Transverse waves

Physics Enhancement Programme for Gifted Students

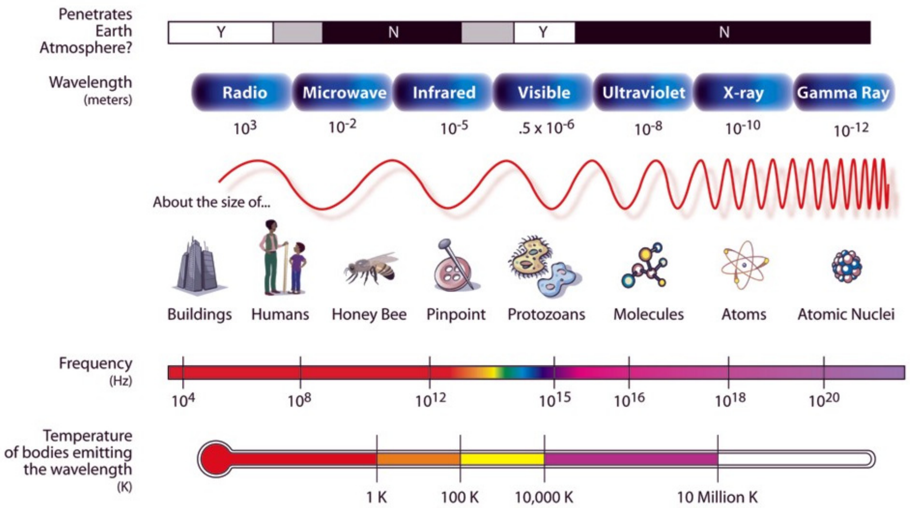
The Hong Kong Academy for Gifted Education
and
Department of Physics, HKBU

Waves

1. **Mechanical waves**
   - e.g. water waves, sound waves, seismic waves, strings in musical instruments
2. **Electromagnetic waves**
   - light (ultraviolet, visible, infrared), mcrowaves, radio waves, television waves, X-rays
3. **Matter (=quantum) waves**
   - electrons, protons, other fundamental particles, atoms and molecules
4. **Gravity waves ← never observed!**

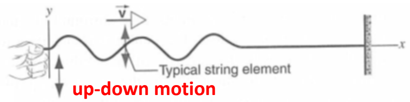
Electromagnetic Spectrum

❖ EM waves are transverse.



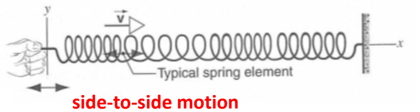
Wave motion

(1) Transverse wave

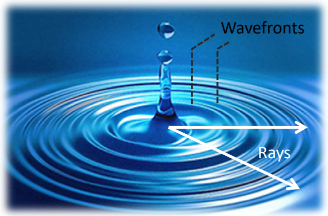


- Sending a transverse wave along a string.
- Each element of the string vibrates at right angles to the propagation direction of the wave.

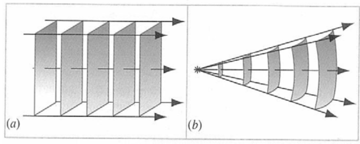
(2) Longitudinal wave



- Sending a longitudinal wave along a spring.
- Each element of the spring vibrates parallel to the propagation direction of the wave.



❖ Wavefronts → represented by ripples
❖ Rays → direction of wave motion, perpendicular (⊥) to the wavefronts



Plane wave Spherical wave

❖ Wavefronts → represented by planes, spaced one wavelength apart.
❖ Rays → direction of wave motion, ⊥ to the wavefronts, not observable!

Extra information

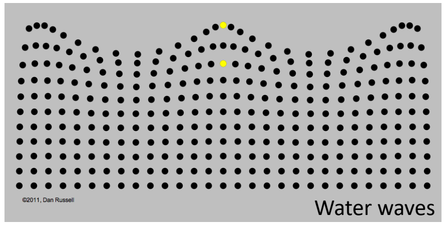
Waves with both longitudinal and transverse motions

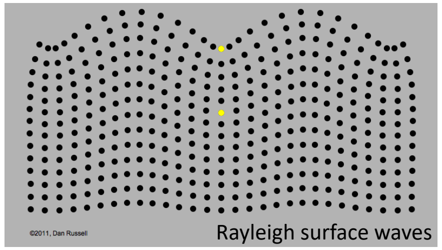
❖ **Water waves**

- In a water wave, all particles travel in clockwise circles .
(see the yellow dots)

❖ **Rayleigh surface waves**

- a type of waves during earthquake, moving in elliptical paths
- particles at the surface trace out a counter-clockwise ellipse; while particles at a depth could trace out clockwise ellipses
(see the yellow dots)



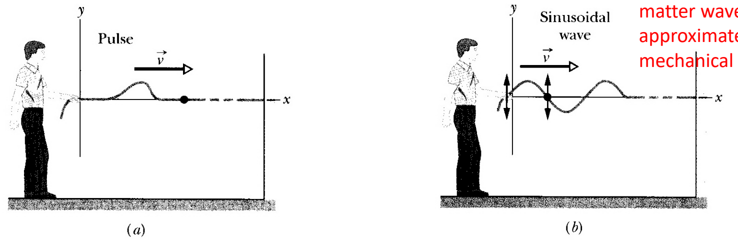


Transverse waves

❖ **Transverse waves**

- the displacement of a point on the string is *perpendicular* (⊥) to the direction of the travelling wave
- up-down motion

ideal wave form (true for EM and matter waves, approximate for mechanical waves)



- A single pulse is sent along a stretched string.
- A typical string element (see the dot) moves up and then down once as the pulse passes.
- element's motion ⊥ wave's direction

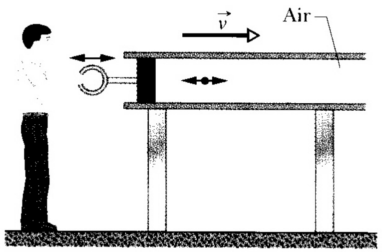
- A "sine" wave is sent along the string.
- A typical string element moves up and then down continuously as the wave passes.
- element's motion ⊥ wave's direction

transverse wave

Longitudinal waves

❖ **Longitudinal waves**

- the displacement of a particle is *parallel* (//) to the direction of travel of the wave
- side-to-side motion of the particle



- A sound wave is set up in an air-filled pipe by moving a piston back and forth.
- The oscillation of an element of the air is back and forth as well.
- element's motion // wave's direction

↓

longitudinal wave

Equation of wave propagation

Suppose that at time $t = 0$, a travelling wave has the form

$$y(x,0) = y_m \sin kx$$

, where y_m is the amplitude and k is the wavenumber.

At time t, the travelling wave will have the same form, except that it is displaced along the positive x direction by a displacement vt, where v is the wave speed.

Hence the displacement at position x and time t is given by

❖ $$y(x,t) = y_m \sin\left[k(x - vt)\right]$$

This is usually written as (for sinusoidal waves)

❖ $$y(x,t) = y_m \sin(kx - \omega t)$$, where $\omega = kv$ or $v = \frac{\omega}{k}$.

phase (= $kx - \omega t$)

Wave equation → a function of position and time which gives the height of the wave at any position x and any time t

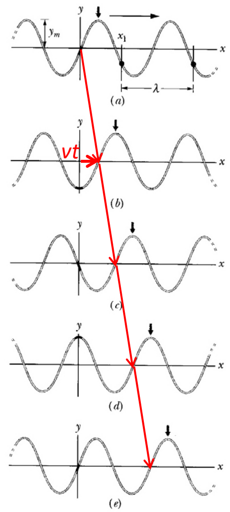


FIG. 16-5 Five "snapshots" of a string wave traveling in the positive direction of an x axis. The amplitude y_m is indicated. A typical wavelength λ, measured from an arbitrary position x_1, is also indicated.

Wavenumber k

Suppose that at $t = 0$, a travelling wave has the form

$$y(x,0) = y_m \sin kx$$

Since the waveform repeats itself when <u>displaced by one wavelength (λ)</u>,

$$y_m \sin kx = y_m \sin\left[k\left(x+\lambda\right)\right] = y_m \sin\left(kx + k\lambda\right)$$

Thus, $k\lambda = 2\pi$, which gives the wavenumber

$$k = \frac{2\pi}{\lambda}$$

related to wavelength (i.e. distance for one wave cycle)

Similar to angular frequency $\omega = \frac{2\pi}{T}$

related to period (i.e. time for one wave cycle)

k is also called angular wavenumber.

Angular frequency ω

At $x = 0$, the wave function becomes

$$y(0,t) = -y_m \sin \omega t$$

Since the waveform repeats itself when <u>delayed by one period (T)</u>,

$$-y_m \sin \omega t = -y_m \sin\left[\omega\left(t+T\right)\right] = -y_m \sin\left(\omega t + \omega T\right)$$

Thus, $\omega T = 2\pi$, which gives the angular frequency

$$\omega = \frac{2\pi}{T}$$ Unit: rad/s

, and the frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$. Unit: 1/s = Hertz = Hz

Wave speed v

Since $k = 2\pi/\lambda$ and $\omega = 2\pi/T$, the wave speed is

$$v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f$$

The wave travels by a distance of one wavelength in one period.

Since $y(x, t) = y_m \sin(kx - \omega t)$, the peak of the travelling wave is described by

$$kx - \omega t = \frac{\pi}{2}$$

In general, any point on the waveform, as the wave moves in space and time, is described by:

$$kx - \omega t = \text{constant}$$

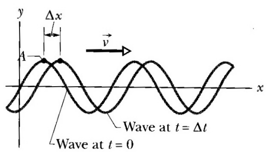


FIG. 16-8 Two snapshots of the wave of Fig. 16-5, at time $t = 0$ and then at time $t = \Delta t$. As the wave moves to the right at velocity $\vec{v}$, the entire curve shifts a distance Δx during Δt. Point A "rides" with the wave form, but the string elements move only up and down.

Wave equation

A wave travelling towards positive x direction is described by

$$y(x,t) = y_m \sin\left[k\left(x - vt\right)\right] = y_m \sin\left(kx - \omega t\right)$$

A point on the waveform, as the wave moves in space and time, is described by $kx - \omega t$ = constant.

A wave travelling towards negative x direction is described by

$$y(x,t) = y_m \sin\left[k\left(x + vt\right)\right] = y_m \sin\left(kx + \omega t\right)$$

A point on the waveform, as the wave moves in space and time, is described by $kx + \omega t$ = constant.

Example 1

A transverse wave travelling along a string is described by

$$y(x,t) = 0.00327\sin(72.1x - 2.72t)$$

, in which the numerical constants are in SI units.

(a) What is the amplitude of this wave?
(b) What are the wavelength, period, and frequency of this wave?
(c) What is the velocity of this wave?
(d) What is the displacement y at x = 22.5 cm and t = 18.9 s?
(e) What is the transverse velocity u of this element of the string at the place time in (d)?
(f) What is the transverse acceleration a_y at the position and time in (d)?

Answers of Example 1

assume sinusoidal waves

$$y(x,t) = \underbrace{0.00327}_{y_m}\sin(\underbrace{72.1}_{k}x - \underbrace{2.72}_{\omega}t)$$

(a) $y_m = 0.00327\text{ m} = 3.27\text{ mm}$

(b) $\lambda = \frac{2\pi}{k} = \frac{2\pi}{72.1} = 0.0871 = 8.71\text{ cm}$, $T = \frac{2\pi}{\omega} = \frac{2\pi}{2.72} = 2.31\text{ s}$, $f = \frac{1}{T} = \frac{1}{2.31} = 0.433\text{ Hz}$

(c) $v = \frac{\omega}{k} = \frac{2.72}{72.1} = 0.0377 = 3.77\text{ cms}^{-1}$

(d) $y(x,t) = 0.00327\sin(72.1\times0.225 - 2.72\times18.9) = 0.00192 = 1.92\text{ mm}$

(e)
$$u = \frac{\partial y}{\partial t} = -\omega y_m\cos(kx - \omega t)$$
$$= -(2.72)(0.00327)\cos(72.1\times0.225 - 2.72\times18.9)$$
$$= 7.20\text{ mms}^{-1}$$

(f)
$$a_y = \frac{\partial u}{\partial t} = -\omega^2 y_m\sin(kx - \omega t) = -\omega^2 y$$
$$= -(2.72)^2(0.00192) = -0.0142 = -14.2\text{ mms}^{-2}$$

Section Summary

❖ Transverse waves
- displacement of a point ⊥ wave propagation direction

❖ Longitudinal waves
- displacement of a point // wave propagation direction

❖ Equation of wave propagation (towards +ve x direction)

$$y(x,t) = y_m\sin[k(x - vt)] \quad \text{or} \quad y(x,t) = y_m\sin(kx - \omega t)$$

❖ Equation of wave propagation (towards -ve x direction)

$$y(x,t) = y_m\sin(kx + \omega t)$$

❖ Wavenumber, angular frequency, frequency, wave speed

$$k = \frac{2\pi}{\lambda} \qquad \omega = \frac{2\pi}{T} \qquad f = \frac{1}{T} = \frac{\omega}{2\pi} \qquad v = \frac{\omega}{k} = f\lambda$$

Wave speed on a Stretched String

Consider the peak of a wave travelling from left to right on the stretched string.
If we observe the wave from a reference frame moving at the wave speed v, the peak becomes stationary, but the string moves from right to left with speed v. Find the speed v.

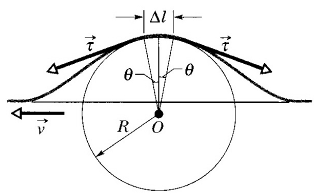


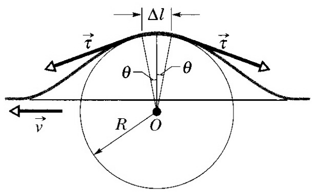
FIG. 16-10 A symmetrical pulse, viewed from a reference frame in which the pulse is stationary and the string appears to move right to left with speed v. We find speed v by applying Newton's second law to a string element of length Δl, located at the top of the pulse.

Wave speed on a Stretched String

Consider a small segment of length Δl at the peak.

Let τ be the tension in the string.

Vertical component of the force on the element:



both sides ↘ small θ ↙

$$F = 2\tau\sin\theta \approx \tau(2\theta) = \tau\frac{\Delta l}{R}$$

, where R is the radius of curvature.

Mass of the segment: $\Delta m = \mu\Delta l$ ← $\mu = \Delta m / \Delta l$ = mass per unit length = linear density

Centripetal acceleration in moving reference frame: $a = \dfrac{v^2}{R}$

Using Newton's second law, $F = \Delta m \times a$, $\tau\dfrac{\Delta l}{R} = (\mu\Delta l)\dfrac{v^2}{R}$

"stretching = lengthening" → causes an elastic restoring force

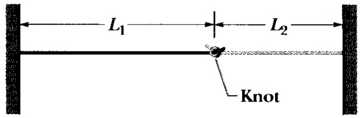
This reduces to $$v = \sqrt{\frac{\tau}{\mu}}$$

Note that τ represents the elastic property of the stretched string, and μ represents its inertial property.

Example 2

Two strings (String 1 and String 2) have been tied together with a knot and then stretched between two rigid supports. The strings have linear densities $\mu_1 = 1.4 \times 10^{-4}$ kgm^{-1} and $\mu_2 = 2.8 \times 10^{-4}$ kgm^{-1}. Their lengths are $L_1 = 3$ m and $L_2 = 2$ m, and String 1 is under a tension of 400 N. Simultaneously, on each string a pulse is sent from the rigid support end, towards the knot. Which pulse reaches the knot first?

FIG. 16-11 Two strings, of lengths L_1 and L_2, tied together with a knot and stretched between two rigid supports.



Analysis → start from simpler case (single string, constant density, same tension); speeds, meeting point?

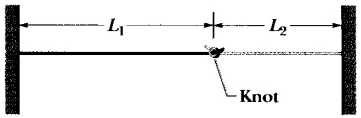
Different strings → effect due to diff. densities or diff. tensions or diff. lengths?

Are tensions different?

Approach → calculate wave speeds on both strings, then calculate arrival times at knot.

Answer of Example 2

FIG. 16-11 Two strings, of lengths L_1 and L_2, tied together with a knot and stretched between two rigid supports.



$$v_1 = \sqrt{\frac{\tau}{\mu_1}}$$

$$t_1 = \frac{L_1}{v_1} = L_1\sqrt{\frac{\mu_1}{\tau}} = 3\sqrt{\frac{1.4\times10^{-4}}{400}} = 1.77\times10^{-3}\ \text{s}$$

$$v_2 = \sqrt{\frac{\tau}{\mu_2}}$$

$$t_2 = \frac{L_2}{v_2} = L_2\sqrt{\frac{\mu_2}{\tau}} = 2\sqrt{\frac{2.8\times10^{-4}}{400}} = 1.67\times10^{-3}\ \text{s}$$

Thus, the pulse on string 2 reaches the knot first.

Energy and Power of a Travelling String Waves

Kinetic energy

Consider a string element of mass dm.

Kinetic energy: $dK = \frac{1}{2}dm\,u^2$

Since $y(x, t) = y_m\sin(kx - \omega t)$,

$$u = \frac{\partial y}{\partial t} = -\omega y_m\cos(kx - \omega t)$$

Since $dm = \mu dx$, ← μ = mass per unit length

$$dK = \frac{1}{2}(\mu dx)(-\omega y_m)^2\cos^2(kx - \omega t)$$

Rate of kinetic energy transmission: ↙ power

$$\frac{dK}{dt} = \frac{1}{2}\mu\omega^2 y_m^2\cos^2(kx - \omega t)\frac{dx}{dt}$$

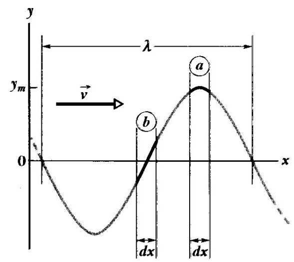


FIG. 16-12 A snapshot of a traveling wave on a string at time $t = 0$. String element a is at displacement $y = y_m$, and string element b is at displacement $y = 0$. The kinetic energy of the string element at each position depends on the transverse velocity of the element. The potential energy depends on the amount by which the string element is stretched as the wave passes through it.

Energy and Power of a Travelling String Waves

Using $v = dx/dt$,

$$\frac{dK}{dt} = \frac{1}{2}\mu v\omega^2 y_m^2 \cos^2(kx - \omega t)$$

<u>Kinetic energy is maximum at the $y = 0$ position</u>.

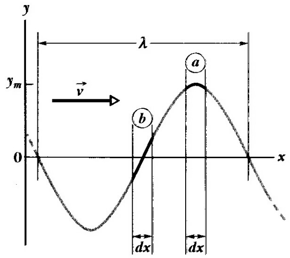


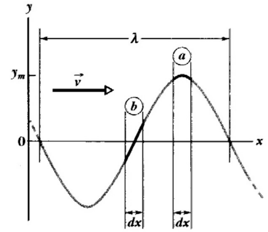
FIG. 16-12 A snapshot of a travelling wave on a string at time $t = 0$. String element a is at displacement $y = y_m$, and string element b is at displacement $y = 0$. The kinetic energy of the string element at each position depends on the transverse velocity of the element. The potential energy depends on the amount by which the string element is stretched as the wave passes through it.

Energy and Power of a Travelling String Waves

Potential energy

Potential energy is carried in the string when it is stretched.
Stretching is largest when the displacement has the largest gradient.
Hence, <u>the potential energy is also maximum at the $y = 0$ position</u>
This is different from the harmonic oscillator, in which the energy is conserved.

Consider the extension Δs of a string element.



$$\Delta s = \sqrt{(dx)^2 + \underbrace{[y(x+dx,t) - y(x,t)]}_{dy}^2} - dx$$

$$\approx \sqrt{(dx)^2 + \left(\frac{\partial y}{\partial x}dx\right)^2} - dx = \left[\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} - 1\right]dx$$

slope

Energy and Power of a Travelling String Waves

Using power series expansion, ($\sqrt{1 + a^2} \sim 1 + \frac{1}{2}a^2$ for small a^2)

$$\Delta s \approx \left[1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 - 1\right]dx = \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx$$

The potential energy of the string element is given by the work done in extending the string element, use work dW = force τ × distance Δs

$$dU = dW = \tau\Delta s \approx \frac{\tau}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx = \frac{\tau}{2}k^2 y_m^2 \cos^2(kx - \omega t)\,dx$$

Rate of potential energy transmission: ← power

$$\frac{dU}{dt} = \frac{\tau}{2}k^2 y_m^2 \cos^2(kx - \omega t)\frac{dx}{dt}$$

Energy and Power of a Travelling String Waves

Since $v = dx/dt$ and $k^2 = \omega^2/v^2 = \omega^2\mu/\tau$,

$$\frac{dU}{dt} = \frac{1}{2}\mu v\omega^2 y_m^2 \cos^2(kx - \omega t) = \frac{dK}{dt}$$

Mechanical energy (power): $\frac{dE}{dt} = \frac{dK}{dt} + \frac{dU}{dt} = \mu v\omega^2 y_m^2 \cos^2(kx - \omega t)$

Average power of transmission: $\langle P\rangle = \left\langle\frac{dE}{dt}\right\rangle = \mu v\omega^2 y_m^2 \left\langle\cos^2(kx - \omega t)\right\rangle$

, where ⟨...⟩ represents averaging over time.

Since $\langle\cos^2(kx - \omega t)\rangle = 1/2$, average power: $\langle P\rangle = \frac{1}{2}\mu v\omega^2 y_m^2$

Energy and Power of a Travelling String Waves

This result can be interpreted in the following way.

Consider the front of a propagating wave along a string.

In a time dt, a string element of length $dx = vdt$ is set into a <u>simple harmonic motion</u>. Its velocity amplitude is ωy_m.

$dm = \mu dx$

Energy of the string element: $dE = \frac{1}{2} dm (\omega y_m)^2$ from E = ½ mv ^{2}

Average power: $\langle P \rangle = \frac{dE}{dt} = \frac{1}{2}\mu\frac{dx}{dt}(\omega y_m)^2 = \frac{1}{2}\mu v \omega^2 y_m^2$

as before

Example 3

A string has a linear density μ of 525 g/m and is stretched with a tension τ of 45 N. A wave whose frequency f and amplitude y_m are 120 Hz and 8.5 mm, respectively, is travelling along the string. At what average rate is the wave transporting energy along the string? Simply applying the equations.

$$\omega = 2\pi f = (2\pi)(120) = 754.0 \text{ rads}^{-1}$$

$$v = \sqrt{\frac{\tau}{\mu}} = \sqrt{\frac{45}{0.525}} = 9.258 \text{ ms}^{-1}$$

$$\langle P \rangle = \frac{1}{2}\mu v \omega^2 y_m^2 = \left(\frac{1}{2}\right)(0.525)(9.258)(754.0)^2(0.0085)^2 = 99.8 \text{ W}$$

Section Summary

❖ Wave speed of a stretched string: $v = \sqrt{\frac{\tau}{\mu}}$

❖ τ → elastic property, tension
❖ μ → inertial property, linear density (i.e. mass per unit length)

❖ Kinetic energy is maximum at the y = 0 position.
❖ Potential energy is also maximum at the y = 0 position.

❖ Average power: $\langle P \rangle = \frac{1}{2}\mu v \omega^2 y_m^2$

Principle of Superposition of Waves

❖ Overlapping waves algebraically add to produce a resultant wave.

Note: not derivative, only sum of waves

$$y'(x,t) = y_1(x,t) + y_2(x,t)$$

❖ Overlapping waves do not in any way alter each other.

❖ True for small amplitudes, but tall waves change each other. e.g. crashing water waves!

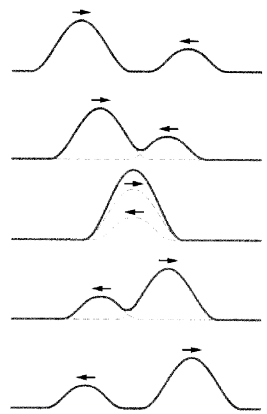


FIG. 16-14 A series of snapshots that show two pulses traveling in opposite directions along a stretched string. The superposition principle applies as the pulses move through each other.

Interference of Waves

Suppose we send two sinusoidal waves of the same wavelength and amplitude in the same direction along a stretched string.

$$y_1(x,t) = y_m \sin(kx - \omega t)$$

delay

$$y_2(x,t) = y_m \sin(kx - \omega t + \phi)$$

ϕ is called the *phase difference* or *phase shift* between the two waves.

Combined displacement:

$$y'(x,t) = y_m \sin(kx - \omega t) + y_m \sin(kx - \omega t + \phi)$$

Interference of Waves

Using the trigonometric identity,

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

we obtain

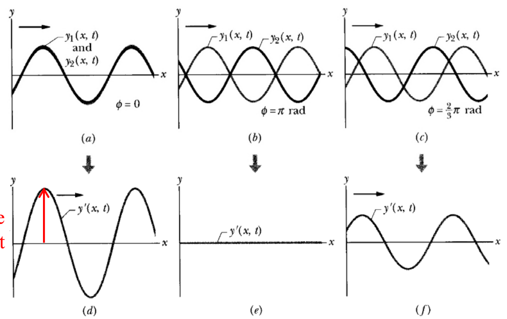
$$y'(x,t) = \left[2y_m \cos\frac{\phi}{2}\right]\sin\left(kx - \omega t + \frac{\phi}{2}\right)$$

amplitude — as before — phase shift

The resultant wave…
(1) is also a travelling wave in the same direction
(2) has a phase constant of $\phi/2$
(3) has an amplitude of $y'_m = 2y_m\cos(\phi/2)$

Interference of Waves

FIG. 16-16 Two identical sinusoidal waves, $y_1(x,t)$ and $y_2(x,t)$, travel along a string in the positive direction of an x axis. They interfere to give a resultant wave $y'(x,t)$. The resultant wave is what is actually seen on the string. The phase difference ϕ between the two interfering waves is (a) 0 rad or 0°, (b) π rad or 180°, and (c) $\frac{2}{3}\pi$ rad or 120°. The corresponding resultant waves are shown in (d), (e), and (f).



double height

Fully constructive interference (maximum amplitude)

If $\phi = 0$, $y'(x,t) = 2y_m \sin(kx - \omega t)$ See fig. (d)

Fully destructive interference (zero amplitude)

If $\phi = \pi$, $y(x,t) = 0$ See fig. (e)

Intermediate interference

If ϕ is between 0 and π or between π and 2π, the amplitude is intermediate. See fig. (f)

Interference of Waves

TABLE 16-1

Phase Difference and Resulting Interference Types[a]

Phase Difference, in Degrees	Radians	Wavelengths	Amplitude of Resultant Wave	Type of Interference
0	0	0	$2y_m$	Fully constructive
120	$\frac{2}{3}\pi$	0.33	y_m	Intermediate
180	π	0.50	0	Fully destructive
240	$\frac{4}{3}\pi$	0.67	y_m	Intermediate
360	2π	1.00	$2y_m$	Fully constructive
865	15.1	2.40	$0.60y_m$	Intermediate

[a]The phase difference is between two otherwise identical waves, with amplitude y_m, moving in the same direction.

Example 4

Two identical sinusoidal waves, moving in the same direction along a stretched string, interfere with each other.
The amplitude y_m of each wave is 9.8 mm, and the phase difference ϕ between them is 100°.

(a) What is the amplitude y'_m of the resultant wave due to the interference, and what is the type of this interference?
(b) What phase difference, in radians and wavelengths, will give the resultant wave an amplitude of 4.9 mm?

Answers to Example 4

(a) $y'_m = \left|2y_m \cos\frac{\varphi}{2}\right| = \left|(2)(9.8\text{ mm})\cos\left(\frac{100^\circ}{2}\right)\right| = 13\text{ mm}$

(b) $y'_m = \left|2y_m \cos\frac{\phi}{2}\right|$

$$4.9 = (2)(9.8)\cos\frac{\phi}{2}$$

$$\phi = 2\cos^{-1}\left(\frac{4.9}{(2)(9.8)}\right) = \pm 2.636\text{ rad} \approx \pm 2.6\text{ rad}$$

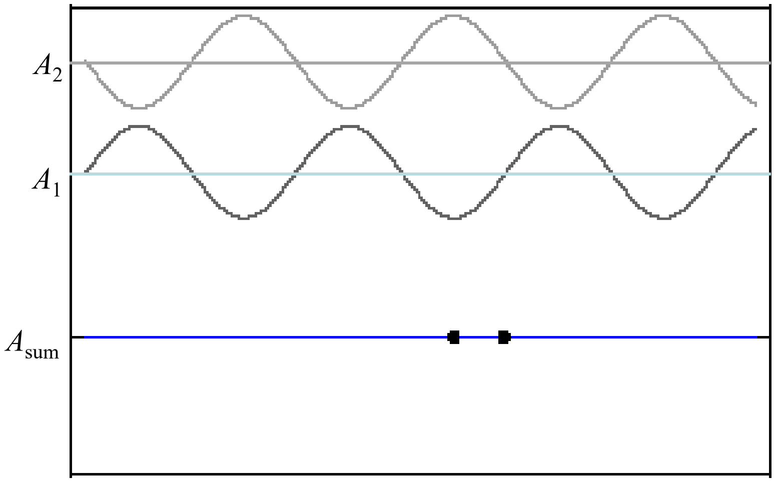
two possible solutions

$\phi = +2.6$ rad: The second wave leads (travels ahead of) the first wave.
$\phi = -2.6$ rad: The second wave lags (travels behind) the first wave.

In wavelengths, the phase difference is $\pm\frac{2.636}{2\pi} = \pm 0.42$ wavelength

Standing Waves

Two opposite traveling waves add to each other, forming a standing wave:



$$A_{sum} = A_1 + A_2$$

Standing Waves

❖ Superposition of two waves of equal wavelength and equal amplitude, travelling in opposite directions

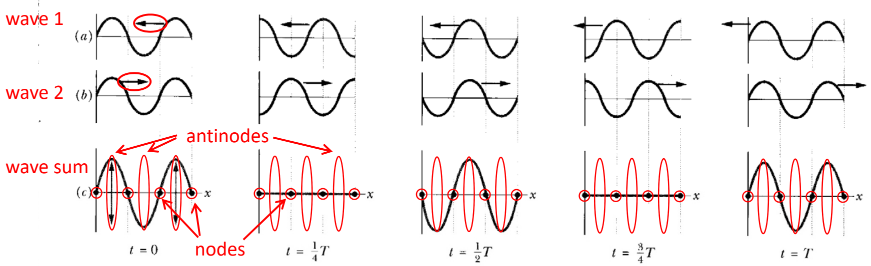


FIG. 16-19 (*a*) Five snapshots of a wave traveling to the left, at the times t indicated below part (*c*) (T is the period of oscillation). (*b*) Five snapshots of a wave identical to that in (*a*) but traveling to the right, at the same times t. (*c*) Corresponding snapshots for the superposition of the two waves on the same string. At $t = 0, \frac{1}{2}T$, and T, fully constructive interference occurs because of the alignment of peaks with peaks and valleys with valleys. At $t = \frac{1}{4}T$ and $\frac{3}{4}T$, fully destructive interference occurs because of the alignment of peaks with valleys. Some points (the nodes, marked with dots) never oscillate; some points (the antinodes) oscillate the most.

Standing Waves

Consider two sinusoidal waves of the same wavelength and amplitude travelling in the opposite direction along a stretched string.

$$y_1(x,t) = y_m \sin(kx - \omega t)$$
$$y_2(x,t) = y_m \sin(kx + \omega t)$$

Combined displacement:

$$y'(x,t) = y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t)$$

Using the trigonometric identity,

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

we obtain

$$y'(x,t) = [2y_m \sin kx]\cos\omega t$$

x and *t* are now decoupled!
→ standing wave

Standing Waves

Properties:

(1) The resultant wave is *not* a travelling wave, but is a standing wave.
e.g. the locations of the maxima and minima do not change.

(2) There are positions where the string is permanently at rest.
They are called **nodes**, and are located at

$\sin(kx) = 0$ → $kx = n\pi$ for $n = 0, 1, 2, \cdots$

$$x = \frac{n\pi}{k} = n\frac{\lambda}{2} \quad \text{for} \quad n = 0, 1, 2, \cdots$$

The nodes are separated by half wavelength.

(3) There are positions where the string has the maximum amplitude.
They are called **antinodes**, and are located at

$\sin(kx) = \pm 1$ → $kx = \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi, \cdots = \left(n+\frac{1}{2}\right)\pi$ for $n = 0, 1, 2, \cdots$

$$x = \left(n+\frac{1}{2}\right)\frac{\lambda}{2} \quad \text{for} \quad n = 0, 1, 2, \cdots$$

The antinodes are separated by half wavelength.

Standing Waves

Energy in standing waves:
- does not travel
- exchanges between kinetic energy K and potential energy U

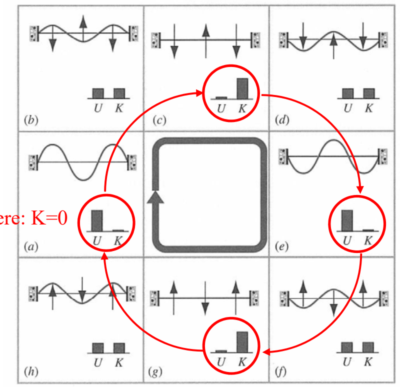


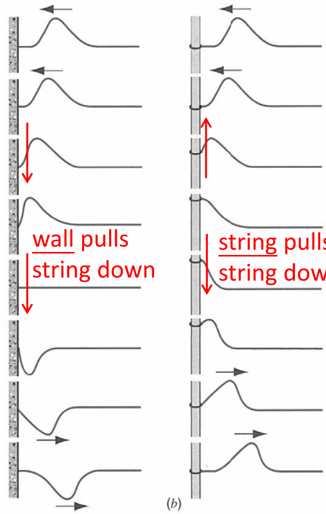
FIGURE 18-18. A standing wave on a stretched string, showing one cycle of oscillation. At (a) the string is momentarily at rest with the antinodes at their maximum displacement. The energy of the string is all elastic potential energy. (b) One-eighth of a cycle later, the displacement is reduced and the energy is partly potential and partly kinetic. The vectors show the instantaneous velocities of particles of the string at certain locations. (c) The displacement is zero; there is no potential energy, and the kinetic energy is maximum. The particles of the string have their maximum velocities. (d–h) The motion continues through the remainder of the cycle, with the energy being continually exchanged between potential and kinetic forms.

Reflections at a Boundary

Fixed end:
(1)The fixed end becomes a node.
(2)The reflected wave vibrates in the opposite transverse direction.

Free end:
(1)The free end becomes an antinode.
(2)The reflected wave vibrates in the same transverse direction.

transverse pulse



At the fixed end:
Displacement = 0

At the free end:
Displacement gradient = 0

See animation
"Reflection of Waves in Physics"
http://www.youtube.com/watch?v=0mZk2vW5rWU

FIGURE 18-19. (a) A transverse pulse incident from the right is reflected by a rigid wall. Note that the phase of the reflected pulse is inverted, or changed by 180°. (b) Here the end of the string is free to move, the string being attached to a loop that can slide freely along the rod. The phase of the reflected pulse is unchanged.

Section Summary

Two sinusoidal waves of the same wavelength and amplitude along a stretched string:

❖ Travelling in the same direction → still travelling wave

$$y'(x,t) = \left[2y_m \cos\frac{\phi}{2}\right]\sin\left(kx - \omega t + \frac{\phi}{2}\right)$$

Fully constructive interference: $\phi = 0$ → maximum amplitude

Fully destructive interference: $\phi = \pi$ → zero amplitude

❖ Travelling in the opposite direction → standing wave, energy does not travel, K.E. ←→ P.E. exchanges

$$y'(x,t) = [2y_m \sin kx]\cos\omega t$$

nodes: always at rest → $x = \frac{n\pi}{k} = n\frac{\lambda}{2}$ for $n = 0, 1, 2, \cdots$

antinodes: max. amplitude→ $x = \left(n + \frac{1}{2}\right)\frac{\lambda}{2}$ for $n = 0, 1, 2, \cdots$

*Fixed end → node → out-of-phase reflection *Free end → antinode → in-phase

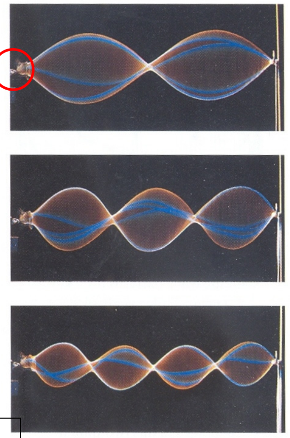
Standing Waves and Resonance

oscillator

2 loops:

3 loops:

4 loops:



1, 5, 6, 7, … loops also possible

FIG. 16-22 Stroboscopic photographs reveal (imperfect) standing wave patterns on a string being made to oscillate by an oscillator at the left end. The patterns occur at certain frequencies of oscillation. *(Richard Megna/Fundamental Photographs)*

oscillator

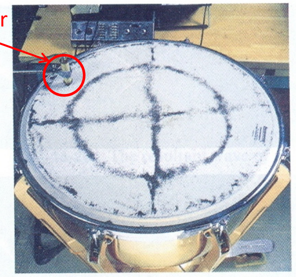


FIG. 16-24 One of many possible standing wave patterns for a kettle-drum head, made visible by dark powder sprinkled on the drumhead. As the head is set into oscillation at a single frequency by a mechanical oscillator at the upper left of the photograph, the powder collects at the nodes, which are circles and straight lines in this two-dimensional example. *(Courtesy Thomas D. Rossing, Northern Illinois University)*

Circular shape: somewhat complicated, special math functions

Rectangular shape: simpler: like two linear patterns superposed

Standing Waves and Resonance

Consider a string with length L stretched between two fixed ends.

Boundary condition: nodes at each of the fixed ends.

When the string is driven by an external force, at a certain frequency the standing wave will fit this boundary condition.

Then this **oscillation mode** will be excited.

The frequency at which the oscillation mode is excited is called the **resonant frequency**.

See animation "Standing Waves Demo"
https://www.youtube.com/watch?v=-gr7KmTOrx0

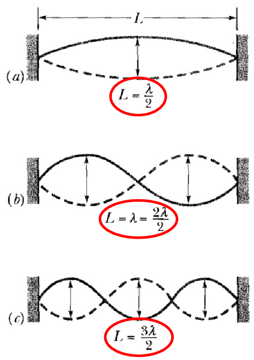


FIG. 16-23 A string, stretched between two clamps, is made to oscillate in standing wave patterns. (*a*) The simplest possible pattern consists of one *loop*, which refers to the composite shape formed by the string in its extreme displacements (the solid and dashed lines). (*b*) The next simplest pattern has two loops. (*c*) The next has three loops.

Standing Waves and Resonance

See Youtube "Resonance Phenomena in 2D on a Plane" and "Millenium Bridge Opening".
https://www.youtube.com/watch?v=Qf0t4qIVWF4 *https://www.youtube.com/watch?v=gQK21572oSU*

Case (a): 2 nodes at the ends, 1 antinode in the middle

$L = \frac{\lambda}{2}$, $\lambda = 2L$.

Resonant frequency: $f = \frac{v}{\lambda} = \frac{v}{2L}$.

Case (b): 3 nodes and 2 antinodes

$L = \lambda$.

Resonant frequency: $f = \frac{v}{\lambda} = \frac{v}{L}$.

Case (c): 4 nodes and 3 antinodes

$L = \frac{3\lambda}{2}$, $\lambda = \frac{2L}{3}$.

Resonant frequency: $f = \frac{v}{\lambda} = \frac{3v}{2L}$.

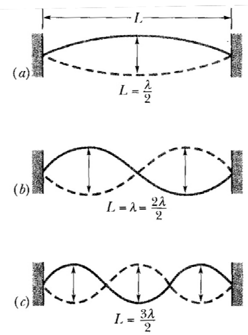


FIG. 16-23 A string, stretched between two clamps, is made to oscillate in standing wave patterns. (*a*) The simplest possible pattern consists of one *loop*, which refers to the composite shape formed by the string in its extreme displacements (the solid and dashed lines). (*b*) The next simplest pattern has two loops. (*c*) The next has three loops.

Standing Waves and Resonance

In general,

$$L = \frac{n\lambda}{2}, \qquad \lambda = \frac{2L}{n}, \qquad \text{for } n = 1, 2, 3, \ldots$$

Resonant frequency:

$$f = \frac{v}{\lambda} = n\frac{v}{2L}, \qquad \text{for } n = 1, 2, 3, \ldots$$

$n = 1$ fundamental mode, or first harmonic
$n = 2$ second harmonic
$n = 3$ third harmonic
etc.

Example 5

A string of mass m = 2.5 g and length L = 0.8 m is under tension τ = 325 N.
(a) What is the wavelength λ of the transverse waves producing the standing-wave pattern in Fig. 16-25, and what is the harmonic number n?
(b) What is the frequency f of the transverse waves and of the oscillations of the moving string elements?
(c) What is the maximum magnitude of the transverse velocity um of the element oscillating at coordinate x = 0.18 m?
(d) At what point during the element's oscillation is the transverse velocity maximum?

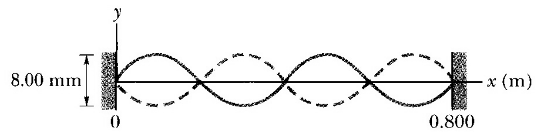


FIG. 16-25 Resonant oscillation of a string under tension.

Answers to Example 5

(a) $\lambda = \frac{L}{2} = \frac{0.8}{2} = 0.4\ \text{m}$

Since there are four loops, $n = 4$.

(b) $v = \sqrt{\frac{\tau}{\mu}} = \sqrt{\frac{325}{0.0025/0.8}} = 322.5\ \text{ms}^{-1}$

$f = \frac{v}{\lambda} = \frac{322.5}{0.4} = 806.2 \approx 806\ \text{Hz}$

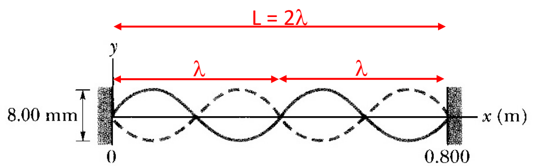


FIG. 16-25 Resonant oscillation of a string under tension.

Answers to Example 5 (continued)

we need the waveform for general location x, not a special location like node or antinode

(c) $y'(x,t) = [2y_m \sin kx]\cos\omega t$ where $y_m = 0.002$ m.

$$u(x,t) = \frac{\partial y}{\partial t} = \frac{\partial}{\partial t}(2y_m \sin kx \cos\omega t) = -2\omega y_m \sin kx \sin\omega t$$

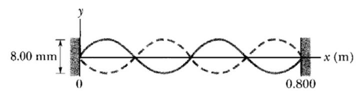


FIG. 16-25 Resonant oscillation of a string under tension.

Magnitude: $u_m = \left|-2\omega y_m \sin kx\right|$

Here, $\omega = 2\pi f = (2\pi)(806.2)$ $\quad k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.4}$

At x=0.18 m, $u_m = \left|-2(2\pi)(806.2)(0.002)\sin\left[\frac{2\pi}{0.4}(0.18)\right]\right| = 6.26\ \text{ms}^{-1}$

(d) The transverse velocity is maximum when $y = 0$.

Example 6

In the arrangement of Fig. 18-23, a motor sets the string into motion at a frequency of 120 Hz. The string has a length of L = 1.2 m, and its linear mass density is 1.6 g/m. To what value must the tension be adjusted (by increasing the hanging weight) to obtain the pattern of motion having four loops?

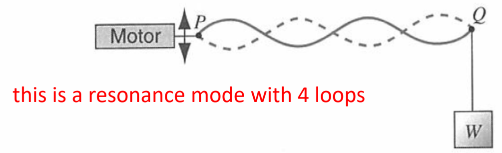


FIGURE 18-23. A string under tension is connected to a vibrator. For a fixed vibrator frequency, standing wave patterns will occur for certain discrete values of the tension in the string.

Answers to Example 6

$$v = \sqrt{\frac{F}{\mu}} \quad \rightarrow F = \mu v^2 \quad \text{... equation (1)}$$

$$f_n = \frac{v}{\lambda_n} = n\frac{v}{2L} \quad (n = 1, 2, 3, ...) \quad \rightarrow v = 2Lf_n/n \quad \text{... equation (2)}$$

To find the tension, we can substitute Eq. (1) into Eq. (2) and obtain

$$F = \frac{4L^2 f_n^{\,2} \mu}{n^2}$$

The tension corresponding to n = 4 (for 4 loops) is found to be

$$F = \frac{4(1.2\text{ m})^2 (120\text{ Hz})^2 (0.0016\text{ kg/m})}{4^2} = 8.3\text{ N}$$

Summary of equations

Travelling waves

$y(x,t) = y_m \sin[k(x - vt)]$ or $y(x,t) = y_m \sin(kx - \omega t)$

Wavenumber

$k = \frac{2\pi}{\lambda}$

Angular frequency

$\omega = \frac{2\pi}{T}$

Frequency

$f = \frac{1}{T} = \frac{\omega}{2\pi}$

Wave velocity

$v = \frac{\omega}{k} = f\lambda$

Travelling wave (opposite direction)

$y(x,t) = y_m \sin(kx + \omega t)$

Stretched string

$v = \sqrt{\frac{\tau}{\mu}}$

Summary of equations

Transmitted power $\langle P \rangle = \frac{1}{2}\mu v \omega^2 y_m^2$

Interference $y'_m = 2y_m \cos\frac{\varphi}{2}$

Standing wave $y(x,t) = [2y_m \sin kx]\cos \omega t$

Reflection at fixed end → node, oscillation at opposite transverse direction
Reflection at free end → antinode, oscillation at same transverse direction

Vibrating string (fixed ends) $f = \frac{v}{\lambda} = n\frac{v}{2L}$