Angular momentum

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Contents

- Operation of vectors
- Angular momentum
- Angular momentum of rigid body
- Conservation of angular momentum
- Examples

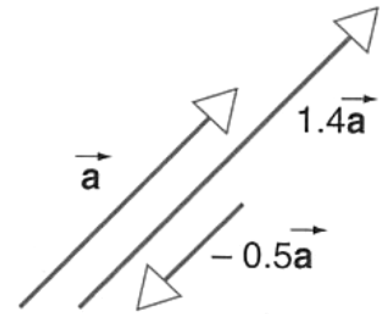
H-4 MULTIPLICATION OF VECTORS

Multiplication of a vector by a scalar:

$$\vec{\mathbf{b}} = c\vec{\mathbf{a}}$$

$$b_x = ca_x \qquad b_y = ca_y$$

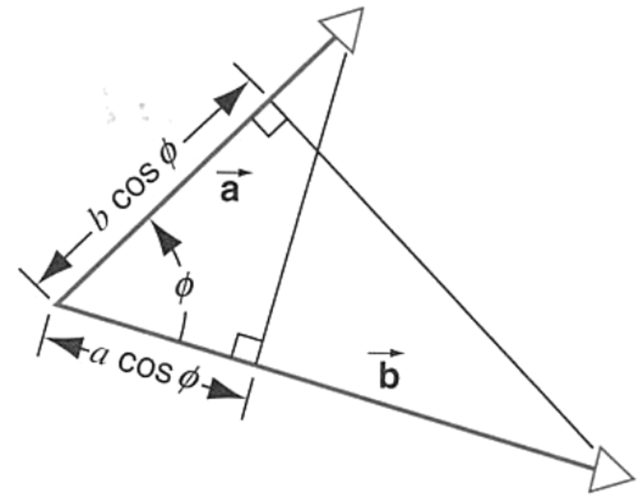
$$b = |c|\, a$$



Dot product (or scalar product) of two vectors:

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = ab\cos\phi = a(b\cos\phi) = b(a\cos\phi)$$

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = \vec{\mathbf{b}}\cdot\vec{\mathbf{a}}$$



$$\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1$$

$$\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{i}}\cdot\hat{\mathbf{k}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = 0$$

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = a_xb_x + a_yb_y + a_zb_z$$

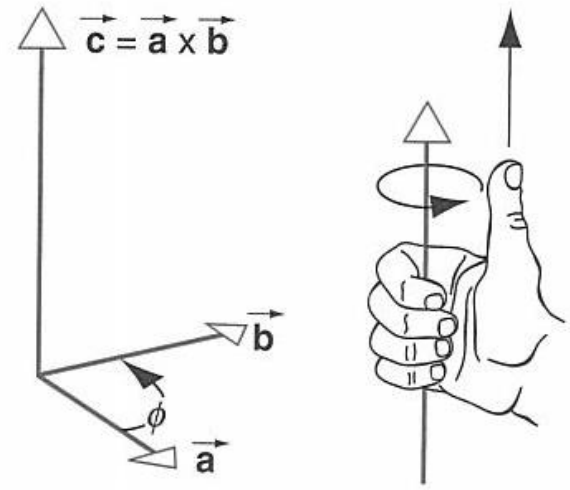
$$\vec{\mathbf{a}}\cdot\vec{\mathbf{a}} = a^2 = a_x^2 + a_y^2 + a_z^2$$

Cross product (or vector product) of two vectors:

$\vec{\mathbf{c}} = \vec{\mathbf{a}} \times \vec{\mathbf{b}}$

$|\vec{\mathbf{c}}| = |\vec{\mathbf{a}} \times \vec{\mathbf{b}}| = ab \sin \phi$

Direction of $\vec{\mathbf{c}}$ is perpendicular to the plane of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, determined by the right-hand rule.



$\vec{\mathbf{b}} \times \vec{\mathbf{a}} = -\vec{\mathbf{a}} \times \vec{\mathbf{b}}$

$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0$

$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \qquad \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}} \qquad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = (a_y b_z - a_z b_y)\hat{\mathbf{i}} + (a_z b_x - a_x b_z)\hat{\mathbf{j}} + (a_x b_y - a_y b_x)\hat{\mathbf{k}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) + (\vec{\mathbf{a}} \times \vec{\mathbf{c}})$

$(s\vec{\mathbf{a}}) \times \vec{\mathbf{b}} = \vec{\mathbf{a}} \times (s\vec{\mathbf{b}}) = s(\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \qquad (s = \text{a scalar}).$

$\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = \vec{\mathbf{b}} \cdot (\vec{\mathbf{c}} \times \vec{\mathbf{a}}) = \vec{\mathbf{c}} \cdot (\vec{\mathbf{a}} \times \vec{\mathbf{b}})$

$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\vec{\mathbf{b}} - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}}$

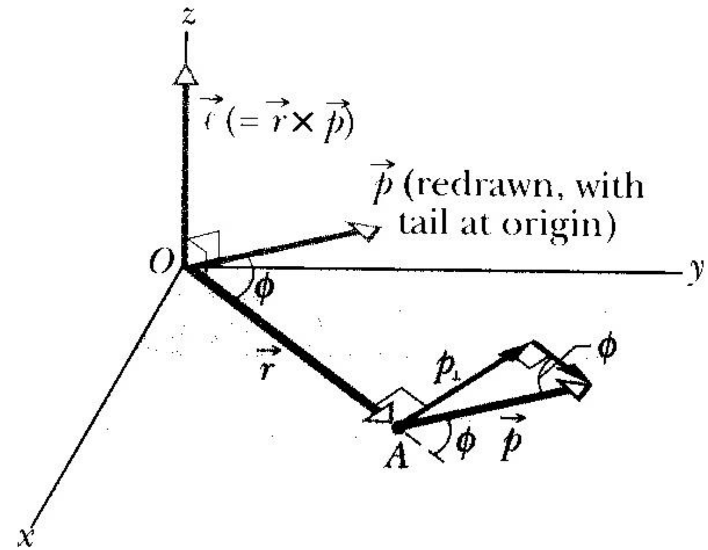
Angular Momentum

$$\vec{l} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$$

$$l = mrv\sin\phi.$$

Alternatively,

$$l = rp_{\perp} = rmv_{\perp} \qquad \text{or} \qquad l = r_{\perp}p = r_{\perp}mv$$

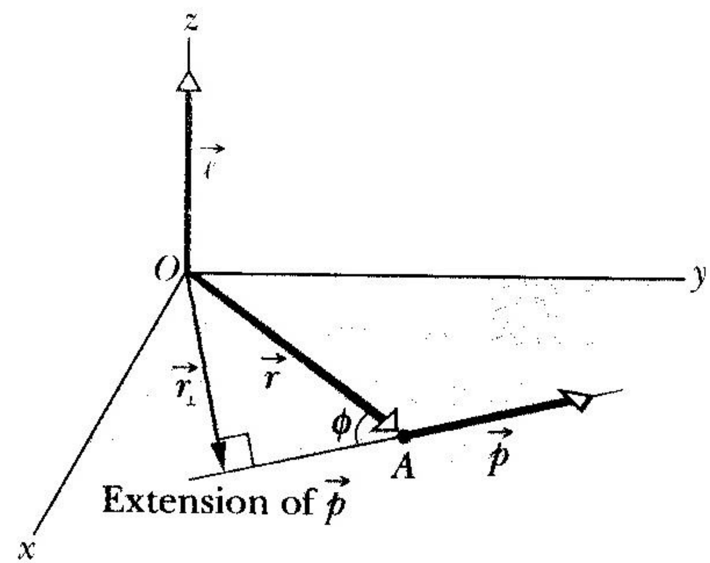


(a)

Newton's Second Law

$$\Sigma\vec{\tau} = \frac{d\vec{l}}{dt}.$$

The vector sum of all the torques acting on a particle is equal to the time rate of change of the angular momentum of that particle.



(b)

Proof

$$\vec{l} = m(\vec{r} \times \vec{v})$$

Differentiating with respect to time,

$$\frac{d\vec{l}}{dt} = m\left(\vec{r} \times \frac{d\vec{v}}{dt} + \frac{d\vec{r}}{dt} \times \vec{v}\right)$$

$$\frac{d\vec{l}}{dt} = m(\vec{r} \times \vec{a} + \vec{v} \times \vec{v})$$

$\vec{v} \times \vec{v} = 0$ because the angle between $\vec{v}$ and $\vec{v}$ is zero.

$$\frac{d\vec{l}}{dt} = m(\vec{r} \times \vec{a}) = \vec{r} \times m\vec{a}$$

Using Newton's law, $\Sigma\vec{F} = m\vec{a}$. Hence

$$\frac{d\vec{l}}{dt} = \vec{r} \times \left(\sum \vec{F}\right) = \sum\left(\vec{r} \times \vec{F}\right).$$

Since $\vec{\tau} = \vec{r} \times \vec{F}$, we arrive at $\sum \vec{\tau} = \frac{d\vec{l}}{dt}$.

The Angular Momentum of a System of Particles

Total angular momentum for *n* particles:

$$\vec{L} = \vec{l}_1 + \vec{l}_2 + \cdots = \sum \vec{l}_i$$

Newtons' law for angular motion:

$$\sum \vec{\tau}_i = \sum \frac{d\vec{l}_i}{dt} = \frac{d}{dt} \sum \vec{l}_i = \frac{d\vec{L}}{dt}.$$

$\sum \vec{\tau}_i$ includes torques acting on all the *n* particles. Both internal torques and external torques are considered.

Using Newton's law of action and reaction, the internal forces cancel in pairs. Hence

$$\sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}.$$

TABLE 10-1 Review and Comparison of Translational and Rotational Dynamics*

Translational Quantity		*Equation Number*	*Rotational Quantity*		*Equation Number*
Velocity	$\vec{\mathbf{v}} = d\vec{\mathbf{r}}/dt$	2-9	Angular velocity	$\vec{\boldsymbol{\omega}} = d\vec{\boldsymbol{\phi}}/dt$	8-3
Acceleration	$\vec{\mathbf{a}} = d\vec{\mathbf{v}}/dt$	2-16	Angular acceleration	$\vec{\boldsymbol{\alpha}} = d\vec{\boldsymbol{\omega}}/dt$	8-5
Mass	m		Rotational inertia	$I = \sum mr^2$	9-10
Force	$\vec{\mathbf{F}}$		Torque	$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$	9-3
Newton's second law	$\sum \vec{\mathbf{F}}_{\text{ext}} = m\vec{\mathbf{a}}$	4-3	Newton's second law for rotations about a fixed axis	$\sum \tau_{\text{ext},z} = I\alpha_z$	9-11
Equilibrium condition	$\sum \vec{\mathbf{F}}_{\text{ext}} = 0$	9-22	Equilibrium condition	$\sum \vec{\boldsymbol{\tau}}_{\text{ext}} = 0$	9-23
Momentum of a particle	$\vec{\mathbf{p}} = m\vec{\mathbf{v}}$	6-1	Angular momentum of a particle	$\vec{\mathbf{l}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$	10-1
Momentum of a system of particles	$\vec{\mathbf{P}} = M\vec{\mathbf{v}}_{\text{cm}}$	7-21	Angular momentum of a system of particles	$\vec{\mathbf{L}} = I\vec{\boldsymbol{\omega}}$	10-12
General form of Newton's second law	$\sum \vec{\mathbf{F}}_{\text{ext}} = d\vec{\mathbf{P}}/dt$	7-23	General form of Newton's second law of rotations	$\sum \vec{\boldsymbol{\tau}}_{\text{ext}} = d\vec{\mathbf{L}}/dt$	10-9
Conservation of momentum in a system of particles for which $\sum \vec{\mathbf{F}}_{\text{ext}} = 0$	$\vec{\mathbf{P}} = \sum \vec{\mathbf{p}}_n = \text{constant}$	6-12	Conservation of angular momentum in a system of particles for which $\sum \vec{\boldsymbol{\tau}}_{\text{ext}} = 0$	$\vec{\mathbf{L}} = \sum \vec{\mathbf{l}}_n = \text{constant}$	10-15

* Some of these equations apply only under certain special conditions. Be sure you understand the conditions before using these equations. Equations that

The Angular Momentum of a Rigid Body

For the *i*th particle, angular momentum:

$$l_i = r_i p_i \sin 90^\circ = r_i \Delta m_i \, v_i.$$

The component of angular momentum parallel to the rotation axis (the *z* component):

$$l_{iz} = l_i \sin\theta = (r_i \sin\theta)(\Delta m_i v_i)$$
$$= r_{i\perp} \Delta m_i v_i.$$

The total angular momentum for the rotating body

$$L_z = \sum_i l_{iz} = \sum_i \Delta m_i v_i r_{i\perp}$$
$$= \sum_i \Delta m_i (\omega r_{i\perp}) r_{i\perp} = \omega\left(\sum_i m_i r_{i\perp}^2\right).$$

This reduces to $L = I\omega.$

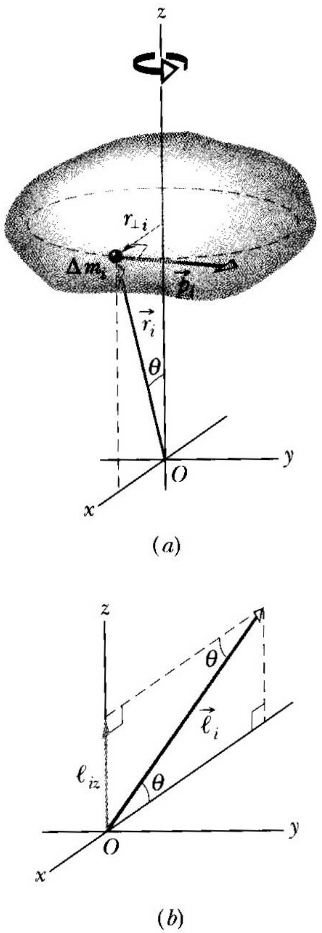


FIG. 11-15 (*a*) A rigid body rotates about a *z* axis with angular speed ω. A mass element of mass Δm_i within the body moves about the *z* axis in a circle with radius $r_{\perp i}$. The mass element has linear momentum $\vec{p}_i$, and it is located relative to the origin *O* by position vector $\vec{r}_i$. Here the mass element is shown when $r_{\perp i}$ is parallel to the *x* axis. (*b*) The angular momentum $\vec{\ell}_i$, with respect to *O*, of the mass element in (*a*). The *z* component ℓ_{iz} is also shown.

Conservation of Angular Momentum

$$\sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$$

If no external torque acts on the system,

$$\frac{d\vec{L}}{dt} = 0$$

$$\vec{L} = \text{constant.}$$

$$\vec{L}_i = \vec{L}_f .$$

Law of conservation of angular momentum.

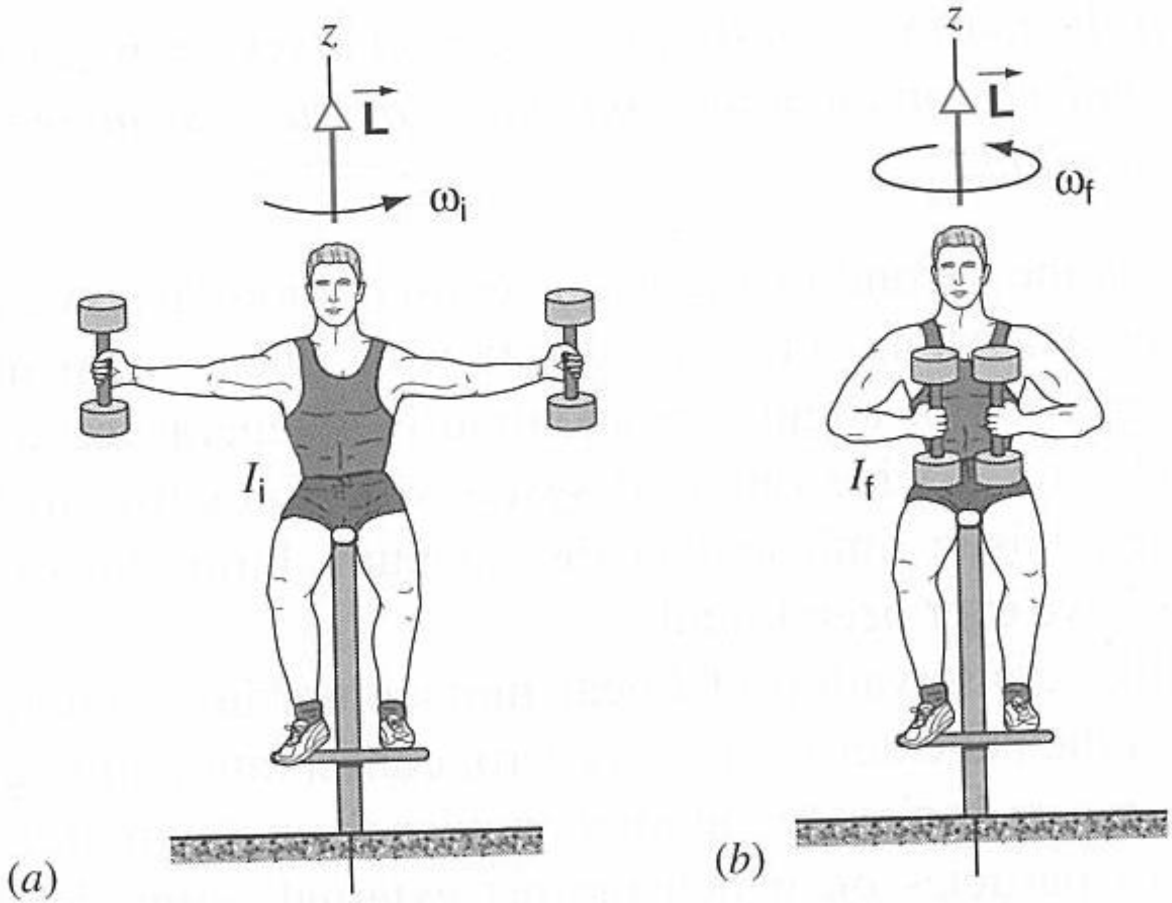


FIGURE 10-12. (*a*) In this configuration, the system (student + weights) has a larger rotational inertia and a smaller angular velocity. (*b*) Here the student has pulled the weights inward, giving a smaller rotational inertia and hence a larger angular velocity. The angular momentum $\vec{\mathbf{L}}$ has the same value in both situations.

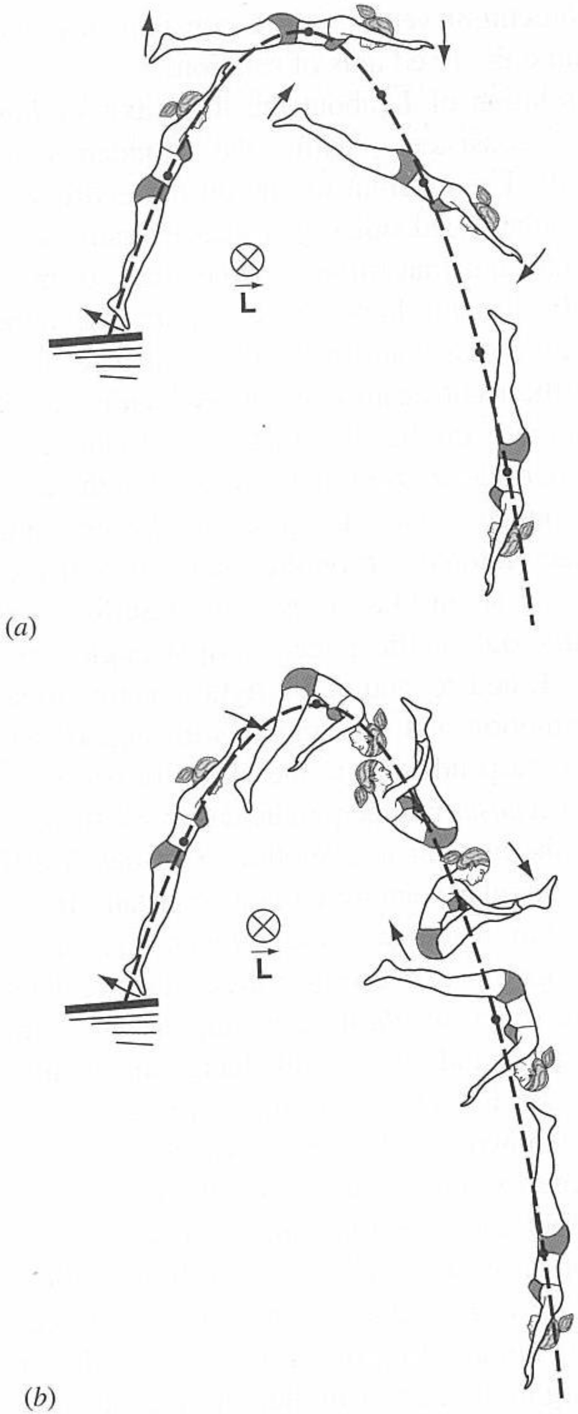
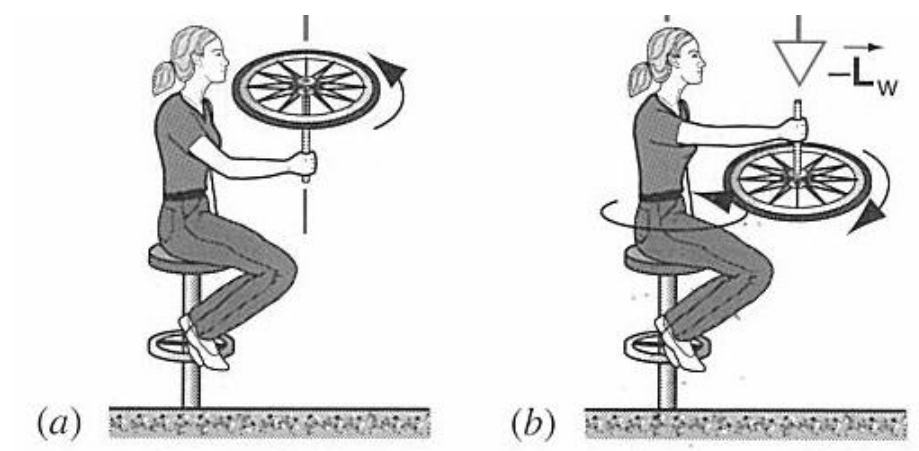


FIGURE 10-13. (*a*) A diver leaves the springboard in such a way that the springboard imparts to her an angular momentum $\vec{\mathbf{L}}$. She rotates about her center of mass (indicated by the dot) by one-half revolution as the center of mass follows the parabolic trajectory. (*b*) By entering the tuck position, she reduces her rotational inertia and thus increases her angular velocity, enabling her to make $1\frac{1}{2}$ revolutions. The external forces and torques on her are the same in (*a*) and (*b*), as indicated by the constant value of the angular momentum $\vec{\mathbf{L}}$.

Examples

A student sits on a stool that can rotate freely about a vertical axis. The student, initially at rest, is holding a bicycle wheel whose rim is loaded with lead and whose rotational inertia *I* about its central axis is 1.2 kgm^2. The wheel is rotating at an angular speed ω_{wh} of 3.9 rev/s; as seen from overhead, the rotation is counterclockwise. The axis of the wheel is vertical, and the angular momentum L_{wh} *of the wheel points vertically upward. The student now inverts the wheel; as a result, the student and stool rotate about the stool axis. The rotational inertia* I_b *of the student + stool + wheel system about the stool axis is 6.8* kgm^2*. With what angular speed* ω_b *and in what direction does the composite body rotate after the inversion of the wheel?*



Using the conservation of angular momentum,

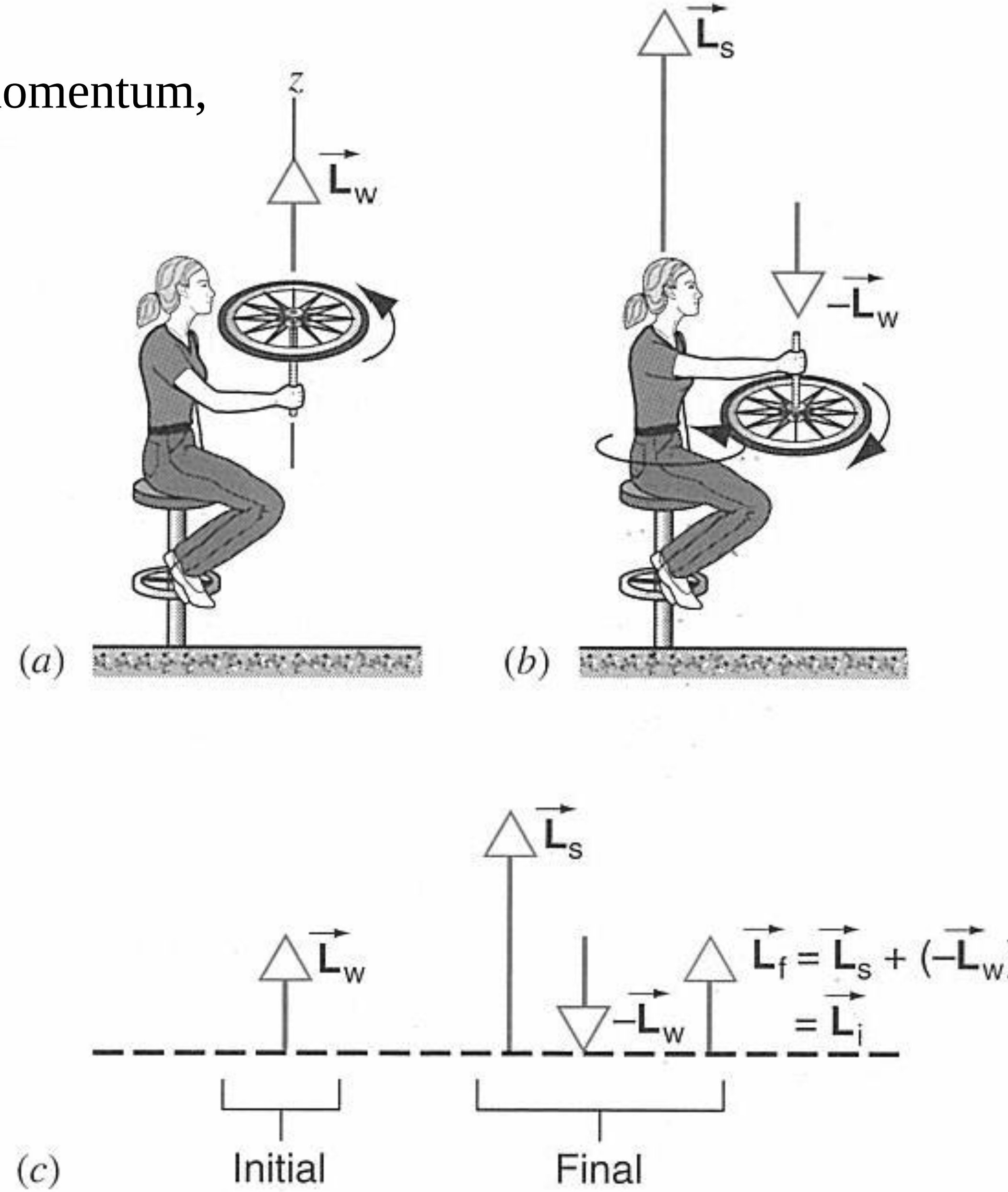
$$L_{\text{wh}} = L + (-L_{\text{wh}})$$

$$L = 2L_{\text{wh}}$$

$$I_b\omega_b = 2I\omega_{\text{wh}}$$

$$\omega_b = \frac{2I\omega_{\text{wh}}}{I_b} = \frac{(2)(1.2)(3.9)}{6.8}$$

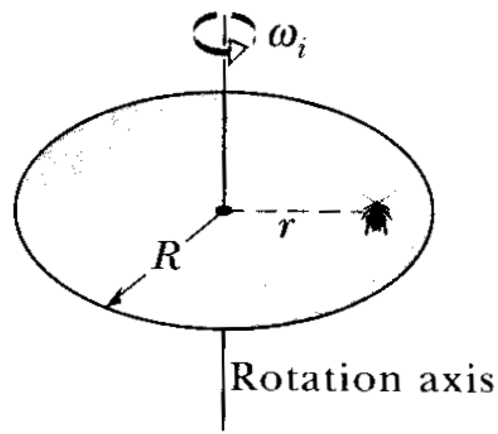
$$= 1.38\ \text{rev s}^{-1}$$



Examples

A cockroach with mass *m* rides on a disk of mass 6*m* and radius *R*. The disk rotates like a merry-go-round around its central axis at angular speed ω_I = 1.5 rad s^{-1}. *The cockroach is initially at radius r* = 0.8*R*, but then it crawls out to the rim of the disk. Treat the cockroach as a particle. What then is the angular speed?

FIG. 11-22 A cockroach rides at radius *r* on a disk rotating like a merry-go-round.



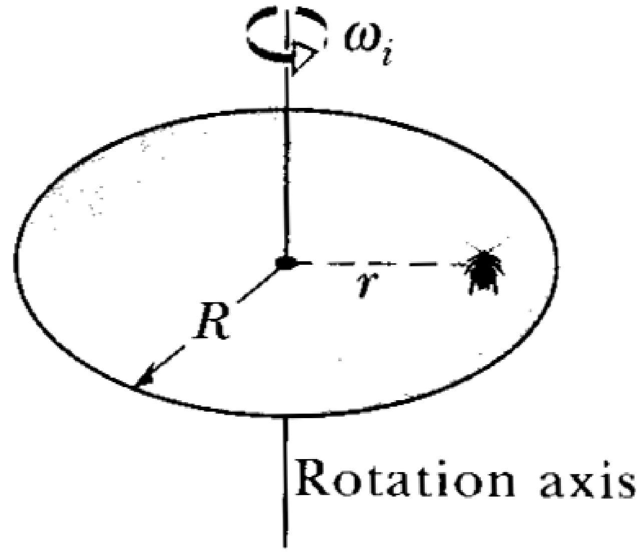
Using the conservation of angular momentum, we have

$$I_f \omega_f = I_i \omega_i$$

Rotational inertia:

The disk:

$$I_d = \frac{1}{2} MR^2 = 3mR^2$$



The cockroach:

$$I_{ci} = m(0.8R)^2 = 0.64mR^2$$

and $$I_{cf} = mR^2$$

$$I_i = I_d + I_{ci} = 3.64mR^2$$

$$I_f = I_d + I_{cf} = 4mR^2$$

Therefore, $$\omega_f = \frac{I_i \omega_i}{I_f} = \frac{(3.64mR^2)(1.5)}{4mR^2} = 1.37 \text{ rad s}^{-1}$$

Precession of a Gyroscope

Torque due to the gravitational force

$$\tau = Mgr\sin 90^{\circ} = Mgr$$

Angular momentum

$$L = I\omega$$

For a rapidly spinning gyroscope, the magnitude of $\vec{L}$ is not affected by the precession,

$$dL = L d\phi \qquad \frac{dL}{dt} = L\frac{d\phi}{dt}$$

Using Newton's second law for rotation,

$$\tau = \frac{dL}{dt}$$

$$Mgr = L\frac{d\phi}{dt} = L\Omega \quad \text{where} \quad \Omega = \frac{d\phi}{dt}$$

is the precession rate $\quad \Omega = \dfrac{Mgr}{L} = \dfrac{Mgr}{I\omega}$

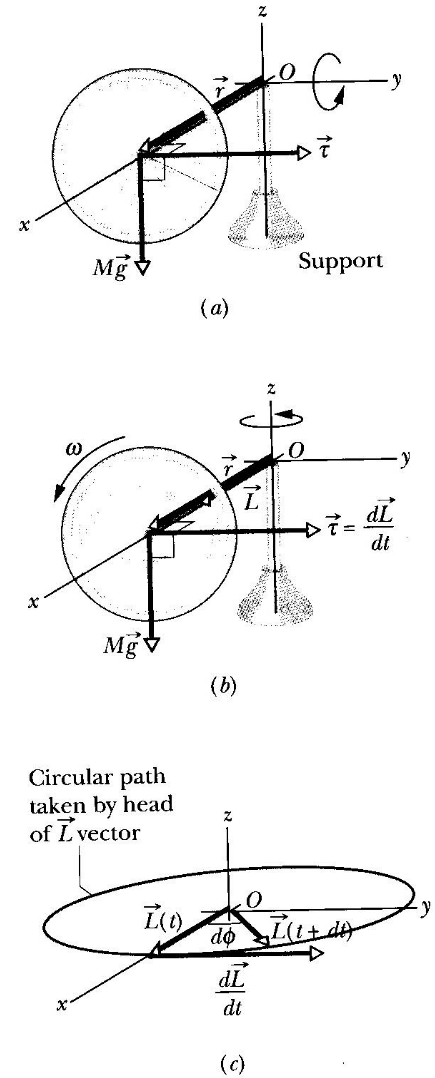


FIG. 11-23 (*a*) A nonspinning gyroscope falls by rotating in an *xz* plane because of torque $\vec{\tau}$. (*b*) A rapidly spinning gyroscope, with angular momentum $\vec{L}$, precesses around the *z* axis. Its precessional motion is in the *xy* plane. (*c*) The change $d\vec{L}/dt$ in angular momentum leads to a rotation of $\vec{L}$ about *O*.

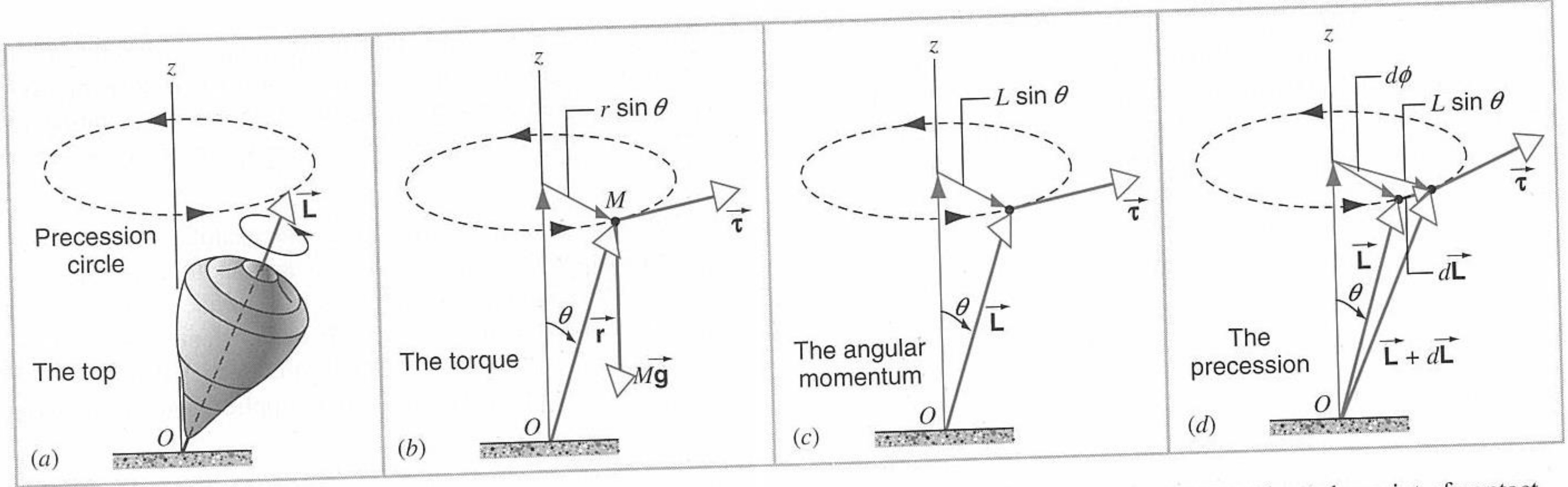


FIGURE 10-18. (*a*) A spinning top precesses about a vertical axis. (*b*) The weight of the top exerts a torque about the point of contact with the floor. (*c*) The torque is perpendicular to the angular momentum vector. (*d*) The torque changes the direction of the angular momentum vector, causing precession.

In a time dt, the axis rotates through an angle $d\phi$ (see Fig. 10-18*d*), and thus the angular speed of precession ω_P is

$$\omega_P = \frac{d\phi}{dt}. \qquad (10\text{-}20)$$

From Fig. 10-18*d*, we see that

$$d\phi = \frac{dL}{L\sin\theta} = \frac{\tau\, dt}{L\sin\theta}. \qquad (10\text{-}21)$$

Thus

$$\omega_P = \frac{d\phi}{dt} = \frac{\tau}{L\sin\theta} = \frac{Mgr\sin\theta}{L\sin\theta} = \frac{Mgr}{L}. \qquad (10\text{-}22)$$

SAMPLE PROBLEM 10-5. A turntable consisting of a disk of mass 125 g and radius 7.2 cm is spinning with an angular speed of 0.84 rev/s about a vertical axis (Fig. 10-17*a*). An identical, initially nonrotating disk is suddenly dropped onto the first. The friction between the two disks causes them eventually to rotate at the same speed. A third identical nonrotating disk is then dropped onto the combination, and eventually all three are rotating together (Fig. 10-17*b*). What is the angular speed of the combination?

Solution This problem is the rotational analogue of the completely inelastic collision, in which objects stick together (see Section 6-5). There is no net vertical external torque, so the vertical (z) component of angular momentum is constant. The frictional force between the disks is an internal force, which cannot change the angular momentum. Thus Eq. 10-17 applies, and we can write $I_i\omega_i = I_f\omega_f$, or

$$\omega_f = \omega_i \frac{I_i}{I_f}.$$

Without doing any detailed calculations, we know that the rotational inertia of three identical disks about their common axis will be three times the rotational inertia of a single disk. Thus $I_i/I_f = \frac{1}{3}$ and

$$\omega_f = (0.84 \text{ rev/s})(\tfrac{1}{3}) = 0.28 \text{ rev/s}.$$

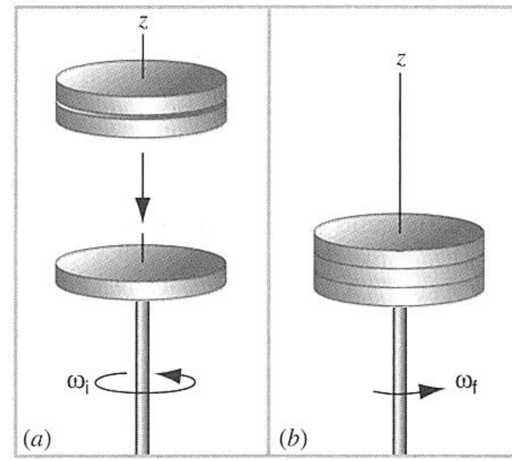


FIGURE 10-17. Sample Problem 10-5. (*a*) A disk is spinning with initial angular velocity ω_i. (*b*) Two identical disks, neither of which is initially rotating, are dropped onto the first, and the entire system then rotates with angular velocity ω_f.

Example Rolling of a Hexagonal Prism (1998 IPhO)

Consider a long, solid, rigid, regular hexagonal prism like a common type of pencil. The mass of the prism is M and it is uniformly distributed. The length of each side of the cross-sectional hexagon is a. The moment of inertia I of the hexagonal prism about its central axis is $I = 5Ma^2/12$.

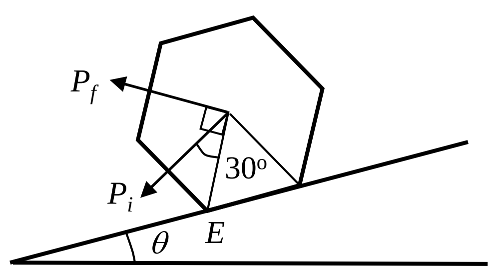
a) The prism is initially at rest with its axis horizontal on an inclined plane which makes a small angle θ with the horizontal. Assume that the surfaces of the prism are slightly concave so that the prism only touches the plane at its edges. The effect of this concavity on the moment of inertia can be ignored. The prism is now displaced from rest and starts an uneven rolling down the plane. Assume that friction prevents any sliding and that the prism does not lose contact with the plane. The angular velocity just before a given edge hits the plane is ω_i while ω_f is the angular velocity immediately after the impact. Show that $\omega_f = s\omega_i$ and find the value of s.

b) The kinetic energy of the prism just before and after impact is K_i and K_f. Show that $K_f = rK_i$ and find r.

c) For the next impact to occur, K_i must exceed a minimum value $K_{i,min}$ which may be written in the form $K_{i,min} = \delta Mga$. Find δ in terms of θ and r.

d) If the condition of part (c) is satisfied, the kinetic energy K_i will approach a fixed value $K_{i,0}$ *as the prism rolls down the incline. Show that* $K_{i,0}$ *can be written as* $K_{i,0} = \kappa Mga$ and find κ.

e) Calculate the minimum slope angle θ_0 *for which the uneven rolling, once started, will continue indefinitely.*



a) Angular momentum about edge *E* before the impact

$$L_i = I_{CM}\omega_i + M(a\omega_i)(a\sin 30^\circ) = \frac{11}{12}Ma^2\omega_i$$

Angular momentum about edge *E* after the impact

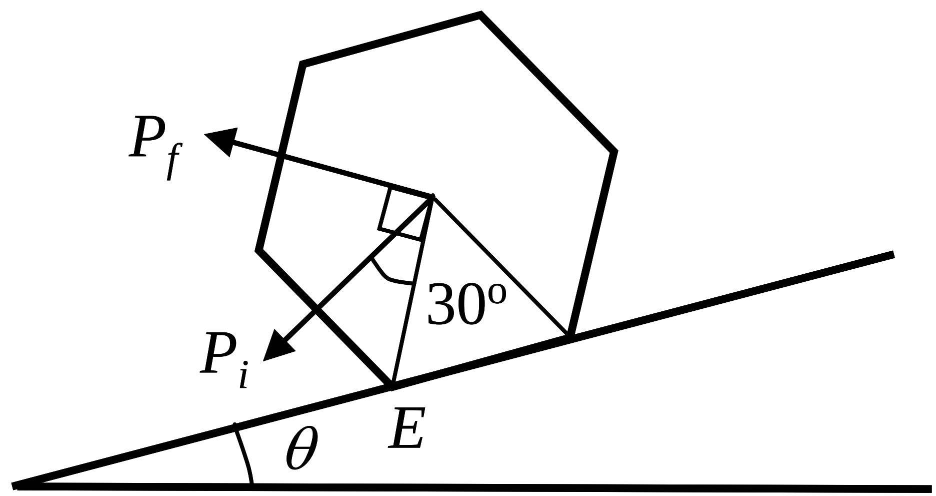
$$L_f = I_E\omega_f = \left(\frac{5}{12}Ma^2 + Ma^2\right)\omega_f = \frac{17}{12}Ma^2\omega_f$$

Using the conservation of angular momentum, $L_i = L_f$

$$\frac{11}{12}Ma^2\omega_i = \frac{17}{12}Ma^2\omega_f$$

$$\omega_f = \frac{11}{17}\omega_i$$

Thus, $s = \frac{11}{17}$



b) The kinetic energy of the prism just before and after impact is K_i and K_f. Show that $K_f = rK_i$ and find r.

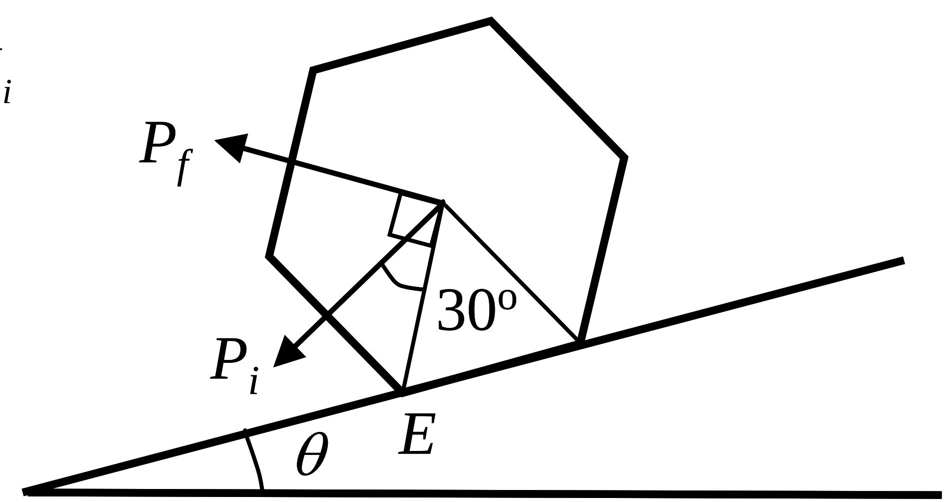
b)

$$K_i = \frac{1}{2}\left(\frac{5}{12}Ma^2 + Ma^2\right)\omega_i^2 = \frac{17}{24}Ma^2\omega_i^2$$

$$K_f = \frac{1}{2}\left(\frac{5}{12}Ma^2 + Ma^2\right)\omega_f^2 = \frac{17}{24}Ma^2\omega_f^2$$

$$K_f = \frac{\omega_f^2}{\omega_i^2}K_i = \left(\frac{11}{17}\right)^2 K_i = \frac{121}{289}K_i$$

$$r = \frac{121}{289}$$



c) For the next impact to occur, K_i must exceed a minimum value $K_{i,min}$ which may be written in the form $K_{i,min} = \delta Mga$. Find δ in terms of θ and r.

c) After the impact, the center of mass of the prism raises to its highest position by turning through an angle $90^o - (\theta + 60^o) = 30^o - \theta$. Hence

$$rK_{i,\min} = Mga - Mga\cos(30^{\circ} - \theta)$$

$$\Rightarrow$$

$$\delta = \frac{1}{r}\left[1 - \cos(30^{\circ} - \theta)\right]$$

d) If the condition of part (c) is satisfied, the kinetic energy K_i will approach a fixed value $K_{i,0}$ *as the prism rolls down the incline. Show that $K_{i,0}$ can be written as $K_{i,0}$ =* κMga and find κ.

d) At the next impact, the center of mass lowers by a height of $a\sin\theta$. Change in the kinetic energy

$$\Delta K = Mga\sin\theta$$

Kinetic energy immediately before the next impact

$$f(K_i) = Mga\sin\theta + rK_i$$

When the kinetic energy approaches $K_{i,0}$,

$$K_{i,0} = Mga\sin\theta + rK_{i,0}$$

$$K_{i,0} = \frac{Mga\sin\theta}{1-r}$$

Thus $$\kappa = \frac{\sin\theta}{1-r}$$

e) For the rolling to continue indefinitely,

$$K_{i,0} \geq K_{i,\min} \qquad \kappa \geq \delta$$

$$\frac{\sin\theta}{1-r} \geq \frac{1}{r}\left[1-\cos(30^\circ-\theta)\right]$$

$$\frac{r}{1-r}\sin\theta \geq 1-\frac{\sqrt{3}}{2}\cos\theta-\frac{1}{2}\sin\theta$$

$$\left(\frac{r}{1-r}+\frac{1}{2}\right)\sin\theta+\frac{\sqrt{3}}{2}\cos\theta \geq 1$$

$$\sqrt{(A+1/2)^2+3/4}\left[\cos u\sin\theta+\sin u\cos\theta\right] \geq 1$$

where $A = r/(1-r) = 121/168$ and

$$\sin u = \frac{\sqrt{3}/2}{\sqrt{(A+1/2)^2+3/4}} = \sqrt{\frac{3}{(205/84)^2+3}} = 0.5788$$

$\Rightarrow u \approx 35.36^\circ$

$$\sin(\theta+u) \geq \frac{1}{\sqrt{(A+1/2)^2+3/4}} = \sqrt{\frac{4}{(205/84)^2+3}} = 0.6683$$

$\Rightarrow \theta + u \geq 41.94^\circ. \quad \Rightarrow \quad \theta_0 \approx 6.58^\circ.$