PEP Assignment 4 Solutions

1 (a)

$$\frac{1}{2}\frac{dy}{dx} + 2y = 0$$

Separate the variables and integrate both sides:

$$\int \quad \frac{1}{4y} dy = \int dx$$

$$\quad \frac{1}{4} \text{Ln } y = x + C$$

General solution:

$$\boldsymbol{y = Ae^{-4x}}$$

Requires that $y(0) = 3$

$$\boldsymbol{A = 3}$$

Specific solution:

$$\boldsymbol{y = 3e^{-4x}}$$

1(b)

$$\frac{d^2y}{dx^2} + y = 2\frac{dy}{dx}$$

Construct an auxillary equation:

$$r^2 \quad 2r + 1 = 0$$

Solving the auxillary equation gives

$$r = 1$$

Since there is only one solution for r, the general solution for the differential equation will be

$$y = (C_1 + C_2 x)e^{rx}$$

where $r = 1$ for this particular problem. Since the question specifies $y(0) = 1$ and $y(1) = 0$, we can then solve for C_1 and C_2. This gives $C_1 = 1$ and $C_2 = \quad 1$.

With C_1 and C_2 are now known, the specific solution is therefore

$$\boldsymbol{y = (1 \quad x)e^x}$$

1(c)

$$\frac{dy}{dx} = e^{(x-2y)}$$

By separating the variables, the general solution is

$$y = \frac{1}{2} \text{Ln } (2e^x + C_1)$$

From the question, $y(0) = 1$

Therefore, $C_1 = e^2 \quad 2$.

With C_1 is now known, the specific solution is therefore

$$\boldsymbol{y = \frac{1}{2} \textbf{Ln } (2e^x + e^2 \quad 2)}$$

4(a)

At time t, the momentum of the rocket and unspent fuel is

$$P(t) = mv$$

At time dt later, the momentum is now

$$P_{\text{rocket}}(t + \text{d}t) = (m + \text{d}m)(v + \text{d}v)$$

Note the quantity $\text{d}m$ is negative.

When the fuel ejected at time $\text{d}t$ later, the mass is $\quad \text{d}m$ and the velocity is $v \quad u$, where u is the exhaust velocity relative to the rocket.

The momentum of the exhaust is

$$P_{\text{exhaust}}(t + \text{d}t) = \quad \text{d}m(v \quad u)$$

The total momentum at time $t + \text{d}t$ is

$$P(t + \text{d}t) \approx mv + m\text{d}v + \text{d}m\, u$$

The change in momentum is

$$P(t + \text{d}t) \quad P(t) = m\text{d}v + \text{d}m\, u$$

By ignoring gravity, the external is zero and therefore the momentum is conserved.

In other words, we have following differential equation

$$\boldsymbol{m\, \textbf{d}v = \quad \textbf{d}m\, u}$$

4(b)

By solving the differential equation in 4(a) using the separation of variables, we have

$$v = \quad u \text{ Ln } m$$

Therefore initially, $v_i = \quad u \text{ Ln } m_i$, where v_i and m_i are the initial speed and initial mass of the rocket + fuel respectively.

Since we are looking for a change in speed (i.e. $v \quad v_i$) , we then have

$$v \quad v_i = u\, Ln \left(\frac{m_i}{m}\right)$$

For $m_i = m_o + m_f + m_2$, $m = m_o + m_2$ and $v_i = 0$ m/s,

The resulting speed v_f is

$$\boldsymbol{v_f = u \,\mathbf{Ln}\left(1 + \frac{m_f}{m_o + m_2}\right)}$$

4(c)

The momentum is conserved between the two rocket parts. Furthermore, there is an initial speed v_f, when the separation occurs. Knowing such facts, we can then write

$$\boldsymbol{v_2 = v_f + \frac{m_0 v_1}{m_2}}$$

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Torque is defined as

$$\tau = I\,\alpha$$

Where I is the moment of inertia and α is the angular acceleration. Furthermore, τ can also be written as

$$\boldsymbol{\tau = r \times F}$$

$$\tau = \quad L_{\mathrm{CM}}\, mg\, \sin\theta$$

Where L_{cm} is the distance between the pivot of the pendulum and the centre of mass of the object, θ is the angular displacement.

Note that there is negative sign, because the torque is opposing the angular displacement.

Now we can construct a differential equation:

$$\tau = I\,\alpha = I\,\frac{d^2\theta}{dt^2}$$

$$I\,\frac{d^2\theta}{dt^2} = \quad L_{\mathrm{CM}}\, mg\, \sin\theta$$

By using small angle approximation:

$$\sin\theta \approx \theta$$

We can rewrite the differential equation as

$$\frac{d^2\theta}{dt^2} = \frac{L_{\mathrm{CM}}\, mg}{I}\,\theta$$

To solve the differential equation, an auxiliary equation is required.

$$r^2 + \frac{L_{\mathrm{CM}}\, mg}{I} = 0$$

$$r_{1,2} = \pm\sqrt{\frac{mg\, L_{\mathrm{CM}}}{I}}$$

For $r_1 = a + jb$ and $r_2 = a \quad jb$,

Here, $a = 0$ and $b = \sqrt{\frac{mg\, L_{\mathrm{CM}}}{I}}$

The general solution for the differential equation is

$$y = e^{ax}(C_1 \cos b\theta + C_2 \sin b\theta)$$

Therefore,

$$y = C_1 \cos\sqrt{\frac{mg\, L_{\mathrm{CM}}}{I}}\,\theta + C_2 \sin\sqrt{\frac{mg\, L_{\mathrm{CM}}}{I}}\,\theta$$

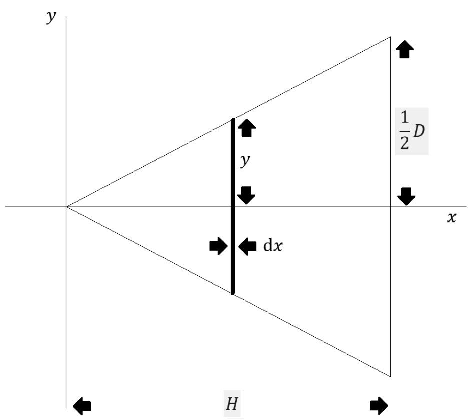
Define $\omega = \sqrt{\frac{mg\, L_{\mathrm{CM}}}{I}}$ to be the angular frequency of the oscillation. The time period T can then be written as

$$T = \frac{2\pi}{\omega}$$

Hence,

$$\boldsymbol{T = 2\pi\sqrt{\frac{I}{mg\, L_{\mathrm{CM}}}}}$$

To find L_{CM}, we need to know the centre of mass location for the triangle in question. Consider the diagram as below:



The centre of mass is defined as

$$x_{\mathrm{CM}} = \frac{\int x\, \mathrm{d}m}{M}$$

First, let's find M (i.e. the mass of the triangle). The area $\mathrm{d}A$ of an infinitesimally small rectangular strip is

$$\mathrm{d}A = 2y\, \mathrm{d}x$$

Therefore,

$$\mathrm{d}m = \rho\, \mathrm{d}A = \rho\, 2y\, \mathrm{d}x$$

where ρ is the density of the triangle.

Relationship between x and y is $y = \frac{D}{2H}\,x$

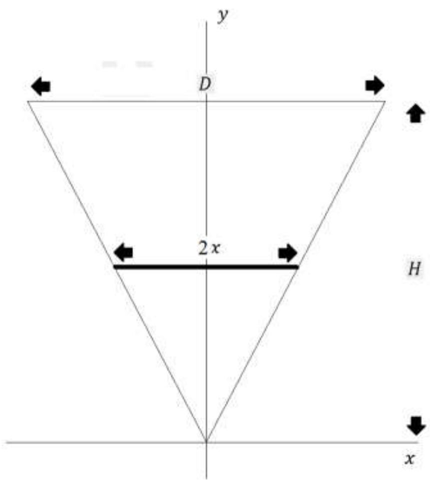
$$M = \int_0^H dm = \left[\frac{\rho D}{H}\frac{x^2}{2}\right]_0^H = \frac{\rho DH}{2}$$

Knowing what $\mathrm{d}m$ and M are, x_{CM} is

$$x_{\mathrm{CM}} = \frac{\frac{\rho D}{H}\int_0^H x^2\,\mathrm{d}x}{M}$$

$x_{\mathrm{CM}} = \frac{2}{3}H$ **from the apex of the triangle.**

Finally, we now need to find the moment of inertia of the triangle with respect to its apex. Consider the diagram as below:



We can think of the triangle is composing of infinitesimally small rods with length of $2x$ and negligible thickness.

The moment of inertia for a rod at its centre of mass is

$$I_{\mathrm{rod}} = \frac{1}{12}M(2x)^2 \rightarrow \mathrm{d}I_{\mathrm{rod}} = \frac{1}{12}\,\mathrm{d}m\,(2x)^2$$

However, the centre of mass of each rod is at a distance y from its apex. We need to use the parallel axis theorem. So that

$$\mathrm{d}I_{\mathrm{rod,apex}} = \mathrm{d}I_{\mathrm{rod}} + \mathrm{d}m\,(y)^2$$

$$\mathrm{d}I_{\mathrm{rod,apex}} = \frac{1}{12}\,\mathrm{d}m\,(2x)^2 + \mathrm{d}m\,(y)^2$$

$$I_{\mathrm{triangle,\,apex}} = \int \mathrm{d}I_{\mathrm{rod,apex}}$$

Given the diagram above, the relationship between y and x is

$$x = \frac{D}{2H}\,y$$

Also,

$$\mathrm{d}m = \frac{M}{A}\,\mathrm{d}A = \frac{4\,M\,x\,\mathrm{d}y}{DH}$$

where M and A are the mass and the area of the triangle respectively.

Combine everything together, we get

$$\mathrm{d}I_{\mathrm{rod,apex}} = \frac{1}{6}\frac{MD^2}{H^4}y^3\,\mathrm{d}y + \frac{2M}{H^2}y^3\,\mathrm{d}y$$

Hence,

$$I_{\mathrm{triangle,\,apex}} = \int_0^H \mathrm{d}I_{\mathrm{rod,apex}}$$

$$I_{\mathrm{triangle,\,apex}} = \int_0^H \frac{1}{6}\frac{MD^2}{H^4}y^3\,\mathrm{d}y + \frac{2M}{H^2}y^3\,\mathrm{d}y$$

The moment of inertia of the triangle with respect to its apex is

$$\boldsymbol{I_{\mathrm{triangle,\,apex}} = \frac{MD^2}{24} + \frac{MH^2}{2}}$$

Therefore, the expression of the time period T in terms of D and H for the physical pendulum is

$$\boldsymbol{T = 2\pi\sqrt{\frac{D^2 + 12\,H^2}{16\,H\,g}}}$$