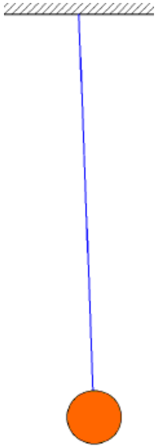
Simple Harmonic Motion

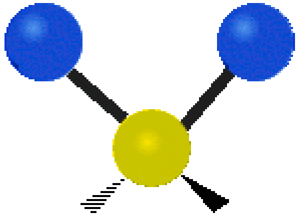
Physics Enhancement Programme for Gifted Students

The Hong Kong Academy for Gifted Education
and
Department of Physics, HKBU

Simple harmonic motion

In mechanical physics, **simple harmonic motion** is a type of *periodic motion* where the restoring force is directly proportional to the displacement. *No matter what the direction of the displacement, the force always acts in a direction to restore the system to its equilibrium position.* It can serve as a mathematical model of a variety of motions, such as the oscillation of a spring, motion of a simple pendulum as well as molecular vibration.





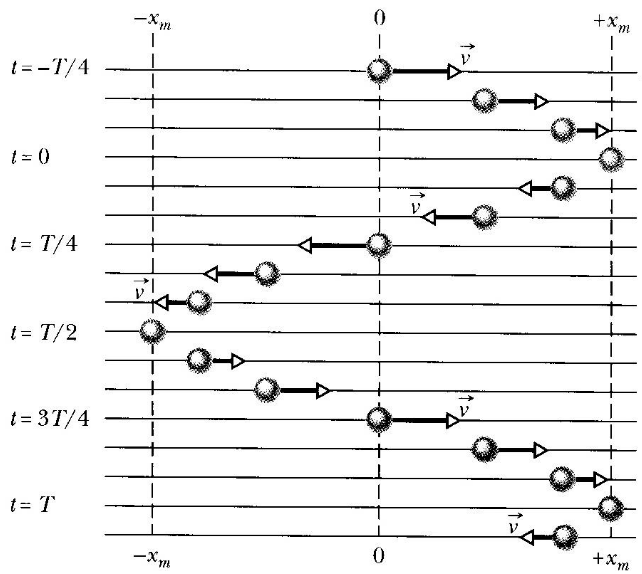
Mathematics of simple harmonic motion

Simple harmonic motion is a type of *periodic motion* which can use mathematical model to express it.

$$x(t) = x_m \cos(\omega t + \phi)$$

$$T = \frac{1}{f}$$

Displacement(position) ***x(t)***
Amplitude $\boldsymbol{x_m}$
Phase $\boldsymbol{\phi}$
Angular frequency **ω**
Frequency ***f***
Period **T**



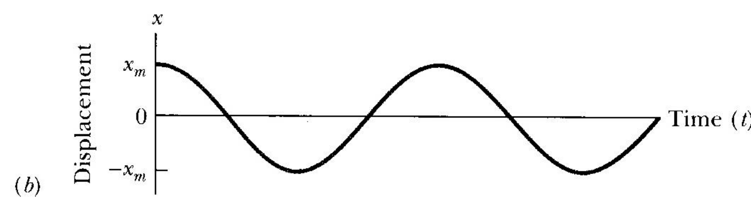


FIG. 15-1 (a) A sequence of "snapshots" (taken at equal time intervals) showing the position of a particle as it oscillates back and forth about the origin of an x axis, between the limits $+x_m$ and $-x_m$. The vector arrows are scaled to indicate the speed of the particle. The speed is maximum when the particle is at the origin and zero when it is at $\pm x_m$. If the time t is chosen to be zero when the particle is at $+x_m$, then the particle returns to $+x_m$ at $t = T$, where T is the period of the motion. The motion is then repeated. (b) A graph of x as a function of time for the motion of (a).

Mathematics of simple harmonic motion

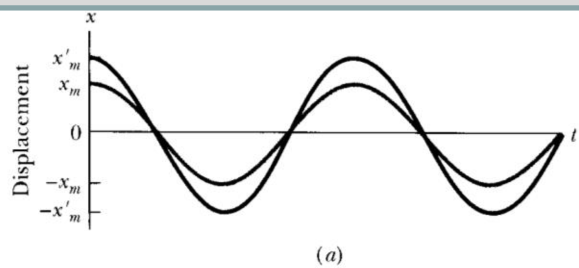
Since the motion returns to its initial value after one period T,

$$x_m \cos(\omega t + \phi) = x_m \cos[\omega(t + T) + \phi)]$$

$$\omega t + \phi + 2\pi = \omega(t + T) + \phi$$

$$\omega T = 2\pi$$

$$\boxed{\omega = \frac{2\pi}{T} = 2\pi f}$$



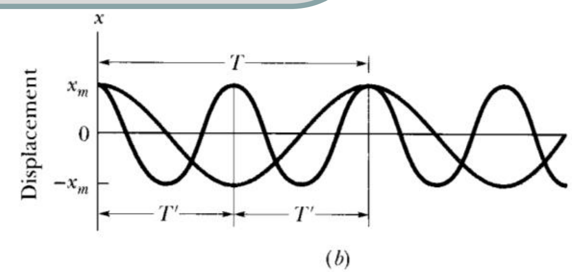
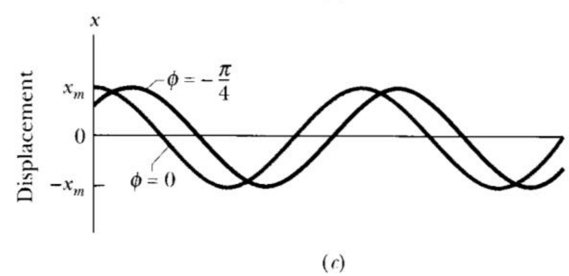


FIG. 15-3 In all three cases, the blue curve is obtained from Eq. 15-3 with $\phi = 0$. (*a*) The red curve differs from the blue curve *only* in that the red-curve amplitude x'_m is greater (the red-curve extremes of displacement are higher and lower). (*b*) The red curve differs from the blue curve *only* in that the red-curve period is $T' = T/2$ (the red curve is compressed horizontally). (*c*) The red curve differs from the blue curve *only* in that for the red curve $\phi = -\pi/4$ rad rather than zero (the negative value of ϕ shifts the red curve to the right).



Mathematics of simple harmonic motion

Velocity

$$v(t) = \frac{dx}{dt} = \frac{d}{dt}[x_m \cos(\omega t + \phi)]$$

$$v(t) = -\omega x_m \sin(\omega t + \phi)$$

Velocity amplitude $\quad v_m = \omega x_m$

Acceleration

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}[-\omega x_m \sin(\omega t + \phi)]$$

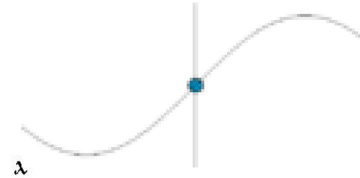
$$a(t) = -\omega^2 x_m \cos(\omega t + \phi) = -\omega^2 x(t)$$

Acceleration amplitude $\quad a_m = \omega^2 x_m$

Equation of motion

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

This equation of motion will be very useful in identifying simple harmonic motion and its frequency.



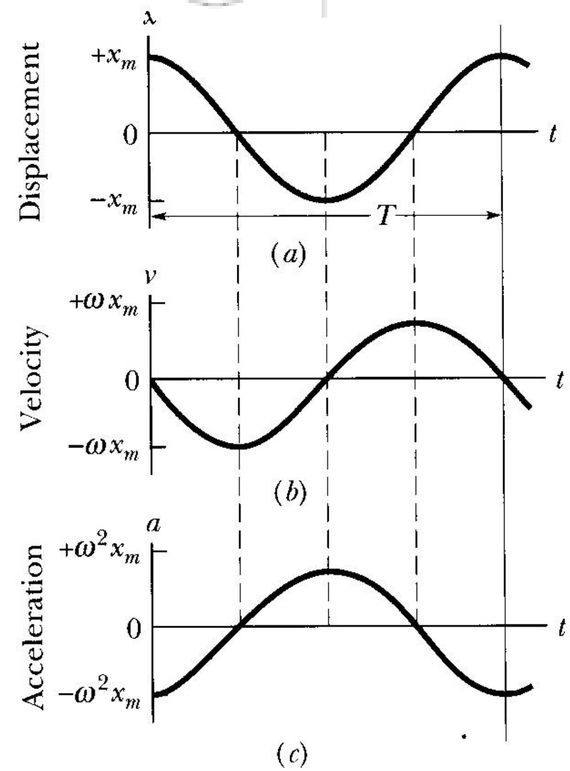


FIG. 15-4 (*a*) The displacement $x(t)$ of a particle oscillating in SHM with phase angle ϕ equal to zero. The period T marks one complete oscillation. (*b*) The velocity $v(t)$ of the particle. (*c*) The acceleration $a(t)$ of the particle.

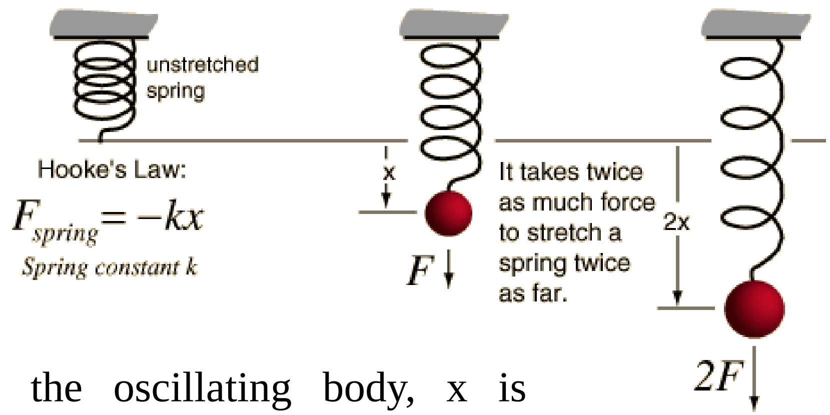
Simple harmonic motion of spring

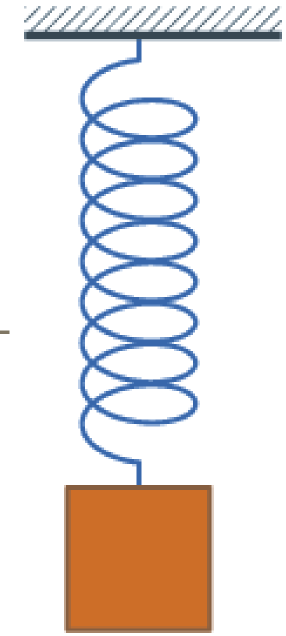
Simple harmonic motion of a mass on a spring is subject to the linear elastic restoring force given by Hooke's Law. The motion is sinusoidal in time and demonstrates a single resonant frequency.

For one-dimensional simple harmonic motion, the equation of motion, which is a second-order linear ordinary differential equation with constant coefficients, could be obtained by means of Newton's second law and Hooke's law.

$$F = -kx$$

$$F = ma = m\frac{d^2x}{dt^2} = -kx$$





where m is the inertial mass of the oscillating body, x is its displacement from the equilibrium, and k is the spring constant.

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

Simple harmonic motion of spring

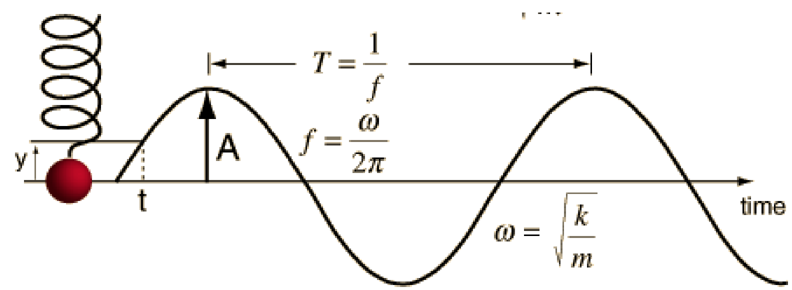
Comparing with the equation of motion for simple harmonic motion,

Angular frequency

$$\because \quad \omega^2 = \frac{k}{m} \qquad \boxed{\omega = \sqrt{\frac{k}{m}}}$$

Period

$$\because \quad T = \frac{2\pi}{\omega} \qquad \boxed{T = 2\pi\sqrt{\frac{m}{k}}}$$



Simple harmonic motion is the motion executed by a particle of mass m subject to a force that is proportional to the displacement of the particle but opposite in sign.

Examples

A block whose mass *m* is 680 g is fastened to a spring whose spring constant *k* is 65 Nm^{-1}. The block is pulled a distance x = 11 cm from its equilibrium position at x = 0 on a frictionless surface and released from rest at t = 0.

What are the angular frequency, the frequency, and the period of the resulting oscillation?

What is the amplitude of the oscillation?

What is the maximum speed of the oscillating block?

What is the magnitude of the maximum acceleration of the block?

What is the phase constant ϕ for the motion?

What is the displacement function x(t)?

(a)

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{65}{0.68}} = 9.78\,\text{rads}^{-1} \qquad f = \frac{\omega}{2\pi} = 1.56\,\text{Hz} \qquad T = \frac{1}{f} = 0.643\,\text{s}$$

(b) $x_m = 11\,\text{cm}$

(c) $v_m = \omega x_m = (9.78)(0.11) = 1.08\,\text{ms}^{-1}$

(d) $a_m = \omega^2 x_m = (9.78)^2(0.11) = 10.5\,\text{ms}^{-2}$

(e) At t = 0, $x(0) = x_m \cos\phi = 0.11$

$$v(0) = -\omega x_m \sin\phi = 0$$

$$\sin\phi = 0 \Rightarrow \quad \phi = 0$$

(f) $x(t) = x_m \cos(\omega t + \phi) = 0.11\cos(9.78t)$

Examples

At t = 0, the displacement of $x(0)$ of the block in a linear oscillator is –8.50 cm. Its velocity $v(0)$ then is –0.920 ms^{-1}, and its acceleration $a(0)$ is +47.0 ms^{-2}.

What are the angular frequency ω?

What is the phase constant ϕ and amplitude x_m?

$$x(t) = x_m \cos(\omega t + \phi) \qquad v(t) = -\omega x_m \sin(\omega t + \phi) \qquad a(t) = -\omega^2 x_m \cos(\omega t + \phi)$$

At t = 0,

$$x(0) = x_m \cos\phi = -0.085 \qquad (1)$$

$$v(0) = -\omega x_m \sin\phi = -0.920 \qquad (2)$$

$$a(0) = -\omega^2 x_m \cos\phi = +47.0 \qquad (3)$$

(3) ÷ (1):

$$\frac{a(0)}{x(0)} = -\omega^2$$

$$\omega = \sqrt{-\frac{a(0)}{x(0)}} = \sqrt{-\frac{47.0}{-0.0850}} = 23.5\,\mathrm{rads^{-1}}$$

(2) ÷ (1): $$\frac{v(0)}{x(0)} = -\omega \frac{\sin\phi}{\cos\phi} = -\omega \tan\phi$$

$$\tan\phi = -\frac{v(0)}{\omega x(0)} = -\frac{-0.920}{(23.51)(-0.085)} = -0.4603$$

$\phi = -24.7^{\circ}$ or $\phi = 180^{\circ} - 24.7^{\circ} = 155^{\circ}$

(1): $$x_m = \frac{x(0)}{\cos\phi}$$

If $\phi = -24.7^{o}$, $$x_m = \frac{-0.085}{\cos 24.7^{\circ}} = -0.094\text{ m} = -9.4\text{ cm}$$

If $\phi = 155^{o}$, $$x_m = \frac{-0.085}{\cos 155^{\circ}} = 0.094\text{ m} = 9.4\text{ cm}$$

Since x_m is positive, $\phi = 155^{o}$ *and* $x_m = 9.4$ cm.

Energy in Simple Harmonic Motion

Potential energy

$$\because \qquad x(t) = x_m \cos(\omega t + \phi)$$

$$U(t) = \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2 \cos^2(\omega t + \phi)$$

Kinetic energy

$$\because \qquad v(t) = -\omega x_m \sin(\omega t + \phi)$$

$$K(t) = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 x_m^2 \sin^2(\omega t + \phi)$$

$$\because \qquad \omega = k / m$$

$$K(t) = \frac{1}{2}kx_m^2 \sin^2(\omega t + \phi).$$

Energy in Simple Harmonic Motion

Mechanical energy

$$E = U + K = \frac{1}{2}kx_m^2\cos^2(\omega t + \phi) + \frac{1}{2}kx_m^2\sin^2(\omega t + \phi)$$

$$= \frac{1}{2}kx_m^2[\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)]$$

$$\because \quad [\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)] = 1$$

$$E = U + K = \frac{1}{2}kx_m^2$$

The mechanical energy is conserved !

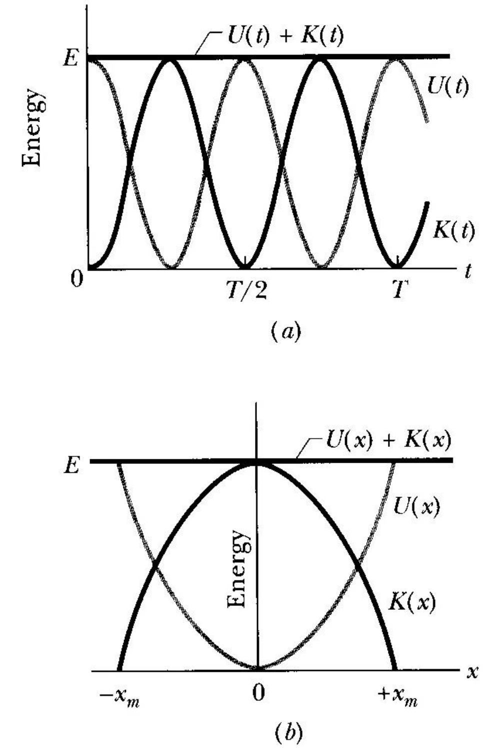


FIG. 15-6 (*a*) Potential energy $U(t)$, kinetic energy $K(t)$, and mechanical energy E as functions of time t for a linear harmonic oscillator. Note that all energies are positive and that the potential energy and the kinetic energy peak twice during every period. (*b*) Potential energy $U(x)$, kinetic energy $K(x)$, and mechanical energy E as functions of position x for a linear harmonic oscillator with amplitude x_m. For $x = 0$ the energy is all kinetic, and for $x = \pm x_m$ it is all potential.

Examples

Suppose the damper of a tall building has mass $m = 2.72 \times 105$ kg and is designed to oscillate at frequency $f = 10$ Hz and with amplitude x_m = 20 cm.

(a) What is the total mechanical energy E of the damper?

(b) What is the speed of the damper when it passes through the equilibrium point?

(a) $k = m\omega^2 = m(2\pi f)^2$

$$= (2.72\times10^5)(20\pi)^2 = 1.073\times10^9 \text{ N}$$

The energy:

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$= 0 + \frac{1}{2}(1.073\times10^9)(0.2)^2$$

$$= 2.147\times10^7 \text{ J} \approx 21.5\,\text{MJ}$$

(b) Using the conservation of energy,

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$2.147\times10^7 = \frac{1}{2}(2.72\times10^5)v^2 + 0$$

$$v = 12.6\,\text{ms}^{-1}$$

An Angular Simple Harmonic Oscillator

When the suspension wire is twisted through an angle θ, the torsional pendulum produces a restoring torque given by

$$\tau = -\kappa\theta.$$

κ is called the torsion constant.

Using Newton's law for angular motion,

$$\tau = I\alpha,$$

$$-\kappa\theta = I\alpha,$$

$$\frac{d^2\theta}{dt^2} + \frac{\kappa}{I}\theta = 0.$$

Comparing with the equation of motion for simple harmonic motion,

$$\omega^2 = \frac{\kappa}{I}.$$

Since $T = \frac{2\pi}{\omega}$, $T = 2\pi\sqrt{\frac{I}{\kappa}}.$

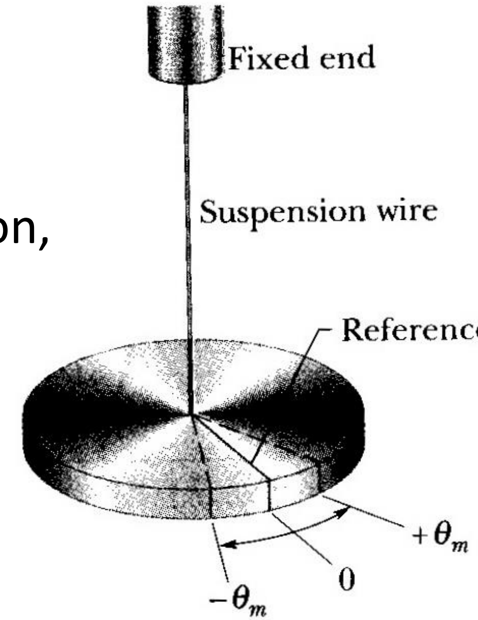
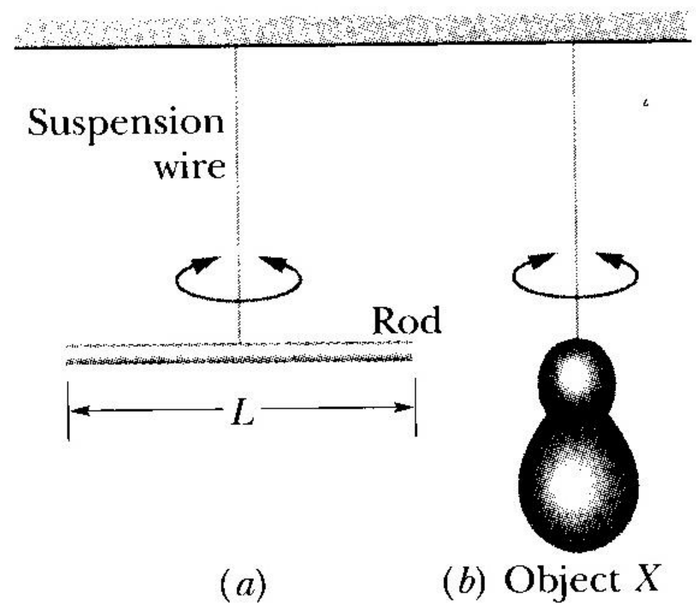


FIG. 15-7 A torsion pendulum is an angular version of a linear simple harmonic oscillator. The disk oscillates in a horizontal plane; the reference line oscillates with angular amplitude θ_m. The twist in the suspension wire stores potential energy as a spring does and provides the restoring torque.

Example

A thin rod whose length L is 12.4 cm and whose mass m is 135 g is suspended at its midpoint from a long wire. Its period T_a of angular SHM is measured to be 2.53 s. An irregularly shaped object, which we call X, is then hung from the same wire, and its period T_b is found to be 4.76 s. What is the rotational inertia of object X about its suspension axis?



IG. 15-8 Two torsion endulums, consisting of a) a wire and a rod and b) the same wire and an tregularly shaped object.

Rotational inertia of the rod about the center

$$= I_a = \frac{1}{12}ML^2$$

$$= \left(\frac{1}{12}\right)(0.135)(0.124)^2$$

$$= 1.7298 \times 10^{-4} \text{ kgm}^2$$

Since $T_a = 2\pi\sqrt{\frac{I_a}{\kappa}}$ and $T_b = 2\pi\sqrt{\frac{I_b}{\kappa}}$

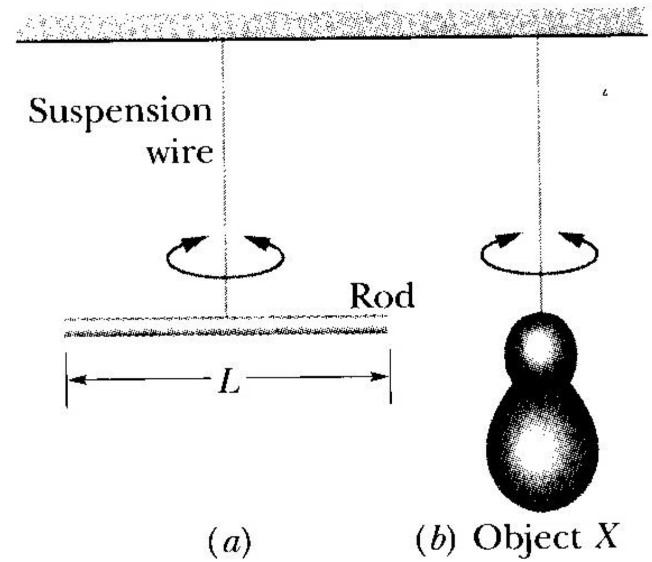
Thus $\frac{T_a}{T_b} = \sqrt{\frac{I_a}{I_b}}$

Therefore,

$$I_b = \left(\frac{T_b}{T_a}\right)^2 I_a$$

$$= \left(\frac{4.76}{2.53}\right)^2 (1.73 \times 10^{-4}) = 6.12 \times 10^{-4} \text{ kgm}^2$$

IG. 15-8 Two torsion endulums, consisting of a) a wire and a rod and b) the same wire and an rregularly shaped object.



The Simple Pendulum

The restoring torque about the point of suspension is $\tau = -mg\sin\theta L$.

Using Newton's law for angular motion, $\tau = I\alpha$,

$$-mg\sin\theta L = mL^2\alpha,$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0.$$

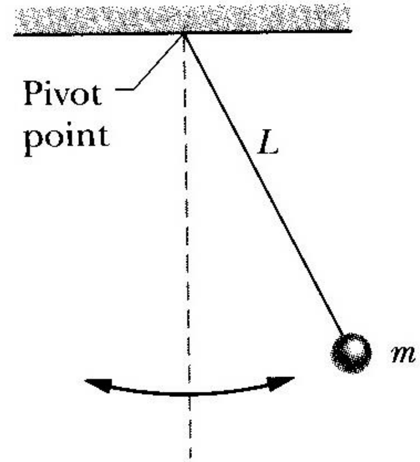
When the pendulum swings through a small angle, $\sin\theta \approx \theta$. Therefore

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0.$$

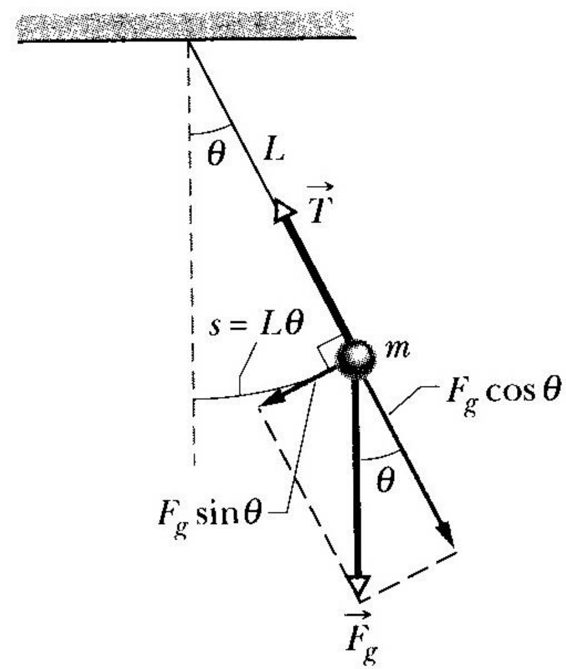
Comparing with the equation of motion for simple harmonic motion,

$$\omega^2 = \frac{g}{L}.$$

Since $T = \dfrac{2\pi}{\omega}$, $\quad T = 2\pi\sqrt{\dfrac{L}{g}}.$



(a)



(b)

The Physical Pendulum

The restoring torque about the point of suspension is $\tau = -mg\sin\theta\, h$.

Using Newton's law for angular motion,

$\tau = I\alpha$,

$$-mg\sin\theta\, h = I\alpha,$$

$$\frac{d^2\theta}{dt^2} + \frac{mgh}{I}\sin\theta = 0.$$

When the pendulum swings through a small angle, $\sin\theta \approx \theta$. Therefore

$$\frac{d^2\theta}{dt^2} + \frac{mgh}{I}\theta = 0.$$

Comparing with the equation of motion for simple harmonic motion,

$$\omega^2 = \frac{mgh}{I}.$$

Since $T = \dfrac{2\pi}{\omega}$, $\quad T = 2\pi\sqrt{\dfrac{I}{mgh}}$.

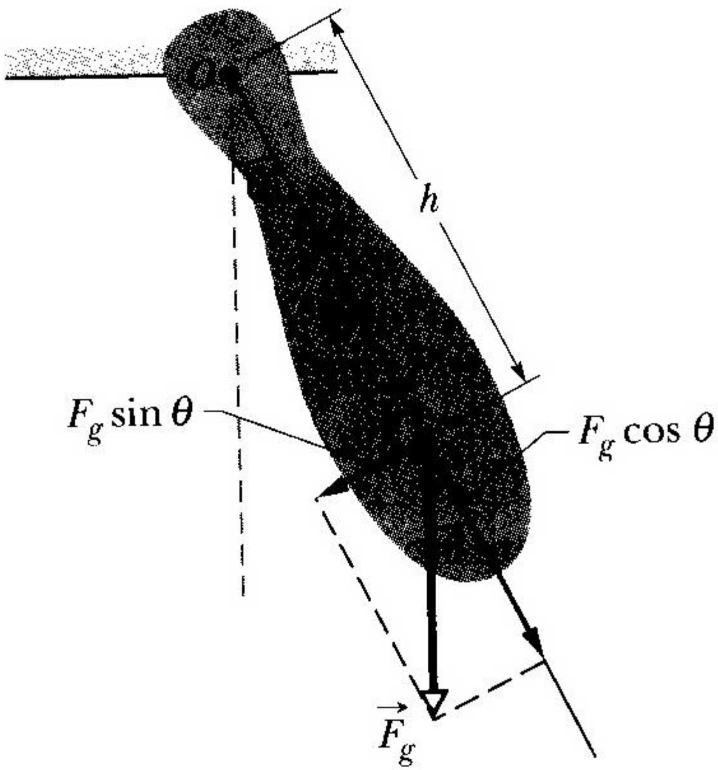


FIG. 15-10 A physical pendulum. The restoring torque is $hF_g \sin\theta$. When $\theta = 0$, center of mass C hangs directly below pivot point O.

If the mass is concentrated at the center of mass *C*, such as in the simple pendulum, then

$$T = 2\pi\sqrt{\frac{I}{mgh}} = 2\pi\sqrt{\frac{mL^2}{mgL}} = 2\pi\sqrt{\frac{L}{g}}.$$

We recover the result for the simple pendulum.

Examples

A meter stick, suspended from one end, swings as a physical pendulum.

(a) What is its period of oscillation T?

(b) A simple pendulum oscillates with the same period as the stick. What is the length L_0 of the simple pendulum?

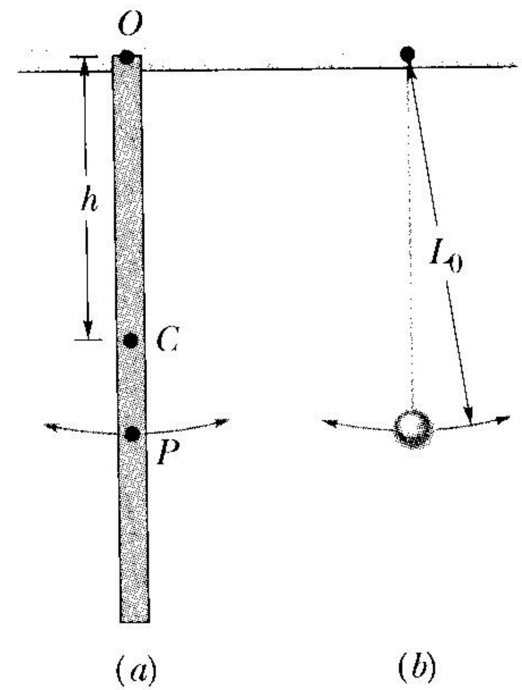


FIG. 15-11 (*a*) A meter stick suspended from one end as a physical pendulum. (*b*) A simple pendulum whose length L_0 is chosen so that the periods of the two pendulums are equal. Point *P* on the pendulum of (*a*) marks the center of oscillation.

(a) Rotational inertia of a rod about one end

$$= \frac{1}{3}ML^2$$

Period $$T = 2\pi\sqrt{\frac{I}{mgh}}$$

$$= 2\pi\sqrt{\frac{mL^2/3}{mgL/2}}$$

$$= 2\pi\sqrt{\frac{2L}{3g}}$$

(b) For a simple pendulum of length L_0,

$$T = 2\pi\sqrt{\frac{L_0}{g}}$$

$$2\pi\sqrt{\frac{L_0}{g}} = 2\pi\sqrt{\frac{2L}{3g}}$$

$$\Rightarrow \quad L_0 = \frac{2}{3}L = 66.7\ \text{cm}$$

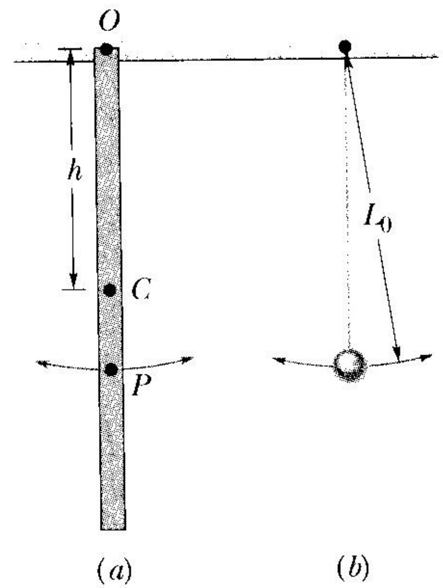


FIG. 15-11 (*a*) A meter stick suspended from one end as a physical pendulum. (*b*) A simple pendulum whose length L_0 is chosen so that the periods of the two pendulums are equal. Point *P* on the pendulum of (*a*) marks the center of oscillation.

Examples

A physical pendulum has a radius of gyration k. When it is suspended at distances l and l' from the center of mass, the periods of oscillation are the same. (a) Find the relation between l and l'. (b) This has been used to determine g accurately. Find an expression for g.

(a) When it is suspended at a distance *l* from the center of mass,

$$I = Mk^2 + Ml^2$$

$$T = 2\pi\sqrt{\frac{Mk^2 + Ml^2}{Mgl}} = 2\pi\sqrt{\frac{k^2 + l^2}{gl}}$$

Similarly, $$T' = 2\pi\sqrt{\frac{k^2 + l'^2}{gl'}}$$

Equating *T* and *T'*, $$\frac{k^2 + l^2}{l} = \frac{k^2 + l'^2}{l'}$$

$$l'k^2 + l^2 l' = lk^2 + ll'^2$$

$$ll' = k^2$$

(b) Substituting into the expression of *T*,

$$T = 2\pi\sqrt{\frac{k^2 + l^2}{gl}} = 2\pi\sqrt{\frac{l + l'}{g}}$$

$$g = 4\pi^2 \frac{l + l'}{T^2}$$

Examples

A diver steps on the diving board and makes it move downwards. As the board rebounds back through the horizontal, she leaps upward and lands on the free end just as the board has completed 2.5 oscillations during the leap. (With such timing, the diver lands when the free end is moving downward with greatest speed. The landing then drives the free end down substantially, and the rebound catapults the diver high into the air.) Modeling the spring board as the rod-spring system (Fig. 15-12(d)), what is the required spring constant *k*? Given *m* = 20 kg, diver's leaping time t_{fl} = 0.62 s.

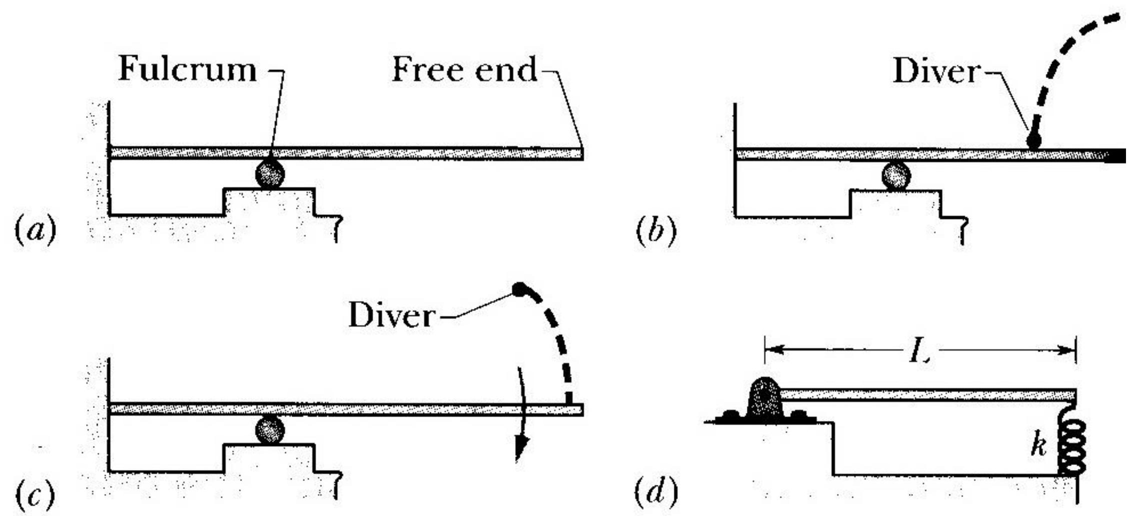


FIG. 15-12 (*a*) A diving board. (*b*) The diver leaps upward and forward as the board moves through the horizontal. (*c*) The diver lands 2.5 oscillations later. (*d*) A spring-oscillator model of the oscillating board.

When the board is displaced by an angle θ,

The restoring torque:

$$\tau = -kxL = -kL^2 \sin\theta$$
$$\approx -kL^2\theta$$

Using Newton's law for angular motion,

$$\tau = I\alpha$$

$$-kL^2\theta = \frac{1}{3}mL^2\frac{d^2\theta}{dt^2}$$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} + \frac{3k}{m}\theta = 0$$

Comparing with the equation of motion for simple harmonic motion,

$$\omega^2 = \frac{3k}{m} \qquad \Rightarrow \qquad k = \frac{m\omega^2}{3} = \frac{m}{3}\left(\frac{2\pi}{T}\right)^2$$

The period should be $T = \frac{t_{fl}}{2.5} = \frac{0.62}{2.5}$

Therefore $k = \frac{20}{3}\left(\frac{2\pi}{0.62/2.5}\right)^2 = 4280\ \mathrm{Nm}^{-1}$

Damped Simple Harmonic Motion

The liquid exerts a damping force proportional to the velocity. Then,

$F_d = -bv$, b = damping constant.

Using Newton's second law,

$$-bv - kx = ma.$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0.$$

Solution: $x(t) = x_m e^{-bt/2m}\cos(\omega' t + \phi)$,

where $\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$.

If $b = 0$, ω' reduces to $\omega = \sqrt{k/m}$

of the undamped oscillator.

If $b << \sqrt{km}$, then $\omega' \approx \omega$.

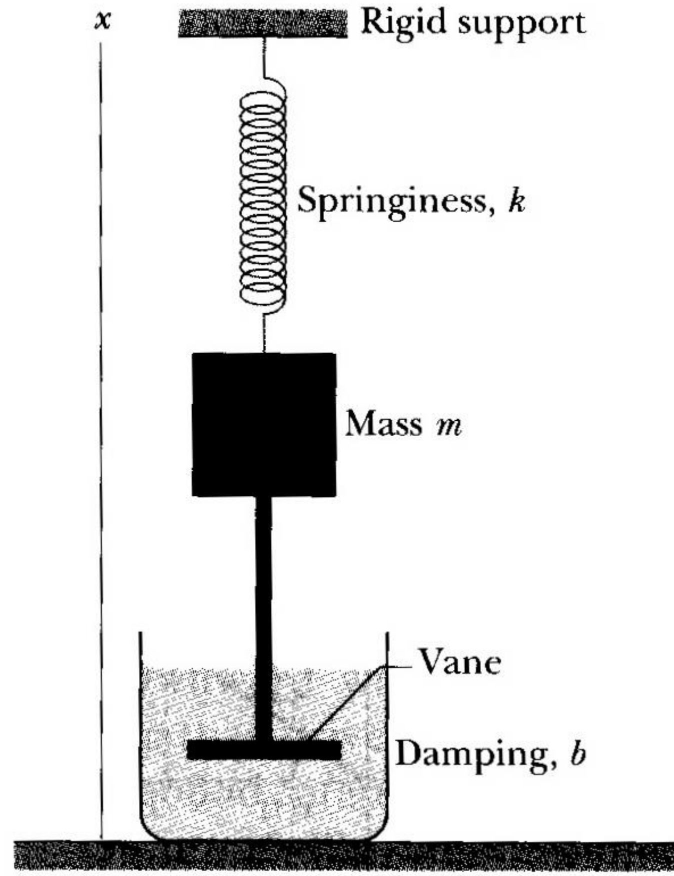


FIG. 15-15 An idealized damped simple harmonic oscillator. A vane immersed in a liquid exerts a damping force on the block as the block oscillates parallel to the x axis.

The amplitude, $x(t) = x_m e^{-bt/2m}$, gradually decreases with time.

The mechanical energy decreases exponentially with time.

$$E(t) = \frac{1}{2} k x_m^2 e^{-bt/m}.$$

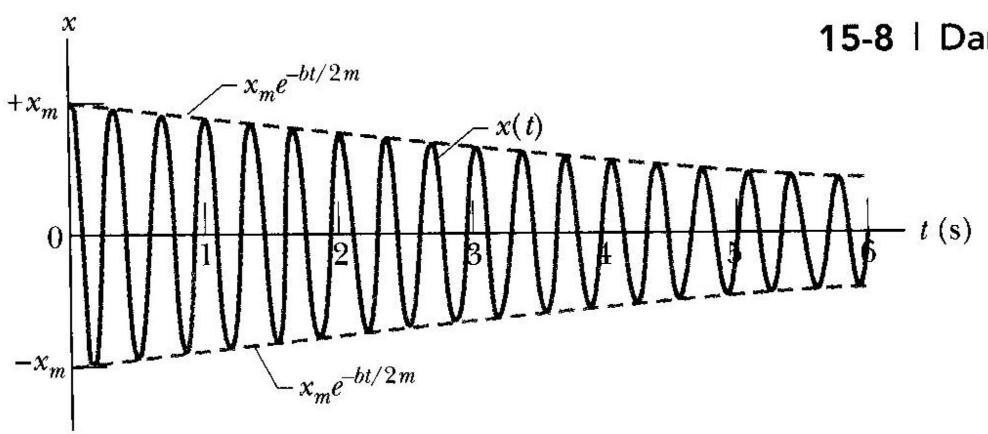


FIG. 15-16 The displacement function $x(t)$ for the damped oscillator of Fig. 15-15, with values given in Sample Problem 15-7. The amplitude, which is $x_m e^{-bt/2m}$, decreases exponentially with time.

Example

For the damped oscillator with m = 250 g, k = 85 Nm^{-1}, and b = 70 gs^{-1}.

(a) What is the period of the motion?

(b) How long does it take for the amplitude of the damped oscillations to drop to half its initial value?

(c) How long foes it take for the mechanical energy to drop to half its initial value?

(a) $$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.25}{85}} = 0.34\,\text{s}$$

(b) When the amplitude drops by half, $$x_m e^{-bt/2m} = \frac{1}{2}x_m$$

$$e^{-bt/2m} = \frac{1}{2}$$

Taking logarithm, $$-\frac{bt}{2m} = \ln\frac{1}{2} = -\ln 2$$

$$t = \frac{2m\ln 2}{b} = \frac{(2)(0.25)(\ln 2)}{0.07} = 4.95\,\text{s}$$

(c) When the energy drops by half,

$$\frac{1}{2}kx_m^2 e^{-bt/m} = \frac{1}{2}\left(\frac{1}{2}kx_m^2\right)$$

$$e^{-bt/m} = \frac{1}{2}$$

Taking logarithm, $-\frac{bt}{m} = \ln\frac{1}{2} = -\ln 2$

$$t = \frac{m\ln 2}{b} = \frac{(0.25)(\ln 2)}{0.07} = 2.48\,\text{s}$$

Forced Oscillations and Resonance

When a simple harmonic oscillator is driven by a periodic external force, we have *forced oscillations* or *driven oscillations*.
Its behavior is determined by *two* angular frequencies:

(1) the *natural* angular frequency $\omega = \sqrt{k/m}$

(2) the angular frequency ω_d of the external driving force.
The motion of the forced oscillator is given by $x(t) = x_m \cos(\omega_d t + \phi)$.

Substituting into the equation of motion,

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F\cos\omega_d t,$$

$$x_m\left[(k - m\omega_d^2)\cos(\omega_d t + \phi) - b\omega_d \sin(\omega_d t + \phi)\right] = F\cos\omega_d t.$$

Using the identity $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$$\sqrt{(k - m\omega_d^2)^2 + b^2\omega_d^2}\, x_m \cos(\omega_d t + \phi + \alpha) = F\cos\omega_d t,$$

where $\cos\alpha = \dfrac{b\omega_d}{\sqrt{(k - m\omega_d^2)^2 + b^2\omega_d^2}}$.

Hence $x_m = \dfrac{F}{\sqrt{m^2(\omega^2 - \omega_d^2)^2 + b^2\omega_d^2}}$, and $\phi = -\alpha$.

(1) It oscillates at the angular frequency ω_d of the external driving force.

(2) Its amplitude x_m is greatest when $\omega_d = \omega$.

This is called **resonance**.

See Youtube "Tacoma Bridge Disaster".

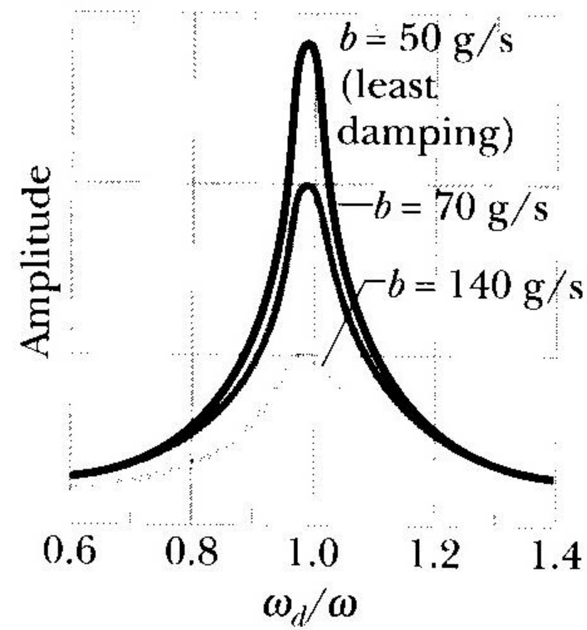
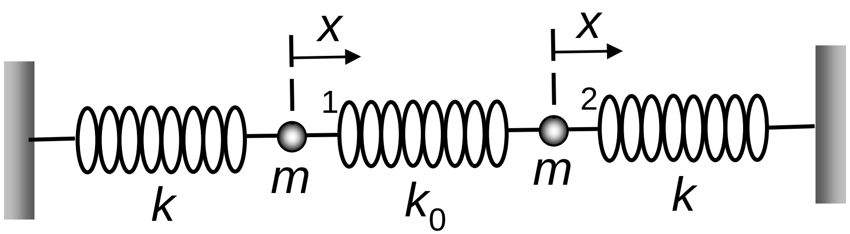


FIG. 15-17 The displacement amplitude x_m of a forced oscillator varies as the angular frequency ω_d of the driving force is varied. The curves here correspond to three values of the damping constant b.

Two Coupled Oscillators and Normal Coordinates



Using Newton's second law,

$$m\ddot{x}_1 = -kx_1 + k_0(x_2 - x_1),$$

$$m\ddot{x}_2 = -kx_2 + k_0(x_1 - x_2).$$

Possible solution, $x_1 = \mathrm{Re} A e^{-i\omega t}$ and $x_2 = \mathrm{Re} B e^{-i\omega t}$ (*A* and *B* are complex.)

It is convenient to adopt the third trial solution. Then

$$(k + k_0 - m\omega^2)A - k_0 B = 0,$$

$$(k + k_0 - m\omega^2)B - k_0 A = 0.$$

For non-trivial solutions, we have $\dfrac{A}{B} = \dfrac{k_0}{k + k_0 - m\omega^2} = \dfrac{k + k_0 - m\omega^2}{k_0}.$

More generally, we can use the matrix form: $\begin{pmatrix} k + k_0 - m\omega^2 & -k_0 \\ -k_0 & k + k_0 - m\omega^2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$

and for non-trivial solutions, $\begin{vmatrix} k+k_0-m\omega^2 & -k_0 \\ -k_0 & k+k_0-m\omega^2 \end{vmatrix} = 0.$

Either way, we arrive at a *secular equation*, $(k+k_0-m\omega^2)^2 - k_0^2 = 0,$

$$\omega^2 = \frac{k}{m}, \quad \text{or} \quad \omega^2 = \frac{k+2k_0}{m}.$$

If $\omega = \omega_1 \equiv \sqrt{\frac{k}{m}},$ $A = B,$

If $\omega = \omega_2 \equiv \sqrt{\frac{k+2k_0}{m}},$ $A = -B.$

Hence we obtain two solutions. In each solution, the two particles oscillate with the same frequency. They are called *normal modes*. Their frequencies are called *normal frequencies*. Any other solutions are combinations of the normal modes.

Symmetric mode: $\omega_1 = \sqrt{\frac{k}{m}}$ and $x_1 = x_2 = A\cos(\omega t + \phi)$

Antiymmetric mode: $\omega_1 = \sqrt{\frac{k + 2k_0}{m}}$ $x_1 = A\cos(\omega t + \phi)$ and $x_2 = -A\cos(\omega t + \phi)$

In general, the mode that has the highest symmetry will have the lowest frequency, while the antisymmetric mode has the highest frequency.

The symmetric mode can be excited by pulling the two particles from their equilibrium positions by equal amounts in the same direction so that

$x_1(0) = x_2(0) = A$ and $\dot{x}_1(0) = \dot{x}_2(0) = 0$

The antisymmetric mode can be excited by pulling apart the two particles equally in opposite directions and then released, so that

$x_1(0) = -x_2(0) = A$ and $\dot{x}_1(0) = \dot{x}_2(0) = 0$

Examples

Find the frequencies of small oscillations of a double pendulum.
Tangential component of forces acting on the upper particle:

$$F_t = -mg\sin\theta_1 + T\sin(\theta_2 - \theta_1)$$

For small oscillations, $T \approx mg$,

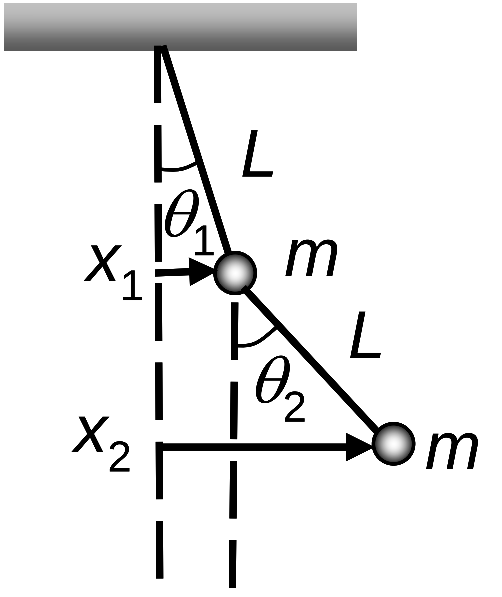
$$\sin\theta_1 \approx \frac{x_1}{L} \qquad \sin\theta_2 \approx \frac{x_2 - x_1}{L} \qquad \sin(\theta_2 - \theta_1) \approx \frac{x_2 - 2x_1}{L}$$

Using Newton's second law,

$$mL\ddot{\theta}_1 = -F_t \quad \Rightarrow \quad m\ddot{x}_1 \approx -\frac{mg}{L}x_1 + \frac{mg}{L}(x_2 - 2x_1)$$

$$mL\ddot{\theta}_2 = -mg\sin\theta_2 \quad \Rightarrow \quad m\ddot{x}_2 = -\frac{mg}{L}(x_2 - x_1)$$

$$\begin{pmatrix}\ddot{x}_1 \\ \ddot{x}_2\end{pmatrix} = \begin{pmatrix}-3g/L & g/L \\ g/L & -g/L\end{pmatrix}\begin{pmatrix}x_1 \\ x_2\end{pmatrix}$$



Let $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathrm{Re}\begin{pmatrix} A \\ B \end{pmatrix} e^{-i\omega t}$

Then $\begin{pmatrix} 3g/L-\omega^2 & -g/L \\ -g/L & g/L-\omega^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

For non-trivial solutions, $\begin{vmatrix} 3g/L-\omega^2 & -g/L \\ -g/L & g/L-\omega^2 \end{vmatrix} = 0$

Symmetric mode: $\omega^2 = (2-\sqrt{2})g/L$ and $B = \sqrt{2}A$

Antisymmetric mode: $\omega^2 = (2+\sqrt{2})g/L$ and $B = -\sqrt{2}A$

Examples

Consider two pendula of length *L* and mass *m* coupled by a spring with force constant *k*. Find the normal frequencies, the normal modes and the general solution.

Symmetric mode:

$$\omega_1^2 = \frac{g}{L}$$, *A* = *B*

Antisymmetric mode:

$$\omega_2^2 = \frac{g}{L} + \frac{2k}{m}$$, *A* = –*B*