HK IPhO Training class (mechanics)

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Lecture I: Examples of applications of Newton's Law

(1)Projectile motion

(2)Harmonic Oscillator

(3)Conservation Laws

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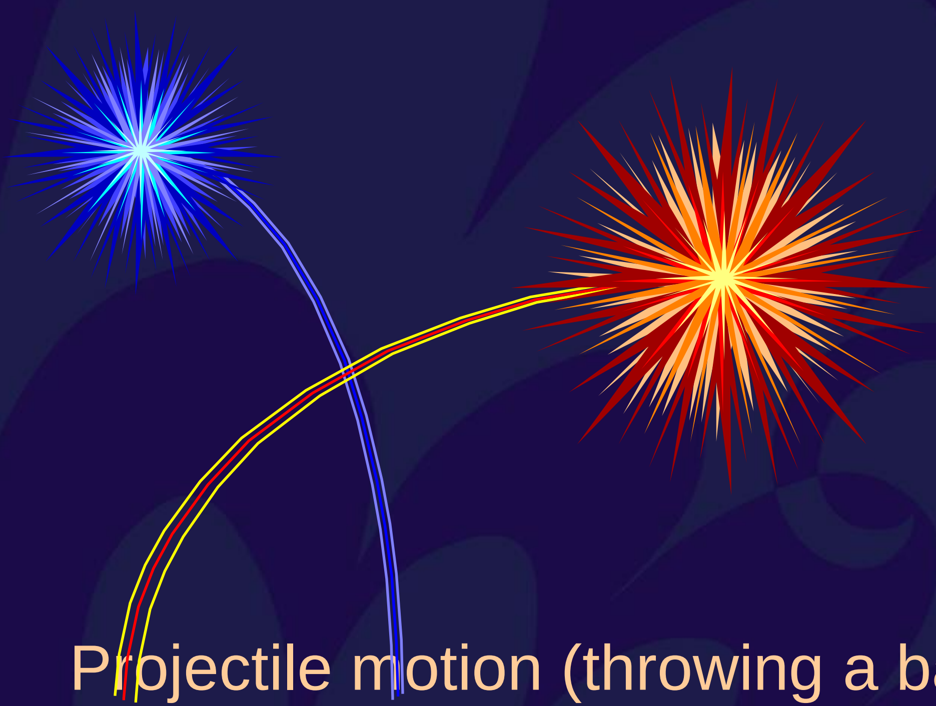
Examples of **Newton's Second Law:**



Projectile motion (throwing a ball across space)

Mathematics of Projectile motion

- Trick: solve Newton's equation in x- and y-direction separately!
- $F_x = ma_x,\quad F_y = ma_y$

- For *constant* F_x and F_y, we have

Mathematics of Projectile motion

$$x(t) = x_o + v_{xo}t + \frac{1}{2}\left(\frac{F_x}{m}\right)t^2$$

$$y(t) = y_o + v_{yo}t + \frac{1}{2}\left(\frac{F_y}{m}\right)t^2$$

Usually F_y=-mg (gravitational force)

$F_x=0, v_{xo}\,(v_{yo}) \neq 0$

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Mathematics of Projectile motion

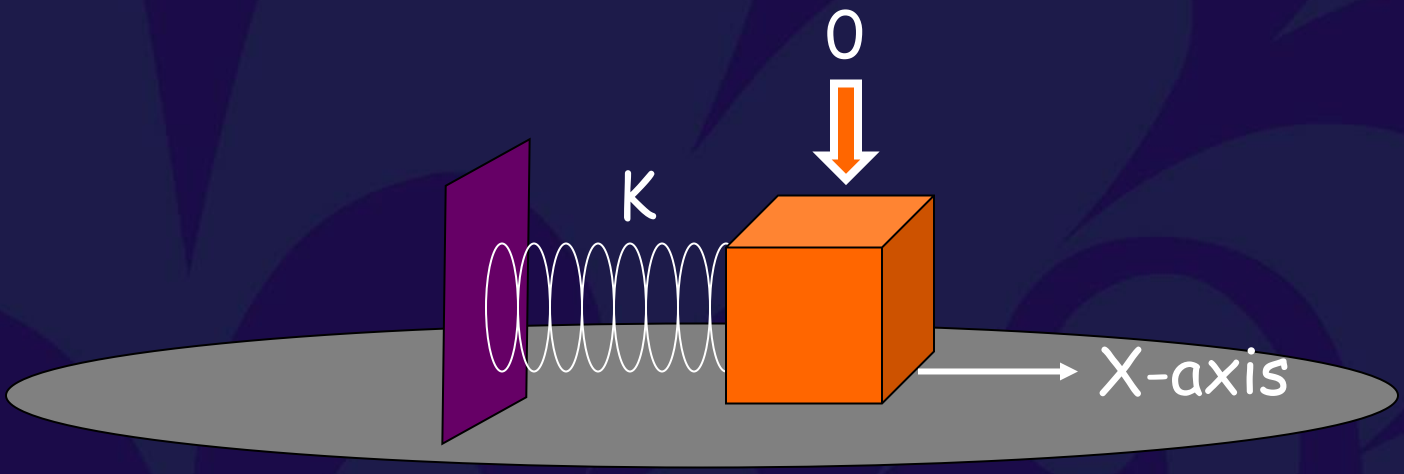
Example: throwing a ball with initial velocity *v* at 45º to horizontal.

$$x(t) = 0 + v\cos(45^o)t + \frac{1}{2}\left(\frac{(0)}{m}\right)t^2$$

$$y(t) = 0 + v\sin(45^o)t + \frac{1}{2}\left(\frac{-mg}{m}\right)t^2$$

Exercise: when and where will the ball hit the ground?

Example(1) Spring + attachment



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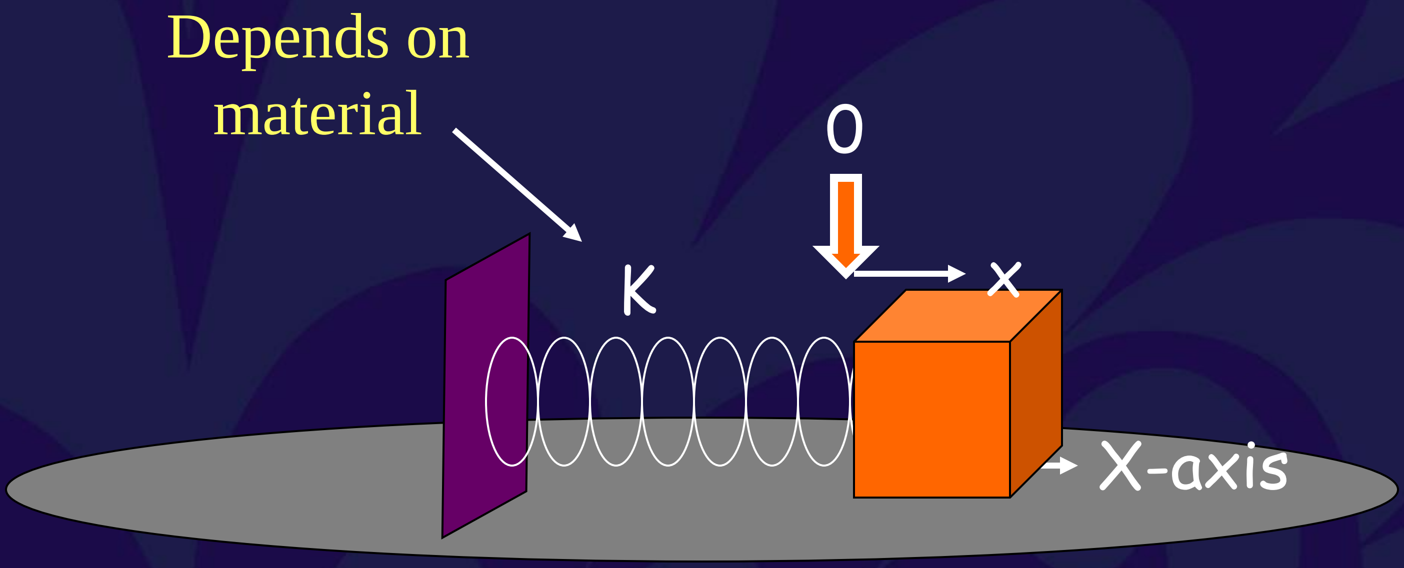
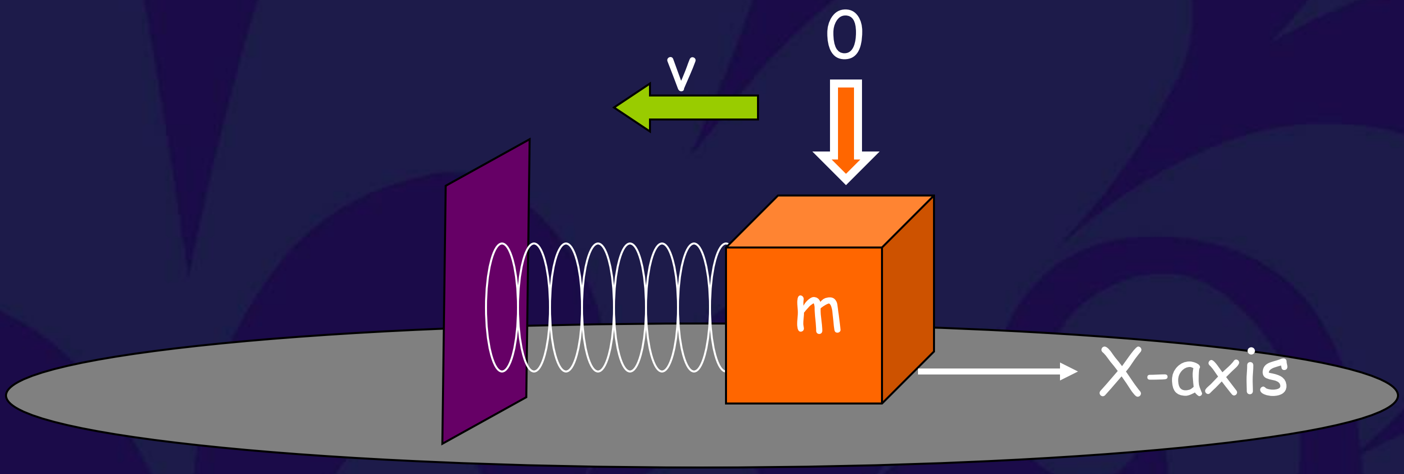
Example(1) Spring + attachment



Hook's Law: need force $F=-Kx$ to stretch (or compress) the spring.

Example(1) Spring + attachment



When the spring is released, the mass begins to move!

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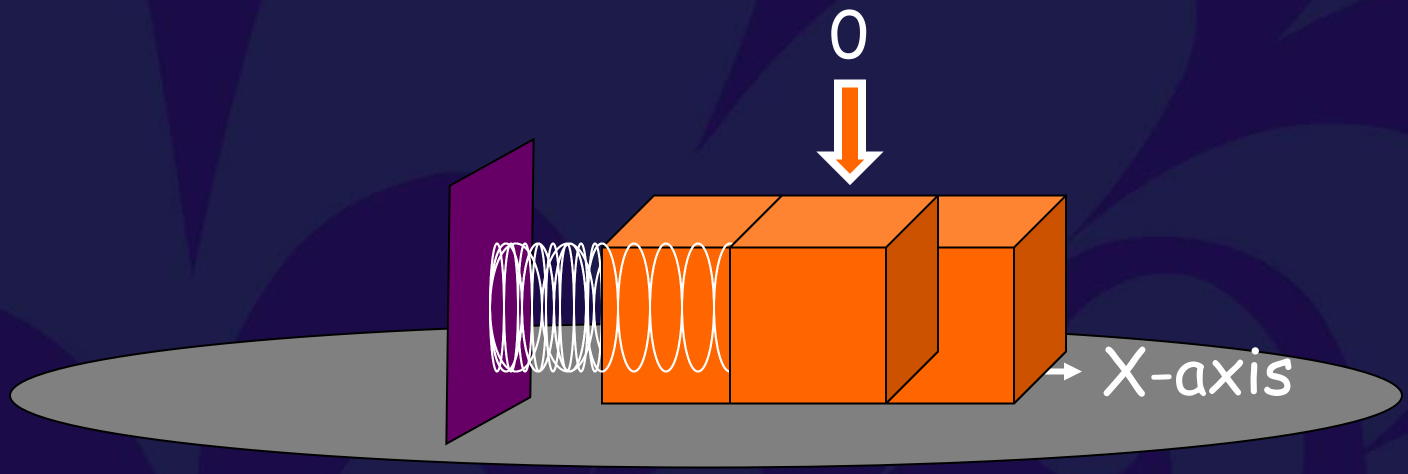
Example(1) Spring + attachment



This is an example of Simple Harmonic Motion

Mathematics of Simple Harmonic Motion of Spring+ load

- *Force* acting on the mass = *-Kx* =ma
- \- the acceleration of the mass when it is at position x is $a = -Kx/m$!
- *Question: can we solve the mathematical problem of how the position of the mass changes with time (x(t)) with this information?*

Mathematics of Simple Harmonic Motion of Spring+ load

- *Answer: Yes!* with help of *calculus*
- The equation $a = -Kx/m$ is called a *differential equation* and can be solved.
- *(Notice that although calculus is not "required" in IPhO, you will find the questions much easier if you know it)*

Mathematics of Simple Harmonic Motion of Spring+ load

- *Anyway, let me try a solution of form $x(t)=a\cos(\omega t)$*
- A, ω are numbers to be determined from the equation.
- *To show that x(t) is a solution, let us calculate v(t) and a(t)*

Mathematics of Simple Harmonic Motion of Spring+ load

First we calculate *v(t)*

$$x(t) = A\cos(\omega t); \qquad x(t+\Delta t) = A\cos(\omega t + \omega\Delta t)$$

$$x(t+\Delta t) = A[\cos(\omega t)\cos(\omega\Delta t) - \sin(\omega t)\sin(\omega\Delta t)]$$

$$\xrightarrow{\Delta t \to 0} A\cos(\omega t)(1 - O(\Delta t)^2) - A\omega\Delta t\sin(\omega t)$$

$$\therefore v(t) = \frac{x(t+\Delta t) - x(t)}{\Delta t} \to \frac{-A\omega\sin(\omega t)\Delta t}{\Delta t}$$

$$= -A\omega\sin(\omega t)$$

Mathematics of Simple Harmonic Motion of Spring+ load

Now we calculate *a(t)*

$$v(t) = -A\omega\sin(\omega t); \qquad v(t+\Delta t) = -A\omega\sin(\omega t + \omega\Delta t)$$

$$v(t+\Delta t) = -\omega A\left[\sin(\omega t)\cos(\omega\Delta t) + \cos(\omega t)\sin(\omega\Delta t)\right]$$

$$\xrightarrow{\Delta t \to 0} -\omega A\sin(\omega t)(1 - O(\Delta t)^2) - A\omega^2\Delta t\cos(\omega t)$$

$$\therefore a(t) = \frac{a(t+\Delta t) - a(t)}{\Delta t} \to \frac{-A\omega^2\cos(\omega t)\Delta t}{\Delta t}$$

$$= -A\omega^2\cos(\omega t) = -\omega^2 x(t)$$

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Mathematics of Simple Harmonic Motion of Spring+ load

Compare with equation

$$a(t) = -Kx(t)/m \Rightarrow$$

$$\boxed{\omega^2 = \frac{K}{m}}$$

i.e., the frequency of oscillation of the load is determined by the spring constant K and mass of the load m

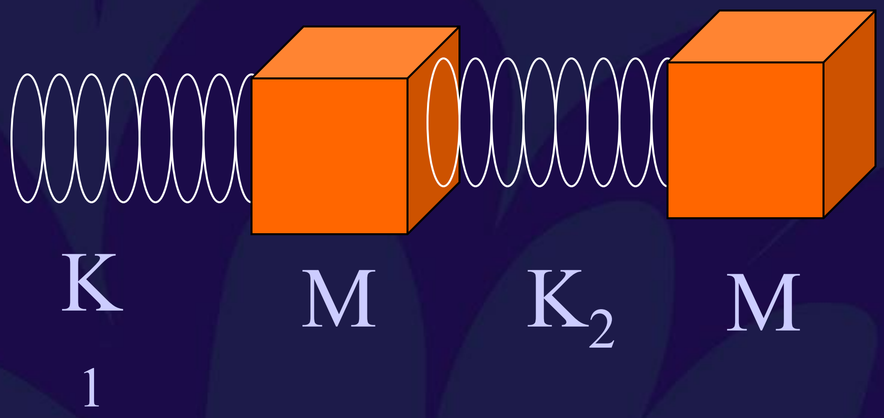
Mathematics of Simple Harmonic Motion of Spring+ load

Exercise: Show that $x(t)=B\sin(\omega t)$ is also a solution of the equation $a(t) = -Kx(t)/m$.

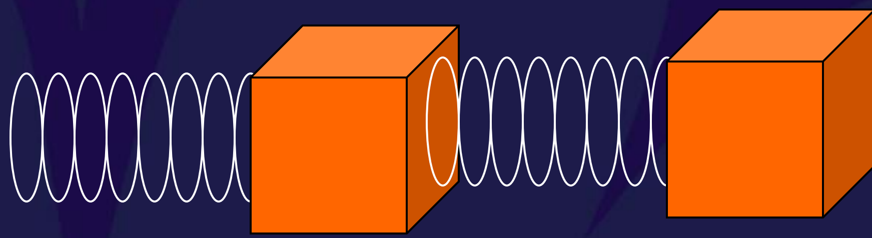
Questions: Can you find more solutions? What determines A (or B)?

Example: spring in series

Exercise: What is the oscillation frequency(ies) of the following spring configuration?



Example: spring in series



K_1 M K_2 M

$$\Rightarrow m\ddot{x}_1 = -K_1 x_1 - K_2(x_1 - x_2),$$

$$m\ddot{x}_2 = -K_2(x_2 - x_1)$$

Can you determine the frequency of oscillation from these equations?

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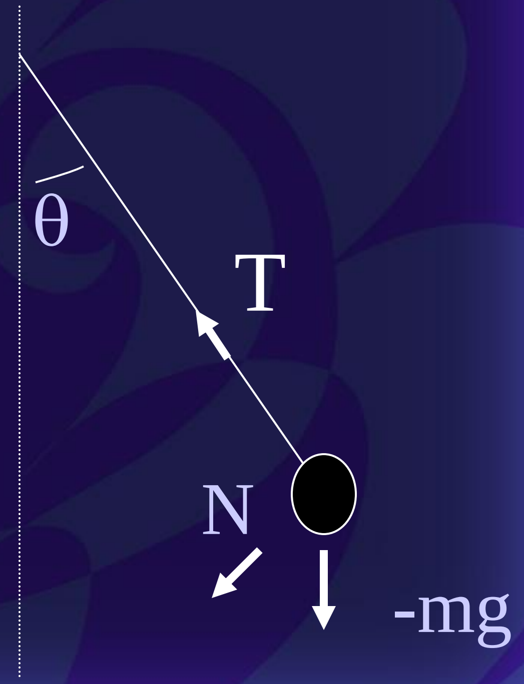
Another example: *swing*

Mathematics of Swing = Harmonic Oscillator under gravity

- $T = mg\cos(\theta(t))$
- Net force (N)= $-mg\sin(\theta(t))$
- Notice: (1)both magnitude and direction of force changes with time &
- (2)the length of the string *l*, is fixed when θ small.



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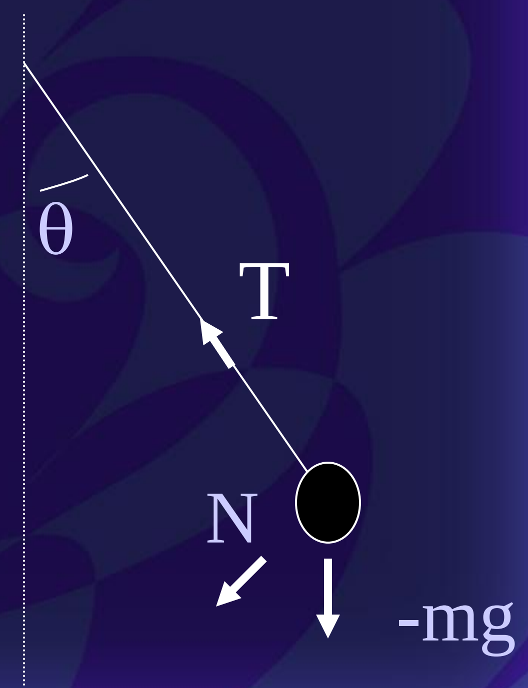
Mathematics of Swing = Harmonic Oscillator under gravity

- *$\Rightarrow$ trick to solve the problem when θ is small!*
- *Notice:*

$$x(t) = l\sin(\vartheta(t)), \qquad y(t) = -l\cos(\vartheta(t))$$

$$N_x(t) = -mg\sin(\vartheta(t))\cos(\vartheta(t)),$$

$$N_y(t) = -mg\sin(\vartheta(t))\sin(\vartheta(t))$$



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Mathematics of Swing = Harmonic Oscillator under gravity

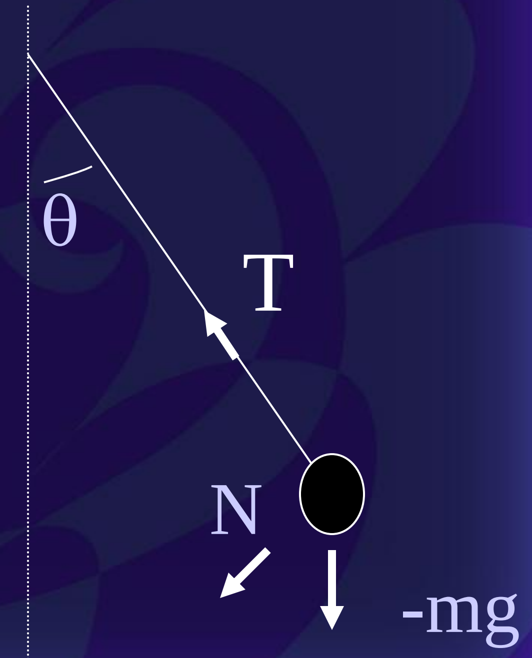
- *when θ is small!*

$$\cos(\vartheta) \approx 1, \qquad \sin(\vartheta) \approx \vartheta$$

$$x(t) \approx l\vartheta(t), \qquad y(t) \approx -l$$

$$N_x(t) \approx -mg\vartheta(t),$$

$$N_y(t) \approx 0$$

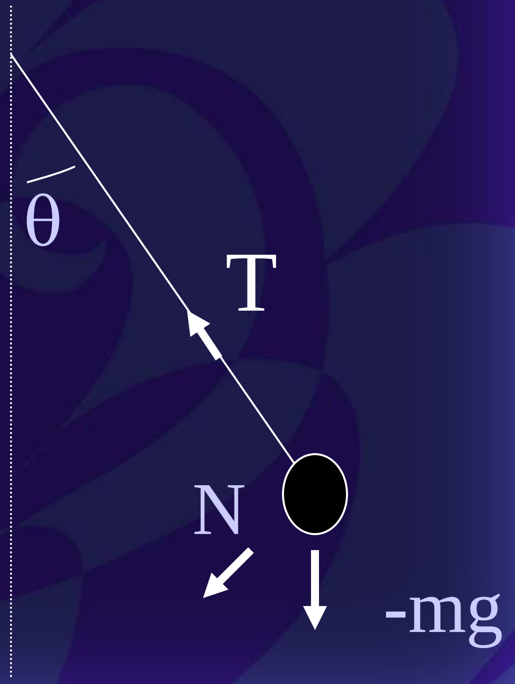


Mathematics of Swing = Harmonic Oscillator under gravity

- $\therefore$ *we have approximately*

$$(N_y(t) \approx 0)$$

$$N_x(t) \approx -\frac{mg}{l}x(t),$$

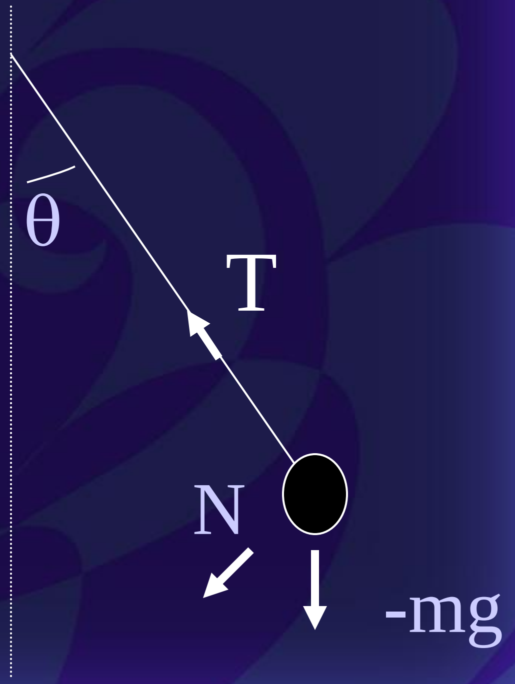


Mathematics of Swing = Harmonic Oscillator under gravity

- $\therefore$ *we have approximately in x-direction*

$$\boxed{ma(t) \approx -\frac{mg}{l}x(t),}$$

which is same as equation for spring+load system except $K/m \rightarrow g/l$



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II. Conservation Laws

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(1) Conservation of momentum

Consider a group of masses m_i with forces F_{ij} between them and external forces F_i acting on each of them, i.e. Newton's Law is

$$m_i a_i(t) = F_i^{ext} + \sum_{j \neq i} F_{ij}, \qquad \forall i$$

Notice $F_{ii}=0$, why?

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Let us study what happens to the CM coordinate

$$\vec{X} = \frac{1}{M}\sum_i m_i \vec{x}_i, \qquad \left(M = \sum_i m_i = total \quad mass\right)$$

$$\Longrightarrow \vec{V} = \frac{1}{M}\sum_i m_i \vec{v}_i$$

$$\Longrightarrow \vec{A} = \frac{1}{M}\sum_i m_i \vec{a}_i = \frac{1}{M}\sum_i \left[F_i^{ext} + \sum_{j\neq i} F_{ij} \right]$$

$$= \frac{1}{M}\left[\sum_i F_i^{ext} + (0) \right] = \frac{F_{tot}^{ext}}{M}$$

Newton's third Law

Let us study what happens to the CM coordinate

In particular, when total external force=0, we have

$$\vec{A} = \frac{F_{tot}^{ext}}{M} = 0$$

$$\Rightarrow M\vec{V} = \sum_i m_i \vec{v}_i = const.$$

Total momentum of the system is a constant of motion (Law of conservation of momentum)

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Recall for a rigid body

- The center of mass is a special point in a rigid body with position defined by

$$\vec{X} = \frac{1}{M}\sum_i m_i \vec{x}_i, \qquad (M = \sum_i m_i = total \quad mass)$$

This point *stays at rest* or in uniform motion when there is no net force acting on the body

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An Example of application

- *Two cars of same mass M are resting side by side on a frictionless surface. A person with mass m stands on one car originally. He jump to the other car and jump back. Can we tell anything about the end velocities of the two cars?*

$$(M+m)v_1 + Mv_2 = (M+m)(0) + M(0)$$

$$\Rightarrow \frac{v_1}{v_2} = -\frac{M}{M+m}$$

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An Example of application

- *Two cars of same mass M are resting side by side on a frictionless surface. A person with mass m stands on one car originally. He push the other car away. Can we tell anything about the end velocities of the two cars?*

$$(M+m)v_1 + Mv_2 = (M+m)(0) + M(0)$$

$$\Rightarrow \frac{v_1}{v_2} = -\frac{M}{M+m}$$

An Example of application

- *Two cars of same mass M are moving side by side on a frictionless surface with speed v. A person with mass m stands on one car originally. He push the other car away. Can we tell anything about the end velocities of the two cars?*

$$(M+m)\vec{v}_1 + M\vec{v}_2 = (M+m)(\vec{v}) + M(\vec{v})$$

$$\Rightarrow \frac{\vec{v}_1 - \vec{v}}{\vec{v}_2 - \vec{v}} = -\frac{M}{M+m}$$

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Conservation of angular momentum

$$\vec{L} = \sum_i m_i \vec{x}_i \times \vec{v}_i$$

We shall discuss this when we discuss circular motion

Conservation of Energy

First question: what is *energy*?

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After working for a long time, we start to feel tired.

We said that we are running out of *energy*.

The term energy is often used to describe how long we can sustain our usage of force (or work).

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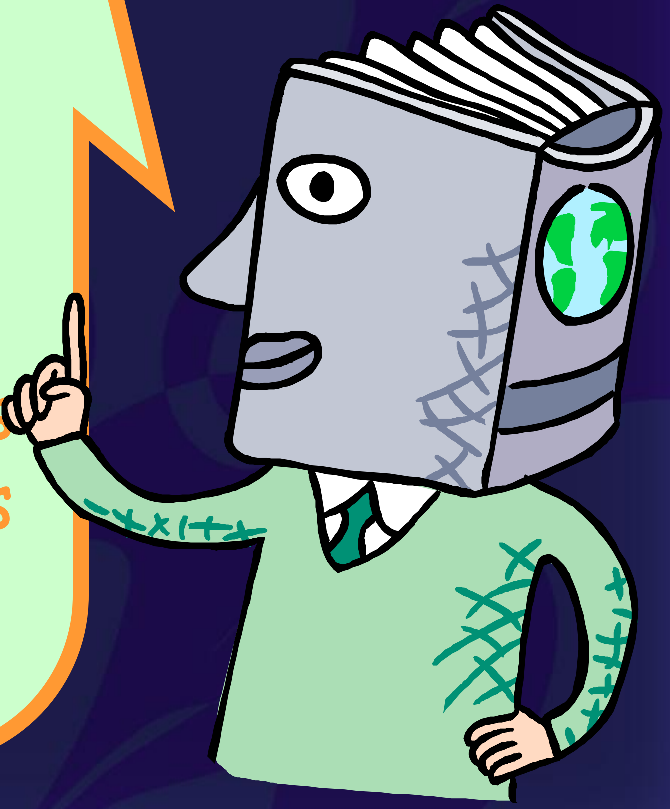
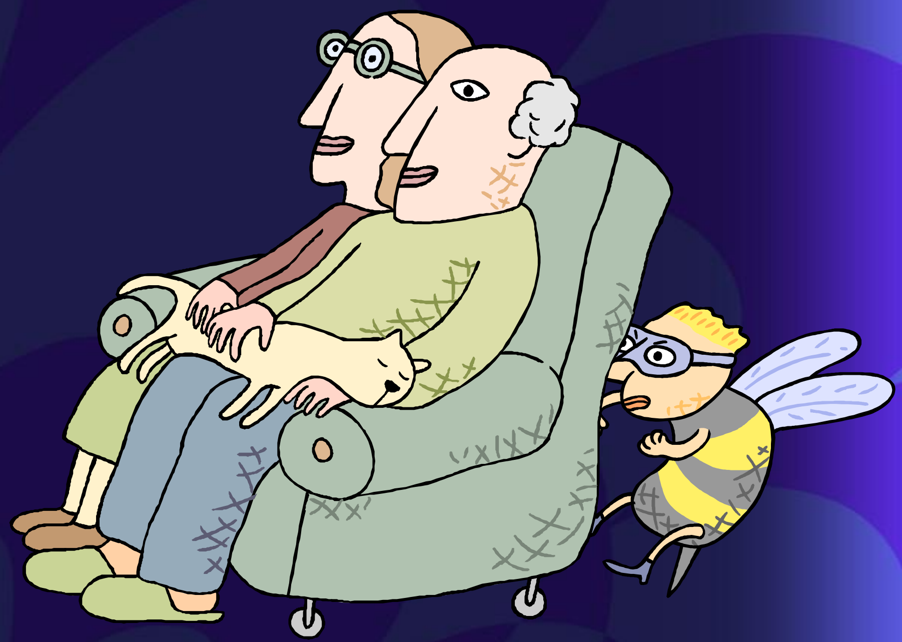
In physics, the terms **work and energy** have similar qualitative meaning as we use them in everyday life, except that rigorous mathematical definitions are given to these terms in Newtonian mechanics.

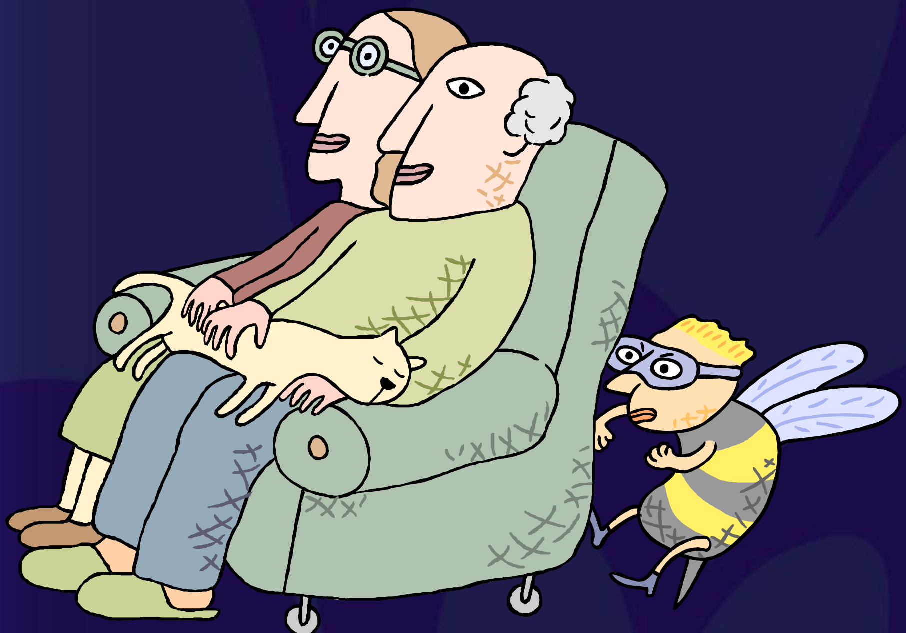


Work Done

Imagine you have to move a piece of heavy furniture from position A to position B in a room.



Imagine you have to move a piece of heavy furniture from position A to position B in a room.



B

A

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Afterward, when you are chatting with your friend, you try to explain to him/her how much hard work you have done. Well, suppose your friend wants to know whether you are just exaggerating or whether you have really done a lot of work.

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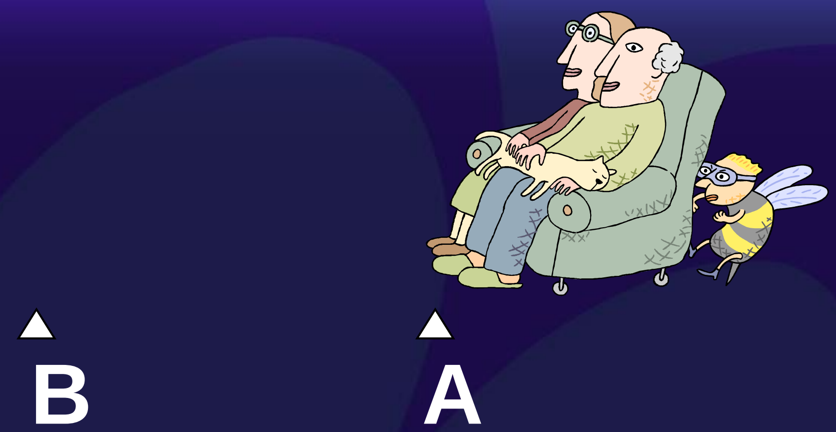
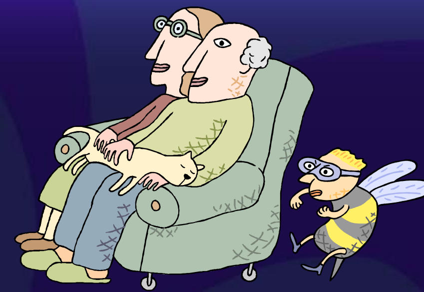
So come the question:

Is there a consistent way to measure how much work one has done in situation like the above?



We can start by listing the factors we believe which determines 'work done' in the above example:

1. How big and heavy the furniture is.
2. How long you have spent on moving the furniture.
3. How far is the distance between A and B.
4. Friction between ground and furniture.



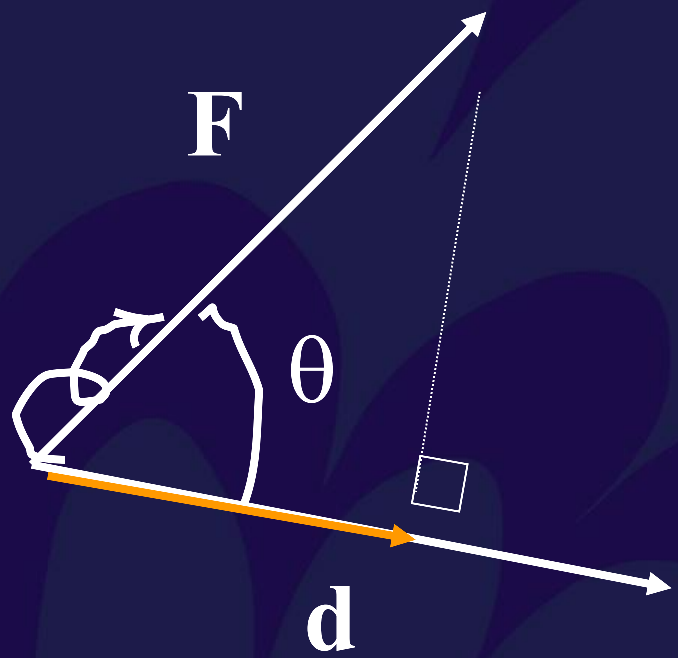
B A

Questions:

Do you think these are reasonable factors affecting work done? Can you think of other factors? Can you build up a scientific method of measuring work done based on the above factors?

1. How big and heavy the furniture is.
2. How long you have spent on moving the furniture.
3. How far is the distance between A and B.
4. Friction between ground and furniture.

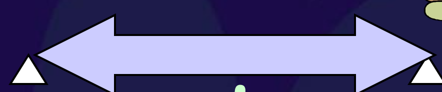
In Mechanics the work done by a constant force F *on an object* is equal to F·d. d is the distance where the object has moved under the force.



$$\mathbf{F}\cdot\mathbf{d} = Fd\cos\theta$$

Let's try to apply the formula. First let us assume that $\theta = 0$ and the ground is flat.

$$W = Fd_{AB}$$



B d_{AB} A

$$\underbrace{F = F_{friction}}_{\text{We expect!}}$$

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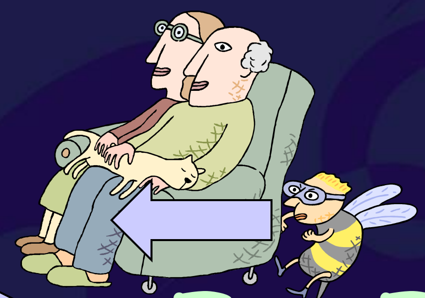
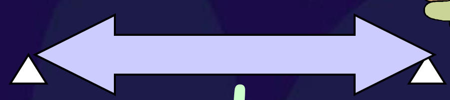
The formula looks OK. Agree?

$$W = Fd_{AB}$$



$$F = F_{friction}$$



B d_{AB} A

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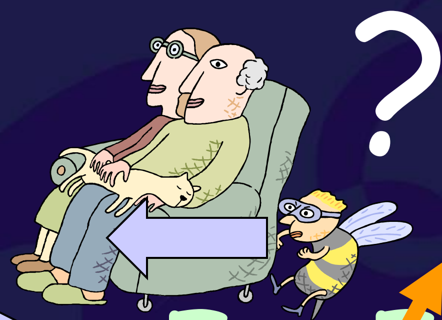
But there is a problem. Imagine what happens if the ground is frictionless, $F_{friction}=0$ (e.g. on top of ice).

$$W = \overset{0}{\cancel{F}} d_{AB}$$

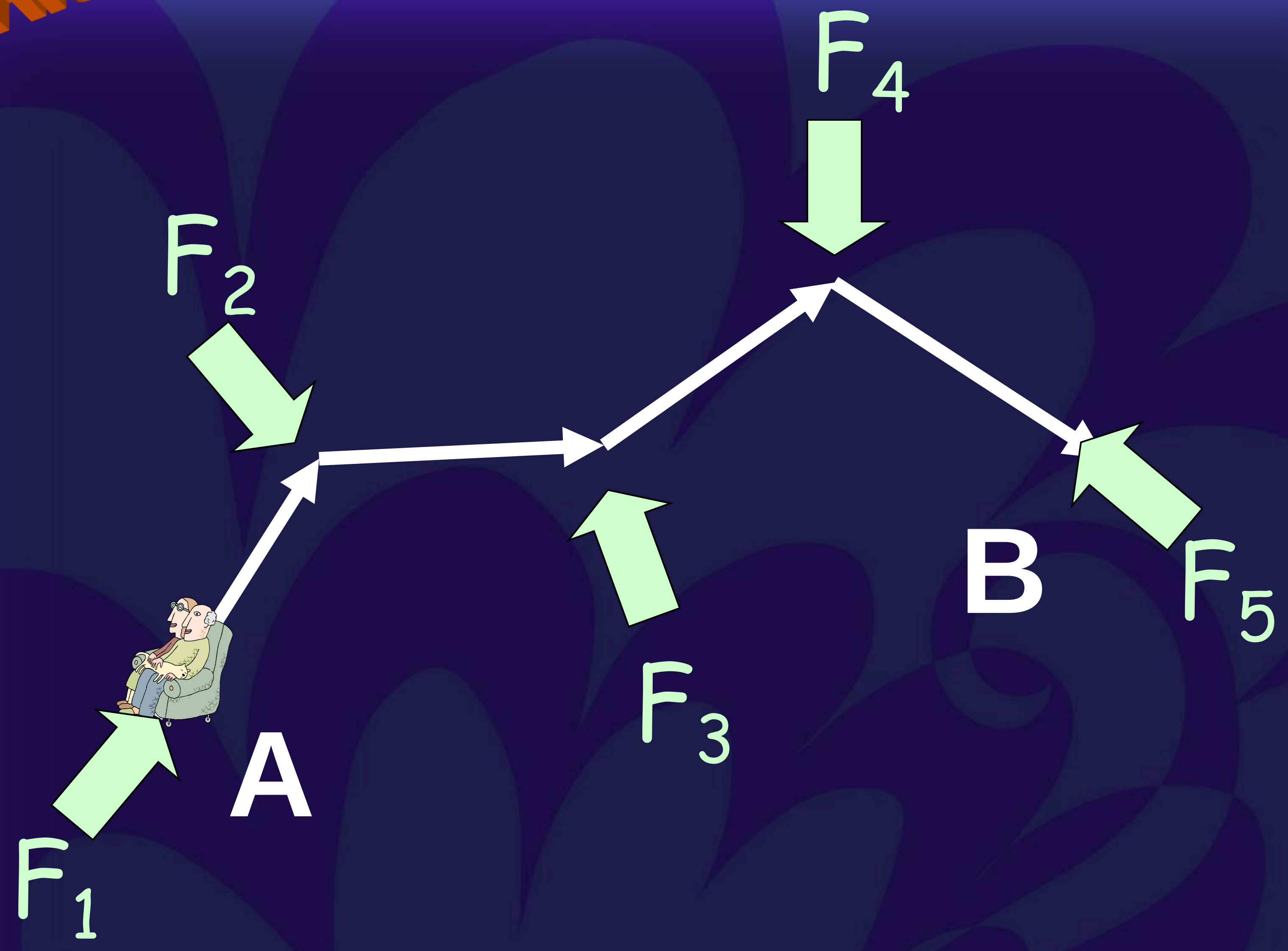


B d_{AB} A

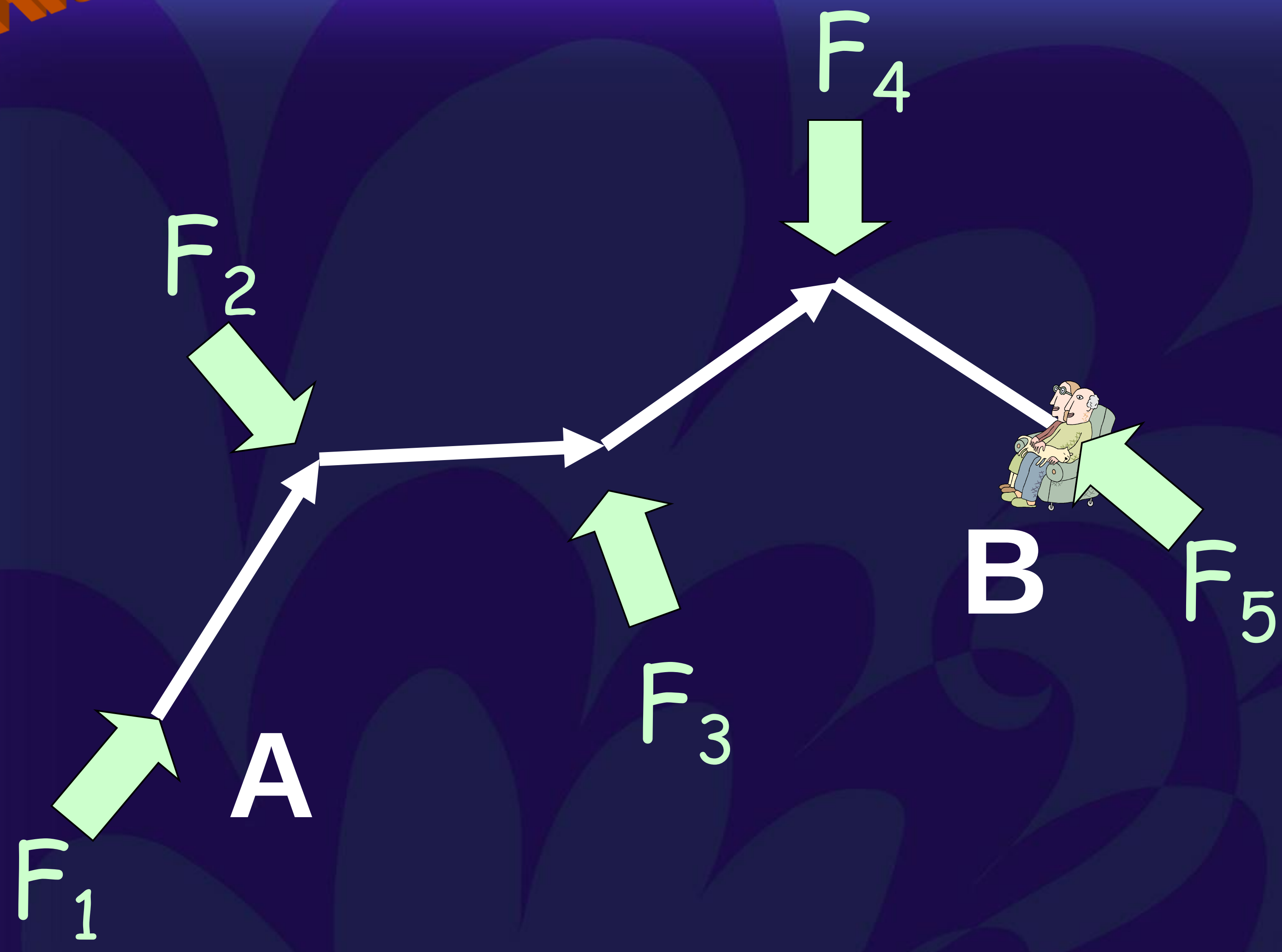
$$F = \overset{0}{\cancel{F}}_{friction}$$

It seems that you don't have to do any work to move the furniture in this case! Can this be right?

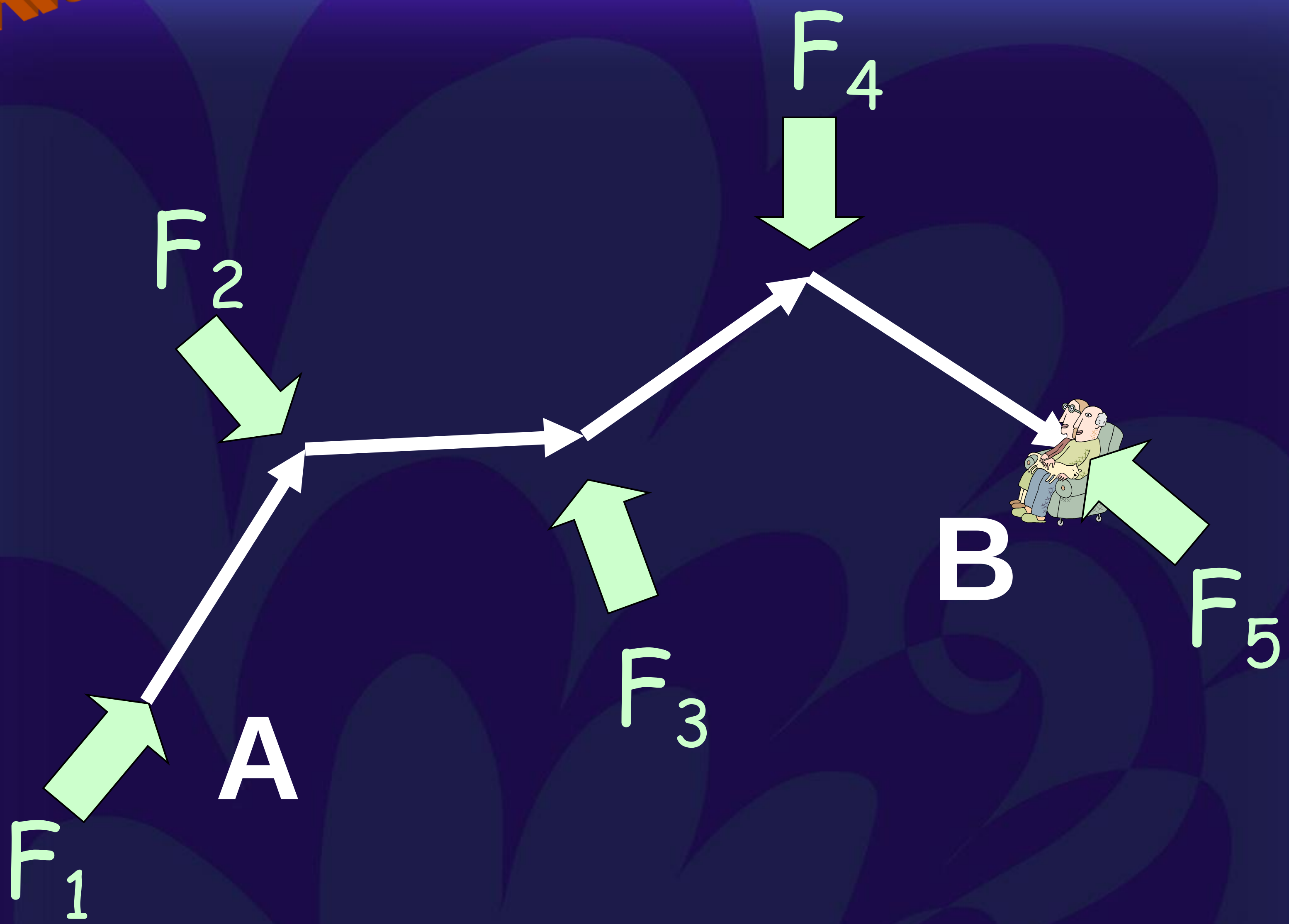

A
B
F_1



F2
F4
F3
F5
B
A
F1



F_4
F_2
F_3
F_5
F_1
A
B


F2
F4
F1
A
F3
B
F5

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In fact, you are doing more than just that if you think about Newton's third law.

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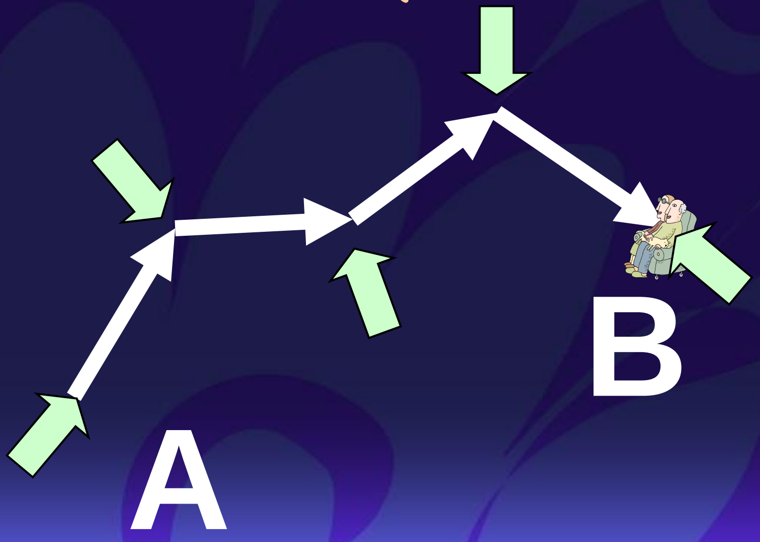
Do you need to do work to “stand still”?

three different work done:

(1)the work done by you; and

(2)the work done on the furniture to overcome friction, and

(3)the work done on the furniture to change the velocity of the furniture (initial and final pushes).



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Question:

If used more appropriately, do you think the formula **$W = F \cdot d$** can still be applied to describe ALL the work done? And How?

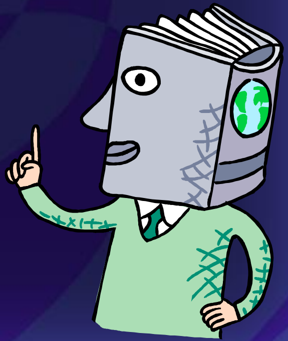
Work done to change the state of motion:

kinetic energy

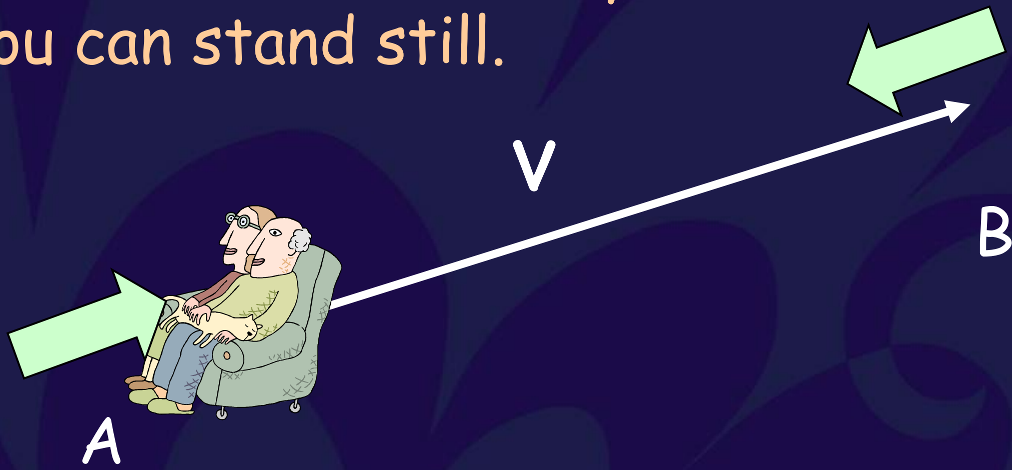
Question:

where does your energy go? Do they just vanish?

Physics provide a rather surprising answer: *Energy can never vanish, they can just be transformed from one form into another.*

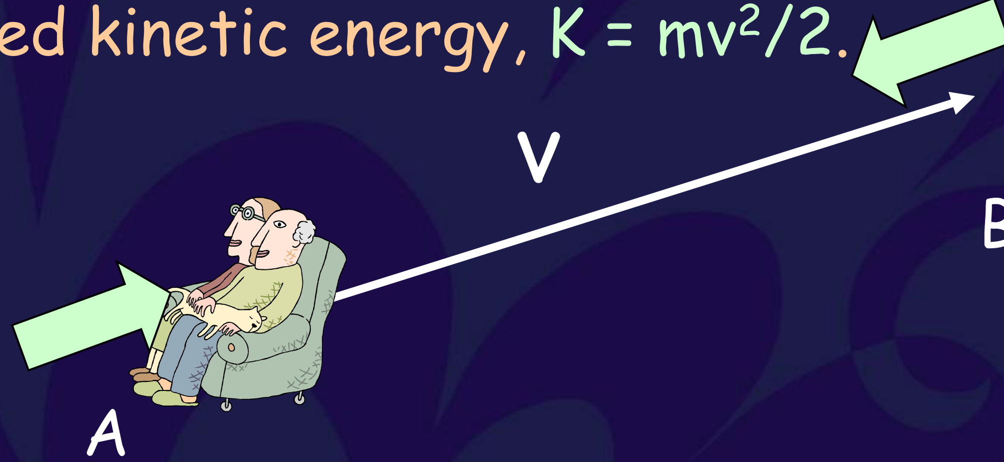


Let us go back to the furniture problem and ask in what way our energy are transformed. Let me assume for simplicity that the surface between furniture and ground is frictionless, but there is enough friction between you and the ground so that you can stand still.



Therefore, all you have to do is just an initial push, the object (furniture) slides by itself from point A to point B and is stopped by another push.

In this case, we have done work at two instances:
(1)At the beginning, when we do work on the object to start it moving with velocity **v**. Using his equations, Newton found that in this case, we have transferred our energy to the object in forms of so called kinetic energy, $K = mv^2/2$.



This result can be understood roughly as follows:

Assume that the force is constant and has act on the object for a period of (short) time t_D. During this time, the distance traveled by the object is D.

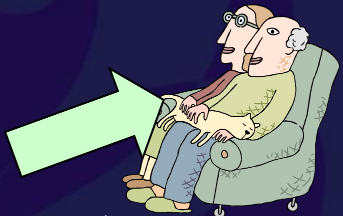
Using Newton's Law, we find that

(1) between $t_D > t > 0$, the velocity of the object is

$v(t) = at = (F/m)t$,

and displacement is

$x(t) = at^2/2$.



A

This result can be understood roughly as follows:

Assume that the force is constant and has act on the object for a period of (short) time t_D. During this time, the distance traveled by the object is D.

Using Newton's Law, we find that

(2)for $t > t_D$, the velocity is

$v = at_D$.

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This result can be understood roughly as follows:

Assume that the force is constant and has act on the object for a period of (short) time t_D. During this time, the distance traveled by the object is D.

Using Newton's Law, we find that

Using the displacement equation, we obtain

$D = at_D{}^2/2 \Rightarrow v = at_D = (2Da)^{1/2}$

and $mv^2/2 = m(2Da)/2 = DF$ = work done!

i.e. Kinetic energy is equal to the work we have done on the object to make it move with velocity v.

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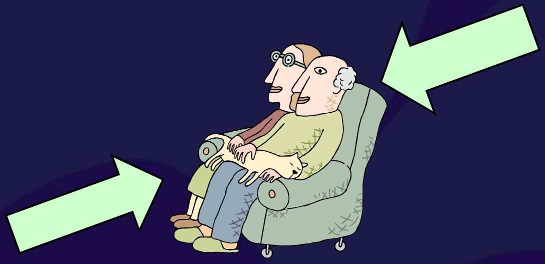
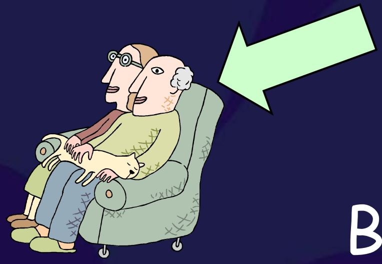
Question: Is this just a mathematical trick?



If kinetic energy is a form of "energy". Can it be used to do work?

Let's see what happens when the object is stopped at position B



B

Unless there exists a large friction between G and the ground, otherwise G itself will be set into motion by the object, i.e. the furniture has acquired the ability to do work!

Potential Energy

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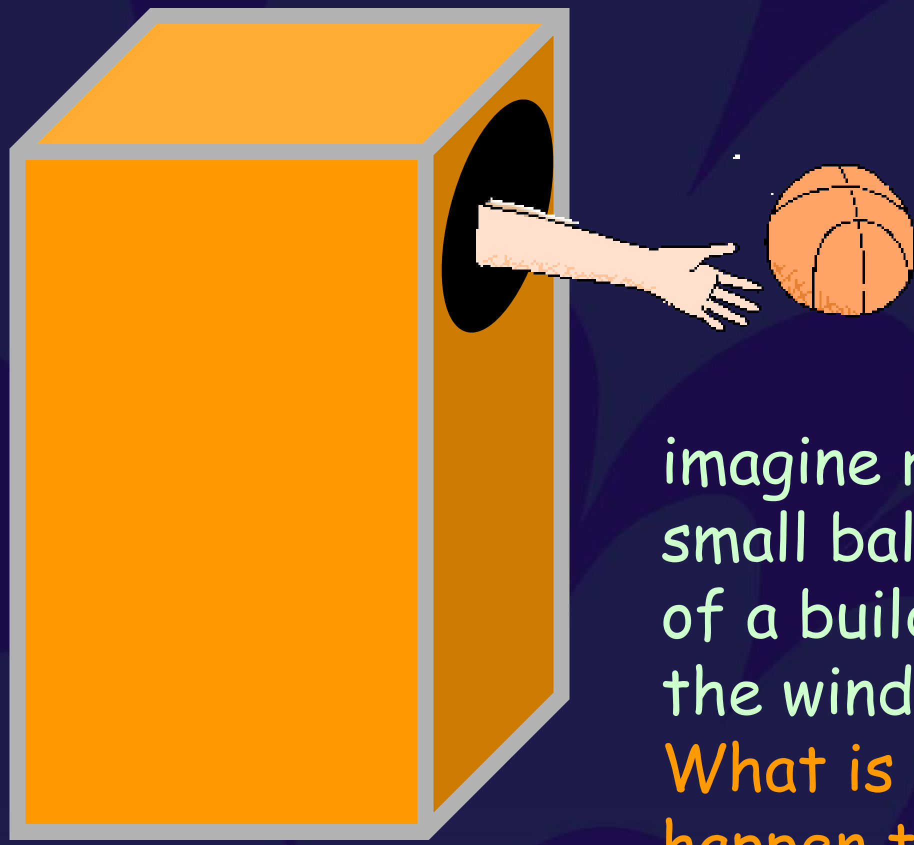
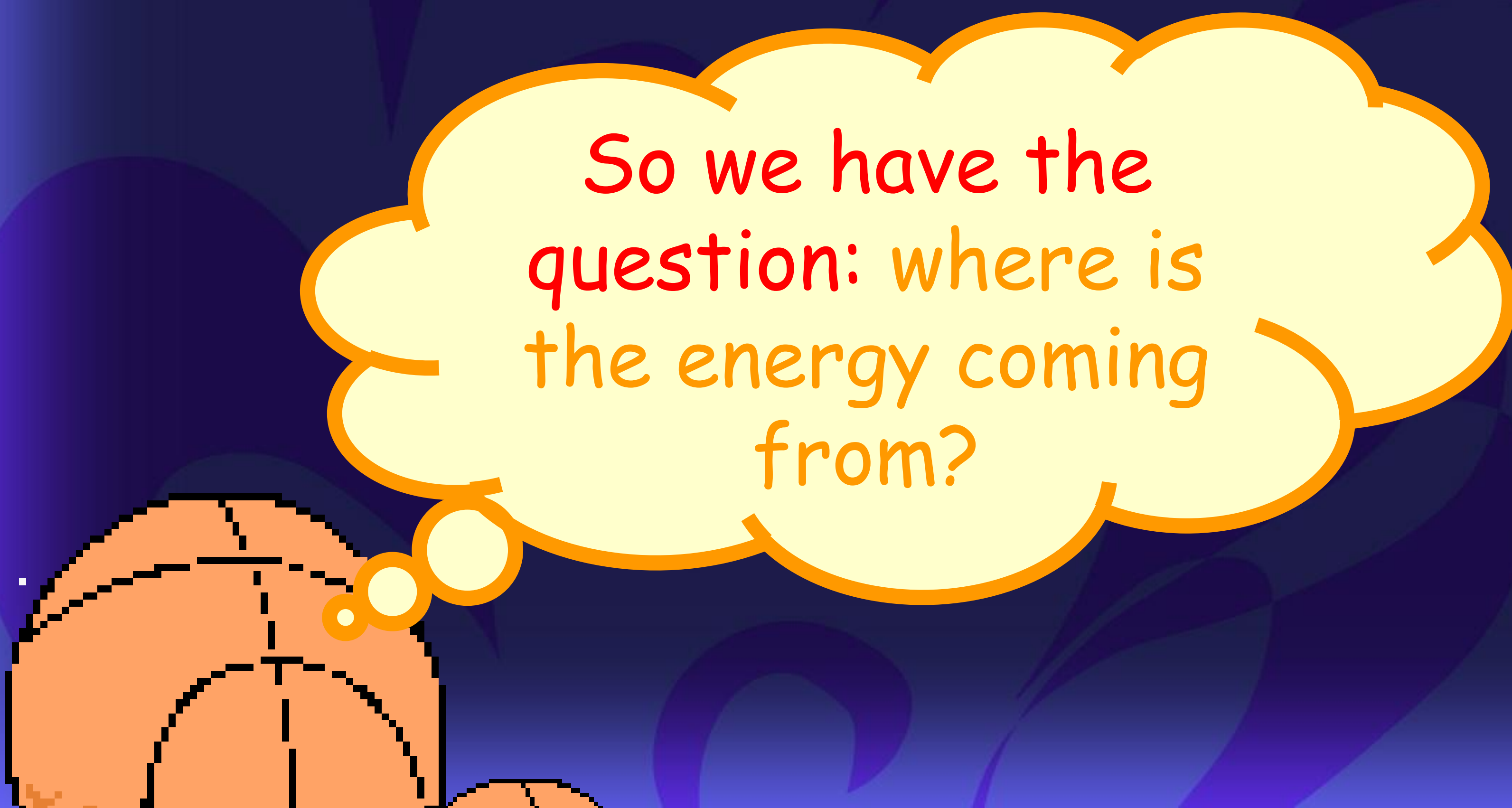
The concept of potential energy can be understood by a simple question:



imagine releasing a small ball at the top of a building outside the window.

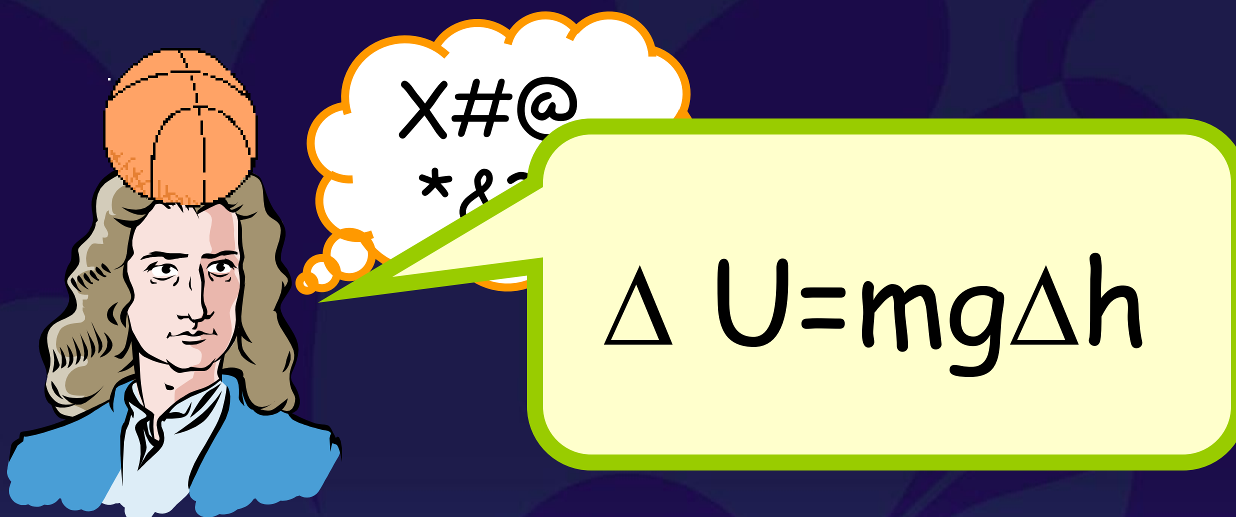
What is going to happen to the ball?

Of course we all know that the ball will fall down with *increasing speed* because of gravitational force F = mg. The fact that the ball's speed is increasing means that it's kinetic energy is increasing.



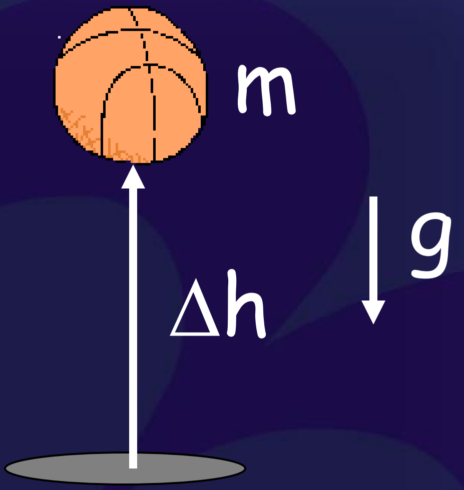
Newton found that…

according to his equations, the source of this energy can be assigned to the gravitational force, in the form…



HKIPhO

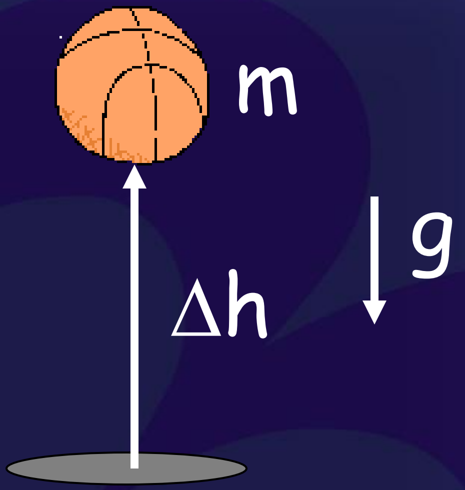
$$\Delta U = mg\Delta h$$



where ΔU is the change in gravitational energy when the object goes through a change in height Δh. Notice that ΔU and Δh are negative if the object's final height is less than the initial height.

HKIPhO

$\Delta U = mg\Delta h$

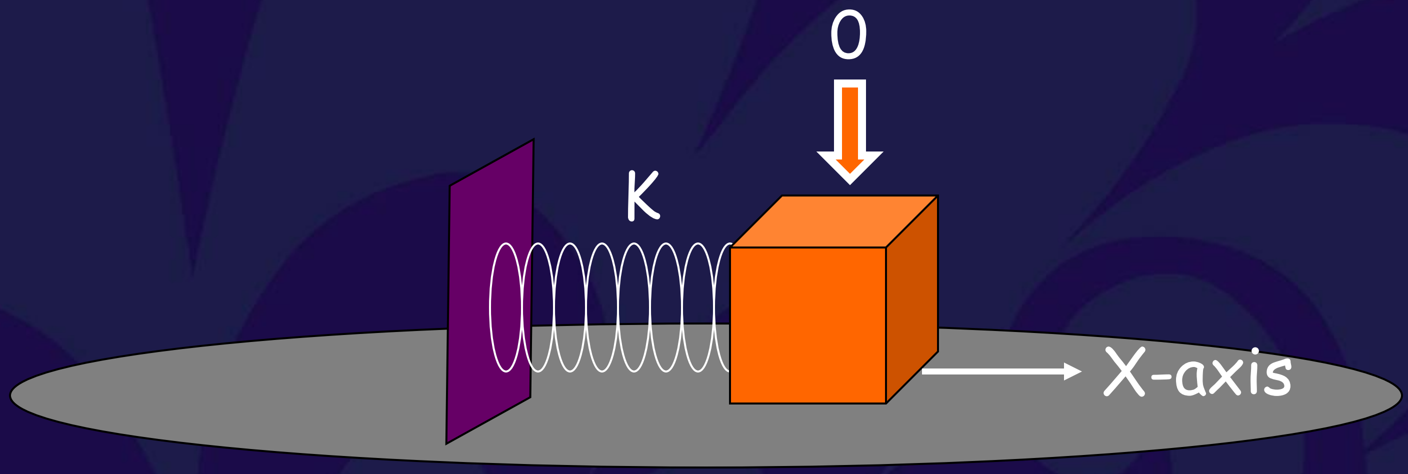


Exercise: Prove that the total energy $E = mv^2/2 + U(h)$ is a constant of motion (conserved) for an object moving under gravitational force.

HKIPhO

Another example of potential energy:

spring (or Harmonic oscillator)



Potential energy = $Kx^2/2$ = energy stored in spring

HKIPhO

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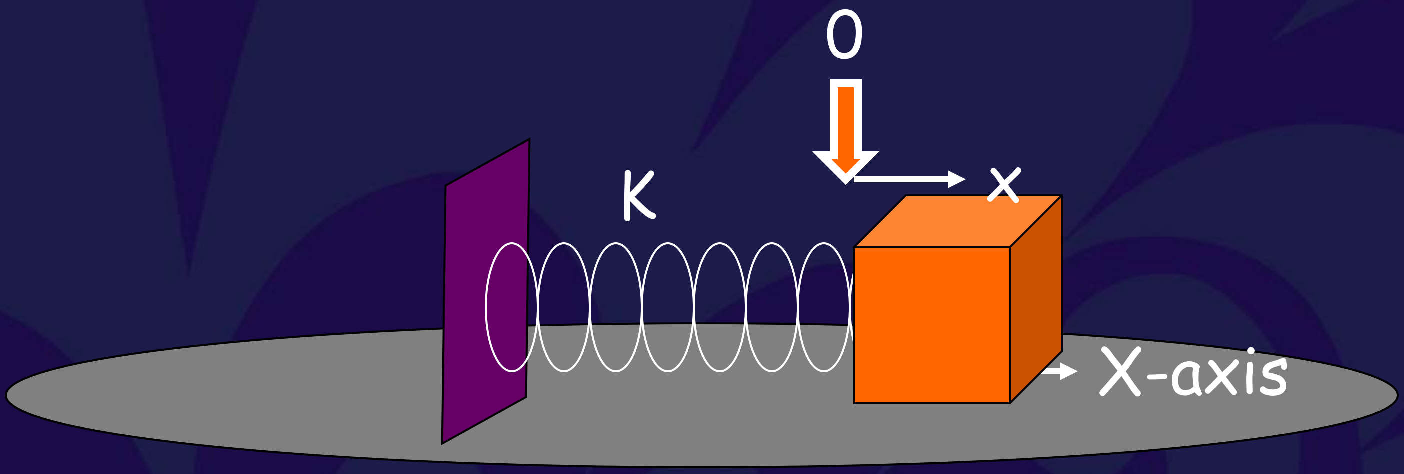

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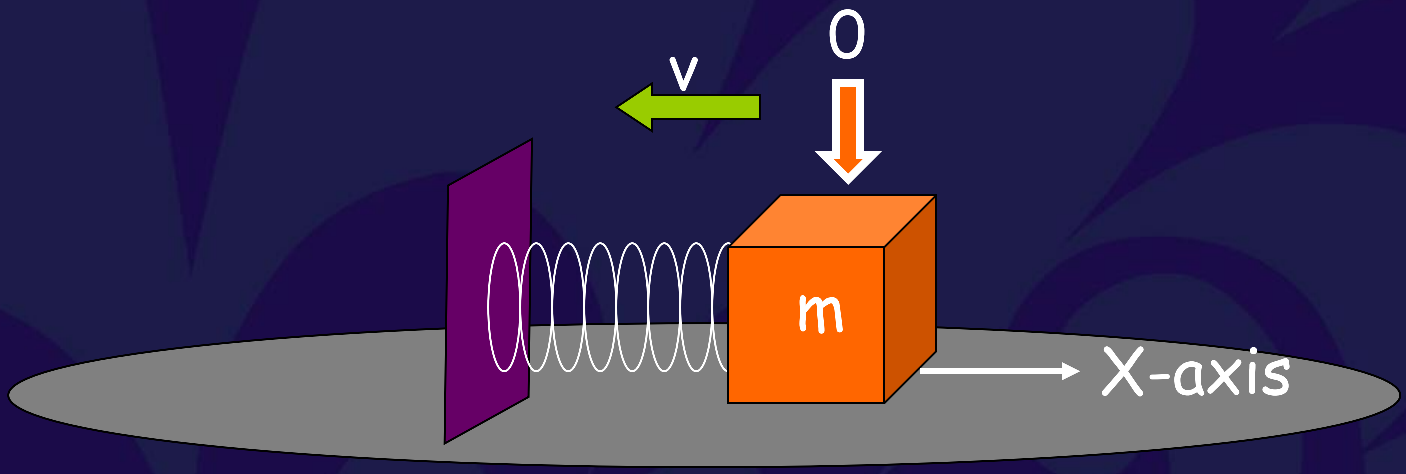


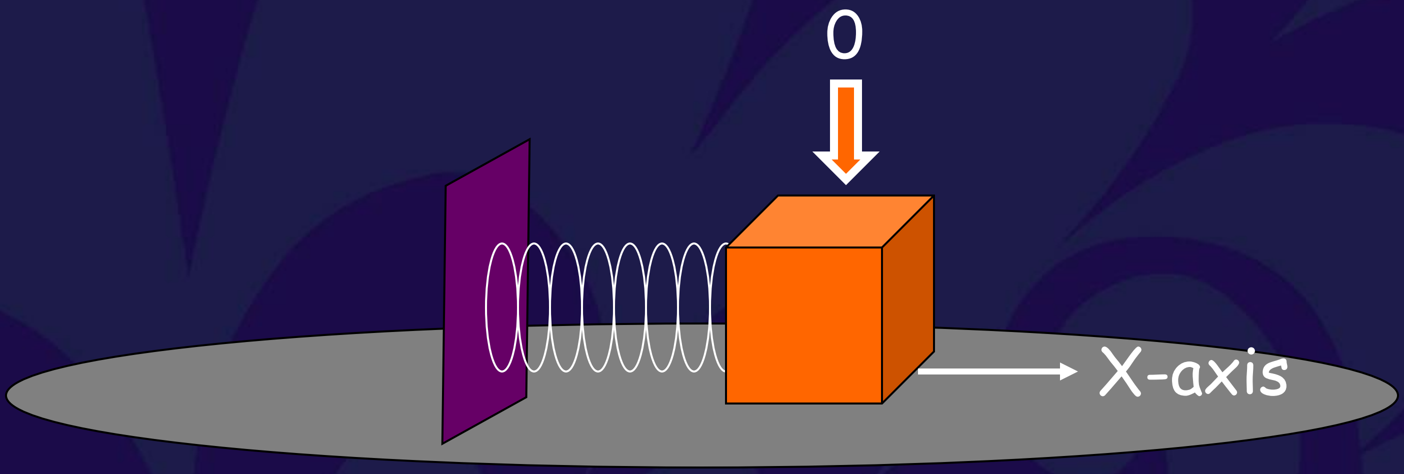
Potential energy = $Kx^2/2$ = energy stored in spring

Kinetic energy = $mv^2/2$ = kinetic energy of mass m.

HKIPhO

Another example of potential energy:

spring (or Harmonic oscillator)



Potential energy = $Kx^2/2$ = energy stored in spring

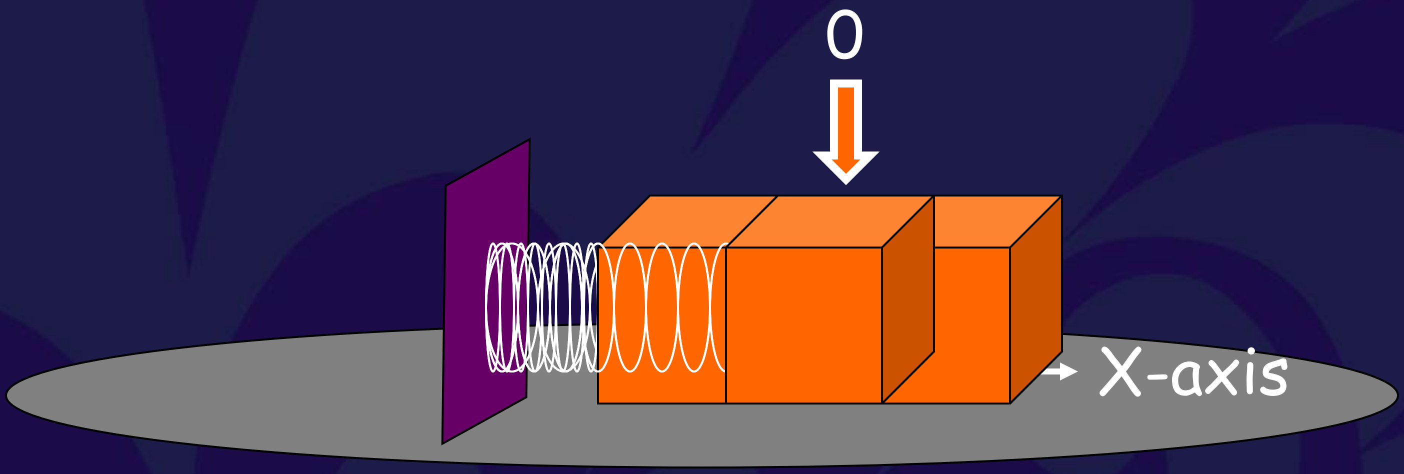
Kinetic energy = $mv^2/2$ = kinetic energy of small ball.

Ball oscillate => Potential energy $\Leftrightarrow$ Kinetic energy

HKIPhO

Another example of potential energy:

spring (or Harmonic oscillator)



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HKIPhO

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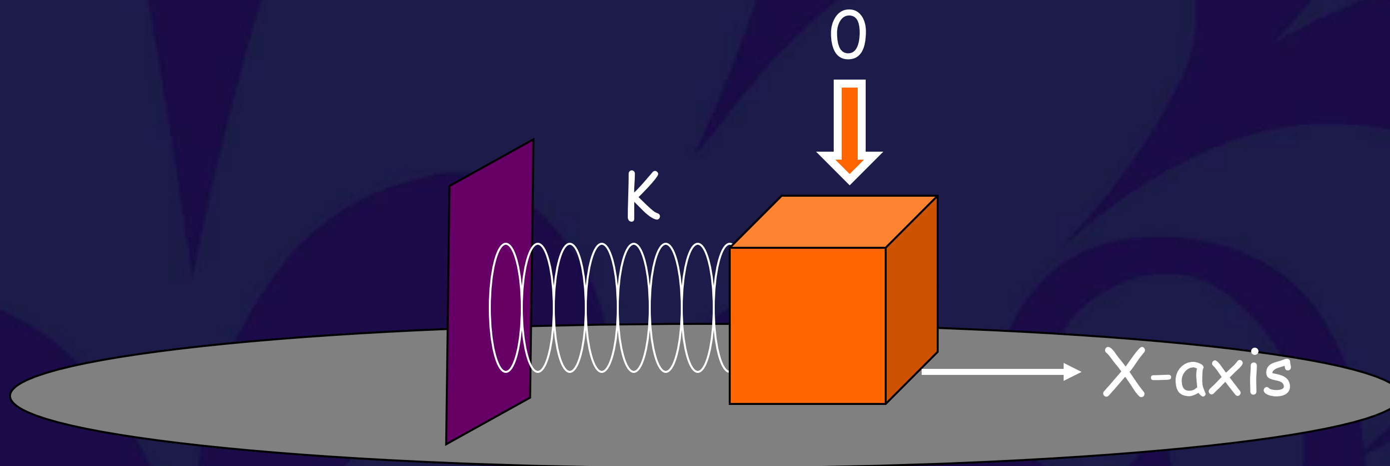
spring (or Harmonic oscillator)



Exercise: Using the solution of Newton's equation, show that *P.E.+ K.E.* is a constant of motion for Harmonic oscillator

HKIPhO

Mathematics of the Conservation Energy

Consider a particle of mass m moving under a conservative force

$$\vec{F} = -\nabla U(\vec{x});$$

$$\left(F_\alpha = -\frac{d}{dx_\alpha} U(x, y, z); \quad \alpha = x, y, z\right)$$

Mathematics of the Conservation Energy

Consider a particle of mass m moving under a conservative force

$$\therefore m\frac{d\vec{v}}{dt} = -\nabla U(\vec{x});$$

$$\Rightarrow m\vec{v}.\frac{d\vec{v}}{dt} = -\frac{d\vec{x}}{dt}.\nabla U(\vec{x});$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}m\vec{v}^{2}\right) = -\frac{d}{dt}U(\vec{x})$$

HKIPhO

Mathematics of the Conservation Energy

Consider a particle of mass m moving under a conservative force

$$\frac{d}{dt}U(\vec{x}) = \frac{dx}{dt}\frac{\partial U}{\partial x} + \frac{dy}{dt}\frac{\partial U}{\partial y} + \frac{dz}{dt}\frac{\partial U}{\partial z} = \frac{d\vec{x}}{dt}.\nabla U$$

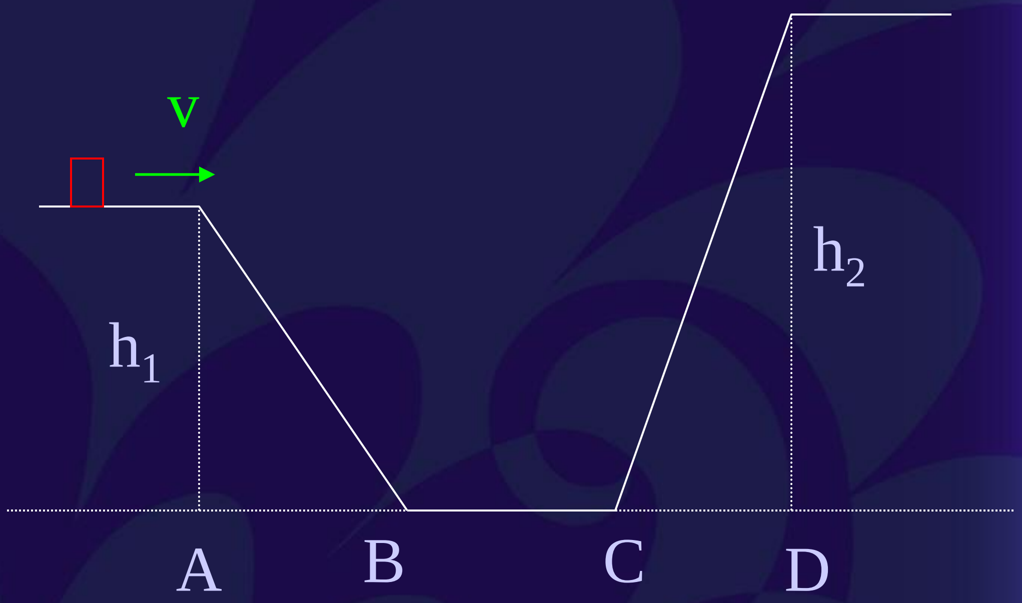
$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}m\vec{v}^{2} + U(\vec{x})\right) = 0;$$

$$or \quad \frac{1}{2}m\vec{v}^{2} + U(\vec{x}) = const.$$

HKIPhO

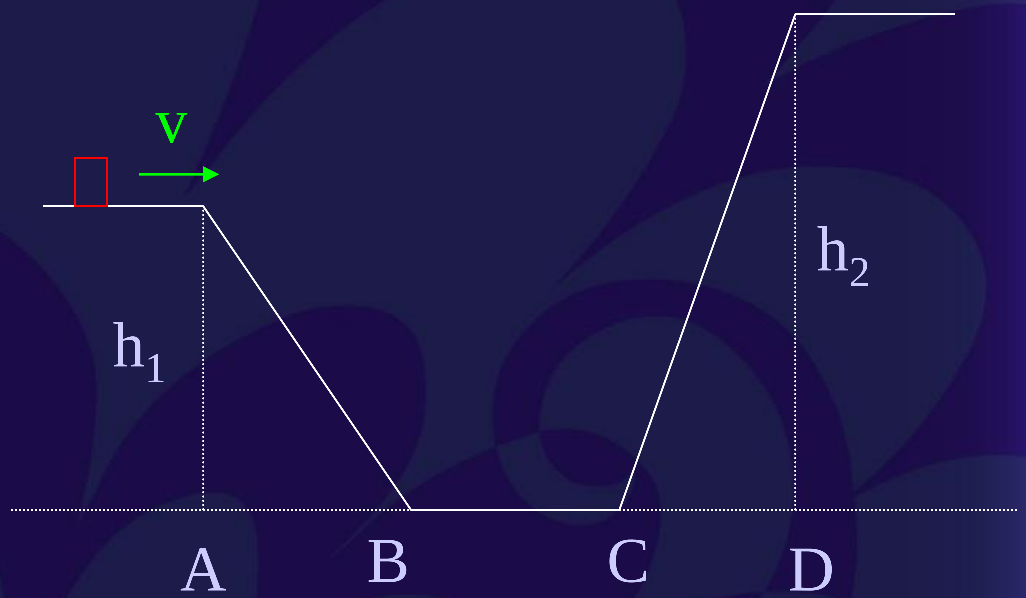
Example of Application

- Consider the figure.
- What is the minimum value of v needed for the block to travel to point D?



Example of Application

- Ans:
- $mg(h_2 - h_1) = \frac{1}{2} mv^2$
- (*assuming no friction*)



Friction

- In previous examples *mechanical energy* of a system is conserved.
- This is not true in presence of frictional force.
- In this case energy is converted into heat, sound, etc.
- But *total energy* is still conserved.

End of lecture II