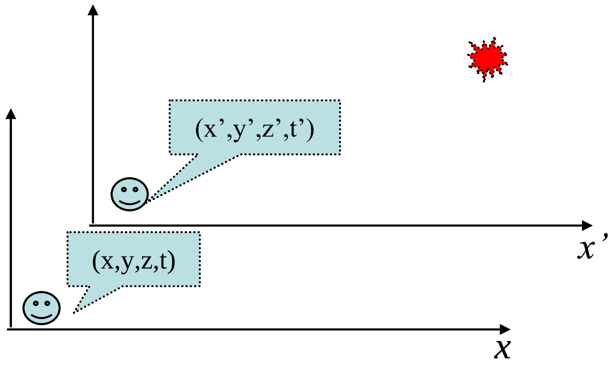
Relativity

1 Conceptual relativistic mechanics

1.1 Space-time

Suppose I am moving at constant velocity **U** relative to you, such as I am riding on a train, you have your coordinate system (X, Y, Z) and clock to record time *t*, and I have my coordinate system (X', Y', Z') and my clock *t*'. Your XYZ axes directions are the same as my X'Y'Z', and the velocity ***u*** is along the Z-direction. Neither of us is accelerating so we both are **inertial reference systems**.

An event (anything you can think of, such as you open a door, someone next to you hits you, etc.,) takes place at a particular place with coordinate (*x, y, z*) and at time *t*, according to you using your coordinate system and clock. I see the same event happens, and I record its coordinate (*x', y', z'*) and time *t'*.



Let's now change the notation a little bit for convenience.

The *space-time 4-vector* that describes the place and time the event takes place is $\tilde{x} = (x_1, x_2, x_3, ict) = (\bar{x}, ict)$, according go you, and $\tilde{x}' = (x'_1, x'_2, x'_3, ict') = (\bar{x}', ict')$ according to me. Here *c* is the speed of light in vacuum, and $i \equiv \sqrt{-1}$.
(Notation is slightly different from different textbooks. The term *ict* is used here instead of *ct*)

The relation between what I observe and what you observe of the same event is in classic physics (Galileo transformation)

$$\begin{pmatrix} x_1' = x_1 \\ x_2' = x_2 \\ x_3' = x_3 - ut \\ t' = t \end{pmatrix}. \qquad (1)$$

Example-1 Speed of a particle as seen by you and me.

Suppose a particle is moving along the z-direction at speed *V* as seen by you. What is its speed seen by me?

Answer:

Let's consider two events. You see:
Event-1: the particle is at *z* = 0 at *t* = 0.
Event-2: the particle is at *z* = *VT* at *t* = *T*. The speed of the particle you see is
(*VT* – 0)/*T* = *V*

I see the two events too. According to me and the Galileo transformation,
Event-1: the particle is at $z' = 0$ at $t' = 0$ (this is just for convenience)
Event-2: the particle is at $z' = z - ut$, at $t' = t = T$. The speed of the particle I see is
$V' = (z' - 0)/t' = (z - uT)/T = V - u$. This is the 'speed addition rule' which is consistent with common sense.

The accelerations are $a' = \frac{dV'}{dt'} = \frac{dV}{dt} - \frac{du}{dt} = \frac{dV}{dt} - 0 = \frac{dV}{dt} = a$. So $F = ma$, and $F' = m'a'$, if $F' = F$ and $m' = m$, which means Newton's Laws are true in both reference systems.

But in Einstein's Special Relativity

$$\begin{pmatrix} x_1' = x_1 \\ x_2' = x_2 \\ x_3' = \gamma\, x_3 - \beta\gamma ct = \gamma\, x_3 - \gamma\, ut \\ ict' = \gamma ict - i\beta\gamma x_3 \\ or \\ t' = \gamma t - \frac{\gamma u x_3}{c^2} \end{pmatrix} \qquad (2)$$

$$\begin{aligned} \gamma &\equiv 1/\sqrt{1-\beta^2} \\ \beta &\equiv u/c \end{aligned} \qquad (3)$$

In matrix form:

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ ict' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ ict \end{pmatrix} \qquad (4)$$

or in short, $\tilde{x}' = \lambda \tilde{x}$, where λ is the Lorentz Transformation matrix.

Example-2 Reconsider the case in Example-1 using Relativity transformation.

Using eq. (2) and after some algebra we have $V' = \frac{V-u}{1 - Vu/c^2}$. The result is the same as in Example-1 if both V and u are $<< c$.

Note also that $V' = c$ if $V = c$, i. e., speed of light in vacuum is constant, regardless of the speed of the light source or observer.

Einstein's two postulates:

- **The principle of relativity**. The laws of physics apply in all inertial reference systems.

- **The universal speed of light**. The speed of light in vacuum is the same for all inertial observers, regardless of the motion of the source.

Einstein's principle of relativity requires that all physics laws be the same in any coordinates (frames) that are not accelerating. For example, if in your frame, $\nabla \cdot \vec{B} = 0$*, then in my frame (I am still riding on a train),I should have* $\nabla' \cdot \vec{B}' = 0$

If some laws do not follow that rule, such as Newton's law, then they have to be modified. But first, we have to find out how physical quantities other than space-time change (transform) from one frame to another. For example, you see a magnetic field $\vec{B}$ at a particular place and at a particular time, I see the same field at the same space-time (although its coordinates and time must be Lorentz transformed) but its value is $\vec{B}'$. What is the relationship between $\vec{B}$ and $\vec{B}'$?

Equation 3 brings some interesting relativistic effects, one of which is the dependence of *t'* on x_3, i. e., time spends on space. Let us look at a specific case. Suppose two events took place at (x_1, ict_1) and (x_2, ict_2) according to S. Event-2 took place after event-1 so $t_2 > t_1$. To S', the time interval between the two events is $t_2' - t_1' = \gamma(t_2 - t_1 - \frac{u}{c^2}(x_2 - x_1))$.

Note that if $t_2 - t_1 < \frac{u}{c^2}(x_2 - x_1)$, then $t_2' - t_1'$ becomes negative, i. e., to S', event-1 took place after event-2.

Let us now see how large $x_2 - x_1$ must be in order to reverse the time sequence in S'. Since the maximum *u* is *c*, we see that when $x_2 - x_1 > c(t_2 - t_1)$ then this can happen. However, the two events that satisfy such condition cannot be logically correlated, i. e., event-2 happens because event-1 happens. This is because the distance between the two events is too large even for light to travel from event-1 to event-2 in time before event-2 takes place.

1.2 Proper time

$\tilde{x} \cdot \tilde{x} \equiv \tilde{x}^2 = \vec{x}^2 - c^2 t^2$ is an invariant, that means $\tilde{x} \cdot \tilde{x} = \tilde{x}' \cdot \tilde{x}'$ (Give a 2-D example)
(Exercise: Verify $\tilde{x} \cdot \tilde{x} = \tilde{x}' \cdot \tilde{x}'$ using the Lorentz Transformation)

Consider a particle moving at speed *u* according to you. According to the coordinate system on the particle (and therefore moving with the particle) the space-time of the particle is $X = (0, ic\tau)$, and according to you it is $\tilde{x} = (x_1, x_2, x_3, ict) = (\vec{x}, ict)$. X and $\tilde{x}$ are related by Lorentz Transformation. Furthermore, $X \cdot X = \tilde{x} \cdot \tilde{x}$

Now consider a small increment of time, then $dX = (0, icd\tau)$, and
$d\tilde{x} = (dx_1, dx_2, dx_3, icdt) = (d\vec{x}, icdt)$

$$dX \cdot dX = d\tilde{x} \cdot d\tilde{x} \qquad (5)$$

Proper time is defined as $d\tau \equiv \frac{-i}{c}\sqrt{dX \cdot dX}$. Using Eq. (5), we have

$$d\tau = -\frac{i}{c}\sqrt{dx_\mu dx_\mu} = \left(dt^2 - \frac{dt^2}{c^2}\frac{dx_j}{dt}\frac{dx_j}{dt}\right)^{1/2} = dt\sqrt{1 - u^2/c^2} = dt/\gamma, \qquad (6)$$

Here the ordinary velocity is $\bar{u} = \frac{d\bar{x}}{dt}$, in your coordinate (often called the laboratory coordinate). Proper time $d\tau$ is an invariant, i. e., I see the same $d\tau$ as you do.

1.3 Other 4-vectors

The way to go is to construct 4-vectors, because 4-vectors transform from one frame to another just like the space-time 4-vector (x_1, x_2, x_3, ict) .

Four-velocity $\tilde{U} \equiv \frac{d\tilde{x}}{d\tau} = (\frac{d\bar{x}}{d\tau}, ic\frac{dt}{d\tau}) = (\gamma\bar{u}, \gamma ic)$ (7)

$\tilde{U}$ is a 4-vector because $d\tilde{x}$ is a 4-vector and $d\tau$ is an invariant.

Momentum 4-vector $\tilde{P} = m\tilde{U} = (\vec{p}, iW/c)$, where *m* is the mass of the particle. *Mass m is an invariant. W* is (*assigned as*) the total energy of the particle, = γmc^2. This is the famous Energy = Mc^2 formula, where *M* is the 'moving mass' = γm. $\bar{p} = \gamma m\bar{u}$ is now the momentum. Compared with the classic momentum, this relativistic momentum has an extra factor γ, or one could say it still maintains the form of $M\bar{u}$ with the moving mass $M = \gamma m$.

In particle's frame, the particle is always at rest so its 4-vector momentum is $\tilde{P}_0 = (\vec{0}, imc) = (\vec{0}, iW_0/c)$, so the rest energy $W_0 = mc^2$.

Now $\tilde{P}_0$ is the 4-vector momentum of the particle seen in particle's frame, and $\tilde{P}$ is its momentum seen in the laboratory frame, so $\tilde{P}_0 \cdot \tilde{P}_0 = \tilde{P} \cdot \tilde{P}$. This leads to another famous formula

$$W^2 = (pc)^2 + m^2c^4 \qquad (8).$$

Here *p* is the momentum of the particle.

The kinetic energy of a particle is defined as the difference of $KE = W(p) - W(p=0)$. Using Eq. (8)

$$KE = \sqrt{(pc)^2 + m^2c^4} - mc^2 \qquad (9).$$

When $pc << mc^2$, Eq. (9) becomes $KE = 0.5p^2/m$, which is the same as in classic mechanics.

For photon which is massless, $W = pc$, according to Eq. (8). Since a photon has energy $W = hf$, where *h* is the Planck constant and *f* the frequency, the momentum of a photon is hf/c.

Lorentz Transformation

$$\tilde{x}' = \lambda\tilde{x}$$
$$\tilde{U}' = \lambda\tilde{U}$$
$$\tilde{P}' = \lambda\tilde{P}$$

In fact all four-vectors follow Lorentz transformation.

Conservation laws: The relativistic momentum and energy are conserved in an isolated system.

Example-3 A number of particles with momentums $\{p_j\}$ and energies $\{E_j\}$ as measured in the laboratory reference frame, all along *z*-axis. Find the speed of the center of mass of the particles.

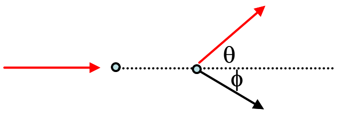
Answer: The 4-vector momentum of the particle system is $\tilde{P} = (\sum p_j, i\sum E_j / c)$. In the center of mass frame S', the total momentum of the particles is zero.
i. e., $\tilde{P}' = (0, iW'/c)$. We need to find the Lorentz transformation that make $\tilde{P}' = \lambda\tilde{P}$. Using Eq. (4) with x and ct replaced by $\sum p_j$ and $\sum E_j / c$, we get $\beta = \dfrac{c\sum p_j}{\sum E_j}$, which is the speed of the center of mass divided by c.

Example-4 Two particles of rest mass m are moving at speed 0.6c in opposite directions towards each other in a collision course. After collision they form a lump at rest. Find the mass of the lump M.

Answers: The total momentum of the two particle system is zero before and after. So it is trivial.

The energy of each particle before collision is γmc^2, with $\beta = 0.6$.
By energy conservation, the total energy of the system after collision is $Mc^2 = 2\gamma mc^2$. This results in $M = 2.5\ m$, i. e., the mass of the lump is increased by 0.5 m which comes from the kinetic energy before the collision.

Example 5 As seen in the figure, a photon of energy E_0 collides with a particle (an electron) of mass m at rest. After the collision the photon is moving at angle θ relative to its original motion direction and its energy has changed to E. Find E.



(before) (after)

Answer: Let the momentum of the electron after collision to be p_e, that of the photon p_p. Now apply momentum (two-dimensional) and energy conservation.

In Y-direction, $0 = p_e \sin\phi - p_p \sin\theta$ (i)

$E = cp_p$ (ii)

In X-direction, $E_0 / c = p_p \cos\theta + p_e \cos\phi$ (iii)

Energy before collision = $E_0 + mc^2$.

Energy after collision = $E + \sqrt{p_e^2 c^2 + m^2 c^4}$. So

$$E + \sqrt{p_e^2 c^2 + m^2 c^4} = E_0 + mc^2. \qquad \text{(iv)}$$

In the above four equations there are four unknowns: E, ϕ, p_p, and p_e. Using (ii) to replace p_p by E in (i) and (iii), and combine (i) and (iii) to eliminate ϕ. Then eliminate p_e to get the final answer

$$E = 1/(\frac{1-\cos\theta}{mc^2} + \frac{1}{E_0}) .$$

In terms of wavelength $E = hc/\lambda$, we get $\lambda = \lambda_0 + \frac{h}{mc}(1-\cos\theta)$.

This is the Compton Scattering experiment. The wavelength is ~ Å (X-ray). Ans.

Dynamics

$\vec{F} = \frac{d\vec{p}}{dt}$, where $\vec{p} = m\gamma\,\vec{u} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}}$ (10)

2 Doppler Effect

Consider a plane wave $\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-wt)}, \quad \vec{E}' = \vec{E}_0' e^{i(\vec{k}'\cdot\vec{r}'-w't')}$.

The phase must be an invariant, that is, at the place and time $\vec{E}$ reaches maximum, $\vec{E}'$ must do too.

So $$\vec{k}\cdot\vec{r} - \omega t = \vec{k}'\cdot\vec{r}' - \omega' t' = \tilde{k}\cdot\tilde{x} \qquad (11)$$

Where $\tilde{x} = (\vec{r}, ict)$ is the normal space-time 4-vector and $\tilde{k} = (\vec{k}, i\omega/c)$ is a new four-vector describing the wave propagation.

$$\begin{pmatrix} k_1' \\ k_2' \\ k_3' \\ i\omega'/c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ i\omega/c \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \\ \gamma(k_3 - \beta\omega/c) \\ \gamma(\omega/c - \beta k_3)i \end{pmatrix}. \qquad (12)$$

Here we take the relative velocity $\vec{v}$ to be along the Z direction.

From Eq. (12), $k_1' = k_1, \quad k_2' = k_2, \quad k_3' = \gamma(k_3 - \beta\omega/c), \quad \omega' = \gamma(\omega - \beta c k_3)$

Let $k_1 = ke_1, \quad k_2 = ke_2, \quad k_3 = ke_3,$ where $k = n\frac{\omega}{c}$, $e_{1,2,3} = \cos(\theta_{1,2,3})$ and $\theta_{1,2,3}$ are the angle between $\vec{k}$ and X, Y, Z axes. In particular, $e_3 = \frac{\vec{v}\cdot\vec{k}}{vk}$.

Then $k_3' = \gamma\frac{\omega}{c}(ne_3 - \beta) = \gamma\frac{\omega}{c}(n\frac{\vec{v}\cdot\vec{k}}{vk} - \beta),$

i) Doppler Effect

$$\omega' = \gamma\omega(1 - n\beta e_3) = \gamma\omega(1 - n\frac{\vec{v}\cdot\vec{k}}{ck}) \qquad (13)$$

So $\begin{array}{ll} \vec{v} \perp \vec{k}, & \omega' = \gamma\omega \\ \vec{v} // \vec{k}, & \omega' = \omega\gamma(1 - n\beta) \end{array}$.

ii) Moving Medium (How does refractive index *n* change with speed?)

Note $\tilde{k} \cdot \tilde{k} = k^2 - \omega^2 / c^2 = k'^2 - \omega'^2 / c^2$

In frame S: $k = n\frac{\omega}{c}$, and in S': $k' = n'\frac{\omega'}{c}$.

$$\text{So } n'^2 = \frac{\omega^2}{\omega'^2}(n^2 - 1) + 1 \qquad (14),$$

and $\frac{\omega}{\omega'}$ is given by Doppler Effect (Eq. 13), and depends on $\vec{v} \cdot \vec{k}$.

In vacuum, *n* = 1, thus *n*' = 1 => speed of light is always *c* in vacuum.

3 Relativistic Electrodynamics

Current density 4-vector

$$\tilde{J} = (\vec{J},\ ic\rho) = \rho_0 \tilde{U} = \rho_0(\gamma\vec{u},\ i\gamma c) \qquad (15)$$

ρ_0 is the charge density in the reference frame where the charge is stationary.

$\rho = \gamma\rho_0$ due to Lorentz contraction and total charge conservation.

$$\begin{array}{ll} \vec{E}'_{\parallel} = \vec{E}_{\parallel} & \vec{E}'_{\perp} = \gamma\left[\vec{E}_{\perp} + \vec{u} \times \vec{B}\right] \\ \vec{B}'_{\parallel} = \vec{B}_{\parallel}, & \vec{B}'_{\perp} = \gamma\left[\vec{B}_{\perp} - \frac{\vec{u} \times \vec{E}}{c^2}\right] \end{array} \qquad (16)$$

Example 6 A straight neutral wire with electric current *I* following through, as seen in frame S. Find the current and charge density in frame S' which is moving at speed *u* along the wire.

Answer: Take the wire direction as *z* and positive *z* is in the motion direction of S', we can roughly write the 4-vector current density $\tilde{J} = (I,\ 0)$.

$\tilde{J}' = \lambda\tilde{J} = (\gamma I,\ -i\beta\gamma I)$. So in S' there is a net line charge $-\beta\gamma I / c$, which will produce an electric field as well.

Some more examples

Example 7 Uniform line charge in motion.

In rest frame $\vec{E}_{\parallel} = \vec{0}$, $\qquad \vec{E}_{\perp} = \frac{\rho_0}{2\pi\varepsilon_0}\frac{\vec{R}}{R^2}$, $\qquad \vec{B} = \vec{0}$.

In motion frame $\vec{\mathrm{R}}' = \vec{\mathrm{R}}$ because they are $\perp$ to $\vec{\mathrm{n}}$.

$\vec{B}'_{\parallel} = \vec{0}$, $\quad \vec{B}'_{\perp} = -\frac{\gamma}{c^2}\vec{u}\times\vec{E} = -\frac{\gamma}{c^2}\vec{u}\times\frac{\rho_0}{2\pi\varepsilon_0}\frac{\vec{R}}{R^2} = -\frac{\mu_0}{2\pi}\rho_0\gamma\vec{u}\times\frac{\vec{R}}{R^2}$

But $\gamma\rho_0\vec{u} = \rho\vec{u} = -\vec{I}$, so $\vec{B}'_{\perp} = \frac{\mu_0}{2\pi}\vec{I}\times\frac{\vec{R}}{R^2}$, $B'_{\perp} = \frac{\mu_0}{2\pi}\frac{I}{R'}$ same as Ampere's Law

$E'_{\perp} = \gamma E_{\perp} = \frac{\rho_0}{2\pi\varepsilon_0}\frac{\gamma}{R} = \frac{\rho}{2\pi\varepsilon_0 R}$, $\vec{E}_{\parallel} = \vec{0}$

Notice that $\rho = \gamma\rho_0$, or moving charge is denser.

Example 8 A point charge in uniform motion.

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3} = \alpha\frac{\vec{r}}{r^3}, \quad \vec{B} = 0$$

In another frame moving along $\hat{x}_3$ direction,

$E'_1 = \gamma E_1 = \gamma\alpha\frac{x_1}{r^3}$, $E'_2 = \gamma E_2 = \gamma\alpha\frac{x_2}{r^3}$, $E'_3 = E_3 = \alpha\frac{x_3}{r^3}$

Now, we need to change $\vec{r}$ to $\vec{r}'$,

$x'_1 = x_1$, $\quad x'_2 = x_2$, $\quad x'_3 = \gamma x_3 - \beta c\gamma t$, $\quad t' = -\beta\gamma x_3 / c + \gamma t$

$x'_1 = x_1$, $\quad x'_2 = x_2$, $\quad x'_3 = x_3 / \gamma \quad$ (at $t' = 0$)

So $\quad \vec{E}' = \frac{\alpha}{r^3}\left(\gamma x'_1, \gamma x'_2, \gamma x'_3\right) = \frac{\alpha\gamma\vec{r}'}{r^3}$

Change $\quad r^2 = x_1'^2 + x_2'^2 + \gamma^2 x_3'^2 = r'^2(\sin^2\theta + \gamma^2\cos^2\theta) = r'^2\gamma^2(1-\beta^2\sin^2\theta)$

So $\vec{E}' = \frac{q}{4\pi\varepsilon_0}\frac{(1-\beta^2)\vec{r}'}{r'^3(1-\beta^2\sin^2\theta)^{3/2}}$ same as what we get before but much easier to get.

So field transform $\vec{E}(\vec{r}) -> \vec{E}'(\vec{r}')$ both the format and the variables have to change.

$\vec{B}'_{\perp} = -\frac{\gamma}{c^2}\vec{u}\times\vec{E} = -\frac{1}{c^2}\vec{u}\times\vec{E}'$, $\vec{B}'_{\parallel} = 0$