PEP Assignment 1 Solutions

1 (a) $f(x) = x^3$

$$\frac{d}{dx}x^3 = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x}$$

One can then use the binomial theorem to get

$$(x + \Delta x)^3 = (\Delta x)^3 + (6x^2\Delta x) + x^3$$

After simplifying, this leads to

$$\frac{d}{dx}x^3 = \lim_{\Delta x \to 0} (\Delta x)^2 + (3x\Delta x) + 3x^2$$

If $\Delta x = 0$, then

$$\mathbf{\frac{d}{dx}x^3 = 3x^2}$$

1(b) $f(x) = \frac{1}{\sqrt{x}}, x > 0$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\sqrt{x}} = \lim_{\Delta x \to 0} \frac{\frac{1}{\sqrt{x + \Delta x}} - \frac{1}{\sqrt{x}}}{\Delta x}$$

After simplifying, this leads to

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\sqrt{x}} = \lim_{\Delta x \to 0} \frac{-1}{x\sqrt{x + \Delta x} + \sqrt{x}(x + \Delta x)}$$

If $\Delta x = 0$, then

$$\mathbf{\frac{d}{dx}\frac{1}{\sqrt{x}} = -\frac{1}{2x^{\frac{3}{2}}}}$$

1(c) $f(x) = \sec x = \frac{1}{\cos x}$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \lim_{\Delta x \to 0} \frac{\frac{1}{\cos(x + \Delta x)} - \frac{1}{\cos x}}{\Delta x}$$

After simplifying, this leads to

$$\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \lim_{\Delta x \to 0} \frac{\cos x - \cos(x + \Delta x)}{\Delta x \cos x \cos(x + \Delta x)}$$

Using trig. Identity, one can get

$$\cos(x + \Delta x) = \cos x \cos \Delta x - \sin x \sin \Delta x$$

This leads to

$$\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x \cos(x + \Delta x)} + \left(\frac{\sin x}{\cos x}\right)\frac{\sin \Delta x}{\Delta x \cos(x + \Delta x)}$$

If $\Delta x = 0$, then

$\sin \Delta x = 0$ $\qquad$ $\cos \Delta x = 1$

1(c) Continued …

Knowing that

$$\lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} = 0 \qquad \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1$$

One can then get

$$\frac{\mathrm{d}}{\mathrm{d}x}\sec x = 0 + \frac{1}{\cos x}\tan x$$

$$\mathbf{\frac{d}{dx}\sec x = \sec x \tan x}$$

2(a) $f(x) = x^2 + x + 3$

One can use a combination of the power rule and the sum rule to get

$$\mathbf{\frac{d}{dx}f(x) = 2x + 1}$$

2(b) $f(x) = \sin x \cos x$

One can use the product rule

$$\frac{\mathrm{d}}{\mathrm{d}x}[g(x)h(x)] = g(x)\frac{\mathrm{d}}{\mathrm{d}x}h(x) + h(x)\frac{\mathrm{d}}{\mathrm{d}x}g(x)$$

$g(x) = \sin x$ $\qquad$ $\frac{\mathrm{d}}{\mathrm{d}x}g(x) = \cos x$

$h(x) = \cos x$ $\qquad$ $\frac{\mathrm{d}}{\mathrm{d}x}h(x) = -\sin x$

Putting all the terms together to get

$$\mathbf{\frac{d}{dx}f(x) = -\sin^2 x + \cos^2 x}$$

2(c) $f(x) = \sqrt{1 - x^3}$

Let $u = 1 - x^3$ and $y = \sqrt{u}$

One can then use the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}x}f(x) = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{1}{2\sqrt{u}}$ $\qquad$ $\frac{\mathrm{d}u}{\mathrm{d}x} = -3x^2$

Putting all the terms together to get

$$\mathbf{\frac{d}{dx}f(x) = -\frac{3x^2}{2\sqrt{1 - x^3}}}$$

2(d) $f(x) = \sin\sqrt{1-x^2}$

Let $k = 1 - x^2$, $c = \sqrt{k}$ and $y = \sin c$

One can use the chain rule

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}c}\frac{\mathrm{d}c}{\mathrm{d}k}\frac{\mathrm{d}k}{\mathrm{d}x}$$

$$\frac{\mathrm{d}k}{\mathrm{d}x} = -2x \qquad \frac{\mathrm{d}c}{\mathrm{d}k} = \frac{1}{2\sqrt{k}} \qquad \frac{\mathrm{d}y}{\mathrm{d}c} = \cos c$$

Putting all the terms together to get

$$\boldsymbol{\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{\sqrt{1-x^2}}\cos\sqrt{1-x^2}}$$

2(e) $f(x) = \frac{x+2}{x-2}$

Let $g(x) = x + 2$ and $h(x) = x - 2$

One can use the quotient rule

$$\frac{d}{dx}\frac{g(x)}{h(x)} = \frac{g(x)\frac{d}{dx}h(x) + h(x)\frac{d}{dx}g(x)}{[h(x)]^2}$$

$$g(x) = x + 2 \qquad \frac{d}{dx}g(x) = 1$$

$$h(x) = x - 2 \qquad \frac{d}{dx}h(x) = 1$$

Putting all the terms together to get

$$\boldsymbol{\frac{d}{dx}f(x) = -\frac{4}{(x-2)^2}}$$

3(a)

Knowing that

$$V = \pi H\left(\frac{D}{2}\right)^2 \quad (1)$$

$$A = \pi DH + 2\pi\left(\frac{D}{2}\right)^2 \quad (2)$$

A = Total surface area of the cylindrical tank

V = Total volume of the cylindrical tank

H = Height of the cylindrical tank

D = Diameter for the base of the cylindrical tank

From (1), we get

$$H = \frac{16}{\pi D^2} \quad (3)$$

Substitute (3) into (2)

$$A = \frac{16}{D} + \frac{\pi}{2}D^2 \quad (4)$$

To get a minimum

$$\frac{dA}{dD} = 0$$

$$-\frac{16}{D^2} + \pi D = 0 \quad (5)$$

Solving (5) for D

$$\boldsymbol{D = \left(\frac{16}{\pi}\right)^3}\ \textbf{meters}$$

OR

$$\boldsymbol{D = 1.721}\ \textbf{meters}$$

* Substitute the value of D into (1), should find that $\boldsymbol{H = D}$

Substitute the value of D into (2), the total surface area A of the metal sheet required is

$$\boldsymbol{A = \frac{3\pi}{2}D^2}\ \textbf{sq.meters}$$

OR

$$\boldsymbol{A = 13.949}\ \textbf{sq.meters}$$

3(b)

Recall the relationship between V, H and D from **3(a)**:

$$H = D \quad (1)$$

$$V = \pi H\left(\frac{D}{2}\right)^2 \quad (2)$$

Substitute (1) into (2)

$$V = \pi\left(\frac{D}{2}\right)^3 \quad (3)$$

Differentiate (3) with respect to D gives

$$\frac{\mathrm{d}V}{\mathrm{d}D} = \frac{3}{4}\pi D^2 \quad (4)$$

3(b) continued …

If ΔD is the error in D (and in H), then (4) can be rewritten as

$$\Delta V \approx \frac{3}{4}\pi D^2\, \Delta D$$

Therefore, the relative error in V is

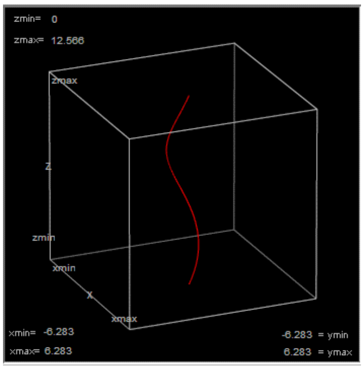
$$\frac{\Delta V}{V} \approx 3\frac{\Delta D}{D}$$

If the relative error in D is 2%, then

$$\boldsymbol{\frac{\Delta V}{V} \approx 6\%}$$

4(a)

$$\mathbf{r} = \cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}} + 2t\hat{\boldsymbol{k}}$$



The trajectory is a spiral.

Differentiate r with respect to time t to get the velocity $\boldsymbol{v}$

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = -\sin t\hat{\boldsymbol{\imath}} + \cos t\hat{\jmath} + 2\hat{\boldsymbol{z}}$$

Obtain the second derivative of $\boldsymbol{r}$ to get the acceleration $\boldsymbol{a}$

$$\boldsymbol{a} = \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\cos t\hat{\boldsymbol{\imath}} - \sin t\,\hat{\boldsymbol{\jmath}}$$

5(a)

The position vector of the projectile at any time is

$$r = v\cos(\theta+\alpha)t\hat{\boldsymbol{\imath}} + \left[v\sin(\theta+\alpha)t - \frac{1}{2}gt^2\right]\hat{\jmath}$$

Therefore, the x and y component of the position vector are

$$x = v\cos(\theta+\alpha)\,t \qquad (1)$$

$$y = v\sin(\theta+\alpha)\,t - \frac{1}{2}gt^2 \qquad (2)$$

5(a) continued …

The equation of the incline is

$$y = x\tan\theta \qquad (3)$$

Sometime later, the trajectory and the incline will meet. This leads to (2) being the same as (3), so that

$$\sin(\theta+\alpha)t - \frac{1}{2}gt^2 = x\tan\theta$$

$$\sin(\theta+\alpha)t - \frac{1}{2}gt^2 = v\cos(\theta+\alpha)\,t\tan\theta \qquad (4)$$

Rearrange (4) to get t

$$t = \frac{2v(\sin(\theta+\alpha)\cos\theta - \sin\theta\cos(\theta+\alpha))}{g\cos\theta} \qquad (5)$$

Can use the following trig. Identity to simplify (5),

$$\sin(\theta+\alpha-\theta) = \sin(\theta+\alpha)\cos\theta - \cos(\theta+\alpha)\sin\theta$$

Then we get

$$t = \frac{2v\sin\alpha}{g\cos\theta} \qquad (6)$$

Using the geometry outlined in the problem,

$$R = x\sec\theta \qquad (7)$$

Substitute (1) and (6) into (7), we obtain

$$R = \frac{v^2}{g\cos^2\theta}(\sin(2\alpha+\theta) - \sin\theta) \qquad (8)$$

To find the maximum value of angle α, need to find $\frac{dR}{d\alpha}$ and solve for α when $\frac{dR}{d\alpha} = 0$. Using the sum rule and the chain rule, we then obtain

$$\frac{\mathrm{d}R}{\mathrm{d}\alpha} = 2\cos(2\alpha+\theta)$$

$$2\cos(2\alpha+\theta) = 0 \qquad (9)$$

Finally, we solve (9) for α to obtain

$$\boldsymbol{\alpha = \frac{\pi}{4} = \frac{\theta}{2}}$$