Optics-1

The topics to be covered in this lecture are

- Geometric Optics
- Interference
- Polarization

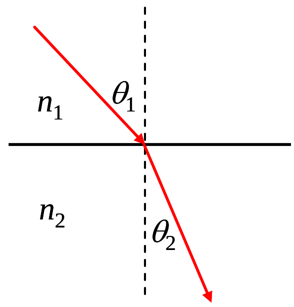
1 Geometric Optics

1A) Principle

In geometric optics, we treat light waves as light rays, or lines, because the length scale of the optical elements we are dealing with, such as lenses and mirrors, is much larger than the wavelength of the light waves. In the sections of Interference and Diffraction we will then treat the EM waves like waves. The surfaces of these optical elements are spherical only (planes can be viewed as spherical with infinitely large radius), and we impose the condition that only the light rays nearly parallel to the optical axis are considered, i. e., the angle θ between the rays and the optical axis is so small that we can use the small angle approximation

$$\sin\theta \approx \tan\theta \approx \theta \qquad (1a),$$
$$\cos\theta \approx 1 \qquad (1b).$$

Such condition plus the law of refraction (Snell's Law) are the foundation of all the lenses and mirrors. When applying Snell's law for curved interface the plane is the tangential plane of the interface at the point the light ray reaches the interface.



The following is an example of using refraction law and small angle approximation to find the image formed by a spherical interface between two media. The refractive indexes of the two media are n_1 and n_2, respectively. The Snell's law states

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \qquad (2a)$$

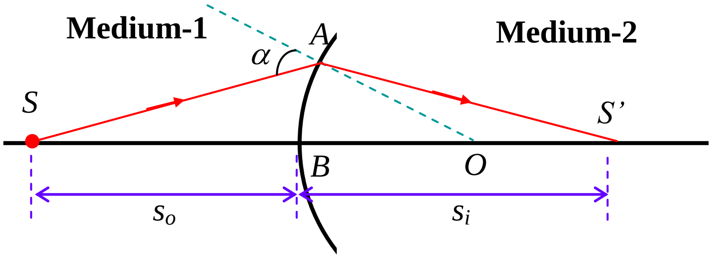
With small angles,

$$n_1\theta_1 = n_2\theta_2 \qquad (2b)$$

1B) Image Formation

Consider the case below. The radius of the spherical interface between Medium-1 and Medium-2 is R. The interface is convex as viewed from Medium-1 to Medium-2. A point light source is placed on the left side of the interface in Medium-1. The line joining the source and the center of the sphere defines the optical axis of the system. Keep in mind that the light beam is actually nearly parallel to the optical axis, and $\overleftrightarrow{AB} << R$, and nearly perpendicular to the optical axis.

Let the distance between the light source S and the interface be s_o. Assume that all the light rays reaching the interface will converge onto a point P which we call the *image* of the source S. The distance from the image to the interface is s_i. The aim is to find the relation between s_o and s_i in terms of the system parameters such as the refractive indexes n_1 and n_2 and the interface curvature R. The strategy is to express the distances in terms of appropriate angles and the length $\overleftrightarrow{AB}$ shared by both sides, and then connect the angles using Snell's law.



$$(\angle ASB) = \frac{\overleftrightarrow{AB}}{s_0} \qquad (3a)$$

$$(\angle AS'B) = \frac{\overleftrightarrow{AB}}{s_i} \qquad (3b)$$

$$(\angle AOB) = \frac{\overleftrightarrow{AB}}{R} \qquad (3c)$$

$$n_1\alpha = n_2(\angle S'AO) \qquad (3d)$$

We now need to express the two angles in Eq. 3d in terms of the three angles in Eqs. 3a-3c. By simple geometry considerations and take triangle ΔASO, we have

$$\alpha = (\angle ASB) + (\angle AOB) \qquad (3e)$$

Take triangle $\Delta AS'O$, we have

$$(\angle AOB) = (\angle AS'B) + (\angle S'AO) \qquad (3f)$$

Putting Eqs. 3a and 3c into Eq. 3e, Eqs. 3b and 3c into Eq. 3f, and finally putting Eqs. 3e and 3f into Eq. 3d, we can find the image distance s_i in terms of the object distance s_o and the parameters of the system n_1, n_2, and R.

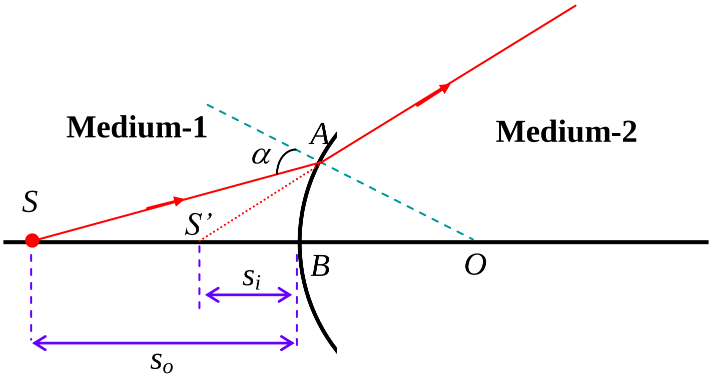
$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{1}{R}(n_2 - n_1) \qquad (3g).$$

Everything in Eq. 3g is positive. Note that the result is independent of $\overleftrightarrow{AB}$. What does that mean? It means that under the small angle approximation all the light rays reaching the interface will converge onto the image point, regardless where they strike. That is the condition for image formation. Otherwise, the image of a point source will be a patch of light, i. e., it is a blur and distorted image.

What if the image is formed on the left side of the interface, as shown below? It is a *virtual* image because the light rays inside Medium-2 seem to come from the point source *S*' inside Medium-1, but there is actually no such source there.

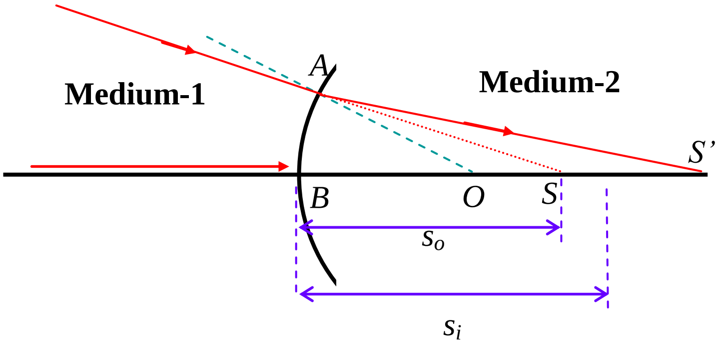
Using the approach similar to the first case, and leaving the actual procedure as an exercise, we can get

$$\frac{n_1}{s_o} + \frac{n_2}{-|s_i|} = \frac{1}{R}(n_2 - n_1) \qquad \text{(3h)}$$



Equation 3h implies Eq. 3g is still applicable in the cases of virtual images, as long as the image distance takes negative values.

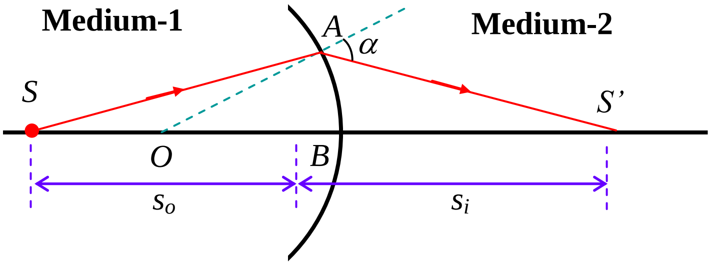
What if the source is virtual, i. e., the rays in Medium-1 would converge onto a point in Medium-2 if there were no refraction at the interface? This can happen if there is another interface to the left of the present interface under consideration, and the image formed by that interface is to the right side of the present interface.



Using the approach similar to the first case, and leaving the actual procedure as an exercise, we can get

$$\frac{n_1}{-|s_o|} + \frac{n_2}{s_i} = \frac{1}{R}(n_2 - n_1) \qquad \text{(3i)}$$

Equation 3i implies that Eq. 3g is still applicable in the cases of virtual sources, as long as the object distance takes negative values.



Finally, in the cases of concave interface shown above, using the approach similar to the first case, and leaving the actual procedure as an exercise, we can get

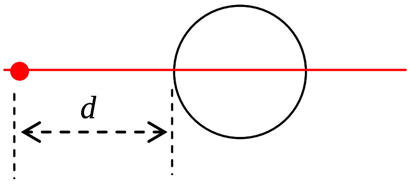
$$\frac{n_1}{s_o}+\frac{n_2}{s_i}=\frac{1}{-|R|}(n_2-n_1) \qquad \text{(3j)}$$

Equation 3j implies Eq. 3g is still applicable in the cases of concave interfaces, as long as the radius takes negative values.

Summarizing the above results, we get the sign convention of the lens formulae Eq. 3g: radius is positive for convex, negative for concave interfaces; object distance is positive if the source is on the left side of the interface, is negative if the *virtual* source is on the right side of the interface; image distance is positive if the image is on the right side of the interface, is negative if the *virtual* image is on the left side of the interface.

Example-1

A point source is placed on the left side of a glass sphere of radius R and refractive index 1.5. The distance between the left surface and the source is d.
Find the final image if (a) d = R, (b) d = 2R, (c) d = 2.5R, (d) If the final image is at the right surface, where should the source be and whether the source is real or virtual?



Solution:

(a) First (left) surface, $s_o=R$, n_1 = 1, n_2 = 1.5, convex,

$\frac{1}{R}+\frac{1.5}{s_i}=\frac{1}{R}(1.5-1)$, which leads to $s_i=-3R$, virtual image.

Second (right) surface, $s_o^{'}=2R-(-3R)=5R$, n_1 = 1.5, n_2 = 1, concave

$\frac{1.5}{5R}+\frac{1}{s_i}=\frac{1}{-R}(1-1.5)$, which leads to $s_i^{'}=5R$. So the real image is at a point 5R from the right surface of the sphere.

(b) First (left) surface, $s_o=2R$, n_1 = 1, n_2 = 1.5, convex,

$\frac{1}{2R}+\frac{1.5}{s_i}=\frac{1}{R}(1.5-1)$, which leads to $s_i=\infty$.

Second (right) surface, $s_o^{'}=\infty$, n_1 = 1.5, n_2 = 1, concave

$\frac{1.5}{\infty}+\frac{1}{s_i}=\frac{1}{-R}(1-1.5)$, which leads to $s_i^{'}=2R$. So the real image is at a point 2*R* from the right surface of the sphere.

(c) First (left) surface, $s_o=2.5R$, n_1 = 1, n_2 = 1.5, convex,

$\frac{1}{2.5R}+\frac{1.5}{s_i}=\frac{1}{R}(1.5-1)$, which leads to $s_i=15R$.

Second (right) surface, $s_o^{'}=2R-15R=-13R$ (virtual source), n_1 = 1.5, n_2 = 1, concave

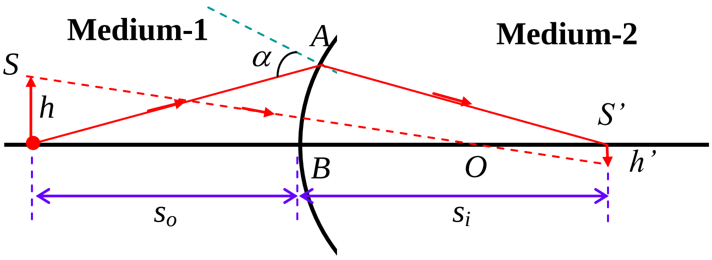
$\frac{1.5}{-13R}+\frac{1}{s_i}=\frac{1}{-R}(1-1.5)$, which leads to $s_i^{'}=\frac{13}{8}R=1.625R$. So the real image is at a point 1.625*R* from the right surface of the sphere.

(d) The image formed by the first (left) surface should be at the second (right) surface.
$s_i=2R$, n_1 = 1, n_2 = 1.5, convex

$\frac{1}{s_o}+\frac{1.5}{2R}=\frac{1}{R}(1.5-1)$, which leads to $s_o=-4R$. So the virtual source should be at 4*R* distance from the left surface.

1C) Magnification

We now consider the magnification of the image. For that we place a second point source at a distance *h* from the optical axis. Its image will still at a distance s_i from the interface, but at a distance *h*' from the axis.



The ray from the second source towards the center of the interface *O* should not change its direction, as shown above. The by simple geometry, we get the magnification is

$$M=\frac{s_i-R}{s_o+R} \qquad (4a)$$

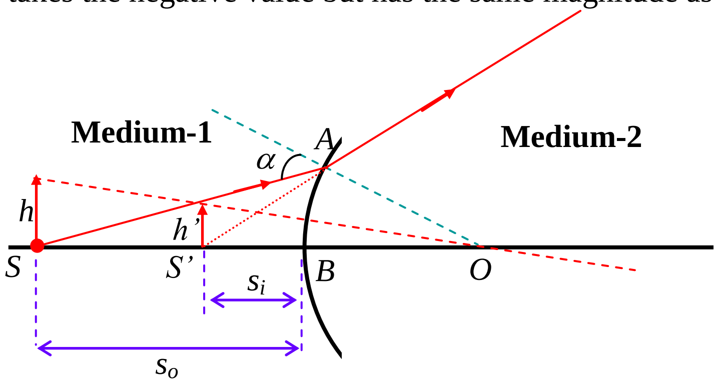
Equation 4a is actually a general expression for all cases, including the ones where the object distance, the image distance, and the radius are negative. The image is real if *M* is positive. The image is a virtual one if *M* is negative. Take the case shown below, the magnification is

$$M=\frac{s_i+R}{s_o+R} \qquad (4b)$$

If we use the sign conventions, the image distance is negative. Put negative image distance to Eq. 4a, we get

$$M = \frac{-|s_i| - R}{s_o + R} = -\frac{|s_i| + R}{s_o + R} \qquad (4c)$$

The result in Eq. 4c takes the negative value but has the same magnitude as in Eq. 4b.



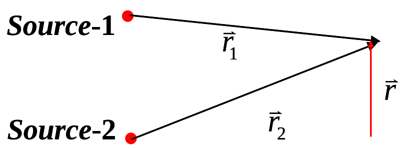
For mirrors, the same lens formulae applies with the difference from the lenses that the focal length = *R*/2, positive for concave and negative for convex; object and image on the left side are positive, on the right side are negative.

2 Interference

2A) Principle

Consider two point sources in space. At *source*-1 the emitted electromagnetic (EM) wave is

$$\tilde{E}_1(t) = E_1 \cos(\omega t).$$



At *source*-2 the emitted EM wave is

$$\tilde{E}_2(t) = E_2 \cos(\omega t)$$

Both sources emit spherical waves of the form

$$\frac{r_0}{r} E \cos(kr - \omega t). \qquad (5a)$$

At an observation position in space $\vec{r}$, the displacement vector from *source*-1 to the position is $\vec{r}_1$, and that from *source*-2 is $\vec{r}_2$. As one can see, both will change with the change of $\vec{r}$. As the change of $\vec{r}_1$ and $\vec{r}_2$ are small, we ignore the amplitude change of the waves at the position due to the denominators in Eq. 5a and treat the amplitudes as constants.

The EM wave from *source*-1 at the observation position is $\tilde{E}_1(\vec{r}, t) = E_1 \cos(kr_1 - \omega t)$.

The EM wave from *source*-2 at the observation position is $\tilde{E}_2(\vec{r}, t) = E_2 \cos(kr_2 - \omega t)$

Suppose the directions of the electric fields of the two EM waves are the same, then according to the superposition principle, the total field at position $\vec{r}$ is

$$\tilde{E}(\vec{r}, t) = \tilde{E}_1(\vec{r}, t) + \tilde{E}_2(\vec{r}, t) = E_1 \cos(kr_1 - \omega t) + E_2 \cos(kr_2 - \omega t). \qquad (5b)$$

As has been shown in the EM wave part of the EM lectures, the light intensity is the time-averaged Poynting vector of the EM wave. So

$$I(\vec{r}) = <\tilde{S}(\vec{r},t)> = c\varepsilon_0 <\tilde{E}(\vec{r},t)^2>$$
$$= c\varepsilon_0\left[E_1^2 <\cos^2(kr_1-\omega t)> + E_2^2 <\cos^2(kr_2-\omega t)> + 2E_1E_2 <\cos(kr_1-\omega t)\cos(kr_1-\omega t)>\right] \quad (5c)$$

The time average is over many periods of the EM wave. Note that the average over one period of EM wave is

$$<\cos^2(kr_1-\omega t)> = \frac{1}{2}<1+\cos(2kr_1-2\omega t)> = \frac{1}{2}, \quad (5d)$$

because the average of any sine or cosine functions over one period is always zero.

$$<\cos(kr_1-\omega t)\cos(kr_1-\omega t)> \frac{1}{2}<\cos(kr_1-kr_2)+\cos(kr_1+kr_2-2\omega t)> = \frac{1}{2}\cos(kr_1-kr_2) \quad (5e)$$

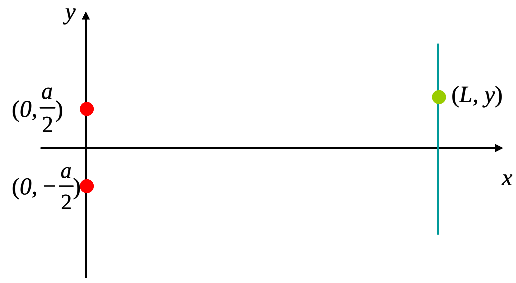
So finally,

$$I(\vec{r}) = \frac{1}{2}c\varepsilon_0\left[E_1^2+E_2^2+2E_1E_2\cos(k\cdot\Delta r(\vec{r}))\right]$$
$$= \frac{1}{2}c\varepsilon_0\left[E_1^2+E_2^2+2E_1E_2\cos(2\pi\cdot\Delta r(\vec{r})/\lambda)\right]. \quad (5f)$$
$$= I_1+I_2+2\sqrt{I_1I_2}\cos(2\pi\cdot\Delta r(\vec{r})/\lambda)$$

Here $\Delta r \equiv |\vec{r}_1|-|\vec{r}_2|$, and λ is the wavelength. Equation 5f is the basic formulae for ideal two-source interference. *For multiple-source cases we should simply add the electric fields of all the sources, and then take the time average of the square of the total field.* Diffraction is in fact a problem of multiple-source interference, so it can be dealt with accordingly.

2B) Optical Path

The optical path difference Δr is a function of observation position $\vec{r}$. A small change in Δr (over a value of wavelength $\lambda/2$) will change the total intensity from
$\frac{1}{2}c\varepsilon_0(E_1^2+E_2^2+2E_1E_2)$ to $\frac{1}{2}c\varepsilon_0(E_1^2+E_2^2-2E_1E_2)$.

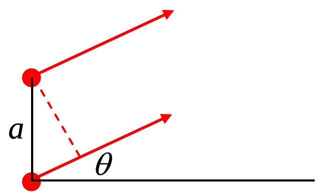


Such change of light intensity over a relatively small distance in space is called the *interference effect* of waves. Below is an example.

Example-2
Two light sources are at the positions (0, a/2) and (0, – a/2), respectively. The observation position is at (L, y), and $L >> a$.
The optical path difference is then

$$\Delta r = \sqrt{L^2+(y+\frac{a}{2})^2} - \sqrt{L^2+(y-\frac{a}{2})^2} \approx \frac{ay}{L}. \quad (6a)$$

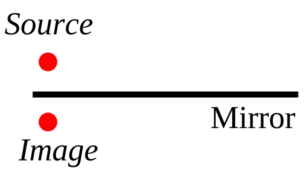
Taylor expansion is used to get the final result. This is equivalent to what is shown in the right figure where the

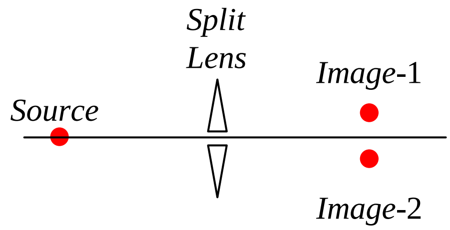


optical path difference is $\Delta r = a\theta$, with $\theta = \frac{y}{L} << 1$. In more general cases, the optical path difference is

$$\Delta r = a\sin\theta \qquad (6b)$$

Some of the other two-source cases are shown below. In the case of the left figure, the image due to the mirror is the second source. In the case of the right figure, the lens is cut in half in the middle. The upper half is lifted a little and the lower part is pulled down a little. As a result, two images of the point source are formed which serve as the two sources. There are many configurations like Newton's ring, thin film interference, which are the cases where the reflection beam of the first surface and that of the second surface interfere. The optical path difference, in normal incidence, is given by 2*dn*, where *n* is the refractive index of the film and *d* the thickness. The calculation of optical path becomes the main task in solving the interference problems. In all these cases, the two sources are actually originated from a single source, so the electric field directions are always the same.





2C) Visibility (Contrast)

Visibility (contract) is defined as

$$A \equiv \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{2E_1E_2}{E_1^2 + E_2^2}\,. \qquad (7)$$

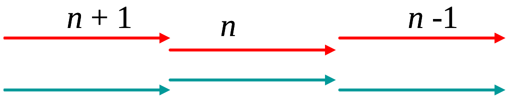
The second equal sign holds only for *ideal* interference cases. *A* takes the maximum value of 1 if $E_1 = E_2$, and takes the minimum value of 0 if $I_{max} = I_{min}$, i. e., there is no interference effect at all. To see how one can get no interference effect, please read the following text concerning realistic interference cases.

2D) Coherent Length of Sources

An ideal source would emit EM waves in the form of $E = E_0\cos(\omega t - \phi)$ forever. A real source emits something similar, except that the phase ϕ changes in a random fashion after a period of time τ, i. e.,

$$\phi(t) = \begin{Bmatrix} \phi_1, & 0 < t \le \tau \\ \phi_2, & \tau < t \le 2\tau \\ \phi_3, & 2\tau < t \le 3\tau \\ & .... \end{Bmatrix} \qquad (8)$$

Here τ is much larger than the period of the wave $2\pi/\omega$, but still much smaller than the observation time. That is, we still need to take time average over a duration much longer than τ. The phases ϕ_1, ϕ_2, …take *random values*. τ is called the *coherent time* of the source, and $L = c\tau$ is called the *coherent length* of the source, where *c* is the speed of light in vacuum. We call each continuous wave with a fixed phase a *wave train*. Obviously, in vacuum the length of such wave train is the coherent length of the source. Sources like ordinary light bulbs usually have coherent length of a few centimeters, while lasers can have coherent length of several meters. A real light source therefore emits a continuous series of wave trains *n* = 1, 2, …, with phases ϕ_1, ϕ_2, …



Consider two sources (say two red laser pointers) emitting EM waves of the same frequency. For simplicity, assume that they are turned on at the same time. Their initial phases are α_1, α_2, …, and β_1, β_2, …, and we choose a position in space where the optical path difference is Δr. The above figure shows the two series of wave trains from the two sources, the red ones from source-1, and the green ones from source-2, at the observation position. For simplicity, we assume $\Delta r \ll L = c\tau$ for the time being. Similar to the ideal interference case, and take time average over a period of $N\tau$, the intensity at the observation position is

$$I = \frac{1}{2} c\varepsilon_0 \left[E_1^2 + E_2^2 + 2E_1E_2 \sum_{n=1}^{N>>1} \cos(k\Delta r + \alpha_n - \beta_n) / N\tau \right]. \qquad (9a)$$

Because the phase differences ($\alpha_n - \beta_n$)'s are all random, the sum over many cosines of them becomes zero. That is:

$$\sum_{n=1}^{N>>1} \cos(k\Delta r + \alpha_n - \beta_n) = 0 . \qquad (9b)$$

We then get

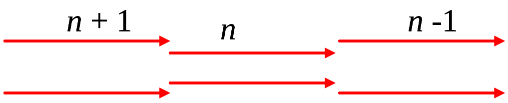
$$I = \frac{1}{2} c\varepsilon_0 (E_1^2 + E_2^2) = I_1 + I_2 , \qquad (9c)$$

i. e., there is no interference effect since the total intensity does not depend on the optical path difference Δr, and the interference contrast is zero.

Note that if the two sources are ideal ones, then the result would be

$$I = \frac{1}{2} c\varepsilon_0 \left[E_1^2 + E_2^2 + 2E_1E_2 \cos(k\Delta r + \alpha - \beta) \right], \qquad (10)$$

where α and β are the initial phases of the two sources, which remain constant all the time for ideal sources. Now the total intensity does depend on Δr, and we would have interference effect. *The reason that we do not get interference from two actual light sources, even though they have the same frequency, is because they are not ideal sources.*

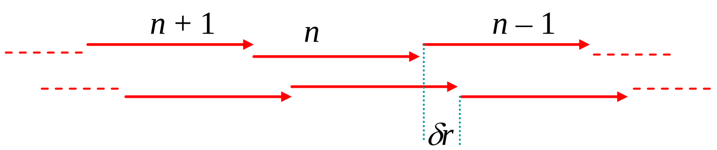


It is now clear that in order to obtain interference, we need to somehow 'split' one real source into two, such as in the case of using a mirror to create an image source to form a two-source scheme, or using the split-lens to create a two-source scheme. The wave trains from such two sources at the observation position are shown in the right figure. Although each wave train has different initial phase, the two wave trains that arrive at the observation point at the same time always have the same initial phase (That is why all the wave trains in the figure are red in color). In such cases, although the initial phase changes, but because both sources have the same initial phase, such change cancels out if the optical path difference $\Delta r << c\tau$. The result is then

$$I = \frac{1}{2} c\varepsilon_0 \left[E_1^2 + E_2^2 + 2E_1E_2 \cos(k\Delta r) \right], \qquad (11)$$

And we have interference like in the ideal source case.

Now we consider the more realistic case that the optical path difference is comparable to the length of the wave trains (the coherent length). The figure below shows the wave trains at the observation point. The two wave trains from the two (split) sources with the same initial phase are not completely overlap in space or time. One train is ahead of the other by a distance δr. The overlapping length of the pair of wave trains with the same initial phase is no longer the entire length of the wave train, but is shortened by δr, i. e., the overlapping length for Eq. 10 is $c\tau$, but in this case it is $(c\tau - \delta r)$.



Similar to the ideal source case, the light intensity is

$$I = \frac{1}{2} c\varepsilon_0 \left\{ E_1^2 + E_2^2 + 2E_1E_2 \left[\frac{N(c\tau - \Delta r)}{Nc\tau} \cos(k\Delta r) + \frac{N\delta r}{Nc\tau} \sum_{n=1}^{N>>1} \cos(\phi_n - \phi_{n+1}) \right] \right\}. \qquad (12a)$$

The last summation term is zero because of the randomness of the phases $(\phi_n - \phi_{n+1})$. So finally

$$I = \frac{1}{2} c\varepsilon_0 \left[E_1^2 + E_2^2 + 2E_1E_2 \frac{(c\tau - \Delta r)}{c\tau} \cos(k\Delta r) \right], \text{ if } \Delta r < c\tau. \qquad (12b)$$

It is also clear based on the discussion above that if $\Delta r > c\tau$, we will have

$$I = \frac{1}{2} c\varepsilon_0 (E_1^2 + E_2^2) = I_1 + I_2 \qquad (12c)$$

That is, if the optical path difference is larger than the coherent length of the sources, then there is no interference effect. One can perform experiments to verify such effect. One way is to introduce some transparent medium of refractive index *n* in one optical path, therefore increasing its value by *n* times. The resulting optical path difference is then increased significantly without even changing position of any other parts, such as the sources and the observation screen. For sources with relatively short coherent length, the interference pattern on the observation screen will then disappear.

3 Polarization of Light

The polarization of light refers to the evolution pattern of the electric field in time at a fixed point in space. Consider the simple case $\vec{D} // \vec{E},\ \nabla \cdot \vec{D} = 0 \ \Rightarrow \vec{E} \perp \vec{k}$. The electric field of the EM wave is therefore within the plane perpendicular to the wave vector $\vec{k}$. The general form of the electric field of a plane EM wave is then

$$\vec{E} = E_{x0}\cos(\omega t - \phi_x)\vec{x}_0 + E_{y0}\cos(\omega t - \phi_y)\vec{y}_0 \qquad (13)$$

at any fixed point in space.

ϕ_x, ϕ_y ---- phase, E_{x0}, E_{y0} are real and positive.

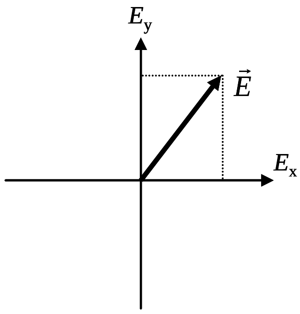
Let

$E_x = E_{x0}\cos(\omega t - \phi_x)$; (14a)

$E_y = E_{y0}\cos(\omega t - \phi_y)$, (14b)

then $\vec{E} = E_x \vec{x}_0 + E_y \vec{y}_0$ (14c)

Consider a position vector $\vec{r} = x(t)\vec{x}_0 + y(t)\vec{y}_0$ of a point in the X-Y coordinate system. As time t goes by the position of the point forms a trajectory. Similar to this, here we plot $\vec{E}$ in Eq. 14c in the E_x, E_y coordinate system. Putting a point at the end of $\vec{E}$, the trajectory of the point in the E_x, E_y coordinate system determines the polarization state of the light beam. If the trajectory is a line, we call it as linearly polarized, or linear polarization. If the trajectory is a circle, we call it circular polarization, and so on.

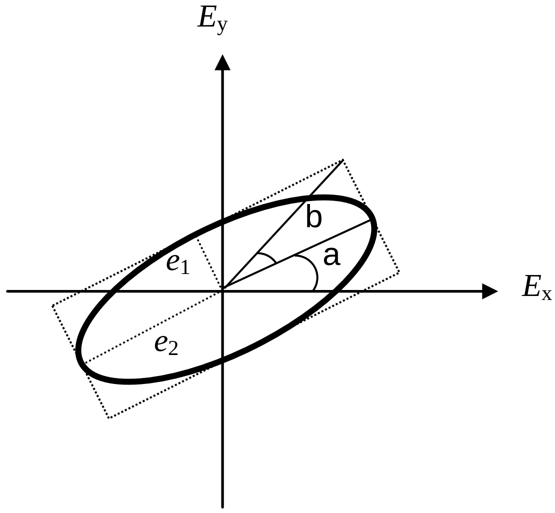


In the special case $\phi_x = \phi_y$, then $\dfrac{E_x}{E_y} = \dfrac{E_{x0}}{E_{y0}}$ or $E_y = \dfrac{E_{x0}}{E_{y0}} \cdot E_x$, leading to a straight line trajectory, and we call such polarization state 'linearly polarized state.'

For more complicated cases one should eliminate time t first, and we get the general form

$$\left(\frac{E_x}{E_{x0}}\right)^2 + \left(\frac{E_y}{E_{y0}}\right)^2 - \frac{2E_xE_y}{E_{x0}E_{y0}}\cos\varphi = \sin^2(\varphi_x - \varphi_y) \qquad (15).$$

This is the equation for an ellipse. So the most general form of polarization is elliptical polarization. The parameters of the ellipse are given below.



$$\tan\beta = \pm\frac{e_1}{e_2}, \quad \begin{matrix} \frac{\pi}{4} \geq \beta \geq 0, right \\ -\frac{\pi}{4} \leq \beta \leq 0, left \end{matrix} \qquad (16a)$$

Right means right hand polarization, i. e., clockwise, and left hand polarization means counter clockwise.

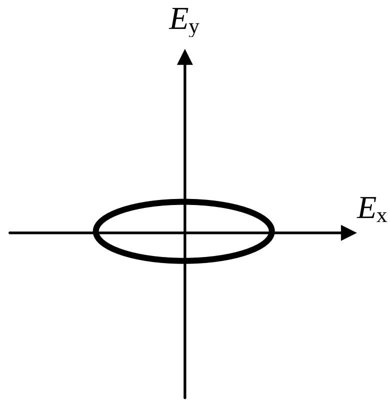
$$\tan 2\alpha = 2E_{x0}E_{y0}\cos\phi/(E_{y0}^{\ 2} - E_{x0}^{\ 2}) \qquad (16b)$$

$$e^2{}_{1,2} = E^2{}_{y0}\cos^2\alpha + E^2{}_{x0}\sin^2\alpha \pm E_{y0}E_{x0}\sin 2\alpha\cos\varphi \qquad (16c)$$

$$e_1e_2 = E_{y0}E_{x0}\sin\phi \qquad (16d)$$

$$\varphi \equiv \varphi_x - \varphi_y. \qquad (16e)$$

- Some special case $\phi = \pi/2 + n\pi, \sin\phi = 1, \cos\phi = 0 \Rightarrow \alpha = 0$. It is an ellipse shown in the right figure.
- Furthermore, if $E_{y0} = E_{x0}$, Circular Polarizations.



Ways to find out where the polarization is right-hand or left-hand

$$\left.\begin{matrix} E_x = A\cos(\omega t) \\ E_y = A\sin(\omega t) \end{matrix}\right\} \quad \begin{matrix} At\ t = 0, E_x = A, E_y = 0 \\ At\ t = \pi/2\omega, E_x = 0, E_y = A \end{matrix}$$

So the polarization is left handed.

Exercise: Determine the polarization state of the EM waves below

i) $E_x = A\cos(\omega t + \phi), E_y = A\sin(\omega t + \phi)$

ii) $E_x = A\cos(\omega t + \phi), E_y = -A\sin(\omega t + \phi)$

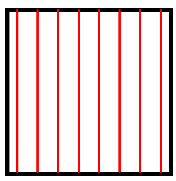
Finally, if the phases ϕ_x, ϕ_y in Eq. 13 change randomly with time, we then get light with no polarization. This is usually the case as the real sources all have finite coherent length (or time).

3B) Polarizing Media

Polarizing media include linear polarizer, birefringence wave plates, and Faraday rotation media.

I Linear Polarizer

Linear polarizers are made of media that allow only the EM waves with the electric field along its axis in the plane perpendicular to the wave propagation direction to pass. For example, devices consisting of equally spaced parallel thin metal wires called (wire grids) can serve such function. For EM waves with the polarization parallel to the wires, the electric field of the wave drives the electrons in the wire to move back and forth, thereby dissipating energy and absorbing the EM wave. For waves with the polarization perpendicular to the wires, no absorption of EM wave occurs as the electrons cannot move in the



Wire grid

direction perpendicular to the wires. Some polymers (long molecule chains) can act like wire grids.

Consider an EM wave $\vec{E}_0 = E_{x0}\vec{x}_0 e^{i(\omega t-\phi_x)} + E_{y0}\vec{y}_0 e^{i(\omega t-\phi_y)}$, after passing through a linear polarizer with its transmitting (passing) axis along the direction $\vec{e} = \vec{x}_0 \cos\theta + \vec{y}_0 \sin\theta$, the resulting electric field is

$$\vec{E} = (\vec{E}_0 \cdot \vec{e})\vec{e} = (E_{x0} \cos\theta e^{i(\omega t-\phi_x)} + E_{y0} \sin\theta e^{i(\omega t-\phi_y)})(\vec{x}_0 \cos\theta + \vec{y}_0 \sin\theta) \qquad (17a)$$

which is linearly polarized because there is no phase difference between the X- and Y-components of the electric field. Using complex number algebra, we can express Eq. 17a as

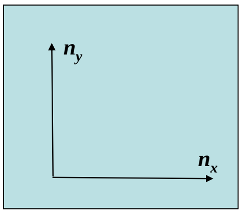
$$\vec{E} = E_0 e^{i(\omega t-\phi)}(\vec{x}_0 \cos\theta + \vec{y}_0 \sin\theta) \qquad (17b).$$

(As an ***exercise***, determine E_0 and ϕ in terms of E_{x0}, E_{y0}, and θ.) If the phases ϕ_x and ϕ_y are randomly changing with time as in the non-polarization case, then it can be shown (the process is left as an exercise) that the intensity of the EM wave is

$$I = \frac{1}{2}c\varepsilon_0 |E_0|^2 = \frac{1}{2}c\varepsilon_0 (E_{x0}^2 \cos^2\theta + E_{y0}^2 \sin^2\theta) \qquad (17c).$$

Equation 17c means that there is no interference effect between the two waves as there is no fixed phase difference, so each one can be viewed as independent. This is similar to the case of waves from two independent sources (Eq. 9c).

II Birefringence plates



Birefringence media are anisotropic ones, i. e., its dielectric constant and therefore refractive index depends on the direction of the electric field in the media. A common example is shown in the figure. For light polarized along the X-axis of the birefringence plate, the refractive index is n_x, while for light polarized along the Y-axis of the birefringence plate, the refractive index is n_y. Suppose the incident wave right before entering the plate (at $z = 0$) is a linearly polarized wave at an angle θ to the X-axis, $\vec{E}(z=0) = E_0(\vec{x}_0 \cos\theta + \vec{y}_0 \sin\theta)e^{i\omega t}$, after passing over a distance d in the plate, the electric field is

$$\vec{E}(z=d) = E_0(\vec{x}_0 e^{-ikn_x d} \cos\theta + \vec{y}_0 e^{-ikn_y d} \sin\theta)e^{i\omega t} = E_0(\vec{x}_0 \cos\theta + \vec{y}_0 e^{-ik(n_y-n_x)d} \sin\theta)e^{i\omega t}. \qquad (18a)$$

The phase difference between the X- and Y-components is

$$\delta\phi = (n_y - n_x)kd = \frac{2\pi}{\lambda}(n_y - n_x)d \qquad (18b).$$

If $\delta\phi$ equals to $\pi/2$, that is $d = \frac{1}{4}(n_y - n_x)\lambda$, and $\theta = 45°$, then $\vec{E}_0(d)$ is circularly polarized. The sense of polarization (left or right) depends on the sign of $(n_y - n_x)$. As an ***exercise***, find other thickness values of the plate that can also turn a linearly polarized wave into a circularly polarized wave.

III Faraday rotation

There is another kind of media that their refractive indexes are different for different sense of circularly polarized waves. Plasma in a magnetic field is such a medium. The refractive index for right circularly polarized wave is n_R, and that for left circularly polarized wave is n_L. Consider an EM wave linearly polarized along the X-axis before entering the medium,

$$\vec{E}(z=0) = E_0\vec{x}_0 e^{i\omega t} = \frac{1}{2}E_0 e^{i\omega t}[(\vec{x}_0 - i\vec{y}_0) + (\vec{x}_0 + i\vec{y}_0)] \qquad \text{(19a).}$$

As an ***exercise***, verify that the first term $(\vec{x}_0 - i\vec{y}_0)$ represents left circular wave and the second one $(\vec{x}_0 + i\vec{y}_0)$ represents right circular wave. Equation 19a means that a linearly polarized wave can be decomposed into two circularly polarized waves of equal amplitude, one being left and the other right.

The wave after passing through the medium by a distance d is then

$$\vec{E}(z=d) = E_0\vec{x}_0 e^{i\omega t} = \frac{1}{2}E_0 e^{i\omega t}[(\vec{x}_0 - i\vec{y}_0)e^{-ikn_L d} + (\vec{x}_0 + i\vec{y}_0)e^{-ikn_R d}] \qquad \text{(19b).}$$

To examine its polarization state, we calculate the ratio of its X-component over the Y-component.

$$\begin{aligned}\frac{E_y}{E_x} &= i\frac{e^{-ikn_R d} - e^{-ikn_L d}}{e^{-ikn_L d} + e^{-ikn_R d}} = i\frac{e^{-2i\phi_R} - e^{-2i\phi_L}}{e^{-2i\phi_R} + e^{-2i\phi_L}} = i\frac{e^{i(\phi_L-\phi_R)} - e^{-i(\phi_L-\phi_R)}}{e^{i(\phi_L-\phi_R)} + e^{-i(\phi_L-\phi_R)}} \\ &= i\frac{2i\sin(\phi_L-\phi_R)}{2\cos(\phi_L-\phi_R)} = \tan(\phi_R-\phi_L) \qquad \text{(19c).}\end{aligned}$$

Since E_y/E_x is real, the phase difference is zero. So the wave is still linearly polarized. However, its direction is now at an angle of $(\phi_R - \phi_L) \equiv \frac{1}{2}(n_R - n_L)kd$ to the X-axis. In other words, the direction of the polarization has rotated by an angle of $\frac{1}{2}(n_R - n_L)kd$ relative to the incident wave.