Astrophysics

1 Rotational reference frame

A) An inertia force is acting on any *stationary* object in a rotational frame

$$\vec{f} = m\omega^2\vec{r} \qquad (1),$$

where r is the distance of the object to the rotation axis. Potential energy of the force is

$$U = -\frac{1}{2}m\omega^2 r^2 \qquad (2),$$

and

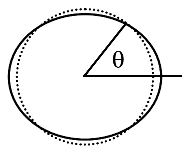
$$\vec{f} = -\frac{dU}{dr}\vec{r}_0 \qquad (3).$$

In spherical coordinate,

$$U = -\frac{1}{2}m\omega^2 r^2\cos^2\theta \qquad (4).$$

(Note that any force that is always pointing to a fixed axis or point can be expressed in the form of Eq. (3), i. e., one can always find a corresponding potential and the force is *conservative*.)

Equal-potential surface = stationary liquid surface, because the net force on any point mass on a liquid surface along the tangential direction of the surface must be zero.
The shape of a rotating liquid drop (could be a neutron star) is then $U_{total} = -\frac{1}{2}\omega^2 r^2\cos^2\theta - G\frac{M}{r} = const.$ (5)



The difference in r, Δr, at θ = 0 and 90 °is then

$\frac{1}{2}\omega^2 r^2 + G\frac{M}{r} = G\frac{M}{r-\Delta r} = G\frac{M}{r}(1+\frac{\Delta r}{r})$, or $\frac{\omega^2 r^4}{2GM} = \Delta r$.

$(1+x)^\alpha = 1+\alpha x$

What is the shape of the surface of a glass of water in an accelerating bus?

B) Additional inertia force on moving objects in a rotational frame.

Consider an object moving radially at speed v_r in a rotating frame. From a truly inertial frame, after δt the angular momentum of the object increases by $\delta L = 2m\omega r\delta r = 2m\omega r v_r \delta t$, which must be brought by a tangential force F to provide the torque. So $rF = \frac{\delta L}{\delta t} = 2m\omega r v_r$, and

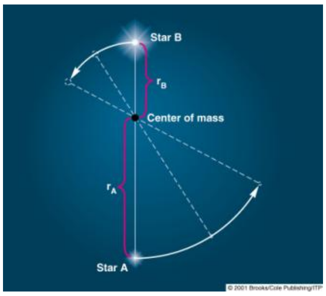
$$F = 2m\omega v_r \qquad (6).$$

Likewise, an object in circular motion (speed v) in the spinning frame is viewed in a truly inertial frame as being in circular motion with orbital speed (v + $r\omega$). The acceleration is then $\frac{(v+r\omega)^2}{r} = \frac{v^2}{r} + \omega^2 r + 2v\omega$. The first term is the *normal* circular motion acceleration seen in the spinning frame, the second term is the inertia force, and the third term is due to motion of the object in the spinning frame.

In general, the *Coriolis* force is

$$\vec{F} = 2m\vec{\omega}\times\vec{v} \qquad (7).$$

2 Binary system



The center of mass C is fixed, and both stars revolve around C. Note that although Star-A is rotating around C at a distance r_A, the G-force that keeps the motion is pointing toward C but its strength is $G\dfrac{m_A m_B}{(r_A + r_B)^2}$. As homework, show that

$$\omega^2 (r_A + r_B)^3 = G(m_A + m_B) \qquad (8).$$

3 Tides (Q3, IPhO-1996, http://www.jyu.fi/tdk/kastdk/olympiads/)

Solution Problem 3

a) With the centre of the earth as origin, let the centre of mass C be located at $\vec{l}$. The distance l is determined by

$$M\,l = M_m\,(L - l),$$

which gives

$$l = \frac{M_m}{M + M_m} L = \underline{\underline{4.63\cdot 10^6\,\mathrm{m}}}, \qquad (1)$$

less than R, and thus inside the earth.
The centrifugal force must balance the gravitational attraction between the moon and the earth:

$$M\omega^2 l = G\frac{MM_m}{L^2},$$

which gives

$$\omega = \sqrt{\frac{GM_m}{L^2 l}} = \underline{\underline{\sqrt{\frac{G(M + M_m)}{L^3}}}} = \underline{\underline{2.67\cdot 10^{-6}\,\mathrm{s}^{-1}}}. \qquad (2)$$

(This corresponds to a period $2\pi/\omega = 27.2$ days.) We have used (1) to eliminate l.

b) The potential energy of the mass point m consists of three contributions:
(1) Potential energy because of rotation (in the rotating frame of reference, see the problem text),

$$-\frac{1}{2} m\omega^2 r_1^2,$$

where $\vec{r}_1$ is the distance from C. This corresponds to the centrifugal force $m\omega^2 r_1$, directed outwards from C.

(2) Gravitational attraction to the earth,

$$-G\frac{mM}{r}.$$

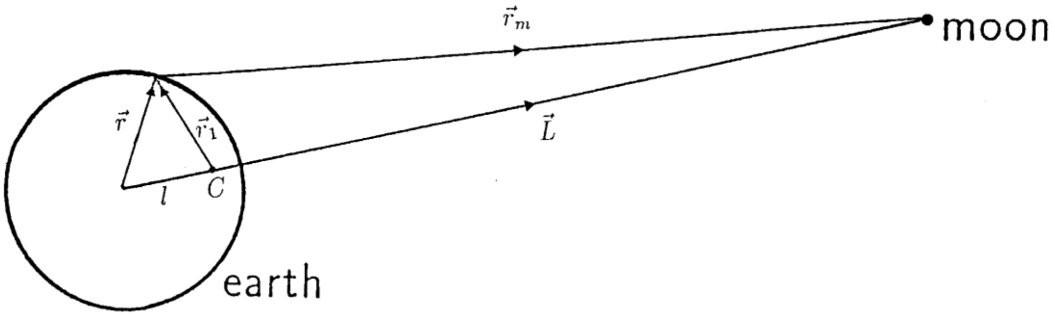
(3) Gravitational attraction to the moon,

$$-G\frac{mM_m}{r_m},$$

where $\vec{r}_m$ is the distance from the moon.

Describing the position of m by polar coordinates r, φ in the plane orthogonal to the axis of rotation (see figure), we have

$$\vec{r}_1^{\,2} = (\vec{r} - \vec{l}\,)^2 = r^2 - 2rl\cos\varphi + l^2.$$



Adding the three potential energy contributions, we obtain

$$V(\vec{r}) = -\frac{1}{2}m\omega^2(r^2 - 2rl\cos\varphi + l^2) - G\frac{mM}{r} - G\frac{mM_m}{\left|\vec{L} - \vec{r}\right|}. \qquad (3)$$

Here l is given by (1) and

$$\left|\vec{r}_m\right| = \sqrt{(\vec{L} - \vec{r})^2} = \sqrt{L^2 - 2\vec{L}\vec{r} + r^2} = L\sqrt{1 + (r/L)^2 - 2(r/L)\cos\varphi},$$

c) Since the ratio $r/L = a$ is very small, we may use the expansion

$$\frac{1}{\sqrt{1 + a^2 - 2a\cos\varphi}} = 1 + a\cos\varphi + a^2\,\tfrac{1}{2}(3\cos^2\varphi - 1).$$

Insertion into the expression (3) for the potential energy gives

$$V(r,\varphi)/m = -\tfrac{1}{2}\omega^2 r^2 - \frac{GM}{r} - \frac{GM_m r^2}{2L^3}(3\cos^2\varphi - 1), \qquad (4)$$

apart from a constant. We have used that

$$m\omega^2 rl\cos\varphi - GmM_m\frac{r}{L^2}\cos\varphi = 0,$$

when the value of ω^2, equation (2), is inserted.

The form of the liquid surface is such that a mass point has the same energy *V everywhere on the surface*. (This is equivalent to requiring no net force tangential to the surface.) Putting

$$r = R + h,$$

where the tide h is much smaller than R, we have approximately

$$\frac{1}{r} = \frac{1}{R+h} = \frac{1}{R}\frac{1}{1+(h/R)} \cong \frac{1}{R}(1-\frac{h}{R}) = \frac{1}{R} - \frac{h}{R^2},$$

as well as

$$r^2 = R^2 + 2Rh + h^2 \cong R^2 + 2Rh.$$

Inserting this, and the value (2) of ω into (4), we have

$$V(r,\varphi)/m = -\frac{G(M+M_m)R}{L^3}h + \frac{GM}{R^2}h - \frac{GM_m r^2}{2L^3}(3\cos^2\varphi - 1), \qquad (5)$$

again apart from a constant.

The magnitude of the first term on the right-hand side of (5) is a factor

$$\frac{(M+M_m)}{M}(\frac{R}{L})^3 \cong 10^{-5}$$

smaller than the second term, thus negligible. If the remaining two terms in equation (5) compensate each other, *i.e.*

$$h = \underline{\frac{M_m r^2 R^2}{2ML^3}(3\cos^2\varphi - 1)},$$

then the mass point m has the same energy everywhere on the surface. Here r^2 can safely be approximated by R^2, giving the tidal bulge

$$h = \frac{M_m R^4}{2ML^3}(3\cos^2\varphi - 1).$$

The largest value $h_{\max} = M_m R^4/ML^3$ occurs for $\varphi = 0$ or π, in the direction of the moon or in the opposite direction, while the smallest value $h_{\min} = -M_m R^4/2ML^3$ corresponds to $\varphi = \pi/2$ or $3\pi/2$. The difference between high tide and low tide is therefore

$$h_{\max} - h_{\min} = \frac{3M_m R^4}{2ML^3} = \underline{\underline{0.54\text{m.}}}$$

(The values for high and low tide are determined up to an additive constant, but the difference is of course independent of this.)

4 Fly-by

Recall that for a spacecraft in a G-field the trajectory is elliptical when total energy E is negative, parabolic when $E = 0$, and hyperbolic when $E > 0$. The fly-by cases are $E > 0$.

Acrobat Document

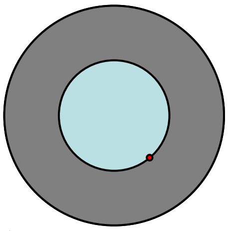
See the file Scattering.pdf

For G-field, replace $\dfrac{Zze^2}{4\pi\varepsilon_0}$ by $(-GmM)$. So

$$\tan\frac{\theta}{2} = -\frac{GM}{v_0^2 b}\,. \qquad (9)$$

For larger scattering angle, one needs small impact parameter b. The speed of the spacecraft before and after encounter remains the same, due to energy conservation.

5 Circular motion with mass distribution in galaxies



Mass enclosed by the orbit of a star can be considered as concentrated at the center. Mass outside the orbit has no net G-force on the star.

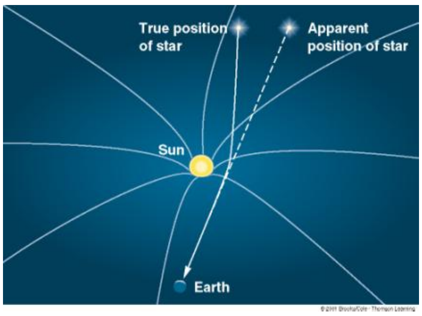
6 Gravitational red shift, G-lens (bending of light by a star)

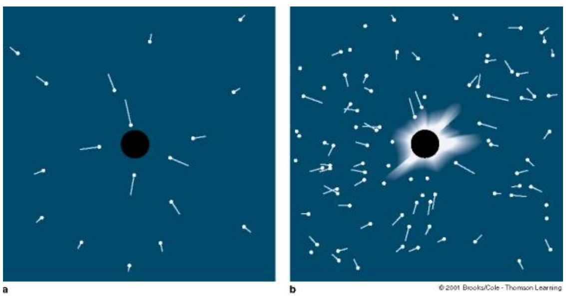
A photon can be regarded as having an effective gravitational mass such that

$$\text{Photon energy} = h\nu = m_p c^2 \qquad (10),$$

where ν is the frequency of the photon and h the Planck constant. At distance r from a star of mass M its gravitation potential energy is

$$U = -G\frac{Mm_p}{r} = -G\frac{Mh\nu}{rc^2} \qquad (11)$$





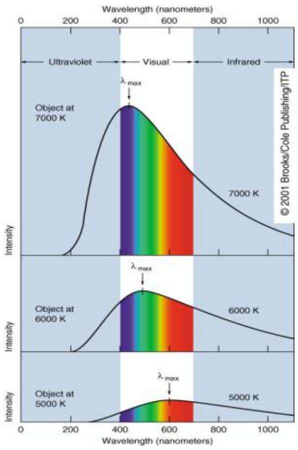
Likewise, the star also exerts G-force on the photon as if it has mass m_p, and the light path becomes curved. There is a problem in the homework to calculate the red shift and bending of light path.

7 Stars

(A) Blackbody radiation

A blackbody reflects no light. When it is kept at temperature T, it emits EM-waves from each unit area of the surface with a spectrum given below.

$$I(\lambda) = \frac{2\pi hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda k_B T} - 1} \qquad (12)$$



Here λ is the wavelength, h is the Planck constant, k_B is the Boltzmann constant. (Recall thermodynamics).

$$\text{Total power/(unit area)} = \int_0^\infty I(\lambda)d\lambda = \sigma T^4 \qquad (13),$$

where $\sigma = 5.67 \times 10^{-8}$ W/(m^{2}K^4)

The wavelength at which $I(\lambda)$ reaches maximum is

$$\lambda_{\max} = (3.0 \times 10^6)/T \text{ (nm)} \qquad (14)$$

For a grey body

$$I_G(\lambda) = (1 - R(\lambda))I_B(\lambda) = \varepsilon(\lambda)I_B(\lambda) \qquad (15)$$

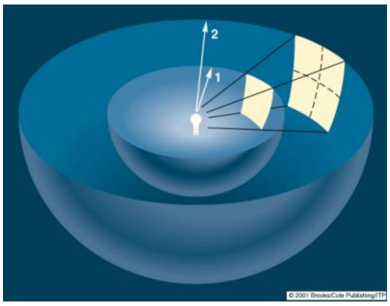
Where $R(\lambda)$ is its reflectivity, and $\varepsilon(\lambda)$ is called emissivity.

(B) Luminosity of a star (L)

Stars can be regarded as blackbody as far as their light emission is concerned. The total energy per second emitted by a star is called its luminosity L. According to Eq. (13),

$$L = 4\pi\sigma T^4 R^2 \qquad (16),$$

where R is the star radius and T its surface temperature.



The light intensity S (Poynting's vector) at a distance r from a star of luminosity L satisfies

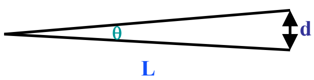
$$S(r) = \frac{L}{4\pi r^2} \qquad (17).$$

So light intensity decreases with $1/r^2$. If L can somehow be determined, then the distance r of the star can be too, since S can always be measured.

(C) Determine distance of stars

Some basic units (for convenience):

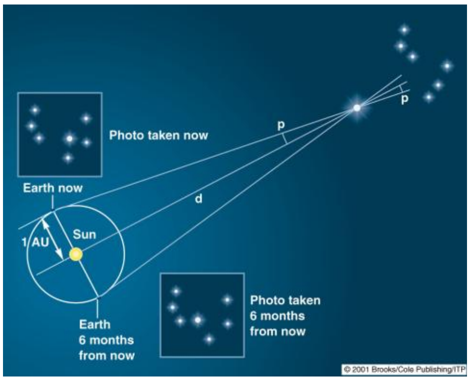
- 1 astronomical unit (AU) = Sun-Earth distance = 1.5×10^{11} m.
- 1 light year = 365 x 24 x 60 x 60 x C = 9.46×10^{15} m = 63,000 AU
- 1 arc second = π /(60*60*180) = 4.85×10^{-6} radian



The small angle formula

$$d = L\theta \qquad (18)$$

Parallax: The apparent position change of *nearby* stars in the sky relative to *far away* background stars due to orbital motion of Earth. If the angular displacement is p, then the distance to the star d can be determined because the orbital size of Earth is known.



Using Eq. (17) the luminosity of these stars can be obtained. They serve as luminosity standards for others. The light spectrum of a far away star is compared with that of the standard ones, and if a match is found, its luminosity L is taken as the same as the matching standard star. Its distance can then be determined, again using Eq. (17).

(D) Resolution of telescopes

$$\theta = 1.22\frac{\lambda}{2a} \qquad (17)$$

For all wavelength from X-ray to infrared, a is the radius of the primary mirror/lens. For micro and radio waves, a is the diameter of the telescope disk for a single telescope, and is the distance between two telescopes if they are connected in phase. For a telescope array, a is the length of the entire array.