Thermal Expansion (Area)

Thermodynamics

$$\alpha = \frac{\Delta L/L}{\Delta T}$$

$$\beta = \frac{\Delta A/A}{\Delta T}$$

$$\gamma = \frac{\Delta V/V}{\Delta T}$$

$$\begin{aligned}
A + \Delta A &= (L + \Delta L)(W + \Delta W) \\
&= (L + \alpha L \Delta T)(W + \alpha W \Delta T) \\
&= LW + \alpha LW \Delta T + \alpha LW \Delta T + \alpha^2 LW \Delta T^2 \quad \text{— } (1) \\
&= LW(1 + 2\alpha \Delta T + \alpha^2 \Delta T^2) \\
&= LW(1 + \alpha \Delta T)^2 \\
&= A(1 + \alpha \Delta T)^2
\end{aligned}$$

$$(1) \Rightarrow \quad \Delta A = 2 \alpha A \Delta T$$

$$\frac{\Delta A}{A} \approx 2 \alpha \Delta T$$

$$\beta = \frac{\Delta A/A}{\Delta T} \approx \frac{2\alpha \Delta T}{\Delta T} = 2\alpha$$

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Thermal Expansion (Volume)

$$V + \Delta V = (L + \Delta L)(W + \Delta W)(H + \Delta H)$$

$$= (L + \alpha L \Delta T)(W + \alpha W \Delta T)(H + \alpha H \Delta T)$$

$$= LWH(1 + \alpha \Delta T)^3$$

$$\frac{\Delta V}{V} = 3\alpha \Delta T + 3(\alpha \Delta T)^2 + (\alpha \Delta T)^3$$

Suppose $(\alpha \Delta T)^2 \ll 1$ and $(\alpha \Delta T)^3 \ll 1$,

$$\Rightarrow \quad \frac{\Delta V}{V} \approx 3 \alpha \Delta T$$

$$\gamma = \frac{\Delta V / V}{\Delta T} = \frac{3 \alpha \Delta T}{\Delta T} = 3\alpha$$

Exercise : Hydrostatic Pressure on Solar Pond

$$P_1 = \rho g h_1 = (1040)(9.8)(0.8)$$

$$= 8.15 \ kN/m^2 \quad (\text{or } 8.15 \ kPa)$$

Remark : $1 \ kN/m^2 = 1 \ kPa$

$$dP = \rho g \ dz$$

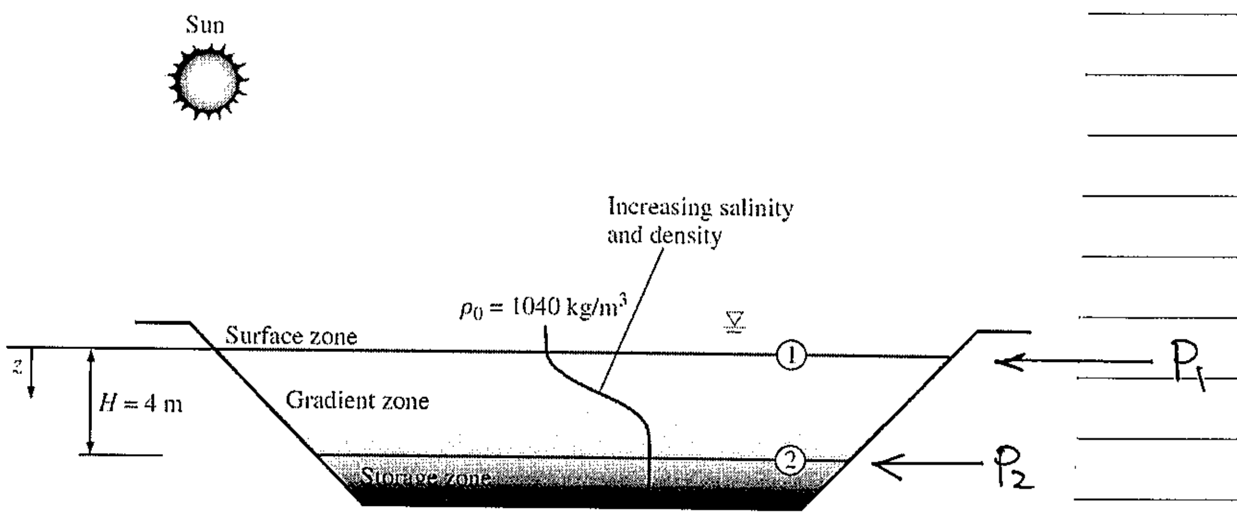
$$P_2 - P_1 = \int_0^z \rho g \ dz$$

$$P_2 = P_1 + \int_0^z \rho_0 \sqrt{1 + \tan^2\left(\frac{\pi}{4}\frac{z}{H}\right)} \, g \ dz$$

$$P_2 = P_1 + \left(\rho_0 g \frac{4H}{\pi}\right) \sinh^{-1}\left(\tan \frac{\pi}{4}\frac{z}{H}\right)$$

$$= 8.15 \ kPa + (1040)(9.8)\left(\frac{4 \times 4}{\pi}\right) \sinh^{-1}\left(\tan \frac{\pi}{4}\frac{4}{4}\right)$$

$$= 53.9 \ kPa$$



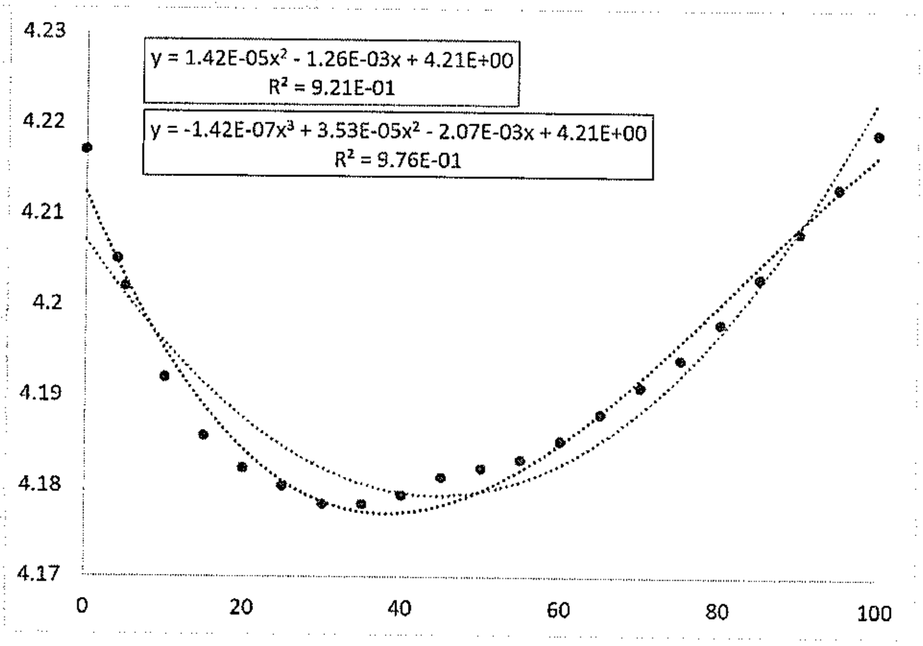
Temperature Dependent Specific Heat

(a) $Q = (1)(4200)(99.99)$

$= \sim 420 \text{ kJ}$

(b)

T	c
0.01	4.217
4	4.205
5	4.202
10	4.192
15	4.1855
20	4.182
25	4.18
30	4.178
35	4.178
40	4.179
45	4.181
50	4.182
55	4.183
60	4.185
65	4.188
70	4.191
75	4.194
80	4.198
85	4.203
90	4.208
95	4.213
100	4.219



$$Q = m\int_{T_i}^{T_f} c(T)\, dT$$

Fitting I : $Q = \int_{0.01}^{100} (1.42\times10^{-5}\,T^2 - 1.26\times10^{-3}\,T + 4.21)\, dT$

$= 419.391 \text{ kJ}$

Fitting II : $Q = \int_{0}^{100} (-1.42\times10^{-7}\,T^3 + 3.53\times10^{-5}\,T^2 - 2.07\times10^{-3}\,T + 4.21)\, dT$

$= 418.825 \text{ kJ}$

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Isothermal Expansion

$$W = -\int_{V_i}^{V_f} P\,dV$$

$$= -\int_{V_i}^{V_f} \frac{nRT}{V}\,dV$$

$$= -nRT \int_{V_i}^{V_f} \frac{dV}{V}$$

$$= nRT \ln\left(\frac{V_i}{V_f}\right)$$

Quick Quiz on Thermal Conductivity

In Series : $R_{eff, series} = R_1 + R_2$

In Parallel : $(R_{eff, parallel})^{-1} = \frac{1}{2A}\left(\frac{A}{R_1} + \frac{A}{R_2}\right)$

$$= \frac{1}{2}\,\frac{R_1 + R_2}{R_1 R_2}$$

$$R_{eff, parallel} = \frac{2 R_1 R_2}{R_1 + R_2}$$

$R_{eff, series} - R_{eff, parallel}$

$$= \frac{2 R_1 R_2}{R_1 + R_2} - R_1 - R_2$$

$$= \frac{2 R_1 R_2 - (R_1 + R_2)(R_1 + R_2)}{R_1 + R_2}$$

$$= \frac{2 R_1 R_2 - (R_1 + R_2)^2}{R_1 + R_2}$$

$$= \frac{R_1^2 + R_2^2}{R_1 + R_2} > 0$$

$\Rightarrow$ Thermal Resistance R(in Series) > R(in parallel)

Exercise on Thermal Conductivity

Two slabs, thickness L_1 and L_2, and thermal conductivities k_1 and k_2, are in thermal contact with each other. The temperatures of their outer surfaces are T_c and T_h, respectively ($T_h > T_c$). Determine the temperature, T, at the interface and the rate of energy transfer under steady-state condition.

Solution:

$$\frac{\Delta Q}{\Delta t} = k A \left(\frac{T_h - T_c}{L} \right)$$

$$\frac{\Delta Q_1}{\Delta t} = k_1 A \left(\frac{T - T_c}{L_1} \right) \quad \text{— (1)}$$

$$\frac{\Delta Q_2}{\Delta t} = k_2 A \left(\frac{T_h - T}{L_2} \right) \quad \text{— (2)}$$

At Steady State, $\frac{\Delta Q_1}{\Delta t} = \frac{\Delta Q_2}{\Delta t}$

$$k_1 A \left(\frac{T - T_c}{L_1} \right) = k_2 A \left(\frac{T_h - T}{L_2} \right)$$

$$\Rightarrow \quad T = \frac{k_1 L_2 T_c + k_2 L_1 T_h}{k_1 L_2 + k_2 L_1}$$

$$\frac{\Delta Q}{\Delta t} = \frac{A \left(T_h - T_c \right)}{\left(\frac{L_1}{k_1} \right) + \left(\frac{L_2}{k_2} \right)}$$

Exercise on Thermal Conductivity II

The following hollow cylinder is filled with water and maintained at temperature T_a, whereas the outer surface is T_b ($T_a > T_b$). The wall of the cylinder has a thermal conductivity k. Find the rate of energy transfer from the inner wall to the outer surface.

Solution:

$$\frac{dQ}{dt} = -kA\frac{dT}{dx}$$

$$\frac{dQ}{dt} = -k(2\pi r L)\frac{dT}{dr}$$

$\frac{dQ}{dt}$ is constant,

$$\Rightarrow \quad dT = \frac{dQ}{dt}\left(\frac{-1}{2\pi k r L}\right) dr$$

$$dT = \frac{dQ}{dt}\left(\frac{-1}{2\pi k L}\right)\frac{dr}{r}$$

$$\int_{T_a}^{T_b} dT = \frac{dQ}{dt}\left(\frac{-1}{2\pi k L}\right)\int_a^b \frac{dr}{r}$$

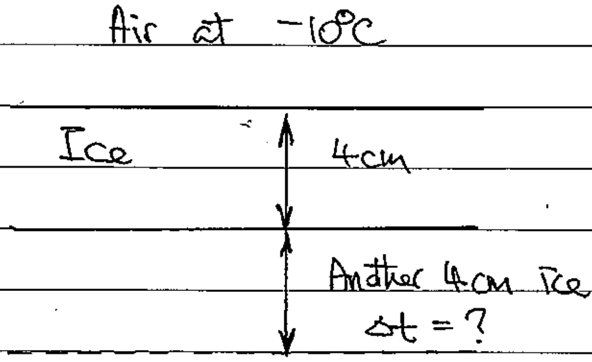
$$T_b - T_a = \frac{dQ}{dt}\left(\frac{-1}{2\pi k L}\right)\ln\left(\frac{b}{a}\right)$$

$$\Rightarrow \quad \frac{dQ}{dt} = \frac{2(T_a - T_b)k\pi L}{\ln(b/a)}$$

Exercise on Thermal Conductivity and Latent Heat

A pond of water at 0°C is covered with a layer of ice 4 cm thick. If the air temperature stays constant at −10°C, how long does it take for the ice thickness to increase to 8cm?

Solution:



$$L \rho A \frac{dx}{dt} = kA\left(\frac{\Delta T}{\Delta x}\right)$$

$$L\rho \int_4^8 x\,dx = k\,\Delta T \int dt$$

$$L\rho \left[\frac{x^2}{2}\right]_4^8 = k\,\Delta T\,\Delta t$$

$$(334000)(917)\left(\frac{0.08^2 - 0.04^2}{2}\right) = (2)(10)\,\Delta t$$

$$\Rightarrow \Delta t = 36753\text{ s} = 10.2\text{ hours}$$

Exercise on Molar Specific Heat

Constant Volume

First Law: $\Delta E_{int} = Q + \cancel{W}^{\,0}$

$$\Delta E_{int} = \frac{3}{2} nRT$$

Also: $\Delta E_{int} = n C_v \Delta T$

$$C_v = \frac{1}{n}\frac{dE_{int}}{dT}$$

$$= \frac{1}{n}\left(\frac{3}{2} nR\right)$$

$$= \frac{3}{2} R$$

Constant Pressure

$$\begin{cases} Q = n C_p \Delta T \\ W = -P \Delta V \end{cases}$$

$$\Delta E_{int} = Q + W$$

$$= n C_p \Delta T + (-p \Delta V)$$

$$= n C_p \Delta T - nR \Delta T$$

$$\Rightarrow \quad n C_v \Delta T = n C_p \Delta T - nR \Delta T$$

$$\Rightarrow \quad C_p - C_v = R$$

Exercise : Heating a Cylinder of Helium Gas

i) $Q = n C_v \Delta T$

$= (3 \text{ moles})(12.5 \text{ J/mole} \cdot \text{K})(200 \text{ K})$

$= 7.5 \text{ kJ}$

ii) $Q = n C_p \Delta T$

$= (3 \text{ moles})(20.8 \text{ J/mole} \cdot \text{K})(200 \text{ K})$

$= 12.48 \text{ kJ}$

Exercise : Adiabatic Process (to be continued)

$$PV = nRT$$

$$PdV + VdP = nR\,dT \quad \text{——— (1)}$$

$$dE_{int} = nC_V\,dT = -PdV \quad \text{——— (2)}$$

Eliminating dT from Eqn. (1) and (2),

$$PdV + VdP = nR\left(\frac{-PdV}{nC_V}\right)$$

$$PdV + VdP = -PR\left(\frac{dV}{C_V}\right)$$

$$PdV + VdP = -PR\,\frac{dV}{C_V}$$

$$VdP = \left(-P - \frac{PR}{C_V}\right)dV$$

$$\frac{dP}{P} = -\left(1 + \frac{R}{C_V}\right)\frac{dV}{V}$$

$$\frac{dP}{P} = -\left(\frac{C_V + R}{C_V}\right)\frac{dV}{V}$$

$$\frac{dP}{P} = -\left(\frac{C_V + C_P - C_V}{C_V}\right)\frac{dV}{V}$$

$$\frac{dP}{P} = -\frac{C_P}{C_V}\,\frac{dV}{V}$$

$$\frac{dP}{P} + \gamma\,\frac{dV}{V} = 0$$

$$\Rightarrow \ln P + \gamma \ln V = \text{Constant}$$

$$\Rightarrow PV^{\gamma} = \text{Constant}$$

Exercise: Adiabatic Process (continued):

$$\begin{cases} P_i V_i^{\gamma} = P_f V_f^{\gamma} \\ P_i = \frac{nRT_i}{V_i} \quad \text{and} \quad P_f = \frac{nRT_f}{V_f} \end{cases}$$

$$\Rightarrow \quad \frac{nRT_i}{V_i}\left(V_i^{\gamma}\right) = \frac{nRT_f}{V_f}\left(V_f^{\gamma}\right)$$

$$\Rightarrow \quad T_i V_i^{\gamma-1} = T_f V_f^{\gamma-1}$$

Exercise : Diesel Engine Cylinder

$$P_f = P_i \left(\frac{V_i}{V_f} \right)^{\gamma} = (1\ \text{atm}) \left(\frac{800}{60} \right)^{1.4}$$

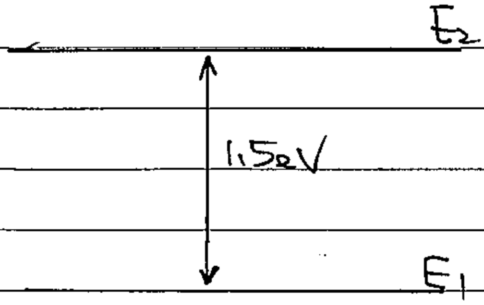
$$= 37.6\ \text{atm}$$

$$\frac{P_i V_i}{T_i} = \frac{P_f V_f}{T_f}$$

$$T_f = \frac{P_f V_f}{P_i V_i} T_i = \frac{(37.6)(60)}{(1)(800)}(293)$$

$$= 826\ \text{K} \quad (553^{\circ}\text{C})$$

Exercise on Number Density



$$\frac{n_v(E_2)}{n_v(E_1)} = \frac{n_0 \exp(-E_2/k_B T)}{n_0 \exp(-E_1/k_B T)}$$

$$= \exp\left(-\frac{E_2 - E_1}{k_B T}\right)$$

$$= \exp\left[-\frac{1.5\,eV}{(1.38\times10^{-23} \times 2500)/1.6\times10^{-19}}\right]$$

$$= \exp(-6.96)$$

$$= 9.49 \times 10^{-4}$$

Exercise on Energy Quality : The Entropy

1 kg water at 20°C is mixed with 1 kg water at 80°C
Calculate the net change in Entropy.

Solution:

Entropy Change $\Delta S = \int_i^f \frac{dQ}{T}$

$= \int_{T_i}^{T_f} \frac{mc\,dT}{T}$

$= mc \ln\left(\frac{T_f}{T_i}\right)$

Final Temperature of the water mixture = 50°C

$\Delta S_{net} = \Delta S_{20°C \to 50°C} + \Delta S_{80°C \to 50°C}$

$= mc \ln\left(\frac{323}{293}\right) + mc \ln\left(\frac{323}{353}\right)$

$= 36.54$ J/K

Exercise: Efficiency of the Carnot Engine

Process A → B, Isothermal Expansion.

Energy transfer by heat from the hot reservoir.

$$|Q_h| = |\Delta E_{int} - W_{AB}| = |0 - W_{AB}| = W_{AB} = nRT_h \ln\left(\frac{V_B}{V_A}\right) \quad \text{—— (1)}$$

Process C → D, Isothermal Compression.

Energy transfer to the cold reservoir.

$$|Q_c| = |\Delta E_{int} - W_{CD}| = |0 - W_{CD}| = W_{CD} = nRT_c \ln\left(\frac{V_C}{V_D}\right) \quad \text{—— (2)}$$

$$\frac{(2)}{(1)} \Rightarrow \frac{|Q_c|}{|Q_h|} = \frac{T_c}{T_h}\left[\frac{\ln\left(\frac{V_C}{V_D}\right)}{\ln\left(\frac{V_B}{V_A}\right)}\right] \quad \text{—— (3)}$$

Adiabatic Process B → C, $\quad T_h V_B^{\gamma-1} = T_c V_C^{\gamma-1}$

Adiabatic Process D → A, $\quad T_h V_A^{\gamma-1} = T_c V_D^{\gamma-1}$

$$\Rightarrow \left(\frac{V_B}{V_A}\right)^{\gamma-1} = \left(\frac{V_C}{V_D}\right)^{\gamma-1}$$

$$\Rightarrow \frac{V_B}{V_A} = \frac{V_C}{V_D} \quad \text{—— (4)}$$

(3) and (4) $\Rightarrow$ $$\frac{|Q_c|}{|Q_h|} = \frac{T_c}{T_h}$$

$$\Rightarrow \eta = 1 - \frac{|Q_c|}{|Q_h|} = 1 - \frac{T_c}{T_h}$$

Carnot Heat Pump COP

$$COP = \frac{|Q_h|}{W} = \frac{|Q_h|}{|Q_h| - |Q_c|}$$

$$= \frac{1}{1 - \frac{|Q_c|}{|Q_h|}}$$

$$= \frac{1}{1 - \frac{T_c}{T_h}}$$

$$= \frac{T_h}{T_h - T_c}$$

Exercise: Efficiency of the Otto Cycle

Process $B \to C$, $|Q_h| = n C_V (T_C - T_B)$

Process $D \to A$, $|Q_c| = n C_V (T_D - T_A)$

$$\eta = 1 - \frac{|Q_c|}{|Q_h|} = 1 - \frac{T_D - T_A}{T_C - T_B} \quad \text{——} \quad (1)$$

Adiabatic Process:

$$A \to B: \quad T_A V_A^{\gamma-1} = T_B V_B^{\gamma-1} \quad \text{——} \quad (2)$$

$$C \to D: \quad T_C V_C^{\gamma-1} = T_D V_D^{\gamma-1} \quad \text{——} \quad (3)$$

Also, $\begin{cases} V_A = V_D = V_1 \\ V_B = V_C = V_2 \end{cases}$

$$(2) \Rightarrow \quad T_A = T_B \left(\frac{V_B}{V_A}\right)^{\gamma-1} = T_B \left(\frac{V_2}{V_1}\right)^{\gamma-1} \quad \text{——} \quad (4)$$

$$(3) \Rightarrow \quad T_D = T_C \left(\frac{V_C}{V_D}\right)^{\gamma-1} = T_C \left(\frac{V_2}{V_1}\right)^{\gamma-1} \quad \text{——} \quad (5)$$

$(5) - (4)$, $$\frac{T_D - T_A}{T_C - T_B} = \left(\frac{V_2}{V_1}\right)^{\gamma-1}$$

$$\Rightarrow \eta = 1 - \left(\frac{V_2}{V_1}\right)^{\gamma-1}$$

$$= 1 - \frac{1}{(V_1/V_2)^{\gamma-1}}$$

Entropy Change in Free Expansion

$$\Delta S = \int_i^f \frac{dQr}{T} = \frac{1}{T}\int_i^f dQr$$

Isothermal Process, according to the 1st law,

$$\int_i^f dQr = -nRT \ln\left(\frac{V_i}{V_f}\right)$$

$$\Rightarrow \Delta S = \frac{1}{T}\left[-nRT \ln\left(\frac{V_i}{V_f}\right)\right]$$

$$= nR \ln\left(\frac{V_f}{V_i}\right)$$

Entropy Generation in a Wall

(i)

$$\underbrace{\dot{S}_{in} - \dot{S}_{out}}_{\text{Rate of net entropy transfer by heat and Mass}} + \underbrace{\dot{S}_{gen}}_{\text{Rate of Entropy Generation}} = \underbrace{\cancel{\frac{dS_{system}}{dt}}}_{\text{Rate of Change in Entropy}}$$

Steady state

$$\left(\frac{\dot{Q}}{T}\right)_{in} - \left(\frac{\dot{Q}}{T}\right)_{out} + \dot{S}_{gen} = 0$$

$$\frac{1035\,W}{291} - \frac{1035\,W}{288} + \dot{S}_{gen} = 0$$

$$\Rightarrow \dot{S}_{gen, wall} = 0.037\ W/K$$

(ii) Rate of total entropy generation:

$$\frac{1035}{293} - \frac{1035}{283} + \dot{S}_{gen} = 0$$

$$\Rightarrow \dot{S}_{gen, total} = 0.125\ W/K$$