Rotational motion

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Physics Enhancement Programme for Gifted Students

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Contents

- Operation of vectors
- Angular displacement, velocity and acceleration
- Torque
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Vectors

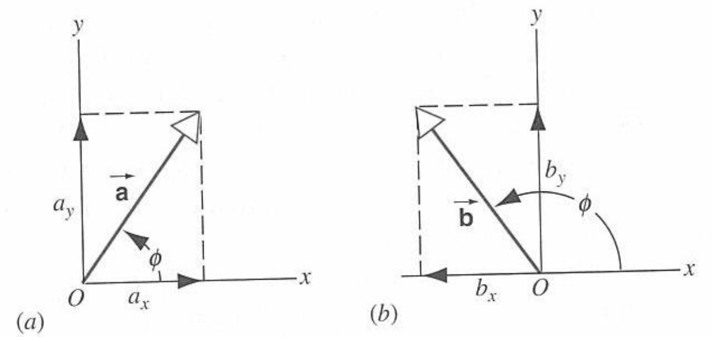


FIGURE 2-2. (*a*) The vector $\vec{\mathbf{a}}$ has component a_x in the x direction and component a_y in the y direction. (*b*) The vector $\vec{\mathbf{b}}$ has a negative x component.

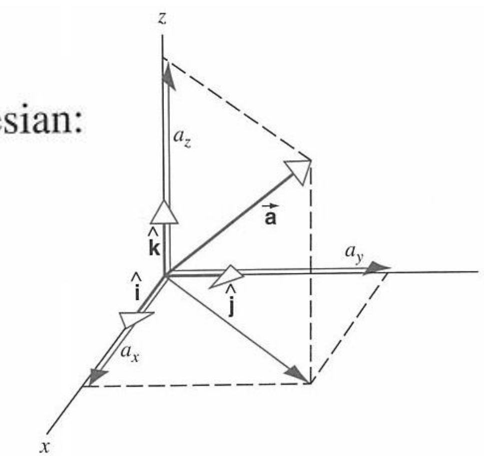
$$a_x = a\cos\phi \quad \text{and} \quad a_y = a\sin\phi.$$

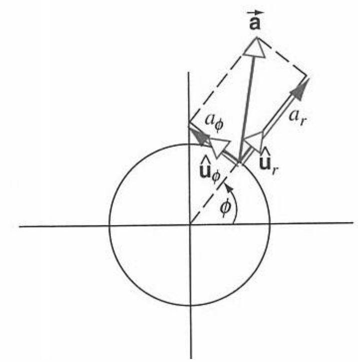
$$a = \sqrt{a_x^2 + a_y^2} \quad \text{and} \quad \tan\phi = a_y/a_x.$$

$$\vec{\mathbf{a}} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}}.$$

Three-dimensional Cartesian:

$$\vec{\mathbf{a}} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$$





Two-dimensional polar:

$$\vec{\mathbf{a}} = a_r\hat{\mathbf{u}}_r + a_\phi\hat{\mathbf{u}}_\phi$$

Adding vectors

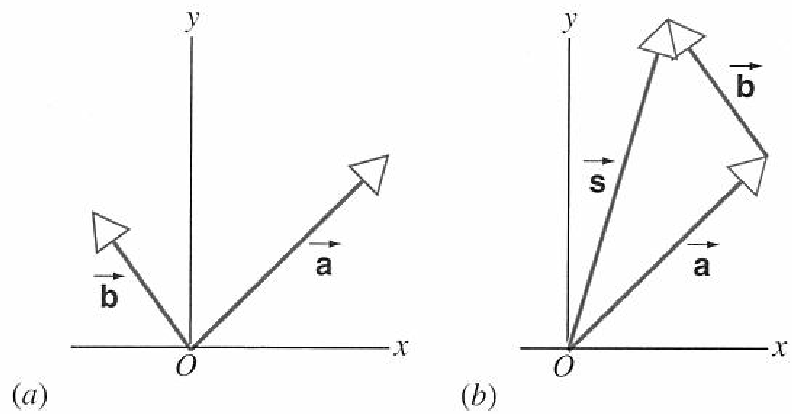


FIGURE 2-4. (*a*) Vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. (*b*) To find the sum $\vec{\mathbf{s}}$ of vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, we slide $\vec{\mathbf{b}}$ without changing its magnitude or direction until its tail is on the head of $\vec{\mathbf{a}}$. Then the vector $\vec{\mathbf{s}} = \vec{\mathbf{a}} + \vec{\mathbf{b}}$ is drawn from the tail of $\vec{\mathbf{a}}$ to the head of $\vec{\mathbf{b}}$.

We can add the vectors graphically

Another way to add vectors is to add their components. That is, $\vec{\mathbf{s}} = \vec{\mathbf{a}} + \vec{\mathbf{b}}$ means

$$s_x\hat{\mathbf{i}} + s_y\hat{\mathbf{j}} = (a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}}) + (b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}})$$
$$= (a_x + b_x)\hat{\mathbf{i}} + (a_y + b_y)\hat{\mathbf{j}}.$$

SAMPLE PROBLEM 2-3. Three vectors in the xy plane are expressed with respect to the coordinate system as

$$\vec{a} = 4.3\hat{i} - 1.7\hat{j},$$

$$\vec{b} = -2.9\hat{i} + 2.2\hat{j},$$

and

$$\vec{c} = -3.6\hat{j},$$

in which the components are given in arbitrary units. Find the vector $\vec{s}$, which is the sum of these vectors.

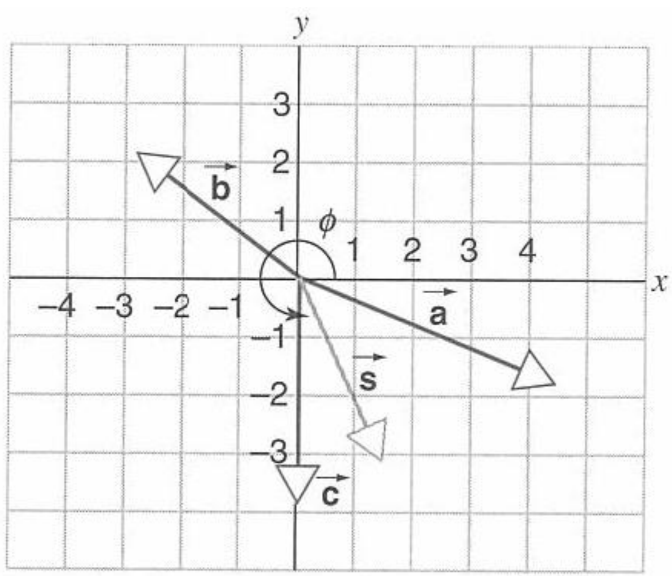


FIGURE 2-10. Sample Problem 2-3.

Solution Generalizing Eqs. 2-4 to the case of three vectors, we have

$$s_x = a_x + b_x + c_x = 4.3 - 2.9 + 0 = 1.4,$$

and

$$s_y = a_y + b_y + c_y = -1.7 + 2.2 - 3.6 = -3.1.$$

Thus

$$\vec{s} = s_x\hat{i} + s_y\hat{j} = 1.4\hat{i} - 3.1\hat{j}.$$

Figure 2-10 shows the four vectors. From Eqs. 2-2 we can calculate that the magnitude of $\vec{s}$ is 3.4 and that the angle ϕ that $\vec{s}$ makes with the positive x axis, measured counterclockwise from that axis, is

$$\phi = \tan^{-1}(-3.1/1.4) = 294°.$$

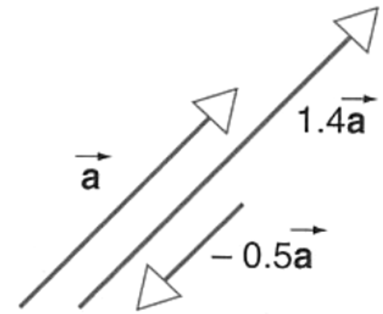
H-4 MULTIPLICATION OF VECTORS

Multiplication of a vector by a scalar:

$\vec{\mathbf{b}} = c\vec{\mathbf{a}}$

$b_x = ca_x \qquad b_y = ca_y$

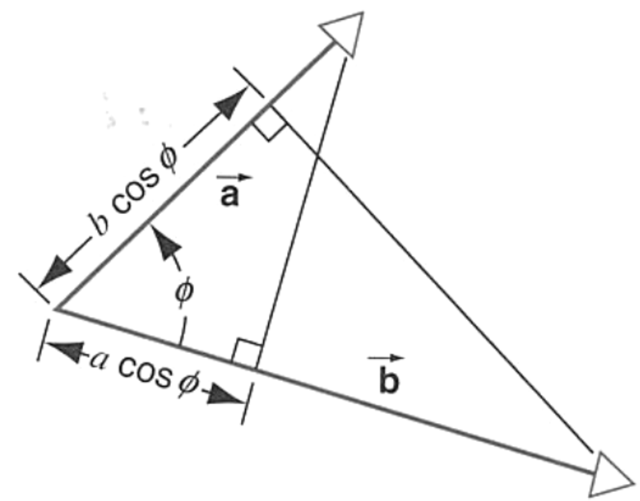
$b = |c|\,a$



Dot product (or scalar product) of two vectors:

$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = ab\cos\phi = a(b\cos\phi) = b(a\cos\phi)$

$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{b}} \cdot \vec{\mathbf{a}}$



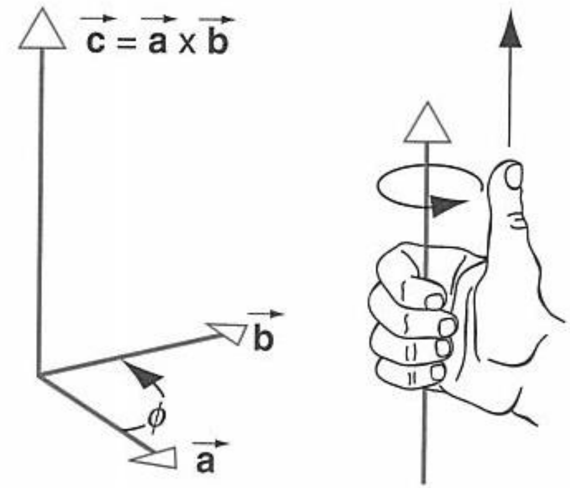
$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$

$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0$

$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = a_x b_x + a_y b_y + a_z b_z$

$\vec{\mathbf{a}} \cdot \vec{\mathbf{a}} = a^2 = a_x^2 + a_y^2 + a_z^2$

Cross product (or vector product) of two vectors:



$$\vec{\mathbf{c}} = \vec{\mathbf{a}} \times \vec{\mathbf{b}}$$

$$|\vec{\mathbf{c}}| = |\vec{\mathbf{a}} \times \vec{\mathbf{b}}| = ab \sin \phi$$

Direction of $\vec{\mathbf{c}}$ is perpendicular to the plane of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, determined by the right-hand rule.

$$\vec{\mathbf{b}} \times \vec{\mathbf{a}} = -\vec{\mathbf{a}} \times \vec{\mathbf{b}}$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \qquad \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}} \qquad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$$

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = (a_y b_z - a_z b_y)\hat{\mathbf{i}} + (a_z b_x - a_x b_z)\hat{\mathbf{j}} + (a_x b_y - a_y b_x)\hat{\mathbf{k}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) + (\vec{\mathbf{a}} \times \vec{\mathbf{c}})$$

$$(s\vec{\mathbf{a}}) \times \vec{\mathbf{b}} = \vec{\mathbf{a}} \times (s\vec{\mathbf{b}}) = s(\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \qquad (s = \text{a scalar}).$$

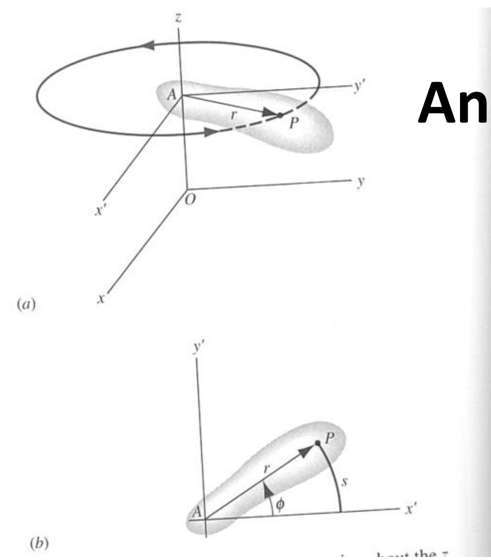
$$\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = \vec{\mathbf{b}} \cdot (\vec{\mathbf{c}} \times \vec{\mathbf{a}}) = \vec{\mathbf{c}} \cdot (\vec{\mathbf{a}} \times \vec{\mathbf{b}})$$

$$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\vec{\mathbf{b}} - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}}$$

Angular Displacement and angular velocity

Angular Displacement

$$\Delta\theta = \theta_2 - \theta_1.$$



Angular Velocity

Average angular velocity

$$\overline{\omega} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}.$$

Instantaneous angular velocity

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}.$$

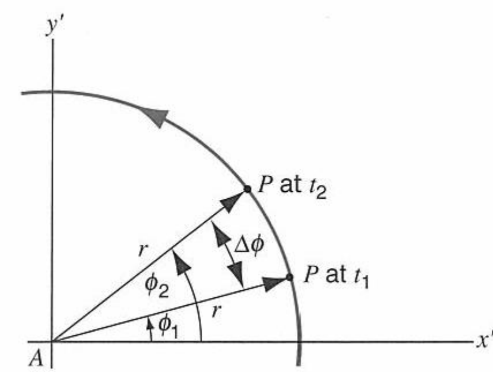


FIGURE 8-4. The reference line AP of Fig. 8-3b is at the angular coordinate ϕ_1 at time t_1 and at the angular coordinate ϕ_2 at time t_2. In the time interval $\Delta t = t_2 - t_1$, the net angular displacement is $\Delta\phi = \phi_2 - \phi_1$.

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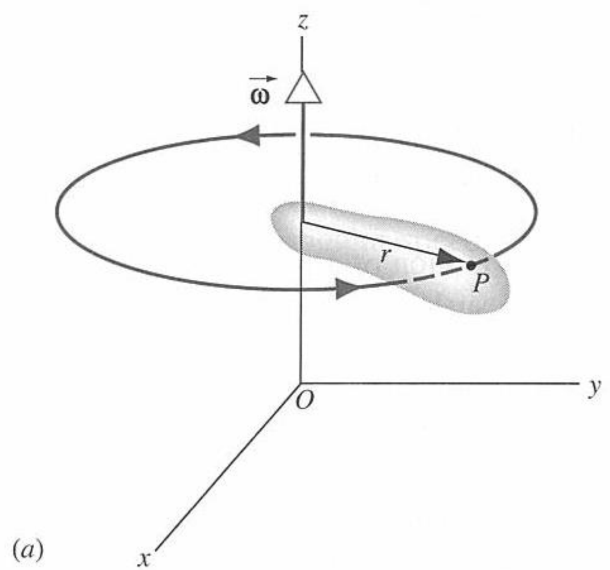
The direction of the angular velocity vector



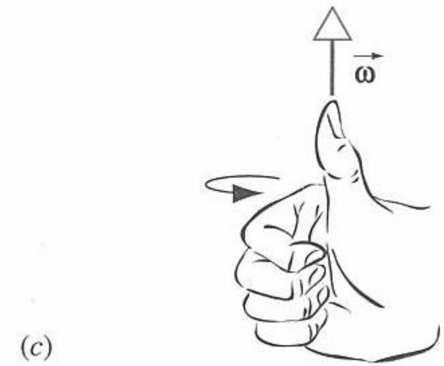


FIGURE 8-6. The angular velocity vector of (*a*) a rotating rigid body and (*b*) a rotating particle, both taken about a fixed axis. (*c*) The right-hand rule determines the direction of the angular velocity vector.

The direction of the vector $\vec{\omega}$ points along the axis of rotation, according to the **right-hand rule**.

Angular Acceleration

Average angular acceleration

$$\bar{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}.$$

Instantaneous angular acceleration

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}.$$

If the rotation with Constant Acceleration, then we have

$$\omega = \omega_0 + \alpha t,$$
$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2,$$
$$\omega^2 = \omega_0^2 + 2\alpha\theta.$$

Relationship between the linear and angular variables

The arc length

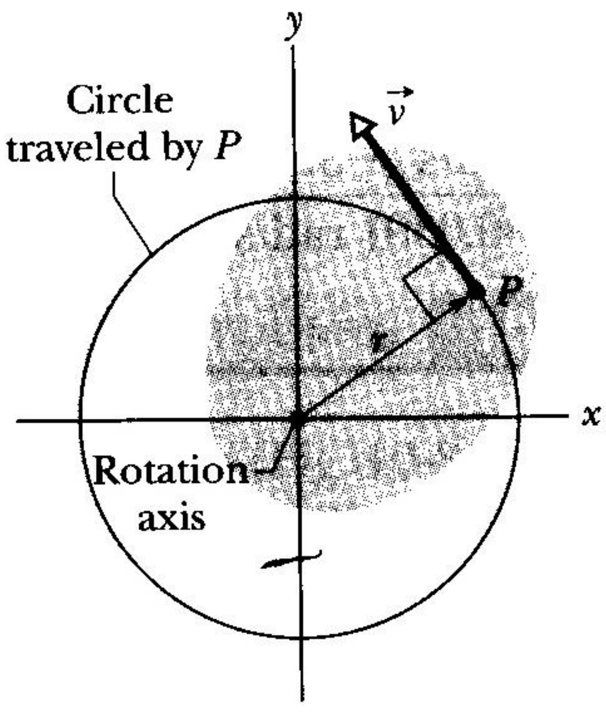
$$s = r\theta$$

The velocity

$$v = r\frac{d\theta}{dt} = r\omega$$

Linear velocity

Angular velocity



(a)

What is the direction of the vector for the angular velocity in this case?

Relationship between the linear and angular variables

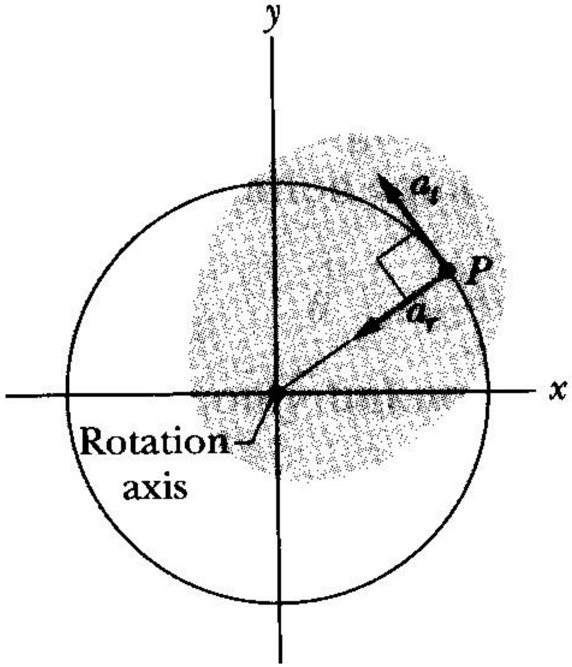
We can separate the acceleration into two components:

Tangential component:

$$a_t = \alpha r$$

Radial component:

$$a_r = \frac{v^2}{r} = \omega^2 r.$$



Kinetic Energy of Rotation

Consider the kinetic energy of a rotating rigid body:

$$K = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \cdots = \sum_i \frac{1}{2} m_i v_i^2.$$

Since $v = \omega r$, and ω is the same for all particles, we have

$$K = \sum_i \frac{1}{2} m_i (\omega r_i)^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2.$$

$$\sum_i m_i r_i^2$$

is called the **rotational inertia**. It tells us how the mass of the rotating body is distributed about its axis of rotation. In summary,

$$I = \sum_i m_i r_i^2 \qquad \text{and} \qquad K = \frac{1}{2} I \omega^2.$$

SAMPLE PROBLEM 9-4. The object shown in Fig. 9-11 consists of two particles, of masses m_1 and m_2, connected by a light rigid rod of length L. (*a*) Neglecting the mass of the rod, find the rotational inertia I of this system for rotations of this object about an axis perpendicular to the rod and a distance x from m_1. (*b*) Show that I is a minimum when $x = x_{cm}$.

Solution (*a*) From Eq. 9-9, we obtain

$$I = m_1x^2 + m_2(L - x)^2.$$

(*b*) We find the minimum value of I by setting dI/dx equal to 0:

$$\frac{dI}{dx} = 2m_1x + 2m_2(L - x)(-1) = 0.$$

Solving, we find the value of x at which this minimum occurs:

$$x = \frac{m_2L}{m_1 + m_2}.$$

This is identical to the expression for the center of mass of the object, and thus the rotational inertia does reach its minimum value at $x = x_{cm}$. This is consistent with the parallel-axis theorem, which requires that I_{cm} be the smallest rotational inertia among parallel axes.

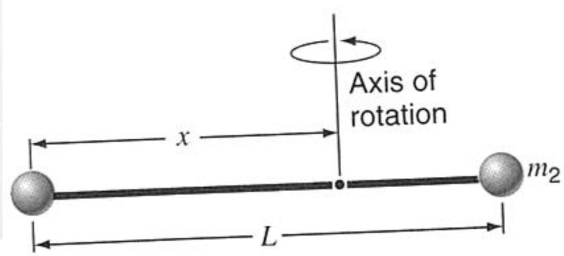


FIGURE 9-11. Sample Problem 9-4. The object is to be rotated about an axis perpendicular to the connecting rod and a distance x from m_1.

SAMPLE PROBLEM 9-3. For the three-particle system of Fig. 9-9, find the rotational inertia about an axis perpendicular to the xy plane and passing through the center of mass of the system.

Solution First we must locate the center of mass:

$$x_{cm} = \frac{\sum m_n x_n}{\sum m_n}$$

$$= \frac{(2.3\text{ kg})(0\text{ m}) + (3.2\text{ kg})(0\text{ m}) + (1.5\text{ kg})(4.0\text{ m})}{2.3\text{ kg} + 3.2\text{ kg} + 1.5\text{ kg}}$$

$$= 0.86\text{ m},$$

$$y_{cm} = \frac{\sum m_n y_n}{\sum m_n}$$

$$= \frac{(2.3\text{ kg})(0\text{ m}) + (3.2\text{ kg})(3.0\text{ m}) + (1.5\text{ kg})(0\text{ m})}{2.3\text{ kg} + 3.2\text{ kg} + 1.5\text{ kg}}$$

$$= 1.37\text{ m}.$$

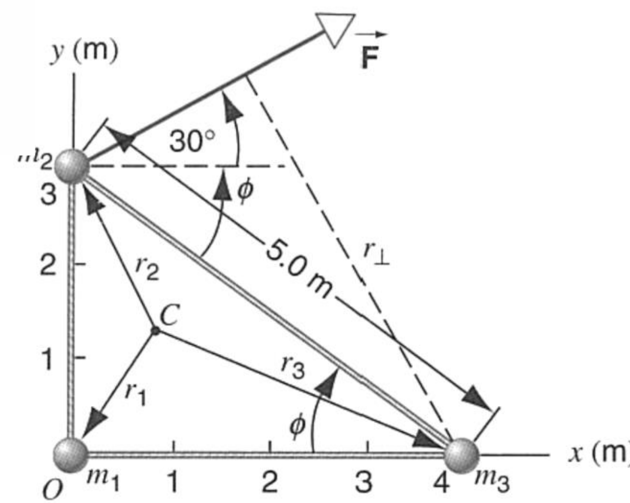


FIGURE 9-9. Sample Problem 9-2. Point C marks the center of mass of the system consisting of the three particles.

The squared distances from the center of mass to each of the particles are

$$r_1^2 = x_{cm}^2 + y_{cm}^2 = (0.86\ \text{m})^2 + (1.37\ \text{m})^2 = 2.62\ \text{m}^2$$

$$r_2^2 = x_{cm}^2 + (y_2 - y_{cm})^2 = (0.86\ \text{m})^2 + (3.0\ \text{m} - 1.37\ \text{m})^2 = 3.40\ \text{m}^2,$$

$$r_3^2 = (x_3 - x_{cm})^2 + y_{cm}^2 = (4.0\ \text{m} - 0.86\ \text{m})^2 + (1.37\ \text{m})^2 = 11.74\ \text{m}^2.$$

The rotational inertia then follows directly from Eq. 9-10:

$$I_{cm} = \sum m_n r_n^2 = (2.3\ \text{kg})(2.62\ \text{m}^2) + (3.2\ \text{kg})(3.40\ \text{m}^2) + (1.5\ \text{kg})(11.74\ \text{m}^2) = 35\ \text{kg}\cdot\text{m}^2.$$

Note that the rotational inertia about the center of mass is the smallest of those we have calculated for this system (compare the values in Sample Problem 9-2). This is a general result, which we shall prove next. It is easier to rotate a body about an axis through the center of mass than about any other parallel axis.

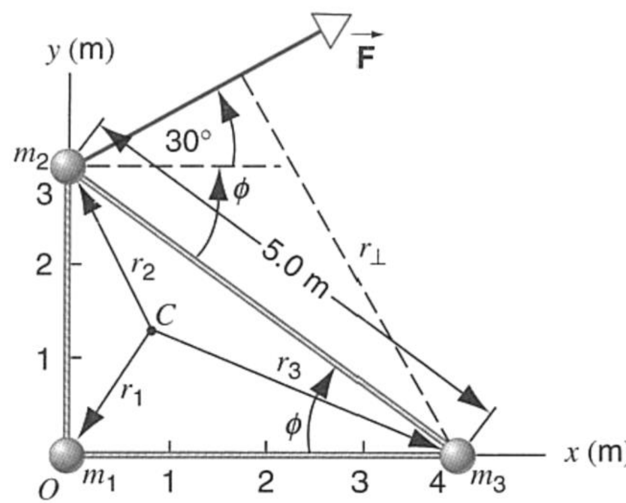


FIGURE 9-9. Sample Problem 9-2. Point C marks the center of mass of the system consisting of the three particles.

Rotational inertia

For continuous bodies, $$I = \int r^2 dm.$$

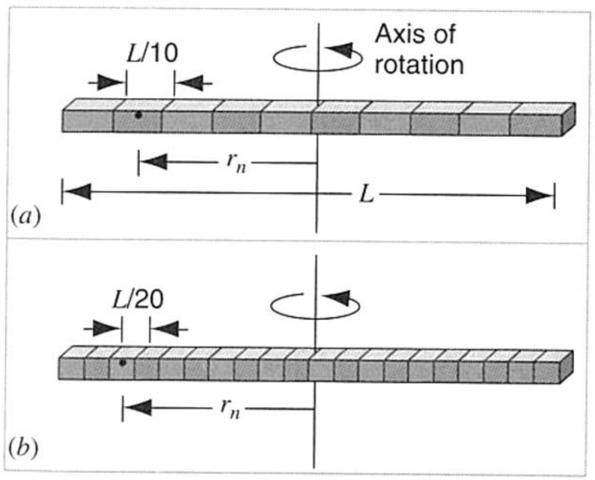


FIGURE 9-12. (*a*) The rotational inertia of a solid rod of length L, rotated about an axis through its center and perpendicular to its length, can be approximately computed by dividing the rod into 10 equal pieces, each of length $L/10$. Each piece is treated as a point mass a distance r_n from the axis. (*b*) A more accurate approximation to the rotational inertia of the rod is obtained by dividing it into 20 pieces.

$$I = \lim_{\delta m_n \to 0} \sum r_n^2 \, \delta m_n ,$$

and in the usual way the sum becomes an integral in the limit:

$$I = \int r^2 \, dm. \qquad (9\text{-}15)$$

Rotational inertia

For continuous bodies, $I = \int r^2 dm.$

Ring

$I = MR^2$ (axis)

$I = \frac{1}{2}MR^2$ (diameter)

Cylinder

$I = \frac{1}{2}MR^2$

Rod

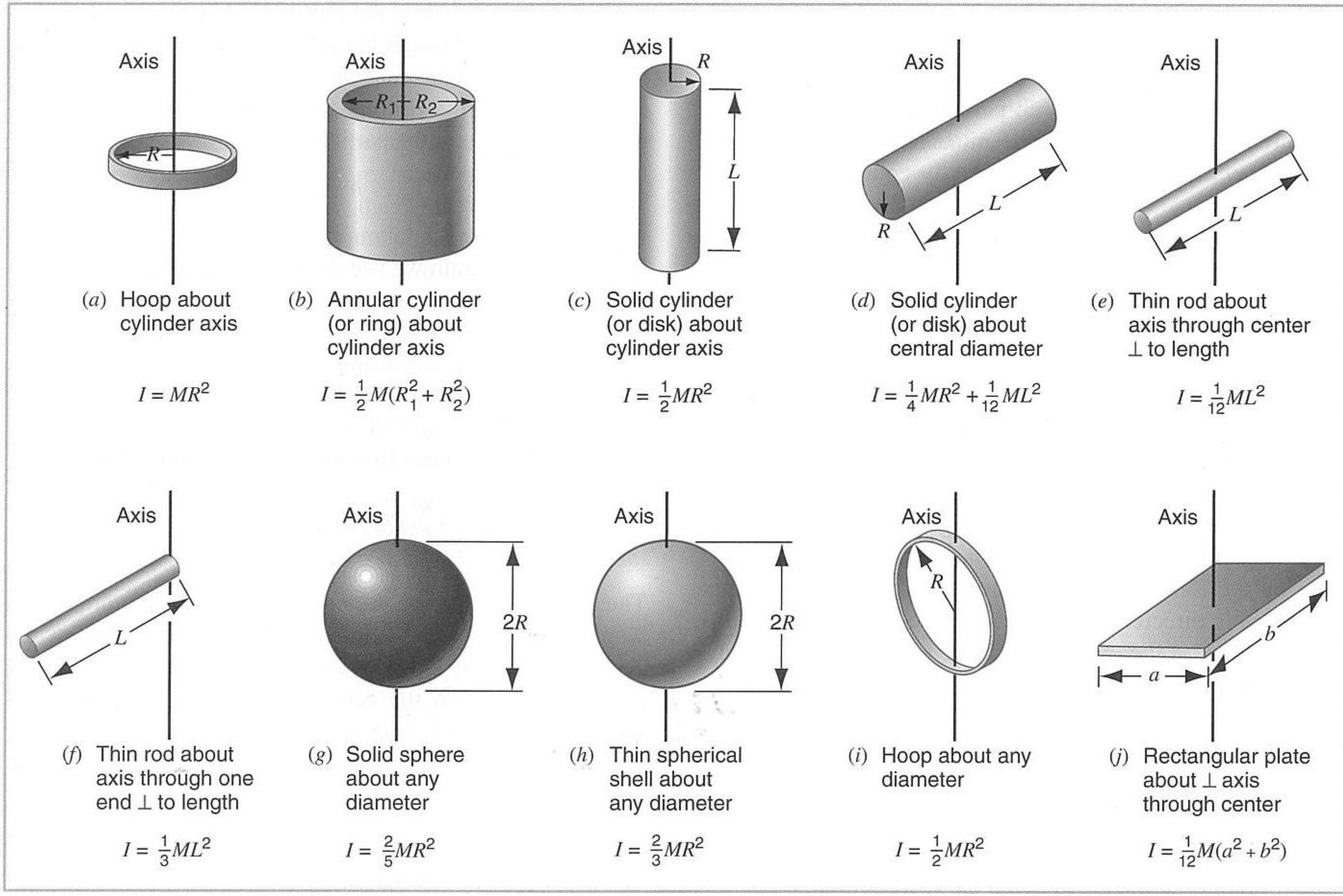
$I = \frac{1}{12}ML^2$ (centre)

$I = \frac{1}{3}ML^2$ (end)

Sphere

$I = \frac{2}{5}MR^2$

Rotational inertia



Parallel Axis Theorem

$$I = I_{\text{cm}} + Mh^2$$

The rotational inertia of a body about any axis is equal to the rotational inertia (= Mh^2) it would have about that axis if all its mass were concentrated at its centre of mass, plus its rotational inertia (= I_{cm}) about a parallel axis through its centre of mass.

Proof

$$I = \int r^2 dm = \int \left[(x-a)^2 + (y-b)^2 \right] dm,$$

which can be written as

$$I = \int (x^2 + y^2) dm - 2a \int x dm - 2b \int y dm + \int (a^2 + b^2) dm.$$

$$I = \int (x^2 + y^2)dm \quad - 2a\int xdm \quad - 2b\int ydm \quad + \int (a^2 + b^2)dm.$$

In the first term, $x^2 + y^2 = R^2$. Hence the first term becomes

$$\int (x^2 + y^2)dm \quad = \int R^2 dm \quad = I_{cm}.$$

In the second and third terms, the position of the centre of mass gives

$$x_{\text{cm}} = \frac{1}{M}\int xdm \quad = 0 \qquad \text{and} \qquad y_{\text{cm}} = \frac{1}{M}\int ydm \quad = 0.$$

Hence these terms vanish.

In the last term, $a^2 + b^2 = h^2$. Hence the last term becomes

$$\int (a^2 + b^2)dm \quad = \int h^2 dm \quad = Mh^2.$$

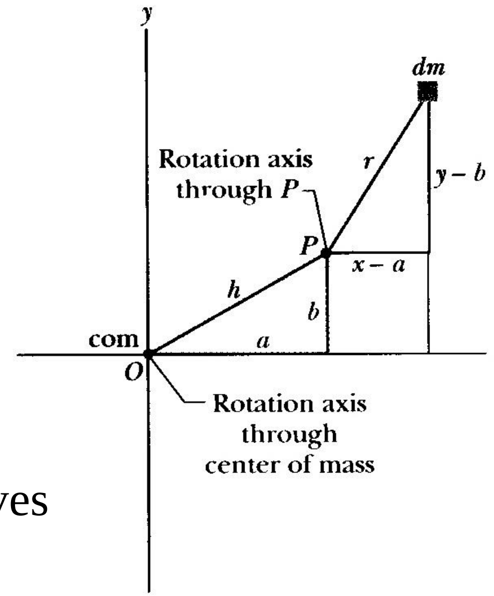


FIG. 10-12 A rigid body in cross section, with its center of mass at O. The parallel-axis theorem (Eq. 10-36) relates the rotational inertia of the body about an axis through O to that about a parallel axis through a point such as P, a distance h from the body's center of mass. Both axes are perpendicular to the plane of the figure.

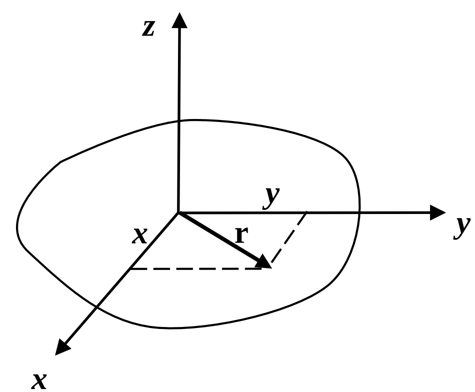
Perpendicular Axis Theorem

The sum of the rotational inertia of a plane lamina about any two perpendicular axes in the plane of the lamina is equal to the rotational inertia about an axis that passes through the point of intersection and perpendicular to the plane of the lamina.

$$I_z = \int (x^2 + y^2) dm$$

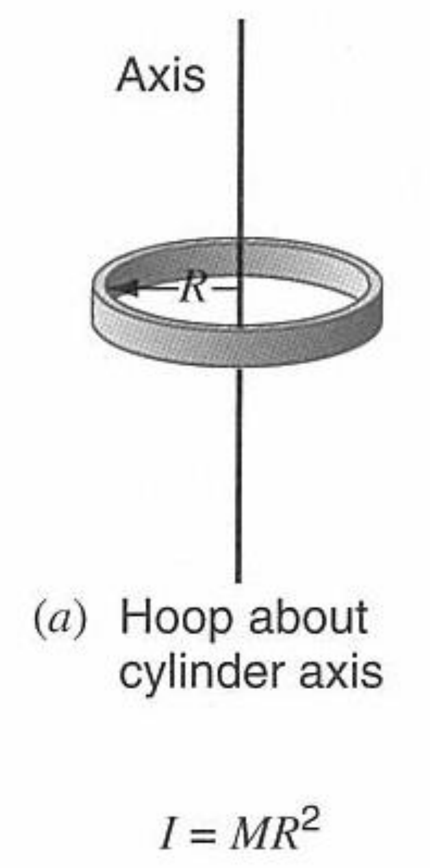
$$= \int x^2 dm + \int y^2 dm$$

$$= I_y + I_x$$



Examples

Given that the rotational inertia of a hoop about its central axis is Ma^2, what is the rotational inertia of a hoop about a diameter?



By symmetry, $I_x = I_y$

Using the perpendicular axis theorem, $I_x + I_y = I_z$.

Since $I_z = Ma^2$,

$$I_x = \frac{1}{2} Ma^2$$

Examples

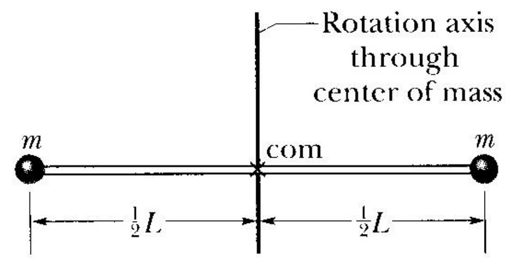
A rigid body consists of two particles of mass *m* connected by a rod of length *L* and negligible mass.

(a) What is the rotational inertia I_{cm} about an axis through the center of mass perpendicular to the rod?

(b) What is the rotational inertia *I* of the body about an axis through the left end of the rod and parallel to the first axis?

(a)

$$I = m\left(\frac{1}{2}L\right)^2 + m\left(\frac{1}{2}L\right)^2 = \frac{1}{2}mL^2.$$

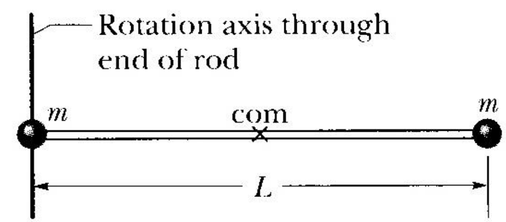


(a)

(b)

Method 1: Direct calculation:

$$I = m(0)^2 + mL^2 = mL^2.$$



(b)

Method 2: Parallel axis theorem:

$$I = I_{cm} + Mh^2 = \frac{1}{2}mL^2 + (2m)\left(\frac{L}{2}\right)^2$$

$= mL^2$

Examples

Consider a thin, uniform rod of mass *M* and length *L*.

(a) What is the rotational inertia about an axis perpendicular to the rod, through its center of mass?

(b) What is the rotational inertia of the rod about an axis perpendicular to the rod through one end?

(a)

$$I = \int r^2 dm$$

$$dm = \frac{m}{L} dx$$

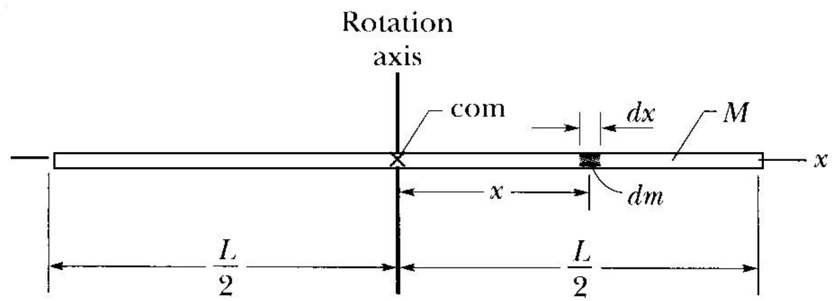


FIG. 10-14 A uniform rod of length *L* and mass *M*. An element of mass *dm* and length *dx* is represented.

$$I = \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \left(\frac{M}{L} dx \right) = \frac{M}{3L} \left[x^3 \right]_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{M}{3L} \left[\left(\frac{L}{2} \right)^3 - \left(-\frac{L}{2} \right)^3 \right] = \frac{ML^2}{12}$$

(b) Using the parallel axis theorem,

$$I = I_{\text{cm}} + Mh^2 = \frac{1}{12} ML^2 + M \left(\frac{L}{2} \right)^2 = \frac{1}{3} ML^2$$

Radius of Gyration

The radius of gyration is that distance from the axis of rotation where we assume all the mass of the body to be concentrated. It is given by

$$I = Mk^2$$

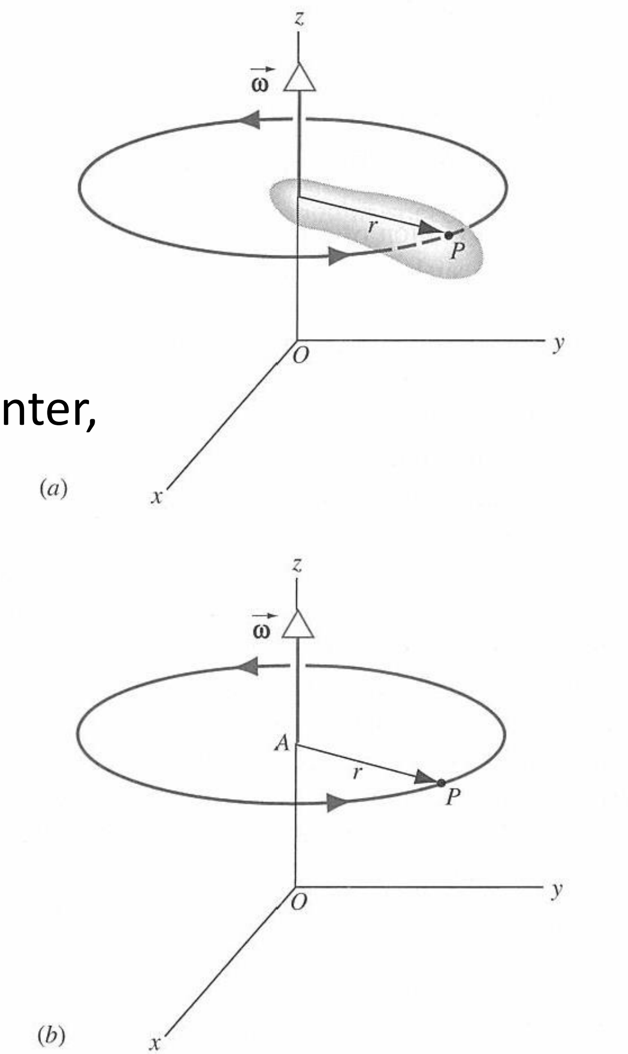
$$k = \sqrt{\frac{I}{M}}$$

For example, for a thin rod rotating about its center,

$$I = \frac{1}{12}ML^2$$

$$\Rightarrow$$

$$k = \frac{L}{\sqrt{12}}$$



Torque

The ability of force $\vec{F}$ to rotate the body depends on:

(1)the magnitude of the tangential component $F_t = F\sin\phi$,

(2)the distance between the point of application and the axis of rotation.

*Define the **torque** as* $\tau = rF\sin\phi.$

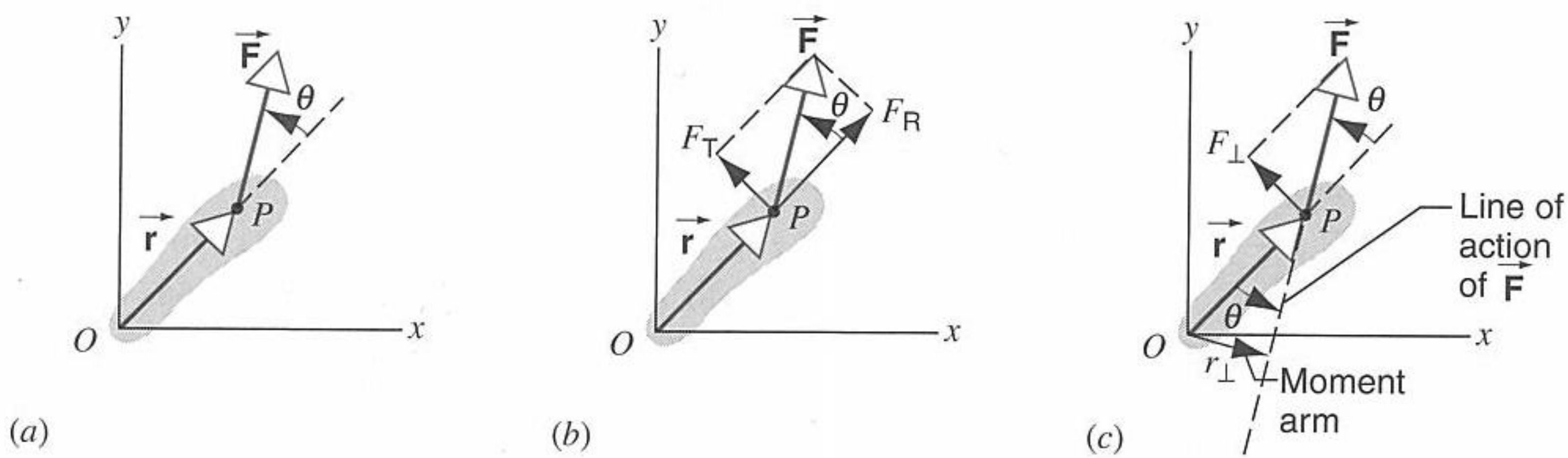


FIGURE 9-3. (*a*) A cross-sectional slice in the *xy* plane of the body shown in Fig. 9-2. The force $\vec{\mathbf{F}}$ is in the *xy* plane. (*b*) The force $\vec{\mathbf{F}}$ is resolved into its radial (F_R) and tangential (F_T) components. (*c*) The component of $\vec{\mathbf{F}}$ perpendicular to $\vec{\mathbf{r}}$ is $F_\perp$ (also identified as the tangential component F_T), and the component of $\vec{\mathbf{r}}$ perpendicular to $\vec{\mathbf{F}}$ (or to its line of action) is $r_\perp$.

It can be considered as either $rF_{\perp}$ or $r_{\perp}F$.

Terms:
line of action
moment arm

τ is positive if it tends to rotate the body counterclockwise.

It is negative if it tends to rotate the body clockwise.
Considering the vector direction,

$$\vec{\tau} = \vec{r} \times \vec{F}.$$

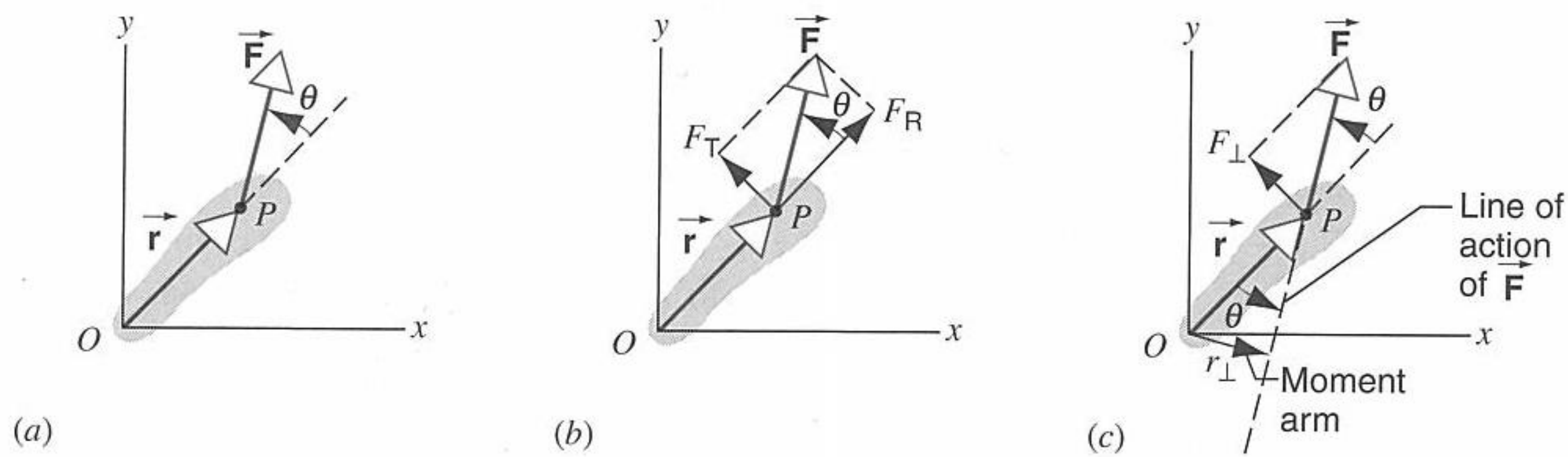


FIGURE 9-3. (*a*) A cross-sectional slice in the *xy* plane of the body shown in Fig. 9-2. The force $\vec{F}$ is in the *xy* plane. (*b*) The force $\vec{F}$ is resolved into its radial (F_R) and tangential (F_T) components. (*c*) The component of $\vec{F}$ perpendicular to $\vec{r}$ is $F_{\perp}$ (also identified as the tangential component F_T), and the component of $\vec{r}$ perpendicular to $\vec{F}$ (or to its line of action) is $r_{\perp}$.

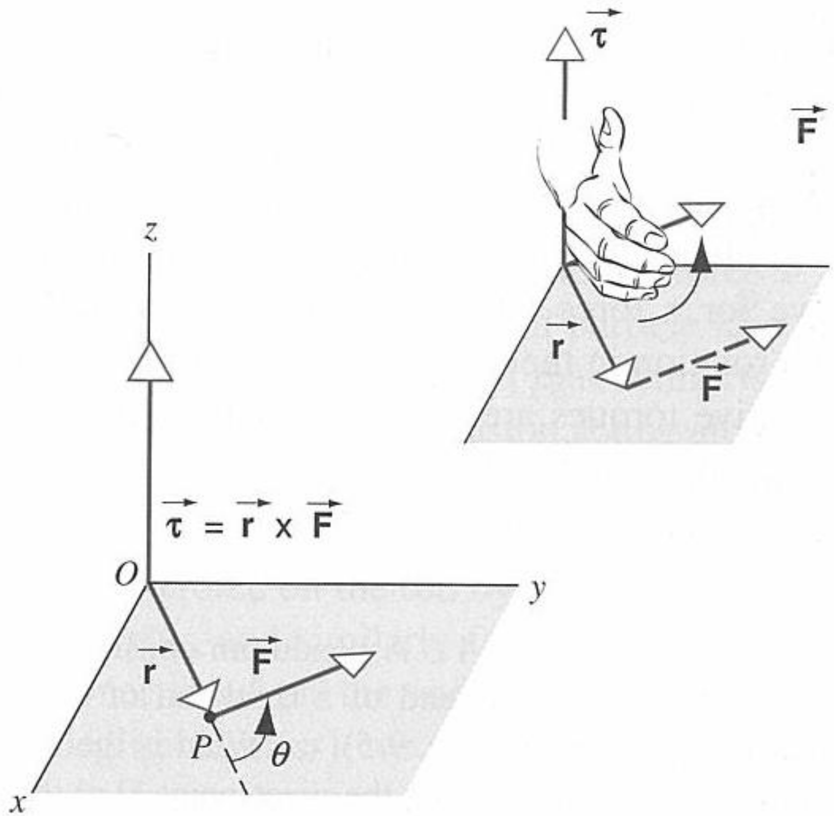


FIGURE 9-4. A force $\vec{\mathbf{F}}$ acts at point P in a rigid body (not shown). This force exerts a torque $\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$ on the body with respect to the origin O. The torque vector points in the direction of increasing z; it could be drawn anywhere we choose, as long as it is parallel to the z axis. The inset shows how the right-hand rule is used to find the direction of the torque. For convenience we can slide the force vector laterally, without changing its direction, until the tail of $\vec{\mathbf{F}}$ joins the tail of $\vec{\mathbf{r}}$.

Newton’s Second Law for Rotation

Newton’s second law:

$$F_t = ma_t.$$

Torque:

$$\tau = F_t r = ma_t r.$$

Since $a_t = \alpha r$, *we obtain*

$$\tau = m(\alpha r)r = (mr^2)\alpha.$$

Conclusion:

$$\tau = I\alpha.$$

If there are more than one forces,

$$\sum \tau = I\alpha.$$

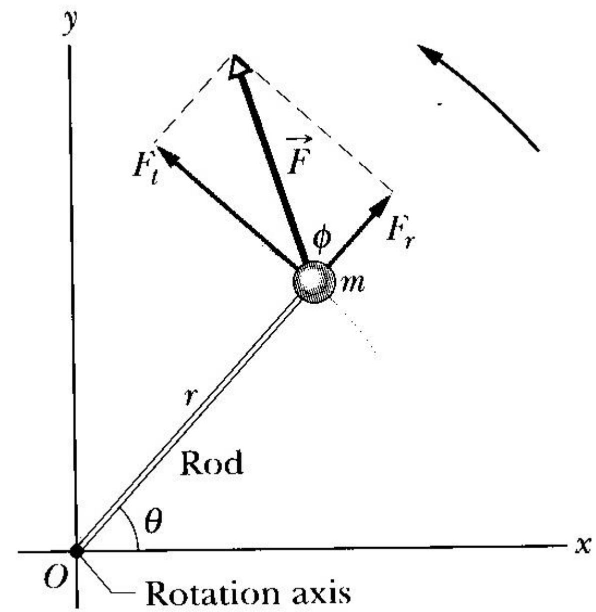
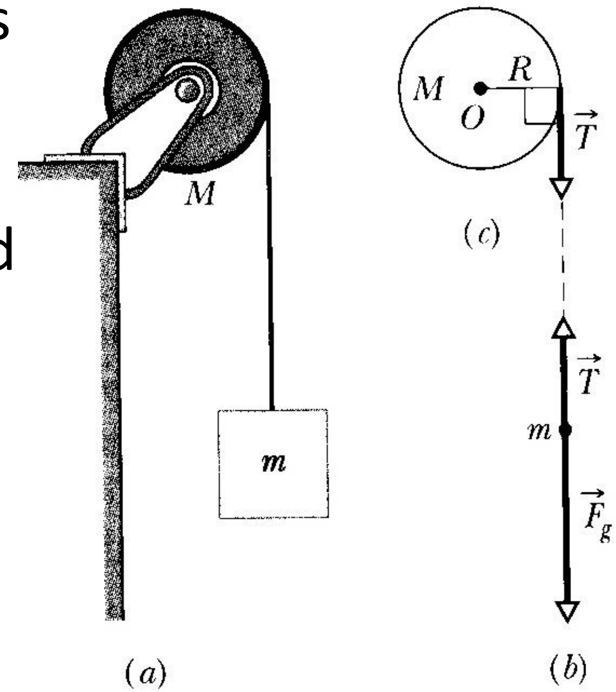


FIG. 10-17 A simple rigid body, free to rotate about an axis through O, consists of a particle of mass m fastened to the end of a rod of length r and negligible mass. An applied force $\vec{F}$ causes the body to rotate.

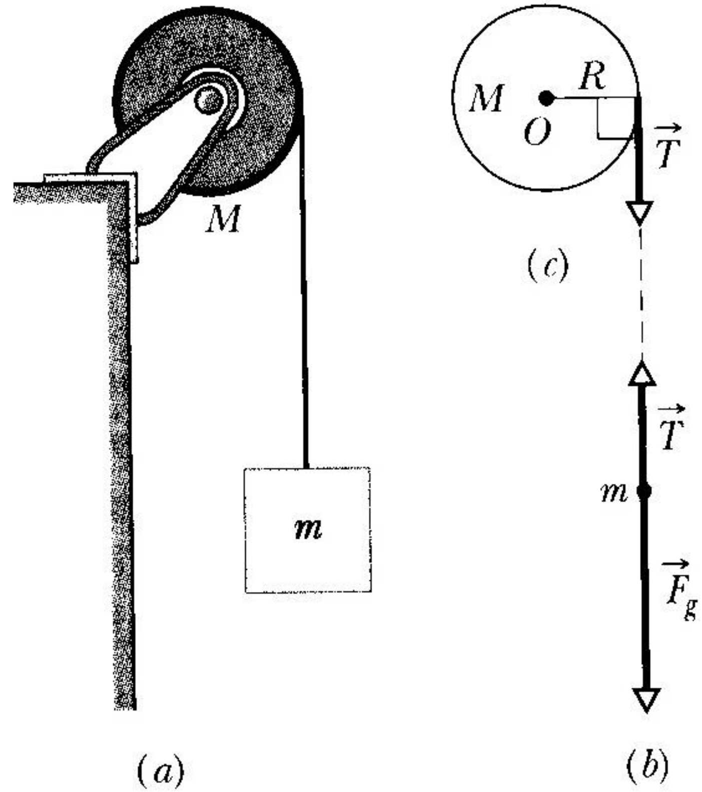
Examples

A uniform disk of mass M = 2.5 kg and radius R = 20 cm is mounted on a fixed horizontal axle. A block whose mass m is 1.2 kg hangs from a massless cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the angular acceleration of the disk, and the tension in the cord. The cord does not slip, and there is no friction at the axle.



Newton's law for the hanging block
(We define downward is positive):

$$mg - T = ma \qquad (1)$$



Newton's law for the rotating disk
(We define clockwise is positive):

$$TR = \frac{1}{2}MR^2\alpha \qquad (2)$$

Since $a = R\alpha$,

From (2): $T = \frac{1}{2}Ma$

From (1): $mg - \frac{1}{2}Ma = ma$

$$mg = \left(m + \frac{M}{2}\right)a$$

$$a = \frac{2mg}{M + 2m} = \frac{(2)(1.2)(9.8)}{2.5 + (2)(1.2)} = 4.8\ \mathrm{ms^{-2}}$$

$$\alpha = \frac{a}{R} = \frac{4.8}{0.2} = 24\ \mathrm{rad\ s^{-2}}$$

$$T = \frac{1}{2}Ma = \left(\frac{1}{2}\right)(2.5)(4.8) = 6\ \mathrm{N}$$

Examples

(Physics of judo) To throw an 80 kg opponent with a basic judo hip throw, you intend to pull his uniform with a force $\vec{F}$ and a moment arm d_1 = 0.30 m from a pivot point (rotation axis) on your right hip, about which you wish to rotate him with an angular acceleration of –6.0 rad s^{-2}, that is, with a clockwise acceleration. Assume that his rotational inertia I is 15 kg m^2.

(a) What must the magnitude of $\vec{F}$ be if you initially bend your opponent forward to bring his centre of mass to your hip?

(b) What must the magnitude of $\vec{F}$ be if he remains upright and his weight $m\vec{g}$ has a moment arm d_2 = 0.12 m from the pivot point?

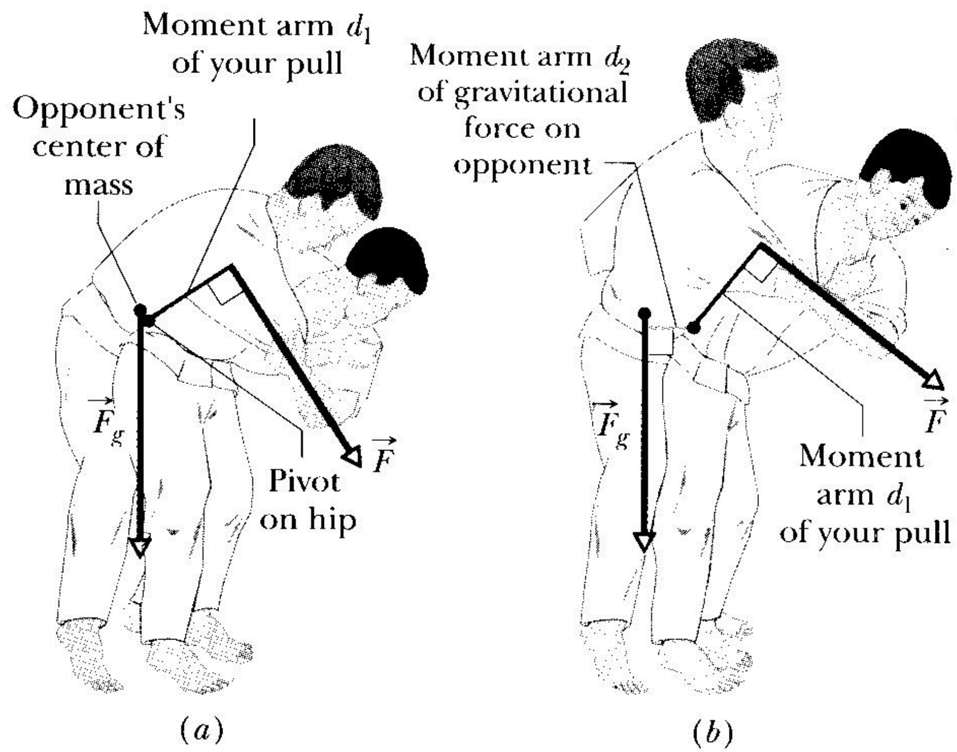


FIG. 10-19 A judo hip throw (*a*) correctly executed and (*b*) incorrectly executed.

(a) Newton's law for the rotating opponent
(we define anticlockwise is positive):

$$\tau = -d_1 F = I\alpha$$

$$F = -\frac{I\alpha}{d_1} = -\frac{(15)(-6)}{0.3}$$

$$= 300\ \text{N}$$

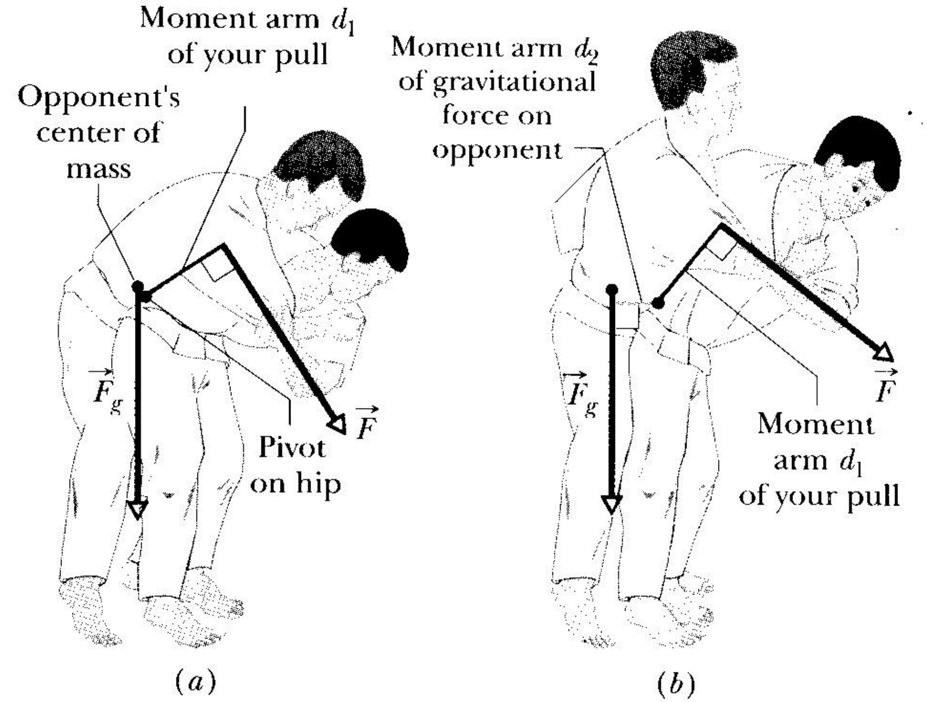


FIG. 10-19 A judo hip throw (*a*) correctly executed and (*b*) incorrectly executed.

(b) $\sum \tau = -d_1 F + d_2 mg = I\alpha$

$$F = -\frac{I\alpha}{d_1} + \frac{d_2 mg}{d_1} = -\frac{(15)(-6)}{0.3} + \frac{(0.12)(80)(9.8)}{0.3} \approx 614\ \text{N}$$

Remark: In the correct execution of the hip throw, you should bend your opponent to bring his center of mass to your hip.

Work and Rotational Kinetic Energy

Work done by the force:

$$dW = \vec{F} \cdot d\vec{s} = F_t ds = F_t r d\theta = \tau d\theta.$$

Total work done:

$$W = \int_{\theta_i}^{\theta_f} \tau d\theta.$$

Work-kinetic energy theorem:

$$\tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\omega\frac{d\omega}{d\theta} = \frac{d}{d\theta}\left(\frac{1}{2}I\omega^2\right).$$

Integrating over the angular displacement,

$$W = \int_{\theta_i}^{\theta_f} \tau d\theta = \int_{\theta_i}^{\theta_f} \frac{d}{d\theta}\left(\frac{1}{2}I\omega^2\right)d\theta = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = \Delta K$$

$$W = \Delta K.$$

Power

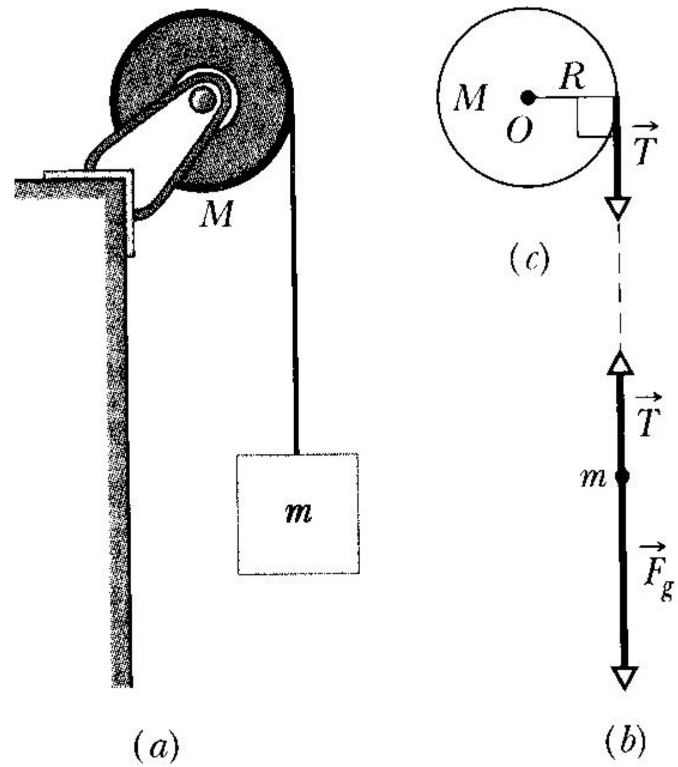
$$P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau\omega.$$

Some Corresponding Relations for Translational and Rotational Motion

Pure Translation (Fixed Direction)		Pure Rotation (Fixed Axis)	
Position	x	Angular position	θ
Velocity	$v = dx/dy$	Angular velocity	$\omega = d\theta/dt$
Acceleration	$a = dv/dt$	Angular acceleration	$\alpha = d\omega/dt$
Mass	m	Rotational inertia	I
Newton's second law	$F_{net} = ma$	Newton's second law	$\tau_{net} = I\alpha$
Work	$W = \int F\,dx$	Work	$W = \int \tau\,d\theta$
Kinetic energy	$K = \frac{1}{2}mv^2$	Kinetic energy	$K = \frac{1}{2}I\omega^2$
Power (constant force)	$P = Fv$	Power (constant torque)	$P = \tau\omega$
Work–kinetic energy theorem	$W = \Delta K$	Work–kinetic energy theorem	$W = \Delta K$

Example

A uniform disk of mass M = 2.5 kg and radius R = 20 cm is mounted on a fixed horizontal axle. A block whose mass m is 1.2 kg hangs from a massless cord that is wrapped around the rim of the disk. What is the rotational kinetic energy K at t = 2.5 s?



Method 1: Use Newton's law directly.

Using Newton's law, we have found α = 24 rad s^{-2}.

$$\omega = \omega_0 + \alpha t = 0 + \alpha t = \alpha t$$

Rotational inertia:

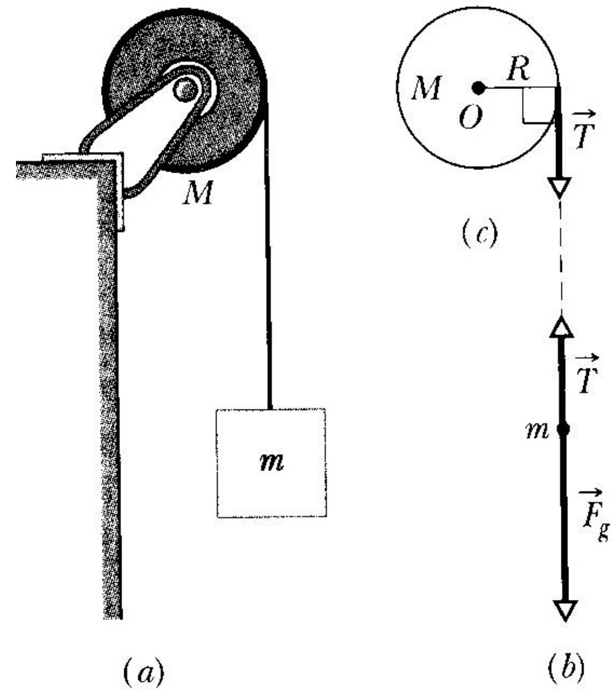
$$I = \frac{1}{2} MR^2$$

Kinetic energy:

$$K = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(\frac{1}{2}MR^2\right)(\alpha t)^2$$

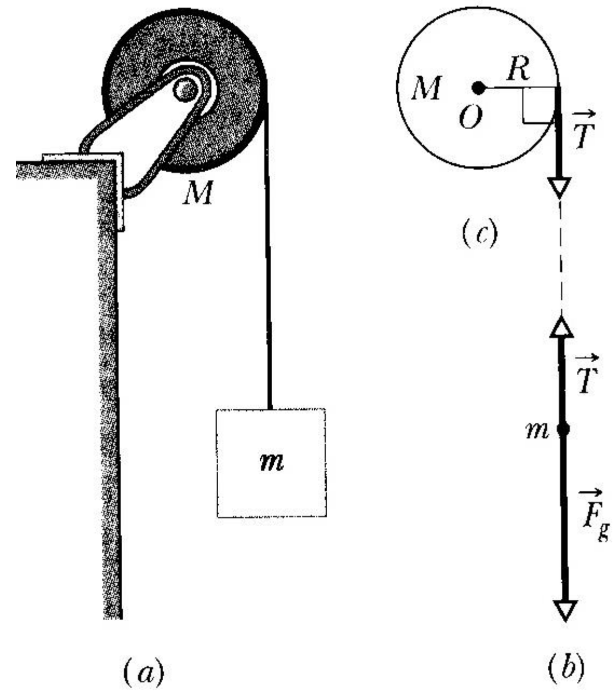
$$= \frac{1}{4} M (R\alpha t)^2$$

$$= \frac{1}{4}(2.5)[(0.2)(24)(2.5)]^2 = 90 \text{ J}$$



Method 2: Use work-kinetic energy theorem.
Word done by the torque:

$$W = \tau\theta = TR\theta$$



Since

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = 0 + \frac{1}{2}\alpha t^2 = \frac{1}{2}\alpha t^2$$

$$W = TR\left(\frac{1}{2}\alpha t^2\right)$$

Using the work-kinetic energy theorem,

$$K = \frac{1}{2}TR\alpha t^2 = \frac{1}{2}(6)(0.2)(24)(2.5)^2 = 90\ \text{J}$$

Example

A tall, cylindrical chimney will fall over when its base is ruptured. Treat the chimney as a thin rod of length L = 55 m. At the instant it makes an angle of $\theta = 35^o$ *with the vertical, what is its angular speed* ω_f?

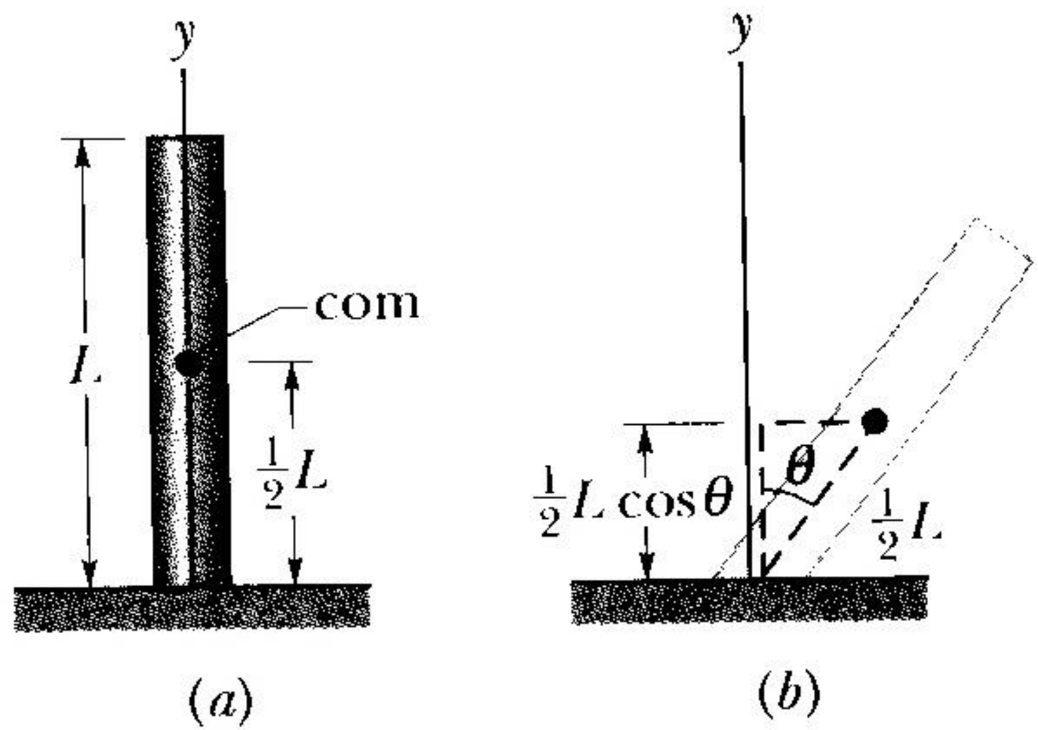


FIG. 10-20 (*a*) A cylindrical chimney. (*b*) The height of its center of mass is determined with the right triangle.

Using the conservation of energy,

$$K_f + U_f = K_i + U_i$$

Rotational inertia about the base:

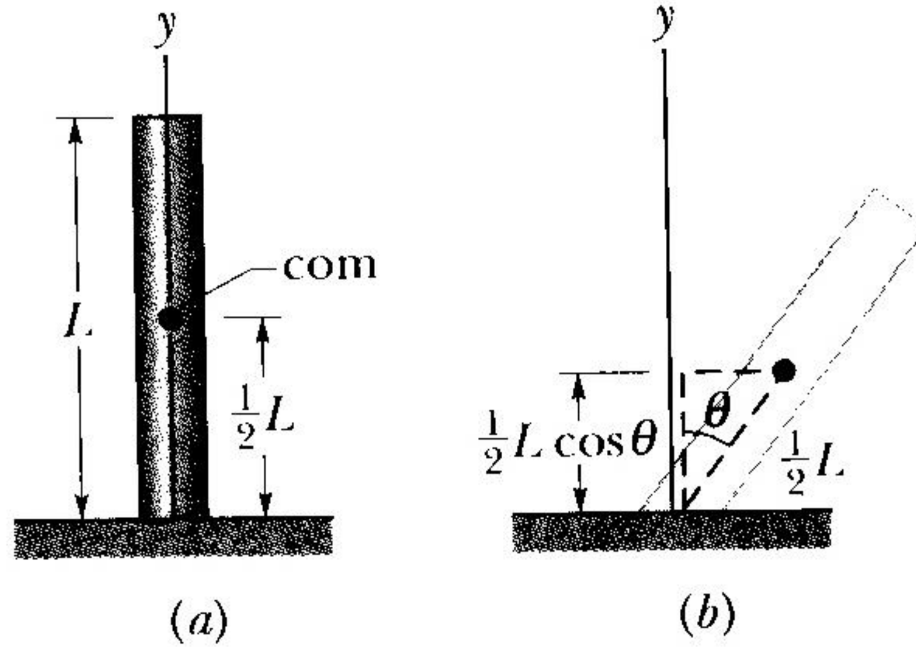
$$I = I_{\mathrm{cm}} + Mh^2 = \frac{1}{12}mL^2 + m\left(\frac{L}{2}\right)^2 = \frac{1}{3}mL^2$$

$$K_i = 0$$

$$K_f = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{3}mL^2\right)\omega^2$$

$$U_i = mg\left(\frac{L}{2}\right)$$

$$U_i = mg\left(\frac{L}{2}\right)\cos\theta$$



Therefore,

$$\frac{1}{6}mL^2\omega^2 + \frac{1}{2}mgL\cos\theta = 0 + \frac{1}{2}mgL$$

$$\omega = \sqrt{\frac{3g}{L}(1-\cos\theta)} = \sqrt{\frac{3(9.8)}{55}(1-\cos 35^{\circ})}$$

$= 0.311$ rad s^{-1}

Rolling

2 points of view:

(1) Combined rotation and translation

(a) Translation: the center of mass moves with velocity v_{cm}.

(b) Rotation: the wheel rotates about the center of mass.

If the wheel *rolls without slipping*, $s = R\theta$, then

$$v_{cm} = \omega R.$$

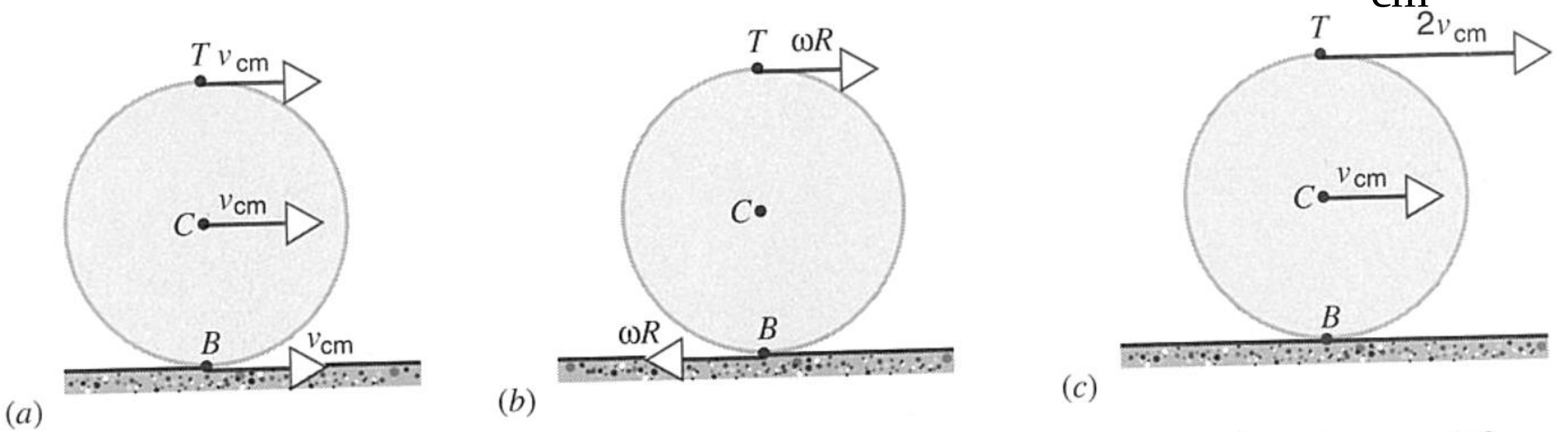


FIGURE 9-30. Rolling can be viewed as a superposition of pure translation and rotation about the center of mass. (*a*) The translational motion, in which all points move with the same linear velocity. (*b*) The rotational motion, in which all points move with the same angular velocity about the central axis. (*c*) The superposition of (*a*) and (*b*), in which the velocities at *T*, *C*, and *B* have been obtained by vector addition of the translational and rotational components.

(2) Pure Rotation

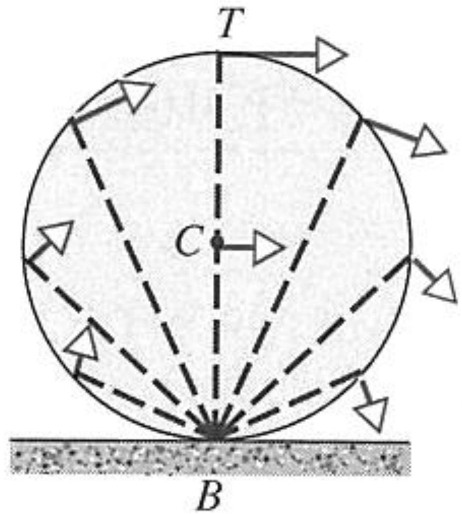


FIGURE 9-31. A rolling body can be considered to be rotating about an instantaneous axis at the point of contact *B*. The vectors show the instantaneous linear velocities of selected points.

Rolling can also be considered as a pure rotation, with angular speed ω, about an axis through the contact point.
e.g. velocity at the top: $v_{top} = (\omega)(2R) = 2(\omega R) = 2v_{cm}$.

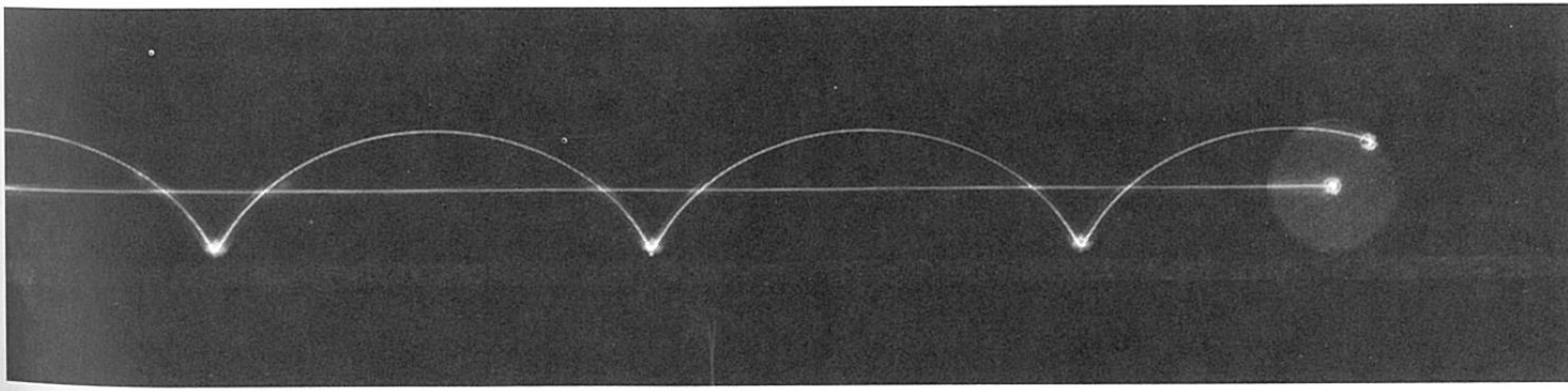


FIGURE 9-28. A time-exposure photo of a rolling wheel. Small lights have been attached to the wheel, one at its center and another at its edge. The latter traces out a curve called a *cycloid*.

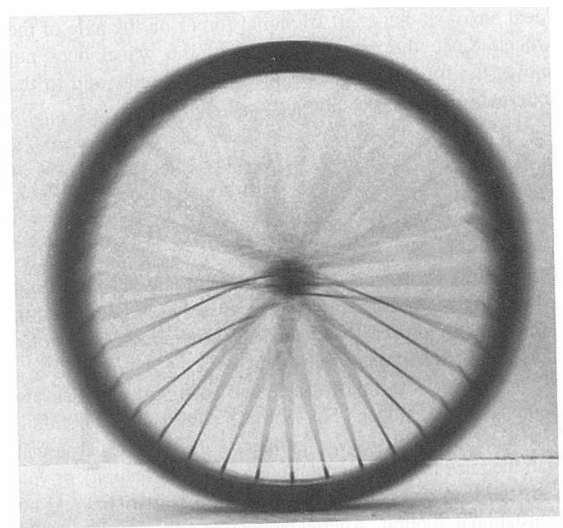


FIGURE 9-29. A photo of a rolling bicycle wheel. Note that the spokes near the top of the wheel are more blurred than those near the bottom. This is because the top has a greater linear velocity.

Relationship between the angular velocity/acceleration and linear velocity/acceleration

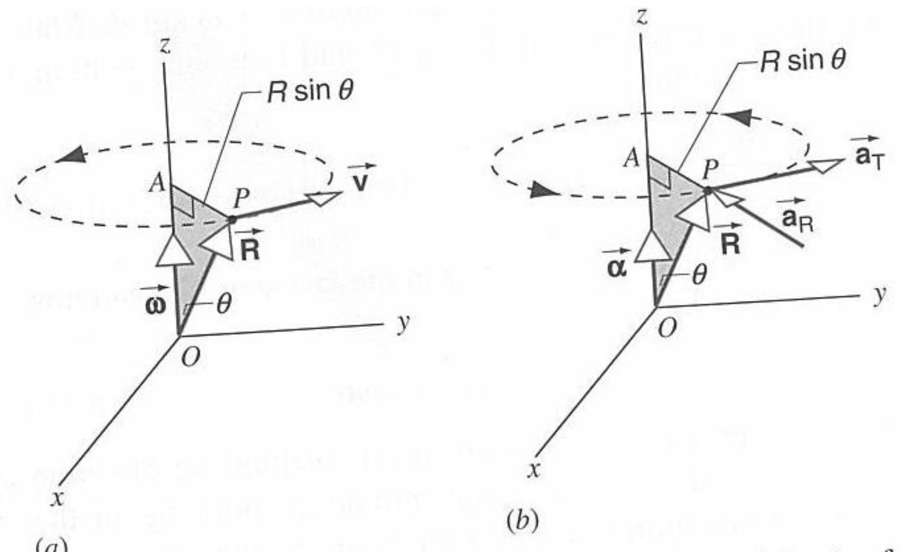


FIGURE 8-11. (*a*) A particle at *P* in the rotating rigid body of Fig. 8-3*a* is located at **R** with respect to the origin *O*. The particle has angular velocity $\vec{\omega}$ (directed along the *z* axis) and tangential velocity $\vec{v}$. (*b*) The particle at *P* has angular acceleration $\vec{\alpha}$ along the *z* axis. The particle also has tangential acceleration $\vec{a}_T$ and radial acceleration $\vec{a}_R$.

$$\vec{v} = \vec{\omega} \times \vec{R}.$$

$$\omega R \sin\theta$$

$$\vec{a} = \vec{\alpha} \times \vec{R} + \vec{\omega} \times \vec{v}.$$

$$\vec{\alpha} \times \vec{R} = \vec{a}_T$$

$$\vec{\omega} \times \vec{v} = \vec{a}_R$$

Kinetic Energy of Rolling

If we consider the motion as a pure rotation about the contact point,

$$K = \frac{1}{2} I_P \omega^2.$$

Using the parallel axis theorem,

$$I_P = I_{\text{cm}} + MR^2.$$

Hence

$$K = \frac{1}{2} I_{\text{cm}} \omega^2 + \frac{1}{2} MR^2 \omega^2$$, and $$K = \frac{1}{2} I_{\text{cm}} \omega^2 + \frac{1}{2} M v_{\text{cm}}^2.$$

The kinetic energy consists of:

(a) the kinetic energy of the translational motion of the center of mass

(b) the kinetic energy of the rotation about the center of mass.

Friction and Rolling

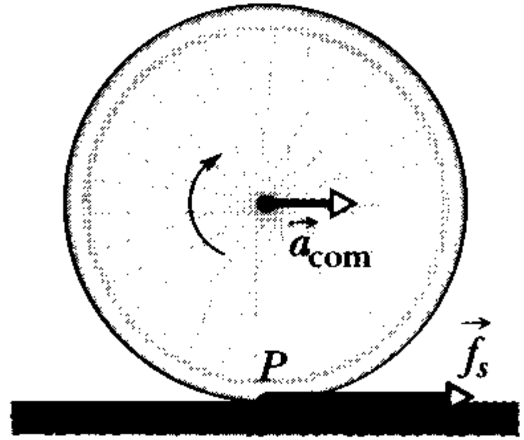


FIG. 11-7 A wheel rolls horizontally without sliding while accelerating with linear acceleration $\vec{a}_{com}$. A static frictional force $\vec{f}_s$ acts on the wheel at P, opposing its tendency to slide.

(a) When the cyclist applies a torque on the wheel intending to make it rotate faster, the bottom of the wheel tends to slide to the left at point *P*. A frictional force at *P*, directed to the right, opposes the tendency to slide.

(b) The frictional force acts on the wheel and produces the acceleration of the bicycle.

Rolling Down a Ramp

The gravitational force tends to make the wheel slide down the ramp. There is a frictional force opposing this sliding, and is thus directed up the ramp. Using Newton's second law for translational motion,

$$Mg\sin\theta - f_s = Ma \quad (1)$$

Using Newton's second law for rotational motion,

$$Rf_s = I_{\text{cm}}\alpha \quad (2)$$

Since $a = R\alpha$, we obtain from (2):

$$f_s = \frac{I_{\text{cm}}}{R^2}a$$

Substituting into (1),

$$a = \frac{g\sin\theta}{1 + I_{\text{cm}}/MR^2}.$$

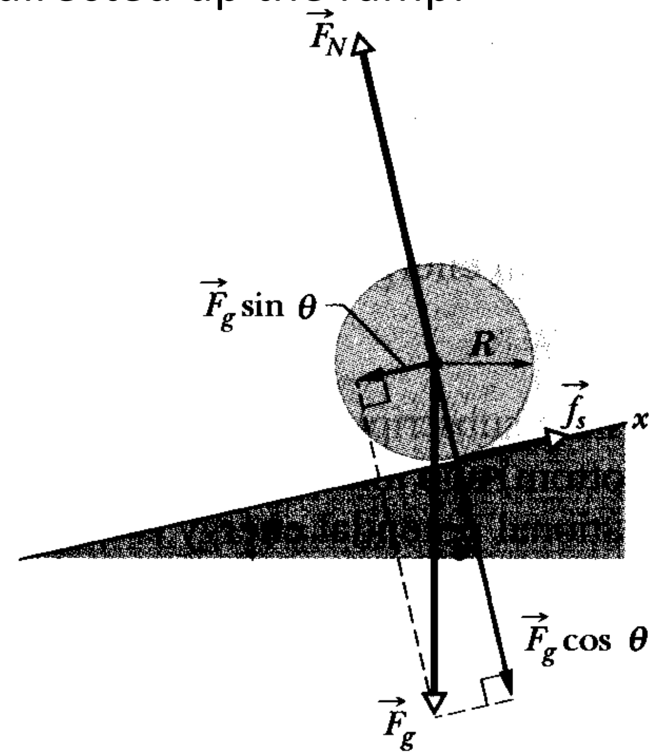


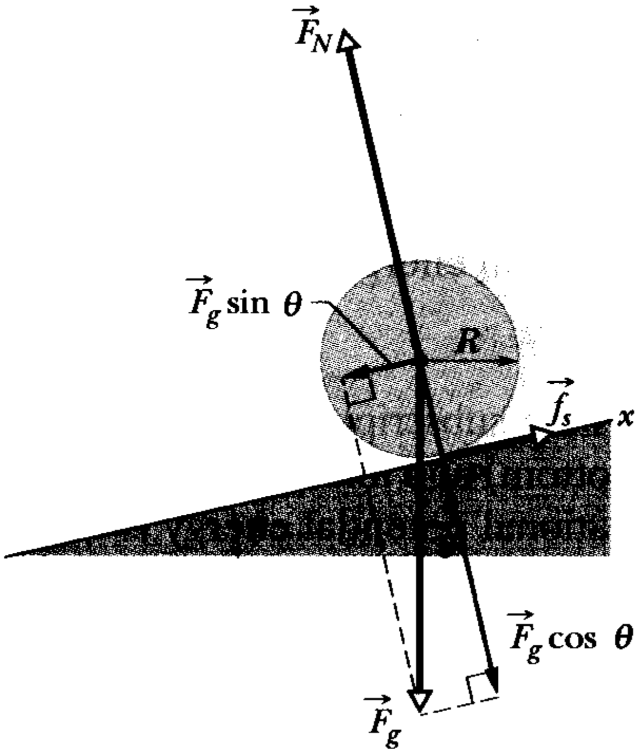
FIG. 11-8 A round uniform body of radius R rolls down a ramp. The forces that act on it are the gravitational force $\vec{F}_g$, a normal force $\vec{F}_N$, and a frictional force $\vec{f}_s$ pointing up the ramp. (For clarity, vector $\vec{F}_N$ has been shifted in the direction it points until its tail is at the center of the body.)

Example

A uniform ball, of mass M = 6.00 kg and radius R, rolls smoothly from rest down a ramp at angle θ = 30.0^o.

(a) The ball descends a vertical height h = 1.20 m to reach the bottom of the ramp. What is its final speed?

(b) What are the magnitude and direction of the frictional force on the ball as it rolls down the ramp?



(a) Method 1: Conservation of energy

$$K_f + U_f = K_i + U_i$$

$$K_f = \frac{1}{2} I_{\mathrm{cm}} \omega^2 + \frac{1}{2} M v_{\mathrm{cm}}^2 = \frac{1}{2}\left(\frac{2}{5} M R^2\right)\frac{v_{\mathrm{cm}}^2}{R^2} + \frac{1}{2} M v_{\mathrm{cm}}^2 = \frac{7}{10} M v_{\mathrm{cm}}^2$$

Other terms: $U_f = K_i = 0$, $U_i = Mgh$. Hence

$$\frac{7}{10} M v_{\mathrm{cm}}^2 + 0 = 0 + Mgh$$

$$v_{\mathrm{cm}} = \sqrt{\frac{10}{7} gh} = \sqrt{\frac{10}{7}(9.8)(1.2)} = 4.1\,\mathrm{ms}^{-1}$$

Method 2: Newton's law

Translational motion:

$$Mg\sin\theta - f_s = Ma \qquad (1)$$

Rotational motion:

$$Rf_s = I_{\mathrm{cm}}\alpha \qquad (2)$$

where $I_{\mathrm{cm}} = \frac{2}{5}MR^2$

Since $a = R\alpha$, (2): $f_s = \frac{I_{\mathrm{cm}}}{R^2}a$

(1): $$a = \frac{g\sin\theta}{1 + I_{\mathrm{cm}}/MR^2} = \frac{5}{7}g\sin\theta$$

$$v^2 = 2a\left(\frac{h}{\sin\theta}\right) = \frac{10}{7}gh$$

$$v = \sqrt{\frac{10}{7}gh} = \sqrt{\frac{10}{7}(9.8)(1.2)} = 4.1\,\mathrm{ms^{-1}}$$

(b) $f_s = \frac{I_{\mathrm{cm}}}{R^2}a = \frac{2}{5}Ma = \frac{2}{7}Mg\sin\theta = \frac{2}{7}(6)(9.8)\sin 30^{\mathrm{o}}$ = 8.4 N

The Yo-Yo

Using Newton's second law for translational motion,

$$Mg - T = Ma. \qquad (1)$$

Using Newton's second law for rotational motion,

$$R_0 T = I_{\mathrm{cm}} \alpha. \qquad (2)$$

Since $a = R_0 \alpha$, we obtain from (2):

$$T = \frac{I_{\mathrm{cm}}}{R_0^2} a$$

Substituting into (1),

$$a = \frac{g}{1 + I_{\mathrm{cm}} / M R_0^2}.$$

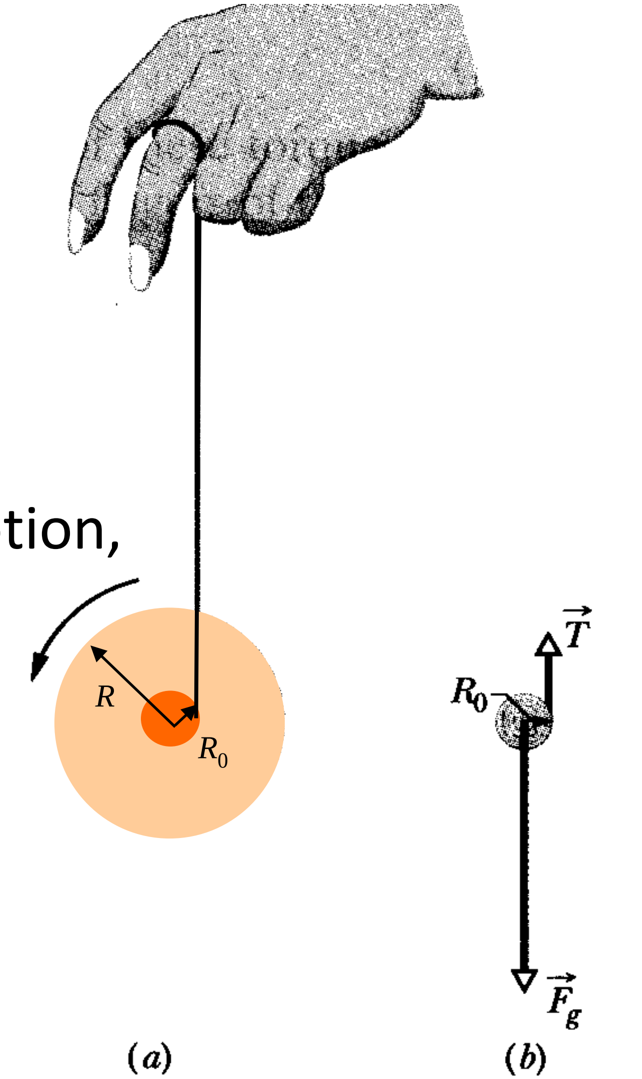


FIG. 11-9 (*a*) A yo-yo, shown in cross section. The string, of assumed negligible thickness, is wound around an axle of radius R_0. (*b*) A free-body diagram for the falling yo-yo. Only the axle is shown.

Uniform circular motion

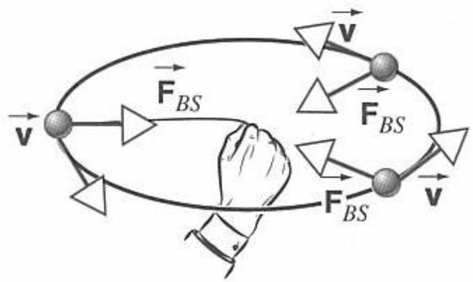


FIGURE 4-14. A ball on a string is whirled in a horizontal circle. Vectors representing the velocity and the force of the string on the ball are shown at three different instants.

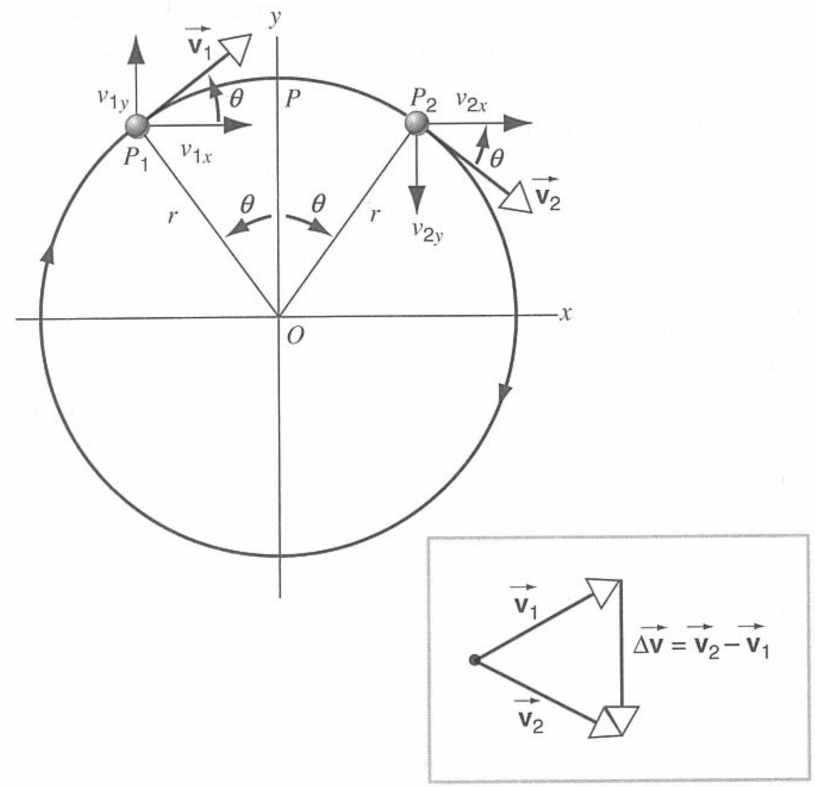


FIGURE 4-16. A particle moves at constant speed in a circle of radius r. It is shown at locations P_1 and P_2, where the radius makes equal angles θ on opposite sides of the y axis. The inset shows the vector $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$; this vector always points toward the center of the circle, no matter where we choose points P_1 and P_2.

$$a_c = \frac{v^2}{r}.$$

SAMPLE PROBLEM 4-5. The Moon revolves about the Earth, making a complete revolution in 27.3 days. Assume that the orbit is circular and has a radius r = 238,000 miles. What is the magnitude of the gravitational force exerted on the Moon by the Earth?

Solution We have $r = 238{,}000 \text{ mi} = 3.82 \times 10^8$ m. From Appendix C, we find the mass of the Moon is $m = 7.36 \times 10^{22}$ kg. The time for one complete revolution, called the period, is $T = 27.3 \text{ d} = 2.36 \times 10^6$ s. The speed of the Moon (assumed constant) is therefore

$$v = \frac{2\pi r}{T} = \frac{2\pi(3.82 \times 10^8 \text{ m})}{2.36 \times 10^6 \text{ s}} = 1018 \text{ m/s}.$$

The centripetal force is provided by the gravitational force on the Moon by the Earth:

$$F_{ME} = \frac{mv^2}{r} = \frac{(7.36 \times 10^{22} \text{ kg})(1018 \text{ m/s})^2}{3.82 \times 10^8 \text{ m}}$$

$$= 2.00 \times 10^{20} \text{ N}.$$

SAMPLE PROBLEM 4-6. A satellite of mass 1250 kg is to be placed in a circular orbit at a height $h = 210$ km above the Earth's surface, where $g = 9.2\ \text{m/s}^2$. (*a*) What is the weight of the satellite at this altitude? (*b*) With what tangential speed must it be inserted into its orbit? The Earth's radius is $R = 6370$ km.

Solution (*a*) The weight of the satellite is

$$W = mg = (1250\ \text{kg})(9.2\ \text{m/s}^2) = 1.15 \times 10^4\ \text{N}.$$

(*b*) The weight is the force of gravity F_{SE} exerted on the satellite by the Earth. Since this is the only force that acts on the satellite,

it must provide the centripetal force. Solving Eq. 4-30 for the tangential speed v, we obtain (with $r = R + h$):

$$v = \sqrt{\frac{F_{SE}r}{m}} = \sqrt{\frac{(1.15 \times 10^4\ \text{N})(6370\ \text{km} + 210\ \text{km})}{1250\ \text{kg}}}$$

$$= 7780\ \text{m/s} = 17{,}400\ \text{mi/h}.$$

At this speed, the satellite completes one orbit every 1.48 h.