1 Electromagnetic Induction

1.1 Lorentz force on moving charge

Point charge moving at velocity $\vec{v}$, $\vec{F} = q\vec{v} \times \vec{B}$ (1)

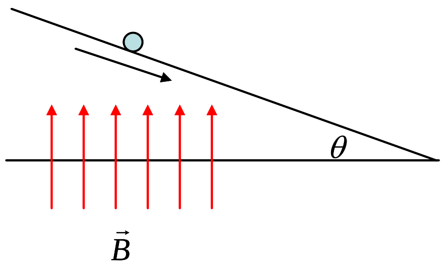
For a section of electric current I in a thin wire $d\vec{l}$ is $Id\vec{l}$, the force is $d\vec{F} = Id\vec{l} \times \vec{B}$ (2)

Electromotive force $\vec{f}_s$ – any force on a charged particle other than that due to other charges.

Electromotance $\xi = \int_a^b \vec{f}_s \cdot d\vec{l}$. Conventionally called emf, but it is not really a force. The Chinese translation 電動勢, is more appropriate.

The emf in a section of wire due to B-field is $\xi = \int_a^b (\vec{v} \times \vec{B}) \cdot d\vec{l}$ (3)

Example-1

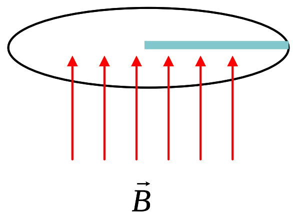


A rod of length L is sliding down a slope in a uniform B-field at speed v. Find emf in the rod.

Solution:

The amplitude of $(\vec{v} \times \vec{B})$ is $vB\cos\theta$ and its direction is pointing straight out of the paper plane, parallel to the length of the rod. So emf = $LvB\cos\theta$

Example-1A



The same rod is spinning at angular speed ω in the B-field around one of its end. The angular momentum is parallel to the B-field. Find emf between the two ends of the rod.

Solution:

Take a small length of the rod dr at distance r from the fixed end. The speed of it is ωr so its emf is $B\omega r dr$. Total emf along the whole length is $\xi = \int_0^L B\omega r dr = \frac{1}{2}B\omega L^2$ <u>ans.</u>

1.2 Faraday's law – A changing magnetic field induces an electric field.

Their relation is given by:

$$\xi = \oint_l \vec{E} \cdot d\vec{l} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} = -\frac{d}{dt}\iint_S \vec{B} \cdot d\vec{S} = -\frac{d\Phi}{dt} \qquad (4),$$

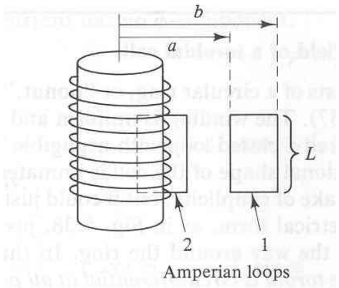
where 'S' is the surface enclosed by the closed line 'l'.

In differential form: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (5)

The total electric field now consists of two 'kind's of electric fields. One is due to charges, and the other is due to change B-field. <u>The direction of the induced E-field is such that it would generate an electric current to counter the change of the B-field.</u>

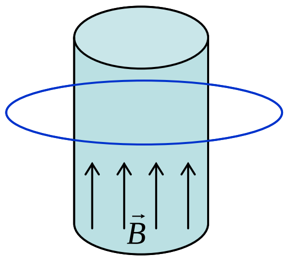
It can be shown that when a wire loop is moving relative to the B-field, Eqs. (3) and (4) are equivalent. However, Eq. (4) is more general, and works even when there is no relative motion, or the wire loop can be at the location where there is no B-field.

Example-2
The current in a long solenoid (N turns per unit length) is decreasing linearly with time t, $I = I_0 - kt$. Find the induced E-field.



Solution

By symmetry argument, we can see that the B-field is along the axis of the solenoid. Take an Ampere's loop-1 outside the solenoid, we find that the B-field is constant. But far away the B-field must be zero, so the B-field outside is zero. Now take loop-2, $BL = \mu_0 NIL$, so $B = \mu_0 N(I_0 - kt)$



Note that outside the solenoid there is no B-field, but the changing B-field induces an E-field so that if the circular wire carries charge, it will spin. Note also that Eq. (4) is of the same in mathematical form as Ampere's law, when $-\frac{\partial \vec{B}}{\partial t}$ is viewed as 'electric current' and the induced E-field as the 'magnetic field'. Applying Eq. (4), the left hand side = $2\pi rE$, where r is the radius of the loop, and the right hand side = $\mu_0 NkA$, where A is the cross section area of the solenoid. So $E = \frac{\mu_0}{2\pi}\frac{kNA}{r}$. <u>Ans.</u>

1.3 Self and mutual inductance (of coils)

Consider a coil (a loop of wire) carrying current I. If I changes with time, its magnetic field, and therefore the magnetic flux Φ through the coil (See Eq. (4)) will change, inducing an emf. As the field (hence flux) is proportional to I, we can define a quantity L which depends only on the geometrics of the coil, called self-inductance, such that

$\Phi = LI$ (6),

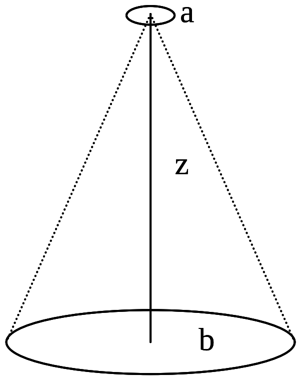
because $\xi = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}$ (6A).

The mutual inductance between two coils, M_{12} and M_{21}, are similarly defined.

$\Phi_2 = M_{12} I_1$, and $\Phi_1 = M_{21} I_2$. (Φ_2 is the magnetic flux through coil-2 due to the current I_1 in coil-1).

It can be shown that $M_{12} = M_{21}$, and they depend only on the coils geometrics.
Example-3

A very small coil of radius *a* is placed at distance *z* above a large coil of radius *b* along its axis, as shown in the figure. If there is current *I* flowing in the small loop, find the magnetic flux through the big loop.



Solution:

The small loop can be treated as a magnetic dipole and its magnetic field can be expressed exactly. The flux through the big loop can then be integrated out. But there is a simpler approach by using $M_{12} = M_{21}$. So let instead the same current *I* flow in the large loop, and find the magnetic field at the center of the small loop, and treat such B-field as uniform over the entire loop.

Take a diagonal pair of small sections of wire on the big loop, as shown, it is easy to see that the combined B-field is pointing along the vertical direction, and the amplitude is

$$dB = \frac{\mu_0 I}{4\pi}\frac{dl}{(b^2+z^2)}\frac{b}{\sqrt{b^2+z^2}} \quad .$$

Everything in the expression is constant so the integral over *dl* is $l = 2\pi b$.

So $B = \dfrac{\mu_0 I}{2}\dfrac{b^2}{(b^2+z^2)^{3/2}}$, $\Phi = B\pi a^2$, and $M = \dfrac{\mu_0}{2}\dfrac{\pi a^2 b^2}{(b^2+z^2)^{3/2}}$. Ans.

Example-4
The magnetic flux through a coil of resistance *R* changes from Φ_1 to Φ_2, Find the total charge passing through any cross section of the wire.

Solution:

From Eq. (6) and (6A) $-\dfrac{d\Phi}{dt} = \xi = RI = R\dfrac{dQ}{dt}$, so $Q = (\Phi_2 - \Phi_1)/R$. ans.

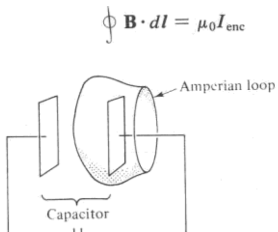
2 Maxwell's Equations

2.1 Maxwell's displacement current

What will the B-field be if $\vec{J}$ is changing with time? So far we assume that the current *I* in wires change with time but *I* is still the same along the wire so there is no charge piling up anywhere. In more precise terms, we assumed that $\nabla\cdot\vec{J} = 0$ still holds. Let us look at the Ampere's law again,

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \iint_S \vec{J}\cdot d\vec{S} = \mu_0 I_{enc}, \text{ or } \nabla\times\vec{B} = \mu_0\vec{J} \qquad (7)$$

There are infinite number of surfaces bound by the loop. Any surface bound by the loop will do as long as $\nabla\cdot\vec{J} = 0$, because the current through any one will be the same as the others. The general charge conservation implies



$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \qquad (8).$$

$\nabla \cdot \vec{J} \neq 0$ implies that there is charge accumulation somewhere, $\frac{\partial \rho}{\partial t} \neq 0$.

Note that $0 \equiv \nabla \cdot (\nabla \times \vec{B}) = \mu_0 \nabla \cdot \vec{J}$, so the Ampere's law must be modified when $\nabla \cdot \vec{J} \neq 0$. Maxwell introduced a second term to Eq. (7),

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad (9).$$

It is easy to show that now the divergence ($\nabla \cdot$) of both sides of Eq. (9) is always zero. This new term in Eq. (9) is called the Maxwell's displacement current. It serves as another source to generate B-field.

2.2 The Maxwell's Equations for E&M fields

(i) $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ (ii) $\nabla \cdot \vec{B} = 0$ (iii) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (iv) $\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

In a medium

$$\vec{D} = \varepsilon \varepsilon_0 \vec{E}, \ \vec{H} = \frac{\vec{B}}{\mu \mu_0} \qquad (10).$$

(i) $\nabla \cdot \vec{D} = \rho$ (ii) $\nabla \cdot \vec{B} = 0$ (iii) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (iv) $\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$ (M')

2.3 Energy of the EM fields

Poynting's vector is the energy flow density (watts/m^2)

In vacuum $\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$ (11)

In medium $\vec{S} = \vec{E} \times \vec{H}$ (11A)

Energy density $W = \frac{1}{2}(\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H})$ (12)

Energy is stored in the field.

Example-4
Current flows in a section of conductor wire.

Solution:

The wire is neutral so the E-field is uniform inside the wire (V/L) and zero outside, where V is the voltage difference between the two end surfaces, and L is the distance between them. The B-field can be found using Ampere's law.

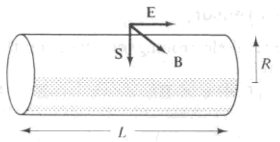


Figure 7.50

Accordingly, the Poynting vector is

$$S = \frac{1}{\mu_0}\frac{V}{L}\frac{\mu_0 I}{2\pi R} = \frac{VI}{2\pi RL}$$

pointing radially inward. The energy per unit time passing in through the surface of the wire is therefore

$$\int \mathbf{S}\cdot d\mathbf{a} = S(2\pi RL) = VI$$

which is exactly what we concluded, on rather more direct grounds, in Section 7.1.

2.4 Boundary conditions

$$\text{(i)} \quad \oint_S \mathbf{D} \cdot d\mathbf{a} = Q_{f_{enc}}$$

$$\text{(ii)} \quad \oint_S \mathbf{B} \cdot d\mathbf{a} = 0$$

over any closed surface S.

$$\text{(iii)} \quad \oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{a}$$

$$\text{(iv)} \quad \oint_L \mathbf{H} \cdot d\mathbf{l} = I_{f_{enc}} + \frac{d}{dt}\int_S \mathbf{D} \cdot d\mathbf{a}$$

for any surface S bounded by the closed loop L.

Applying (i) to a tiny, wafer-thin, Gaussian pillbox extending just a hair into the material on either side of the boundary, we obtain (Fig. 7.44):[12]

$$\mathbf{D}_1 \cdot \mathbf{a} - \mathbf{D}_2 \cdot \mathbf{a} = \sigma_f a$$

(The edge of the wafer contributes nothing in the limit as the thickness goes to zero. Nor does any *volume* density of free charge contribute, in this limit.) Thus, the component of **D** which is perpendicular to the interface is discontinuous in the amount

$$D_{1_\perp} - D_{2_\perp} = \sigma_f \tag{7.57}$$

Identical reasoning, applied to equation (ii), yields

$$B_{1_\perp} - B_{2_\perp} = 0 \tag{7.58}$$

Turning to (iii), a very thin Amperian loop straddling the surface (Fig. 7.45) gives

$$\mathbf{E}_1 \cdot \mathbf{l} - \mathbf{E}_2 \cdot \mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{a}$$

[12]The positive direction for **a** and $D_\perp$ is *from* 2 *toward* 1.

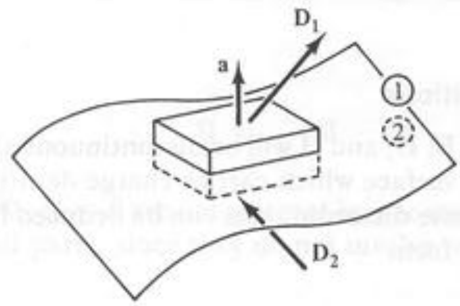


Figure 7.44

2.4 Boundary conditions (continue..)

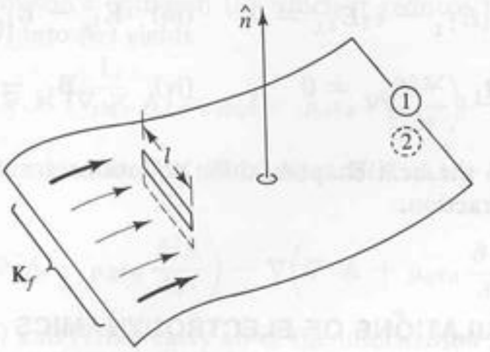


Figure 7.45

But in the limit as the width of the loop goes to zero, the flux vanishes. (I have already dropped the contribution of the two ends to $\oint \mathbf{E} \cdot d\mathbf{l}$, on the same grounds.) Therefore,

$$\boxed{\mathbf{E}_{1_\parallel} - \mathbf{E}_{2_\parallel} = 0} \quad (7.59)$$

That is, the components of **E** *parallel* to the interface are continuous across the boundary. By the same token, (iv) implies

$$\mathbf{H}_1 \cdot \mathbf{l} - \mathbf{H}_2 \cdot \mathbf{l} = I_{f_{\text{enc}}}$$

where $I_{f_{\text{enc}}}$ is the free current passing through the Amperian loop. No *volume* current density will contribute (in the limit of infinitesimal width) but a *surface* current can. In fact, if $\hat{n}$ is a unit vector perpendicular to the interface (pointing from 2 toward 1), so that $(\hat{n} \times \hat{l})$ is normal to the Amperian loop, then

$$I_{f_{\text{enc}}} = \mathbf{K}_f \cdot (\hat{n} \times \hat{l})l = (\mathbf{K}_f \times \hat{n}) \cdot \mathbf{l}$$

and hence

$$\boxed{\mathbf{H}_{1_\parallel} - \mathbf{H}_{2_\parallel} = \mathbf{K}_f \times \hat{n}} \quad (7.60)$$

So the *parallel* components of **H** are discontinuous by an amount proportional to the free surface current density.

Equations (7.57)–(7.60) constitute the general boundary conditions for electrodynamics. In the case of *linear* media, they can be expressed in terms of E and B alone:

$$\left.\begin{array}{llll} \text{(i)} & \epsilon_1 E_{1_\perp} - \epsilon_2 E_{2_\perp} = \sigma_f & \text{(iii)} & \mathbf{E}_{1_\parallel} - \mathbf{E}_{2_\parallel} = 0 \\ \text{(ii)} & B_{1_\perp} - B_{2_\perp} = 0 & \text{(iv)} & \dfrac{1}{\mu_1}\mathbf{B}_{1_\parallel} - \dfrac{1}{\mu_2}\mathbf{B}_{2_\parallel} = \mathbf{K}_f \times \hat{n} \end{array}\right\} \quad (7.61)$$

If there is no free charge or free current at the interface, then

2.5 Plane wave solutions of M's Equations (EM waves)

For plane waves, the fields are of the form $\vec{E}$ or $\vec{B} = \vec{A}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$, where $\vec{k}$ is the wavevector of the wave, and ω is the frequency. They are called plane waves because the equal-phase surface,

$$\vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z = \text{constant}$$

forms a plane. We now look for solutions of Eq. M above. Note that for plane waves, $\nabla\cdot\vec{E} = i\vec{k}\cdot\vec{E}, \nabla\times\vec{E} = i\vec{k}\times\vec{E}$, and $\frac{\partial}{\partial t}\vec{E} = -i\omega\vec{E}$.

In short, $\nabla = i\vec{k}$, and $\frac{\partial}{\partial t} = -i\omega$.

In vacuum, there is no current or charge. The M's equations become

$\vec{k}\cdot\vec{E} = 0$ (13a), $\vec{k}\cdot\vec{B} = 0$ (13b)

$\vec{k}\times\vec{E} = \omega\vec{B}$ (13c), $\vec{k}\times\vec{B} = -\mu_0\varepsilon_0\omega\vec{E}$ (13d)

The equations above imply that $\vec{k} \perp \vec{E} \perp \vec{B}$. Choose $\vec{k} = k\vec{z}_0$, $\vec{E} = E\vec{x}_0$, $\vec{B} = B\vec{y}_0$, where E and B are constants, and apply $\vec{k}\times$ to both sides of Eq. (13c) and put the result in Eq. (13d), we get

$$k = \pm\sqrt{\mu_0\varepsilon_0}\,\omega = \pm\frac{\omega}{c} \qquad (14),$$

where c is the speed of EM-wave propagation in vacuum. (c = 2.99792458 $\times 10^8$ m/s) Equation (14) means that for a given frequency ω, which describes the time variation of the EM fields, the spatial variation parameter k is fixed, as given by Eq. (14). The dependence of k on ω is called the dispersion relation, and is determined by the medium in which EM waves propagate.

In non-conducting medium with no free charge and current, the M's equations become (after replacing $\vec{D}$ and $\vec{H}$ by $\vec{E}$ and $\vec{B}$ using Eq. (10))

$\vec{k}\cdot\vec{E} = 0$ (15a), $\vec{k}\cdot\vec{B} = 0$ (15b)

$\vec{k}\times\vec{E} = \omega\vec{B}$ (15c), $\vec{k}\times\vec{B} = -\mu_0\mu\varepsilon_0\varepsilon\omega\vec{E}$ (15d).

So the results in vacuum still apply except that

$$k = \pm\sqrt{\mu\mu_0\varepsilon\varepsilon_0}\,\omega = \pm\frac{n\omega}{c} \qquad (16).$$

The speed of wave propagation is c/n, where $n \equiv \sqrt{\mu\varepsilon}$ is the refractive index in optics. Notice that the k in Eq. (16) is different from that in Eq. (14), for the same frequency ω. This again shows that dependence of k on ω is determined by the medium in which EM waves propagate.

The energy flow, Poynting's vector *S*, of the EM wave, using Eq. (11), is … (HW-3). *S* is what we usually refer to as the light intensity.

In conducting medium

$\vec{J}_f = \sigma\vec{E}$, while everything else remains the same as in a non-conducting medium (The free charge density due to the free current can be shown to decay exponentially with time, so can be taken as zero if we wait long enough). Equation (15d) becomes

$$i\vec{k}\times\vec{B} = \mu\sigma\vec{E} - i\mu_0\mu\varepsilon_0\varepsilon\omega\vec{E} = -i(i\sigma + \mu_0\mu\varepsilon_0\varepsilon\omega)\vec{E} \qquad (17).$$

The solution is then $k^2 = \mu\mu_0\varepsilon\varepsilon_0\omega^2 + i\mu\mu_0\sigma\omega = \left(k_R^2 - k_I^2 + 2k_Rk_I\right)$

ω is real, so *k* is complex $k = k_R + ik_I = \frac{\omega\sqrt{\mu\varepsilon}}{c}\left(1+\frac{i\sigma}{\omega\varepsilon\varepsilon_0}\right)^{1/2}$ (18)

Solving Eq. (18)

$k_R^2 - k_I^2 = \mu\mu_0\varepsilon\varepsilon_0\omega^2 \qquad 2k_Rk_I = \mu\mu_0\sigma\omega$

$$k_{R,I} = \frac{\omega}{c}\sqrt{\frac{\mu\varepsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\varepsilon_0\omega}\right)^2}\pm 1\right]^{1/2} \qquad (19)$$

The E-wave is $E = E_0e^{i(kz-\omega t)} = E_0e^{-k_I\cdot z}e^{i(k_Rz-\omega t)}$ (20)

Attenuation factor $e^{-k_I\cdot z}$, skin depth $dp = \frac{1}{k_I}$ (penetration depth)

1. Good conductors: $\frac{\sigma}{\varepsilon\varepsilon_0\omega} >> 1$

$$k \approx \frac{\omega}{c}\sqrt{\mu\varepsilon}\left[i\frac{\sigma}{\varepsilon\varepsilon_0\omega}\right]^{1/2} = \sqrt{\frac{\omega\mu\mu_0\sigma}{2}}(1+i) \qquad (21)$$

$$dp = \sqrt{\frac{2}{\omega\mu\mu_0\sigma}}$$

2. Poor conductors: $\frac{\sigma}{\varepsilon\varepsilon_0\omega} << 1$ then $dp = \frac{2}{\sigma}\left(\frac{\varepsilon\varepsilon_0}{\mu\mu_0}\right)^{1/2}$

For $\vec{E} = \vec{e}_1E_0e^{i(kz-\omega t)}$

$$\vec{H} = \frac{i}{\omega\mu\mu_0}\nabla\times\vec{E} = \vec{e}_2\frac{k_R + ik_I}{\omega\mu\mu_0}E_0e^{i(kz-\omega t)}$$ (good conductors , $k_R \approx k_I$)

so $\vec{E}$ & $\vec{H}$ are $\pi/4$ out of phase to each other.

Energy flow

$$\langle\vec{s}\rangle = \frac{1}{2}\mathrm{Re}\{\vec{E}\times\vec{H}^*\} = \frac{1}{2}\frac{k_R\left|E_0^2\right|}{\mu\mu_0\omega}e^{-2k_Iz}\vec{e}_3 \qquad (22),$$

so $\langle\vec{s}\rangle$ attenuates

$$\nabla\cdot\langle\bar{s}\rangle=-\frac{k_R k_I\left|E_0^2\right|}{\mu\mu_0\omega}e^{-2k_I z}=-\frac{1}{2}\frac{\mu\mu_0\sigma\omega}{\mu\mu_0\omega}\left|E_0^2\right|e^{-2k_I z}=-\frac{1}{2}\sigma\left|E_0^2\right|e^{-2k_I z}<0 \qquad (23)$$

E-M wave is loosing energy to media.

Energy loss = heat

Heat = $<\vec{E}\cdot\vec{J}>=\frac{1}{2}\mathrm{Re}(\vec{E}\cdot\vec{J}^*)=\frac{1}{2}\sigma\,\mathrm{Re}(\vec{E}\cdot\vec{E}^*)=\frac{1}{2}\sigma\left|E_0^2\right|e^{-2k_I z}=-\nabla\cdot\langle\bar{s}\rangle$ (24)

Plasma (neutral ionized gas)

Either one type of charge is mobile, free electrons in ideal conductors, or both can move, like in the upper atmosphere. Usually, if electrons are present, they will dominate because of their much lighter mass (1/1840 of a Hydrogen nucleus) than any other ions.
Consider the case with one type of mobile charge particles with charge *q* and mass *m*, and number density *N* per unit volume.
The force due to EM fields on a particle is $\vec{f}=q(\vec{E}+\vec{v}\times\vec{B})$ (25).
But *B = E/c* in vacuum so if *v << c* the B-field force can be neglected.

Newton's law: $q\vec{E}=m\frac{d\vec{v}}{dt}=-i\omega m\vec{v}$ (26)

Current density of plasma $\vec{J}_p=qN\vec{v}=\frac{iq^2N}{m\omega}\vec{E}$ (27)

One can see in Eq. (27) that the particles with lighter mass contribute more to the current.
Applying Ampere's law,

$$i\vec{k}\times\vec{B}=-i\omega\varepsilon\varepsilon_0\vec{E}+\frac{iq^2N}{m\omega}\vec{E}=-i\omega(\varepsilon\varepsilon_0-\frac{q^2N}{m\omega^2})\vec{E} \qquad (28)$$

Here ε is due to the polarization of the particles, such as the immobile ions. Equation (28) is similar to the one in insulating (dielectric) medium except that now the effective dielectric constant is given by

$$\varepsilon(\omega)=(\varepsilon\varepsilon_0-\frac{q^2N}{m\omega^2})=\varepsilon\varepsilon_0(1-\frac{\omega_p^2}{\omega^2}) \qquad (29),$$

where $\omega_p\equiv\sqrt{\frac{q^2N}{\varepsilon\varepsilon_0 m}}$ (30)

is the plasma characteristic frequency.

$$n(\omega)=\sqrt{\varepsilon(\omega)} \qquad (31)$$

The dielectric constant, and refractive index, is now a function of frequency. So we should refer it as dielectric <u>function</u>, instead of dielectric <u>constant</u>.
When the frequency of the EM-wave is below ω_p, the dielectric function $\varepsilon(\omega)$ is negative, so the refractive index $n(\omega)$ is imaginary, and $k = n\omega/c$ is also imaginary. The EM-wave cannot propagate. When $\omega>\omega_p$ the plasma behaves like 'normal' dielectric medium, except that $n(\omega)$ may be less than 1.