PEP Assignment 2 Solutions

1 (a)

$$I = \int \sqrt{x}\,\mathrm{d}x$$

Use the power rule to get

$$\boldsymbol{I = \frac{2x^{\frac{3}{2}}}{3} + C}$$

1(b)

$$I = \int 3\sin^2 x \cos x \,\mathrm{d}x$$

One can use the method of substitution

Let $u = \sin x$, then $\mathrm{d}u = \cos x \ \mathrm{d}x$

$$I = 3\int u^2\,du = 3\left[\frac{u^3}{3}\right] + C$$

$$\boldsymbol{I = 3\left[\frac{sin^3 x}{3}\right] + C}$$

1(c)

$$I = \int \frac{1}{2} - 2x^2\,\mathrm{d}x$$

One can use the difference rule to get

$$\boldsymbol{I = \frac{1}{2}x - \frac{2x^3}{3} + C}$$

1(d)

$$I = \int_0^1 x\sqrt{1-x}\,\mathrm{d}x$$

Let $u = x$ and $\mathrm{d}v = \sqrt{1-x}\,\mathrm{d}x$

And so

$\mathrm{d}u = \mathrm{d}x$ and $v = -\frac{2}{3}(1-x)^{\frac{3}{2}}$

Use integration by parts,

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

Put everything together to get

$$\boldsymbol{I = \left[-\frac{2x}{3}(1-x)^{\frac{3}{2}} - \frac{4}{15}(1-x)^{\frac{5}{2}}\right]_0^1 = \frac{4}{15}}$$

1(e)

$$I = \int_0^{\pi} x^2 \sin x\,\mathrm{d}x$$

Let $u = x^2$ and $\mathrm{d}v = \sin x \ \mathrm{d}x$

And so

$\mathrm{d}u = 2x\,\mathrm{d}x$ and $v = -\cos x$

Use integration by parts,

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

$$\int u\,\mathrm{d}v = -x^2\cos x - \int 2x \ \cos x \ \mathrm{d}x$$

Need to perform integration by parts again for $\int 2x \ \cos x \ \mathrm{d}x$

Let $u = x$ and $\mathrm{d}v = \cos x \ \mathrm{d}x$

And so

$\mathrm{d}u = 1\,\mathrm{d}x$ and $v = \sin x$

$$\int 2x \ \cos x \ \mathrm{d}x = x\sin x + \cos x + \ c$$

Where c is a constant. By putting everything together, we have to following:

$$\boldsymbol{I = [(-x^2 + 2)\cos x + 2x\sin x]_0^{\pi} = \pi^2 - 4}$$

2

To find the area bounded, we will need to solve the following:

$$I = \int_0^{36} (25x)^{\frac{1}{2}}\,\mathrm{d}x$$

$$\boldsymbol{I = \left[\frac{10}{3}x^{\frac{3}{2}}\right]_0^{36} = 720\ \text{sq. unit}}$$

3

To find the arc length s, we will need to use

$$s = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

Where $y = \sqrt{(7-x)(5+x)}$, $a = -5$ and $b = +1$

To find $\frac{\mathrm{d}y}{\mathrm{d}x}$, the use of chain rule is required. This leads to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x+1}{\sqrt{(7-x)(5+x)}}$$

3 Continued …

We will now need to compute

$$s = \int_{-5}^{1} \sqrt{1 + \frac{(-x+1)^2}{(7-x)(5+x)}}\,\mathrm{d}x$$

By completing the square

$(7-x)(5+x) = -(x-1)^2 + 36$

and integrate by substitution using $u = x - 1$, this leads to

$$s = 6\int \frac{1}{\sqrt{36-u^2}}\,\mathrm{d}u$$

(For simplicity, we ignore the integration limits for now …)

Let $u = 6\ \sin v$ and so $\mathrm{d}u = 6\cos v\ \mathrm{d}v$, this leads to

$$s = 6\int \frac{\cos v}{\sqrt{1-\sin^2 v}}\,\mathrm{d}v$$

Using $\sin^2 x + \cos^2 x = 1$, we get

$s = 6\,v$

Since $v = \sin^{-1}\left(\frac{x-1}{6}\right)$, we then need to compute the following

$$s = \left[6\sin^{-1}\left(\frac{x-1}{6}\right)\right]_{-5}^{1} = 3\pi \text{ units}$$

OR

$\boldsymbol{s = 9.42}$ **units**

4

For the equation
$\frac{x^2}{9} + \frac{y^2}{25} = 1,$

we know that $a = 3$ and $b = 5$, where a is the semi minor axis along the x-axis and b is the semi major axis along the y-axis.

Furthermore,

$$y^2 = 25\left(1 - \frac{x^2}{9}\right)$$

To calculate the resulting volume V when the object is revolved by the x axis, we can use

$$V = \pi\int_a^b y^2\,\mathrm{d}x$$

This leads to

$$V = \pi\int_{-3}^{+3} 25\left(1 - \frac{x^2}{9}\right)\mathrm{d}x$$

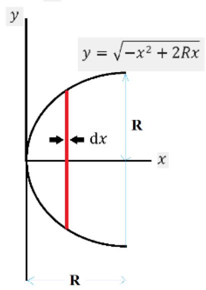
4 Continued …

$$\boldsymbol{V = 25\pi\left[x - \frac{x^3}{27}\right]_{-3}^{+3} = 100\,\pi \text{ cubic units}}$$

OR

$\boldsymbol{V = 314.16}$ **cubic units**

5



Using the method as described in the tutorial notes. Consider a function that describes a circle with its edge located at the origin

$y = \sqrt{-x^2 + 2Rx}$

$\mathrm{d}m = \rho\,\pi y^2\,\mathrm{d}x$, where ρ is the mass per unit volume.

Total mass M is then

$$M = \int_0^R \mathrm{d}m = \int_0^R \rho\pi(-x^2 + 2Rx)\,\mathrm{d}x$$

$$M = \rho\pi\left[\frac{-x^3}{3} + \frac{2Rx^2}{2}\right]_0^R = \rho\pi\left[\frac{2R^3}{3}\right]$$

Based the geometry and by symmetry, we know that its center of mass must be located along the x axis only. The definition of the center of mass is

$$\bar{x} = \frac{\int_0^R x\,\mathrm{d}m}{M}$$

$$\bar{x} = \frac{\int_0^R x\,\rho\pi(-x^2 + 2Rx)\,\mathrm{d}x}{M}$$

$$\bar{x} = \frac{1}{M}\left(\frac{5R^4}{12}\rho\pi\right)$$

$\boldsymbol{\bar{x} = \frac{5}{8}R}$ **from the origin**

OR

$\boldsymbol{\bar{x} = \frac{3}{8}R}$ **from the base of the hemisphere**