PEP Assignment 3 Solutions

1 (a)

$$f(x) = \text{Ln}(1+x)^5$$

		$f(0)$	0
$f'(x)$	$5(1+x)^{-1}$	$f'(0)$	5
$f''(x)$	$-5(1+x)^{-2}$	$f''(0)$	-5
$f'''(x)$	$10(1+x)^{-3}$	$f'''(0)$	10
$f''''(x)$	$-30(1+x)^{-4}$	$f''''(0)$	-30

Therefore, its Taylor series up to the first four terms is

$$f(x) = 5x - \tfrac{5}{2}x^2 + \tfrac{5}{2}x^3 - \tfrac{5}{4}x^4$$

1 (b)

$$f(x) = \cosh(x)$$

		$f(0)$	1
$f'(x)$	$\sinh(x)$	$f'(0)$	0
$f''(x)$	$\cosh(x)$	$f''(0)$	1
$f'''(x)$	$\sinh(x)$	$f'''(0)$	0
$f''''(x)$	$\cosh(x)$	$f''''(0)$	1
$f'''''(x)$	$\sinh(x)$	$f'''''(x)$	0
$f''''''(x)$	$\cosh(x)$	$f''''''(x)$	1

Therefore, its Taylor series up to the first four terms is

$$f(x) = 1 + \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 + \tfrac{1}{720}x^6$$

1 (c)

$$f(x) = \sqrt{1-x}$$

		$f(0)$	1
$f'(x)$	$-\frac{1}{2}(1-x)^{-\frac{1}{2}}$	$f'(0)$	$-\frac{1}{2}$
$f''(x)$	$-\frac{1}{4}(1-x)^{-\frac{3}{2}}$	$f''(0)$	$-\frac{1}{4}$
$f'''(x)$	$-\frac{3}{8}(1-x)^{-\frac{5}{2}}$	$f'''(0)$	$-\frac{3}{8}$

Therefore, its Taylor series up to the first four terms is

$$f(x) = 1 - \tfrac{1}{2}x \quad - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3$$

2(a)

$$Z = 2e^{\frac{i\pi}{4}}$$

$$Z^3 = 8e^{\frac{i3\pi}{4}} = 8\left(\cos\frac{3\pi}{4} + \sin\frac{3\pi}{4}i\right)$$

2(b)

$$Z = \frac{1}{16}e^{i6\pi}$$

$$Z^{\frac{1}{4}} = \frac{1}{2}e^{i\frac{3\pi}{2}} = \frac{1}{2}\left(\cos\frac{3\pi}{2} + \sin\frac{3\pi}{2}i\right)$$

2(c)

$$Z = \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^3$$

$$Z = \left(e^{\frac{i\pi}{3}}\right)^3 = -1$$

2(d)i

$$Z = 5 - 5i$$

$$|Z| = 5\sqrt{2}$$

$$\theta = -\frac{\pi}{4}$$

$$Z = 5\sqrt{2}\, e^{-i\frac{\pi}{4}}$$

2(d)ii

$$Z = 15 - 13i$$

$$|Z| = \sqrt{394}$$

$$\theta = -0.714$$

$$Z = \sqrt{394}\, e^{-i0.714}$$

2(e)

$$Z = \frac{(1+i)^2}{\sqrt{2}(1-i)}$$

Multiply by a factor of $\frac{(1+i)}{(1+i)}$ to get

$$Z = \frac{(1+i)^3}{2\sqrt{2}} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

Therefore,

$$\text{Re}(Z) = -\frac{1}{\sqrt{2}}$$

and

$$\text{Im}(Z) = \frac{1}{\sqrt{2}}$$

(3)

Let's define the following variables:

$\boldsymbol{v_{1,i}}$	Velocity of the object with mass m_1 before the collision
$\boldsymbol{v_{2,i}}$	Velocity of the object with mass m_2 before the collision
$\boldsymbol{v_{1,f}}$	Velocity of the object with mass m_1 after the collision
$\boldsymbol{v_{2,f}}$	Velocity of the object with mass m_2 after the collision
$\boldsymbol{v'_{2,f}}$	Velocity of the object with mass m_2 after the collision with the wall

Elastic collision occurs when the kinetic energy and the momentum are conserved. Therefore, we can write the following equations:

From the conservation of kinetic energy:

$$\frac{1}{2}m_1v_{1,i}^2 = \frac{1}{2}m_1v_{1,f}^2 + \frac{1}{2}m_1v_{2,f}^2$$

From the conservation of momentum:

$$\frac{1}{2}m_1v_{1,i}^2 = \frac{1}{2}m_1v_{1,f}^2 + \frac{1}{2}m_1v_{2,f}^2$$

From these two equations, we get

$$v_{1,f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1,i}$$

and

$$v_{2,f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{1,i}$$

Since the collision between the object with mass m_2 and the wall is considered to be elastic, we can also write

$$v'_{2,f} = -v_{2,f}$$

For the case when $v'_{2,f}$ and $v_{1,f}$ are the same (i.e. no second collision), then

$$\frac{m_2}{m_1} = 3$$

Therefore, in order for the second collision to occur

$$v'_{2,f} > v_{1,f}$$

and so

$$\boldsymbol{\frac{m_2}{m_1} < 3}$$

(4)

Let's define the following variables:

$\boldsymbol{V}$	Velocity of the hemisphere
$\boldsymbol{M}$	Mass of the hemisphere
$\boldsymbol{\theta}$	Angle between the top of the hemisphere and the object sliding down the hemisphere
$\boldsymbol{m}$	Mass of the object sliding down the hemisphere
$\boldsymbol{v'}$	Velocity of the object sliding down the hemisphere

In this scenario, we can write the following equations:

By conservation of momentum:

$$MV = m(v'\cos\theta - V)$$

By conservation of energy:

$$\frac{1}{2}m[(v'\cos\theta - V)^2 + (v'\sin\theta)^2] + \frac{1}{2}MV^2 = mgR(1 - \cos\theta)$$

By Newton's 2nd law (when the object is no longer in contact with the hemisphere):

$$mg\cos\theta = \frac{m(v')^2}{R}$$

With these three equations, we get

$$\boldsymbol{\frac{M}{m} = 2.43}$$