} - v_{2i} = 0 - (-25) = +25\,\text{ms}^{-1}$$

(b) In this case, m_1 is replaced by 1480 kg

$$V = \frac{(1480)(+25) + (1400)(-25)}{1480 + 1400} = 0.694\,\text{ms}^{-1}$$

$$\Delta v_1 = v_{1f} - v_{1i} = 0.694 - (+25) = -24.3\,\text{ms}^{-1}$$

$$\Delta v_2 = v_{2f} - v_{2i} = 0.694 - (-25) = +25.7\,\text{ms}^{-1}$$

Elastic Collisions in One Dimension

Stationary Target

In an elastic collision, the kinetic energy of each colliding body can change, but the total kinetic energy of the system does not change.

In a closed, isolated system, the linear momentum of each colliding body can change, but the net linear momentum cannot change, regardless of whether the collision is elastic.

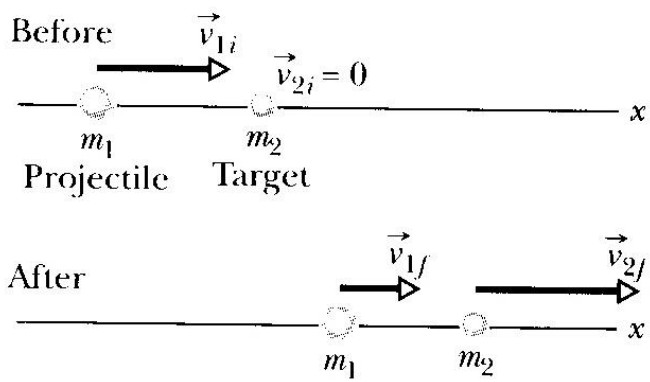


FIG. 9-20 Body 1 moves along an x axis before having an elastic collision with body 2, which is initially at rest. Both bodies move along that axis after the collision.

Conservation of linear momentum:

$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f}.$$

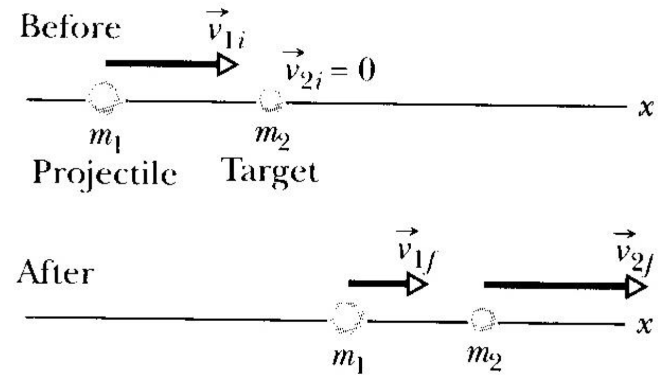


FIG. 9-20 Body 1 moves along an x axis before having an elastic collision with body 2, which is initially at rest. Both bodies move along that axis after the collision.

Conservation of kinetic energy:

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2.$$

Rewriting these equations as

$$m_1 (v_{1i} - v_{1f}) = m_2 v_{2f},$$

$$m_1 (v_{1i}^2 - v_{1f}^2) = m_2 v_{2f}^2.$$

Dividing the second one by the first one,

$$v_{1i} + v_{1f} = v_{2f}.$$

We have two linear equations for v_{1f} and v_{2f}. Solution:

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i},$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}.$$

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \quad \text{and} \quad v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}.$$

Special situations:

1.Equal masses: If $m_1 = m_2$, then $v_{1f} = 0$ and $v_{2f} = v_{1i}$ (pool player's result).

2.A massive target: If $m_2 >> m_1$, then

$$v_{1f} \approx -v_{1i} \quad \text{and} \quad v_{2f} = \left(\frac{2m_1}{m_2}\right) v_{1i} << v_{1i}$$

The light incident particle bounces back and the heavy target barely moves.

3. A projectile: If $m_1 >> m_2$, then

$$v_{1f} \approx v_{1i} \quad \text{and} \quad v_{2f} \approx 2v_{1i}$$

The incident particle is scarcely slowed by the collision.

Example

Two metal spheres, suspended by vertical cords, initially just touch. Sphere 1, with mass m_1 = 3 g, is pulled to the left to height h_1 = 8.0 cm, and then released. After swinging down, it undergoes an elastic collision with sphere 2, whose mass m_2 = 75 g. What is the velocity v_{1f} of sphere 1 just after the collision?

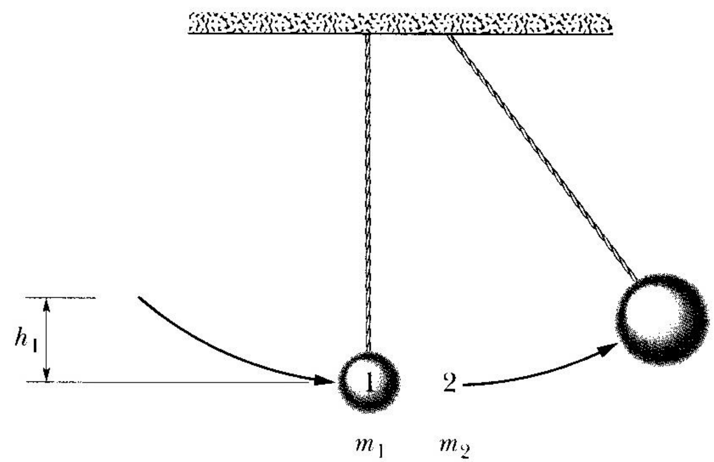


FIG. 9-22 Two metal spheres suspended by cords just touch when they are at rest. Sphere 1, with mass m_1, is pulled to the left to height h_1 and then released.

Using the conservation of energy

$$\frac{1}{2}m_1 v_{1i}^2 = m_1 g h_1$$

$$v_{1i} = \sqrt{2gh_1}$$

$$= \sqrt{(2)(9.8)(0.08)}$$

$$= 1.252 \text{ ms}^{-1}$$

Using the conservation of momentum

$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f}$$

For elastic collisions,

$$\frac{1}{2}m_1 v_{1i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2$$

$$m_1(v_{1i}^2 - v_{1f}^2) = m_2 v_{2f}^2$$

$$m_1(v_{1i} - v_{1f}) = m_2 v_{2f}$$

Dividing the above two,

$$v_{1i} + v_{1f} = v_{2f}$$

$$m_1(v_{1i} - v_{1f}) = m_2(v_{1i} + v_{1f})$$

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} = \left(\frac{0.03 - 0.075}{0.03 + 0.075}\right) 1.252$$

$$= -0.537 \text{ ms}^{-1}$$

Collisions in Two Dimensions

Conservation of linear momentum:

x component:

$$m_1 v_{1i} = m_1 v_{1f} \cos\theta_1 + m_2 v_{2f} \cos\theta_2,$$

y component:

$$0 = -m_1 v_{1f} \sin\theta_1 + m_2 v_{2f} \sin\theta_2.$$

Conservation of kinetic energy:

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2.$$

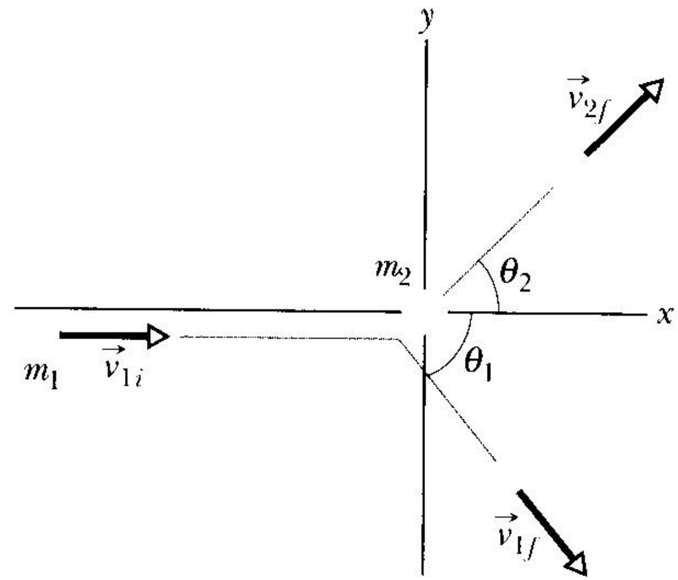


FIG. 9-23 An elastic collision between two bodies in which the collision is not head-on. The body with mass m_2 (the target) is initially at rest.

Typically, we know m_1, m_2, v_{1i} and θ_1. *Then we can solve for* v_{1f}, v_{2f} and θ_2

Systems with Varying Mass: A Rocket

Assume no gravity. By the conservation of linear momentum: $P_i = P_f$.

Initial momentum = Mv
Final momentum of the exhaust
= $(-dM)U$
Final momentum of the rocket
= $(M + dM)(v + dv)$

$$Mv = (-dM)U + (M + dM)(v + dv).$$

Suppose the rocket ejects the exhaust at a velocity v_{rel}

$$U = v + dv - v_{rel}.$$

Substituting and dividing by *dt*,

$$-dMv_{rel} = Mdv,$$

$$-\frac{dM}{dt}v_{rel} = M\frac{dv}{dt}.$$

Since the rate of fuel consumption is $R = -\frac{dM}{dt}$

We have the first rocket equation: $Rv_{rel} = Ma.$

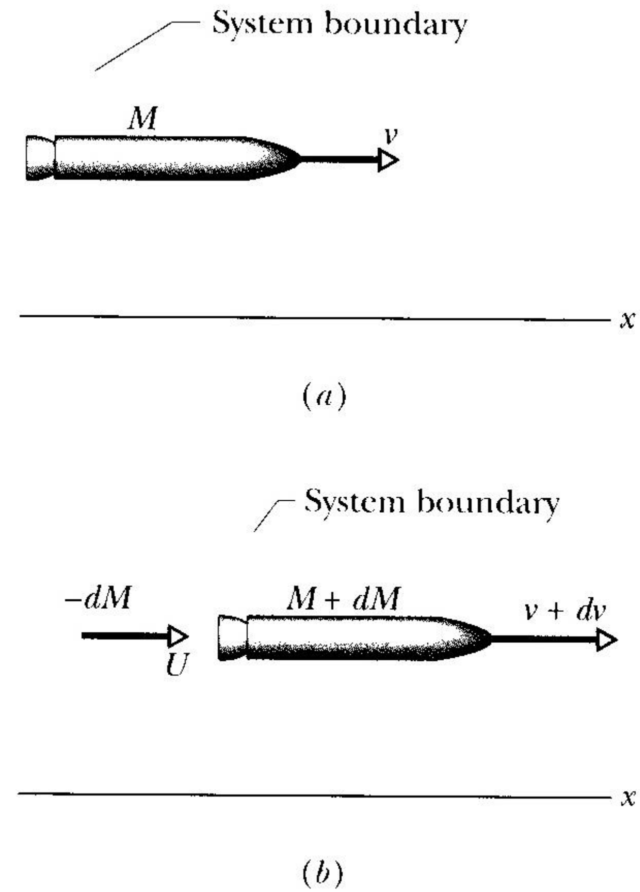


FIG. 9-24 (*a*) An accelerating rocket of mass *M* at time *t*, as seen from an inertial reference frame. (*b*) The same but at time *t* + *dt*. The exhaust products released during interval *dt* are shown.

$T \equiv R v_{\text{rel}}$ is called the **thrust** of the rocket engine. Newton's second law emerges. To find the velocity,

$$dv = -v_{\text{rel}} \frac{dM}{M},$$

Integrating,

$$\int_{v_i}^{v_f} dv = -v_{\text{rel}} \int_{M_i}^{M_f} \frac{dM}{M},$$

$$v_f - v_i = v_{\text{rel}} \ln \frac{M_i}{M_f}. \qquad \text{(second rocket equation)}$$

Remark: Multistage rockets are used to reduce M_f in stages.

Example

A rocket with initial mass M_i = 850 kg consumes fuel at the rate R = 2.3 kgs^{-1}. The speed v_{rel} of the exhaust gases relative to the rocket engine is 2800 ms^{-1}. What thrust does the rocket engine provide? What is the initial acceleration of the rocket?

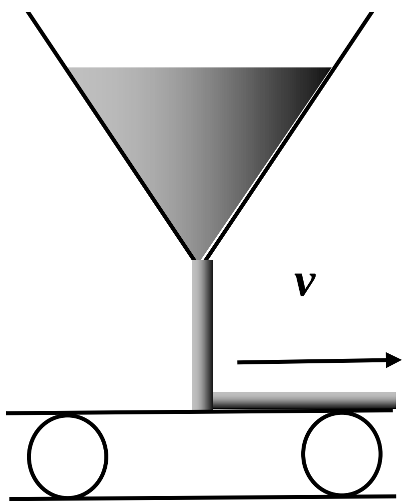
$$T = Rv_{rel} = (2.3)(2800) = 6440 \text{ N}$$

$$a = \frac{T}{M} = \frac{6440}{850} = 7.6 \text{ ms}^{-2}$$

Remark: Since $a < g$, the rocket cannot be launched from Earth's surface.

Advanced Examples

A conveyer belt is driven by a motor to move with horizontal uniform speed v. Sand is continuously dropping on the belt from a stationary hopper at a rate $D = dm/dt$. What is the power supplied by the motor? Explain why it is different from the rate of increase of the kinetic energy of the system.



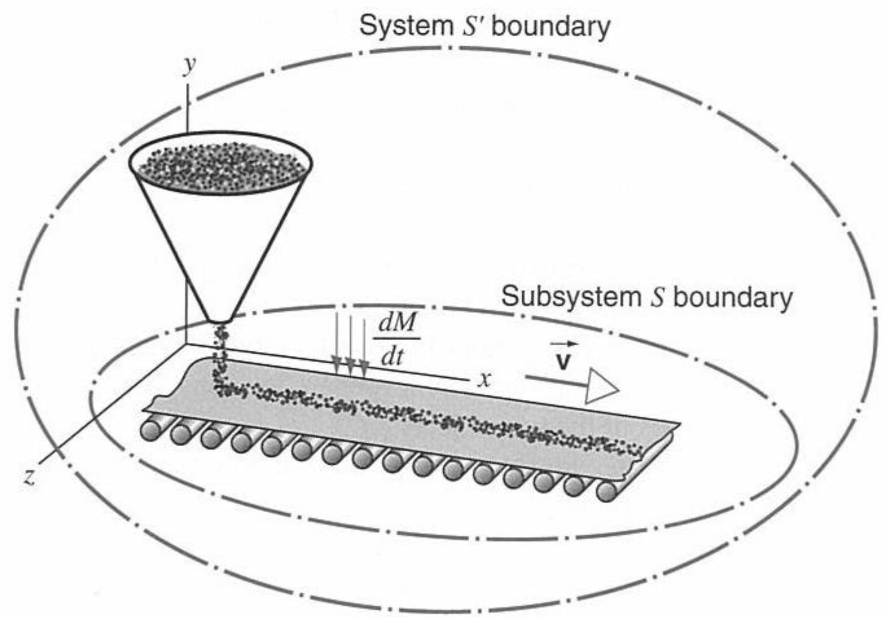


FIGURE 7-20. Sample Problem 7-10. Sand drops from a hopper at a rate dM/dt onto a conveyor belt moving with constant velocity $\vec{v}$ in the reference frame of the laboratory. The hopper is at rest in the reference frame of the laboratory.

Rate of increase in the total momentum:

$$\frac{d}{dt}(mv) = v\frac{dm}{dt} = vD$$

Using Newton's second law,

$$F = vD$$

Power of the motor:

$$P = Fv = v^2 D$$

Rate of increase of the kinetic energy:

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{v^2}{2}\frac{dm}{dt} = \frac{1}{2}v^2 D$$

The missing power is used to do work against friction. To see this, consider the sand falling onto the belt in time Δt.

Mass of the sand falling onto the conveyor belt

$$\Delta m = D\Delta t$$

.

Frictional force acting on them $\Delta f = \mu(\Delta m)g$

By Newton's second law,

$$\mu(\Delta m)g = (\Delta m)a$$

$$\Rightarrow$$

$$a = \mu g$$

Distance traveled by the sand before it reaches the same velocity as the conveyor belt

$$x = \frac{v^2}{2a} = \frac{v^2}{2\mu g}$$

Work done on the sand by the frictional force

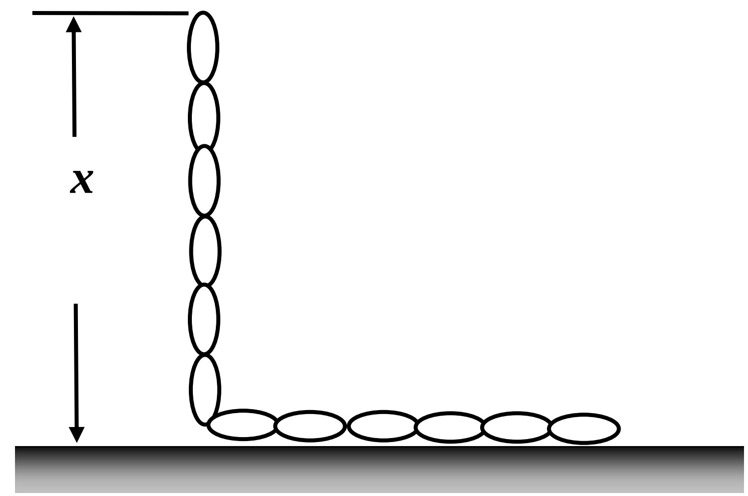
$$\Delta W = (\Delta f)x = \mu(\Delta m)g\left(\frac{v^2}{2\mu g}\right) = \frac{v^2}{2}\Delta m$$

Rate of work done by motor against friction

$$P_f = \frac{v^2}{2}\frac{\Delta m}{\Delta t} = \frac{1}{2}v^2 D$$

Example - Falling Chain

A chain of length L and mass M is held vertically so that the bottom of the chain just touches the horizontal table top. If the upper end of the chain is released, determine the force on the table top when the length of the chain above the table is x, while it is falling.



Weight of the chain resting on the table:

$$W = \left(\frac{L-x}{L}\right)Mg$$

Consider a segment of the chain falling onto the table in time Δt. Velocity of this segment is given by

$$v^2 = 2g(L-x)$$

Mass of this segment

$$\Delta m = \frac{M}{L} v\Delta t$$

Momentum of this segment

$$\Delta p = \left(\frac{M}{L} v\Delta t\right)v = \frac{M}{L} 2g(L-x)\Delta t$$

This momentum is reduced to 0 due to the reaction of the table. Using Newton's second law, force exerted by the table

$$F = \frac{\Delta p}{\Delta t} = 2Mg\left(\frac{L-x}{L}\right)$$

Thus, the total force acting on the table is the weight of the chain already on the table + impulsive force for the segment of the falling chain hit on the table

$$= W + F = 3Mg\left(\frac{L-x}{L}\right)$$

Example - Falling Raindrop

A spherical raindrop falling through fog or mist accumulates mass due to condensation at a rate equal to kAv, where A and v are its cross-sectional area and velocity respectively. Find the acceleration of the raindrop in terms of its radius r and velocity v, density ρ, gravitational acceleration g, and rate constant k. It is assumed that the raindrop starts from rest and has an infinitely small size.

$$\frac{dm}{dt} = kAv$$

Rate of change of momentum of the raindrop:

$$\frac{dp}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt} = ma + kAv^2$$

Using Newton's second law,

$$ma + kAv^2 = mg$$

$$a = g - \frac{kAv^2}{m}$$

Since $m = \rho \frac{4}{3}\pi r^3$ and $A = \pi r^2$

$$a = g - \frac{kAv^2}{m} = g - \frac{3k}{4\rho}\frac{v^2}{r}$$