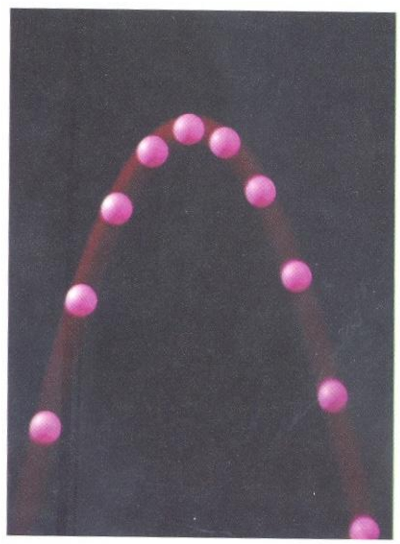
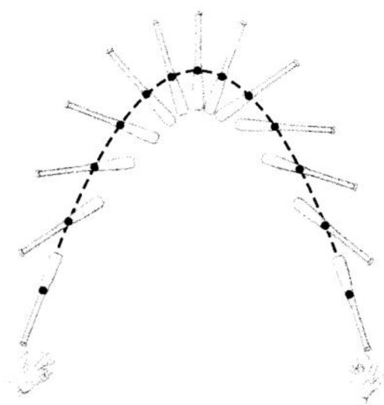
Center of Mass and Momentum

Reading: Chapter 9

The Center of Mass



(a)

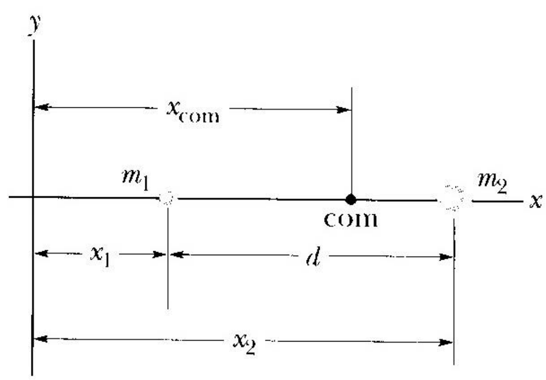


(b)

FIG. 9-1 (*a*) A ball tossed into the air follows a parabolic path. (*b*) The center of mass (black dot) of a baseball bat flipped into the air follows a parabolic path, but all other points of the bat follow more complicated curved paths. *(a: Richard Megna/Fundamental Photographs)*

See animation "An Object Tossed Along a Parabolic Path".

The center of mass of a body or a system of bodies is the point that moves as though all of the mass were concentrated there and all external forces were applied there.



For 2 particles,

$$x_{\text{cm}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m_1 x_1 + m_2 x_2}{M}$$

For *n* particles,

$$x_{\text{cm}} = \frac{m_1 x_1 + \cdots + m_n x_n}{M}$$

In general,

$$\boxed{\begin{aligned} x_{\text{cm}} &= \frac{1}{M}\sum_{i=1}^{n} m_i x_i, \\ y_{\text{cm}} &= \frac{1}{M}\sum_{i=1}^{n} m_i y_i, \\ z_{\text{cm}} &= \frac{1}{M}\sum_{i=1}^{n} m_i z_i. \end{aligned}}$$

In vector form,

$$\boxed{\vec{r}_{\text{cm}} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i.}$$

Solid Bodies

$$x_{\text{cm}} = \frac{1}{M}\int xdm,$$
$$y_{\text{cm}} = \frac{1}{M}\int ydm,$$
$$z_{\text{cm}} = \frac{1}{M}\int zdm.$$

If the object has uniform density,

$$\rho = \frac{dm}{dV} = \frac{M}{V}.$$

Rewriting $dm = \rho dV$ and $m = \rho V$, we obtain

$$x_{\text{cm}} = \frac{1}{V}\int xdV,$$
$$y_{\text{cm}} = \frac{1}{V}\int ydV,$$
$$z_{\text{cm}} = \frac{1}{V}\int zdV.$$

Note:

[1] If the object has a point of symmetry, then the center of mass lies at that point.
If the object has a line of symmetry, then the center of mass lies on that line.

If the object has a plane of symmetry, then the center of mass lies in that plane.

[2] The center of mass of an object need not lie within the object e.g. a doughnut.

Examples

9-1 Three particles of masses m_1 = 1.2 kg, m_2 = 2.5 kg, m_3 = 3.4 kg are located at the corners of an equilateral triangle of edge a = 140 cm. Where is the center of mass?

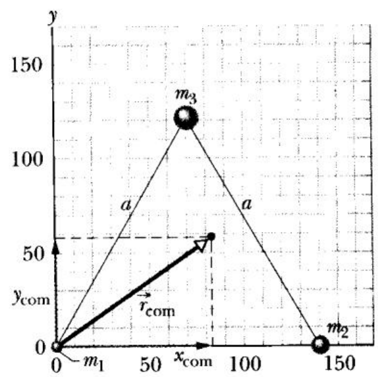


FIG. 9-3 Three particles form an equilateral triangle of edge length a. The center of mass is located by the position vector $\vec{r}_{com}$.

$m_1 = 1.2$	$x_1 = 0$	$y_1 = 0$
$m_2 = 2.5$	$x_2 = 140$	$y_2 = 0$
$m_3 = 3.4$	$x_3 = 140\cos 60^o$	$y_3 = 140\sin 60^o$

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{M}$$

$$= \frac{(1.2)(0) + (2.5)(140) + (3.4)(70)}{7.1} = 83 \text{ cm} \quad \text{(ans)}$$

$$y_{cm} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{M}$$

$$= \frac{(1.2)(0) + (2.5)(0) + (3.4)(121)}{7.1} = 58 \text{ cm} \quad \text{(ans)}$$

9-2 A uniform circular metal plate *P* of radius 2*R* has a disk of radius *R* removed from it. Locate its centre of mass lying on the *x* axis.

Let ρ = density,
t = thickness
Mass:
Object *P*: m_S
$= \pi(2R)^2\rho t - \pi R^2\rho t$
$= 3\pi R^2\rho t$
Object *S*: $m_P = \pi R^2\rho t$
Object *C*: m_C
$= \pi(2R)^2\rho t = 4\pi R^2\rho t$

Center of mass:
Object *P*: x_P = ?
Object *S*: $x_S = -R$
Object *C*: $x_C = 0$

Since

$$x_C = \frac{m_P x_P + m_S x_S}{m_P + m_S} = 0,$$

$$x_P = -\frac{m_S}{m_P} x_S$$

$$= -\frac{\pi R^2 \rho t}{3\pi R^2 \rho t}(-R)$$

$$= \frac{R}{3} \qquad \text{(ans)}$$

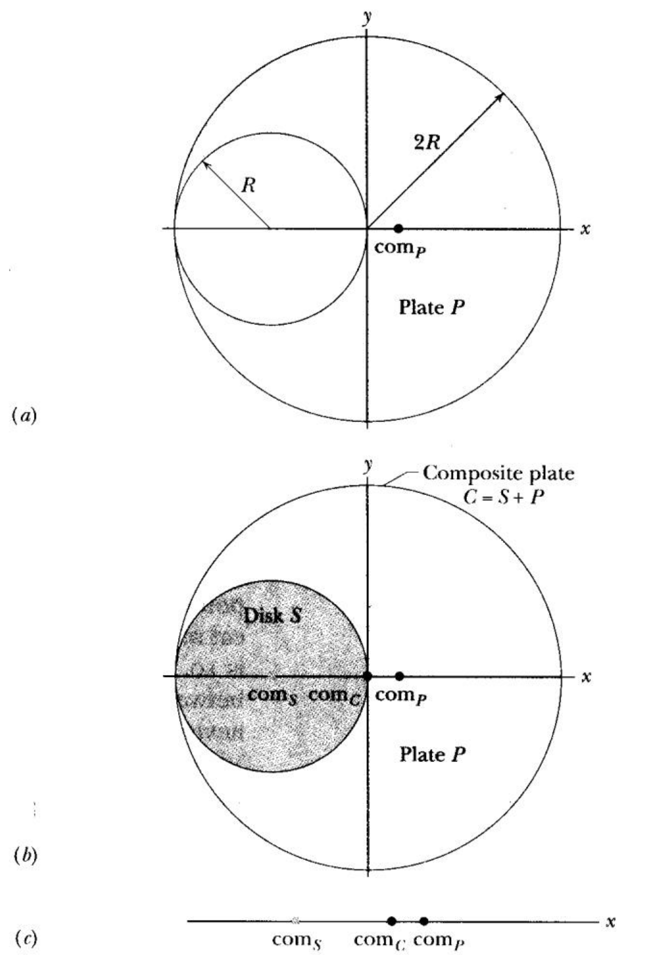


FIG. 9-4 (*a*) Plate *P* is a metal plate of radius 2*R*, with a circular hole of radius *R*. The center of mass of *P* is at point com$_P$. (*b*) Disk *S* has been put back into place to form a composite plate *C*. The center of mass com$_S$ of disk *S* and the center of mass com$_C$ of plate *C* are shown. (*c*) The center of mass com$_{S+P}$ of the combination of *S* and *P* coincides with com$_C$, which is at $x = 0$.

Newton's Second Law for a System of Particles

$$\vec{F}_{\text{net}} = M\vec{a}_{\text{cm}}.$$

In terms of components,

$$F_{\text{net},x} = Ma_{\text{cm},x},$$
$$F_{\text{net},y} = Ma_{\text{cm},y},$$
$$F_{\text{net},z} = Ma_{\text{cm},z}.$$

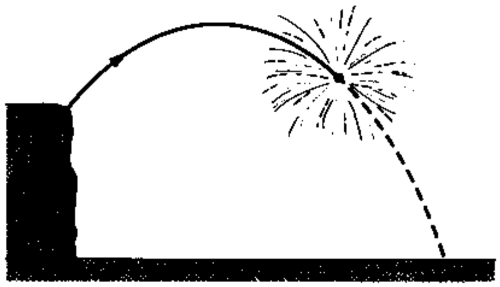
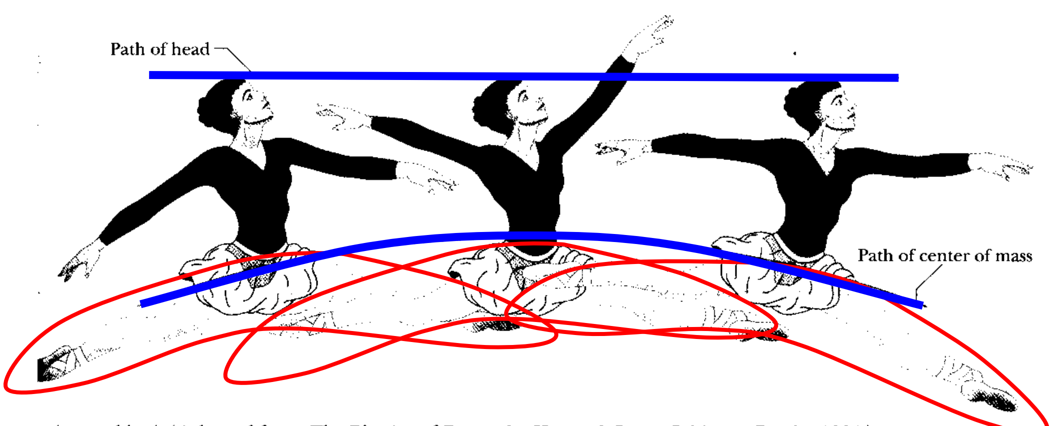


FIG. 9-5 A fireworks rocket explodes in flight. In the absence of air drag, the center of mass of the fragments would continue to follow the original parabolic path, until fragments began to hit the ground.



A grand jeté. (Adapted from *The Physics of Dance,* by Kenneth Laws, Schirmer Books, 1984.)

See Youtube "Zoe Ballet Grand Jeté".

Linear Momentum

For a single particle, the linear momentum is

$$\boxed{\vec{p} = m\vec{v}.}$$

Newton's law:

$$\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt}.$$

For a system of particles, the total linear momentum is

$$\vec{P} = \vec{p}_1 + \cdots + \vec{p}_n = m_1\vec{v}_1 + \cdots + m_n\vec{v}_n.$$

Differentiating the position of the center of mass,

$$M\vec{v}_{\text{cm}} = m_1\vec{v}_1 + \cdots + m_n\vec{v}_n.$$

$$\boxed{\vec{P} = M\vec{v}_{cm}.}$$

The linear momentum of a system of particles is equal to the product of the total mass *M* of the system and the velocity of the center of mass.

Newton's law for a system of particles:

$$\frac{d\vec{P}}{dt} = M\,\frac{d\vec{v}_{\text{cm}}}{dt} = M\vec{a}_{\text{cm}}.$$

Hence

$$\boxed{\vec{F}_{\text{net}} = \frac{d\vec{P}}{dt}.}$$

Collision and Impulse

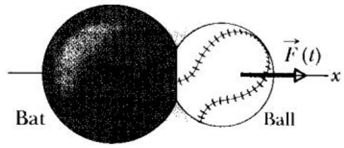


FIG. 9-8 Force $\vec{F}(t)$ acts on a ball as the ball and a bat collide.

Newton's law:

$$\vec{F} = \frac{d\vec{p}}{dt} \;\Rightarrow\; d\vec{p} = \vec{F}(t)dt.$$

Integrating from just before the collision to just afterwards,

$$\int_{\vec{p}_i}^{\vec{p}_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F}(t)dt.$$

Change in linear momentum during the collision:

$$\vec{p}_f - \vec{p}_i = \Delta\vec{p} = \int_{t_i}^{t_f} \vec{F}(t)dt.$$

The integral is a measure of both the strength and the duration of the collision force. It is called the **impulse J** of the collision:

$$\boxed{\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)dt.}$$

Impulse-linear momentum theorem:

$$\boxed{\vec{p}_f - \vec{p}_i = \Delta\vec{p} = \vec{J}.}$$

In terms of components,

$$p_{fx} - p_{ix} = \Delta p_x = J_x,$$
$$p_{fy} - p_{iy} = \Delta p_y = J_y,$$
$$p_{fz} - p_{iz} = \Delta p_z = J_z.$$

Average force:

$$J = \overline{F}\Delta t,$$

where Δt is the duration of the collision.

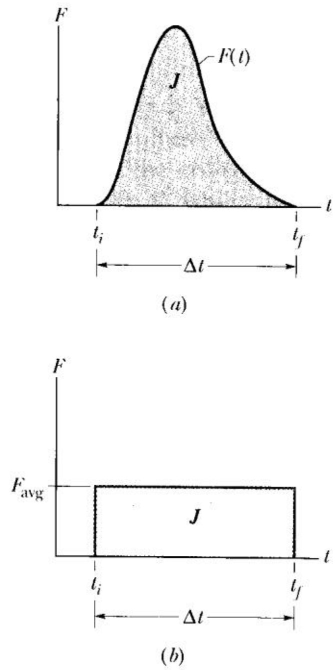


FIG. 9-9 (*a*) The curve shows the magnitude of the time-varying force $F(t)$ that acts on the ball in the collision of Fig. 9-8. The area under the curve is equal to the magnitude of the impulse $\vec{J}$ on the ball in the collision. (*b*) The height of the rectangle represents the average force F_{avg} acting on the ball over the time interval Δt. The area within the rectangle is equal to the area under the curve in (*a*) and thus is also equal to the magnitude of the impulse $\vec{J}$ in the collision.

Series of Collisions

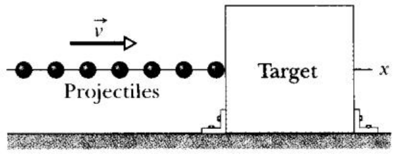


FIG. 9-10 A steady stream of projectiles, with identical linear momenta, collides with a target, which is fixed in place. The average force F_{avg} on the target is to the right and has a magnitude that depends on the rate at which the projectiles collide with the target or, equivalently, the rate at which mass collides with the target.

n particles collide with *R* in time interval Δt.
Total change in momentum = $n\Delta p$.

According to Newton's law,
average force acting on the particles
= rate of change of momentum

$$= \frac{n}{\Delta t}\Delta p = \frac{n}{\Delta t} m\Delta v.$$

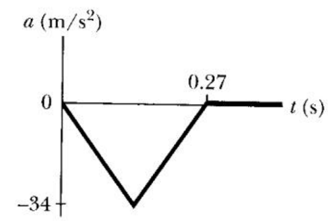
Hence the average force acting on body *R* is:

$$\overline{F} = -\frac{n}{\Delta t} m\Delta v.$$

If the colliding particles stop upon impact, $\Delta v = -v$.
If the colliding particles bounds elastically upon impact, $\Delta v = -2v$.

Examples

9-4 Fig. 9-11 shows the typical acceleration of a male bighorn sheep when he runs head-first into another male. Assume that the sheep's mass is 90.0 kg. What are the magnitudes of the impulse and average force due to the collision?



11 The acceleration
time of a bighorn sheep
a collision with another

Since $a = \dfrac{dv}{dt}$,

$$\Delta v(t) = \int_0^t a(t')dt' \quad .$$

Hence the change in velocity is given by the area enclosed by the *a*-*t* curve.

$$\Delta v = -\frac{1}{2}(0.27)(34) = -4.59\ \mathrm{ms}^{-1}$$

Using the impulse-momentum theorem,

$$J = \Delta p = m\Delta v = (90)(-4.59) = -413\ \mathrm{kg\ ms}^{-1}$$

The magnitude of the impulse is 413 kg ms^{-1}. (ans)

The magnitude of the average force is

$$F_{\mathrm{av}} = \frac{|J|}{\Delta t} = \frac{413}{0.27} = 1530\ \mathrm{N} \qquad \text{(ans)}$$

Remark: The collision time is prolonged by the flexibility of the horns. If the sheep were to hit skull-to-skull or skull-to-horn, the collision duration would be 1/10 of what we used, and the average force would be 10 times of what we calculated!

9-5 Race-car wall collision. A race car collides with a racetrack wall at speeds v_i = 70 ms^{-1} and v_f = 50 ms^{-1} before and after collision respectively (Fig. 9-12a). His mass m is 80 kg.
(a) What is the impulse $\vec{J}$ on the driver due to the collision?
(b) The collision lasts for 14 ms. What is the magnitude of the average force on the driver during the collision?

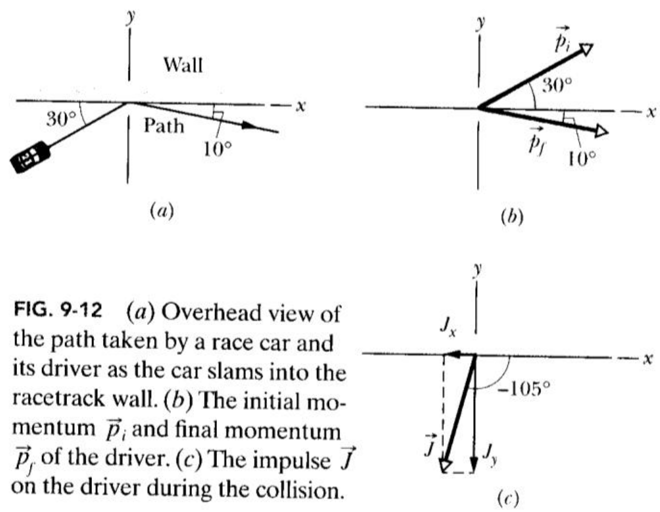


FIG. 9-12 (a) Overhead view of the path taken by a race car and its driver as the car slams into the racetrack wall. (b) The initial momentum $\vec{p}_i$ and final momentum $\vec{p}_f$ of the driver. (c) The impulse $\vec{J}$ on the driver during the collision.

(a) Using the impulse-momentum theorem,

$$\vec{J} = \vec{p}_f - \vec{p}_i = m\vec{v}_f - m\vec{v}_i$$

$$= m(\vec{v}_f - \vec{v}_i)$$

x component:

$$J_x = m(v_{fx} - v_{ix})$$

$$= (80)(50\cos 10^\circ - 70\cos 30^\circ)$$

= –910 kg ms^{-1}

y component:

$$J_y = m(v_{fy} - v_{iy}) = (80)(-50\sin 10^\circ - 70\sin 30^\circ)$$

= –3495 kg ms^{-1}

The impulse is then $\vec{J} = (-910\hat{i} - 3500\hat{j})\ \text{kg ms}^{-1}$ (ans)

Magnitude: $J = \sqrt{J_x^2 + J_y^2} = 3616\ \text{kg ms}^{-1} \approx 3600\ \text{kg ms}^{-1}$

Direction: $\theta = \tan^{-1}\dfrac{J_y}{J_x} = 75.4^\circ - 180^\circ = -105^\circ$ (ans)

(b) $F_{\text{av}} = \dfrac{J}{\Delta t} = \dfrac{3616}{0.014} = 2.583 \times 10^5\ \text{N} \approx 2.6 \times 10^5\ \text{N}$ (ans)

Remark: The driver's average acceleration is $2.583 \times 10^5/80 \approx 3220$ ms^{-2} = 329g – fatal collision!

Conservation of Linear Momentum

If the system of particles is isolated (i.e. there are no external forces) and closed (i.e. no particles leave or enter the system), then

$$\boxed{\vec{P} = \text{constant.}}$$

Law of conservation of linear momentum:

$$\vec{P}_i = \vec{P}_f .$$

Examples

9-7 Imagine a spaceship and cargo module, of total mass M, travelling in deep space with velocity v_i = 2100 km/h relative to the Sun. With a small explosion, the ship ejects the cargo module, of mass 0.20M. The ship then travels 500 km/h faster than the module; that is, the relative speed v_{rel} between the module and the ship is 500 km/h. What then is the velocity v_f of the ship relative to the Sun?

Using the conservation of linear momentum,

$$P_i = P_f$$

$$Mv_i = 0.2M(v_f - v_{\text{rel}}) + 0.8Mv_f$$

$$v_i = v_f - 0.2v_{\text{rel}}$$

$$v_f = v_i + 0.2v_{\text{rel}}$$

= 2100 + (0.2)(500)

= 2200 km/h (ans)

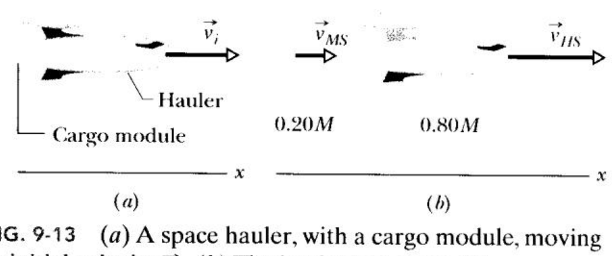


FIG. 9-13 (a) A space hauler, with a cargo module, moving at initial velocity $\vec{v}_i$. (b) The hauler has ejected the cargo module. Now the velocities relative to the Sun are $\vec{v}_{MS}$ for the module and $\vec{v}_{HS}$ for the hauler.

9-8 A firecracker placed inside a coconut of mass *M*, initially at rest on a frictionless floor, blows the fruit into three pieces and sends them sliding across the floor. An overhead view is shown in the figure. Piece *C*, with mass 0.30*M*, has final speed v_{fc}=5.0ms^{-1}.

(a) What is the speed of piece *B*, with mass 0.20*M*?

(b) What is the speed of piece *A*?

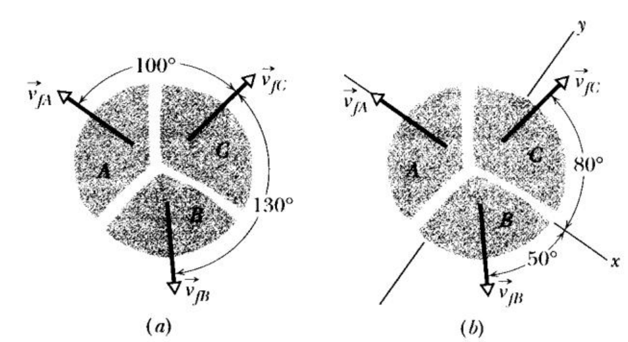


FIG. 9-14 Three pieces of an exploded coconut move off in three directions along a frictionless floor. (*a*) An overhead view of the event. (*b*) The same with a two-dimensional axis system imposed.

(a) Using the conservation of linear momentum,

$$P_{ix} = P_{fx}$$

$$P_{iy} = P_{fy}$$

$$m_C v_{fC} \cos 80^\circ + m_B v_{fB} \cos 50^\circ - m_A v_{fA} = 0 \qquad (1)$$

$$m_C v_{fC} \sin 80^\circ - m_B v_{fB} \sin 50^\circ = 0 \qquad (2)$$

$m_A = 0.5M$, $m_B = 0.2M$, $m_C = 0.3M$.

(2): $0.3Mv_{fC} \sin 80^\circ - 0.2Mv_{fB} \sin 50^\circ = 0$

$$v_{fB} = \left(\frac{0.3\sin 80^\circ}{0.2\sin 50^\circ}\right)5 = 9.64\text{ ms}^{-1} \approx 9.6\text{ ms}^{-1} \qquad \text{(ans)}$$

(b) (1): $0.3Mv_{fC} \cos 80^\circ + 0.2Mv_{fB} \cos 50^\circ = 0.5Mv_{fA}$

$$v_{fA} = \frac{(0.3)(5)\cos 80^\circ + (0.2)(9.64)\cos 50^\circ}{0.5} = 3.0\text{ ms}^{-1} \qquad \text{(ans)}$$

Inelastic Collisions in One Dimension

In an inelastic collision, the kinetic energy of the system of colliding bodies is not conserved.

In a completely inelastic collision, the colliding bodies stick together after the collision.

However, *the conservation of linear momentum still holds*.

$$m_1 v = (m_1 + m_2)V, \quad \text{or} \quad V = \frac{m_1}{m_1 + m_2} v.$$

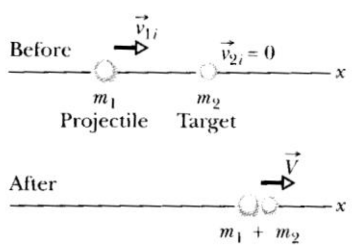


FIG. 9-16 A completely inelastic collision between two bodies. Before the collision, the body with mass m_2 is at rest and the body with mass m_1 moves directly toward it. After the collision, the stuck-together bodies move with the same velocity $\vec{V}$.

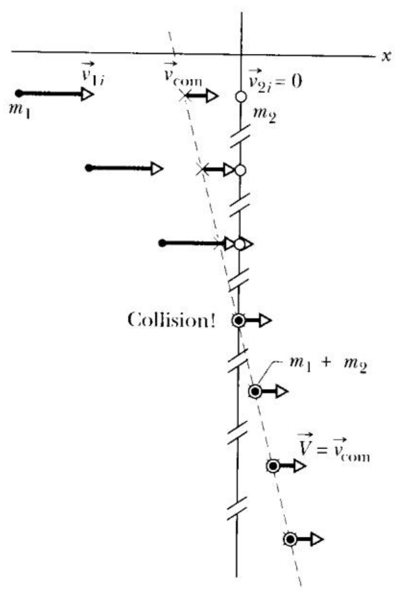


FIG. 9-17 Some freeze-frames of the two-body system in Fig. 9-16, which undergoes a completely inelastic collision. The system's center of mass is shown in each freeze-frame. The velocity $\vec{v}_{com}$ of the center of mass is unaffected by the collision. Because the bodies stick together after the collision, their common velocity $\vec{V}$ must be equal to $\vec{v}_{com}$.

Example

9-9 The *ballistic pendulum* was used to measure the speeds of bullets before electronic timing devices were developed. Here it consists of a large block of wood of mass M = 5.4 kg, hanging from two long cords. A bullet of mass m = 9.5 g is fired into the block, coming quickly to rest. The *block + bullet* then swing upward, their center of mass rising a vertical distance h = 6.3 cm before the pendulum comes momentarily to rest at the end of its arc. What was the speed v of the bullet just prior to the collision?

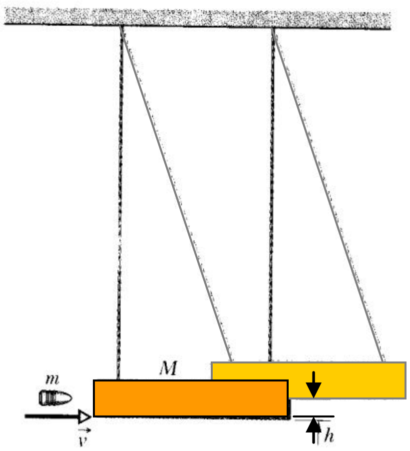


FIG. 9-18 A ballistic pendulum, used to measure the speeds of bullets.

Using the conservation of momentum during collision,

$$mv = (M + m)V \quad (1)$$

Using the conservation of energy after collision,

$$\frac{1}{2}(M + m)V^2 = (M + m)gh \quad (2)$$

$$V = \sqrt{2gh}$$

(1): $v = \frac{M + m}{m}V = \frac{M + m}{m}\sqrt{2gh}$

$$= \frac{5.4 + 0.0095}{0.0095}\sqrt{(2)(9.8)(0.063)} = 630\,\text{ms}^{-1} \quad \text{(ans)}$$

9-10 Consider the collision of cars 1 and 2 with initial velocities v_{1i} = +25 ms^{-1} and v_{2i} = –25 ms^{-1} respectively. Let each car carry one driver. The total mass of cars 1 and 2 are m_1 = 1400 kg and m_2 = 1400 kg respectively.

(a) What are the changes Δv_1 and Δv_2 during their head-on and completely inelastic collision?

(b) Repeat the calculation with an 80 kg passenger in car 1.

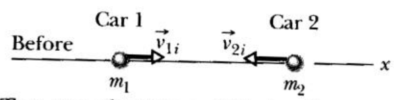


FIG. 9-19 Two cars about to collide head-on.

(a) Using the conservation of momentum,

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

Since the collision is completely inelastic, $v_{1f} = v_{2f} = V$.

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2)V$$

$$V = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2}$$

$$= \frac{(1400)(+25) + (1400)(-25)}{1400 + 1400} = 0$$

$$\Delta v_1 = v_{1f} - v_{1i} = 0 - (+25) = -25\,\text{ms}^{-1} \quad \text{(ans)}$$

$$\Delta v_2 = v_{2f} - v_{2i} = 0 - (-25) = +25\,\text{ms}^{-1} \quad \text{(ans)}$$

(b) In this case, m_1 is replaced by 1480 kg.

$$V = \frac{(1480)(+25) + (1400)(-25)}{1480 + 1400} = 0.694\,\text{ms}^{-1}$$

$$\Delta v_1 = v_{1f} - v_{1i} = 0.694 - (+25) = -24.3\,\text{ms}^{-1} \quad \text{(ans)}$$

$$\Delta v_2 = v_{2f} - v_{2i} = 0.694 - (-25) = +25.7\,\text{ms}^{-1} \quad \text{(ans)}$$

Remark: The risk of fatality to a driver is less if that driver has a passenger in the car!

Elastic Collisions in One Dimension

Stationary Target

In an elastic collision, the kinetic energy of each colliding body can change, but the total kinetic energy of the system does not change.

In a closed, isolated system, the linear momentum of each colliding body can change, but the net linear momentum cannot change, regardless of whether the collision is elastic.

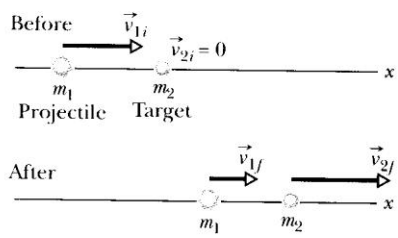


FIG. 9-20 Body 1 moves along an x axis before having an elastic collision with body 2, which is initially at rest. Both bodies move along that axis after the collision.

Conservation of linear momentum:

$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f}.$$

Conservation of kinetic energy:

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2.$$

Rewriting these equations as

$$m_1(v_{1i} - v_{1f}) = m_2 v_{2f},$$
$$m_1(v_{1i}^2 - v_{1f}^2) = m_2 v_{2f}^2.$$

Dividing,

$$v_{1i} + v_{1f} = v_{2f}.$$

We have two linear equations for v_{1f} and v_{2f}. Solution:

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i},$$
$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}.$$

Special situations:

1. *Equal masses:* If $m_1 = m_2$, then $v_{1f} = 0$ and $v_{2f} = v_{1i}$ (pool player's result).

2. *A massive target:* If $m_2 >> m_1$, then

$$v_{1f} \approx -v_{1i} \quad \text{and} \quad v_{2f} = \left(\frac{2m_1}{m_2}\right) v_{1i} << v_{1i}.$$

The light incident particle bounces back and the heavy target barely moves.

3. *A projectile:* If $m_1 >> m_2$, then

$$v_{1f} \approx v_{1i} \quad \text{and} \quad v_{2f} \approx 2v_{1i}.$$

The incident particle is scarcely slowed by the collision.

Example

9-11 Two metal spheres, suspended by vertical cords, initially just touch. Sphere 1, with mass m_1 = 3 g, is pulled to the left to height h_1 = 8.0 cm, and then released. After swinging down, it undergoes an elastic collision with sphere 2, whose mass m_2 = 75 g. What is the velocity v_{1f} of sphere 1 just after the collision?

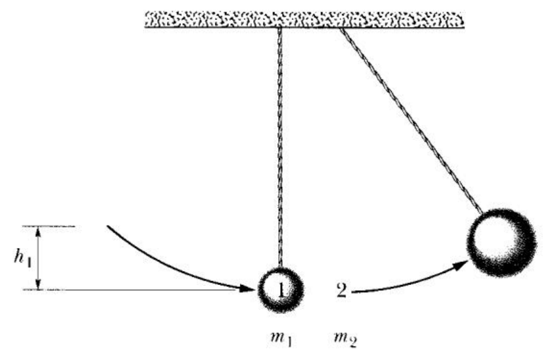


FIG. 9-22 Two metal spheres suspended by cords just touch when they are at rest. Sphere 1, with mass m_1, is pulled to the left to height h_1 and then released.

Using the conservation of energy,

$$\frac{1}{2}m_1 v_{1i}^2 = m_1 g h_1$$

$$v_{1i} = \sqrt{2gh_1}$$

$$= \sqrt{(2)(9.8)(0.08)}$$

$$= 1.252 \text{ ms}^{-1}$$

Using the conservation of momentum,

$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f}$$

For elastic collisions,

$$\frac{1}{2}m_1 v_{1i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2$$

$$m_1(v_{1i} - v_{1f}) = m_2 v_{2f} \qquad (1)$$

$$m_1(v_{1i}^2 - v_{1f}^2) = m_2 v_{2f}^2 \qquad (2)$$

Dividing, $v_{1i} + v_{1f} = v_{2f}$

$$m_1(v_{1i} - v_{1f}) = m_2(v_{1i} + v_{1f})$$

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} = \left(\frac{0.03 - 0.075}{0.03 + 0.075}\right) 1.252$$

$= -0.537 \text{ ms}^{-1} \approx -0.54 \text{ ms}^{-1}$ (ans)

Collisions in Two Dimensions

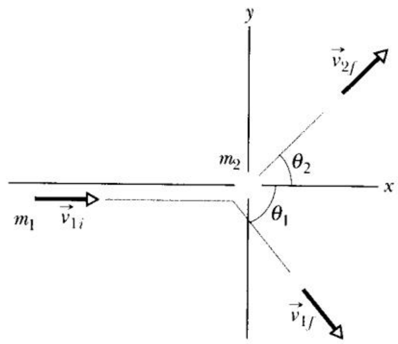


FIG. 9-23 An elastic collision between two bodies in which the collision is not head-on. The body with mass m_2 (the target) is initially at rest.

Conservation of linear momentum:

x component: $m_1 v_{1i} = m_1 v_{1f} \cos\theta_1 + m_2 v_{2f} \cos\theta_2,$

y component: $0 = -m_1 v_{1f} \sin\theta_1 + m_2 v_{2f} \sin\theta_2.$

Conservation of kinetic energy:

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 .$$

Typically, we know m_1, m_2, v_{1i} and θ_1. Then we can solve for v_{1f}, v_{2f} and θ_2.

Systems with Varying Mass: A Rocket

Assume no gravity. Conservation of linear momentum:

$$P_i = P_f .$$

Initial momentum = Mv
Final momentum of the exhaust
= $(-dM)U$
Final momentum of the rocket
= $(M + dM)(v + dv)$

$$Mv = (-dM)U + (M + dM)(v + dv).$$

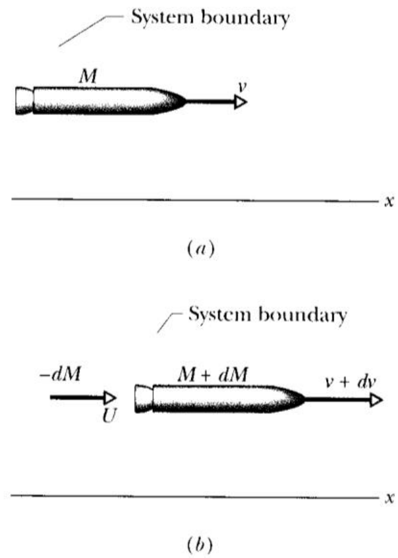


FIG. 9-24 (*a*) An accelerating rocket of mass *M* at time *t*, as seen from an inertial reference frame. (*b*) The same but at time *t* + *dt*. The exhaust products released during interval *dt* are shown.

Suppose the rocket ejects the exhaust at a velocity v_{rel}.

$$U = v + dv - v_{\text{rel}}.$$

Substituting and dividing by dt,

$$-dMv_{\text{rel}} = Mdv,$$
$$-\frac{dM}{dt}v_{\text{rel}} = M\frac{dv}{dt}.$$

Since the rate of fuel consumption is $R = -\dfrac{dM}{dt}$, we have the first rocket equation:

$$Rv_{\text{rel}} = Ma.$$

$T \equiv Rv_{\text{rel}}$ is called the **thrust** of the rocket engine. Newton's second law emerges. To find the velocity,

$$dv = -v_{\text{rel}} \frac{dM}{M},$$

Integrating,

$$\int_{v_i}^{v_f} dv = -v_{\text{rel}} \int_{M_i}^{M_f} \frac{dM}{M},$$

$$v_f - v_i = -v_{\text{rel}} \ln \frac{M_i}{M_f}. \qquad \text{(second rocket equation)}$$

Remark: Multistage rockets are used to reduce M_f in stages.

Example

A rocket with initial mass M_i = 850 kg consumes fuel at the rate R = 2.3 kgs^{-1}. The speed v_{rel} of the exhaust gases relative to the rocket engine is 2800 ms^{-1}. What thrust does the rocket engine provide? What is the initial acceleration of the rocket?

$T = Rv_{\text{rel}}$ = (2.3)(2800) = 6440 N ≈ 6400 N (ans)

$$a = \frac{T}{M} = \frac{6440}{850} = 7.6\,\text{ms}^{-2} \qquad \text{(ans)}$$

Remark: Since $a < g$, the rocket cannot be launched from Earth's surface.