Conservation of Energy

Reading: Chapter 8

Potential Energy

The energy associated with the configuration (or arrangement) of a system of objects that exert a force on one another.
e.g. Gravitational potential energy – associated with the state of separation between objects, which attract one another via the gravitational force.
e.g. elastic potential energy – associated with the state of compression or extension of an elastic object.

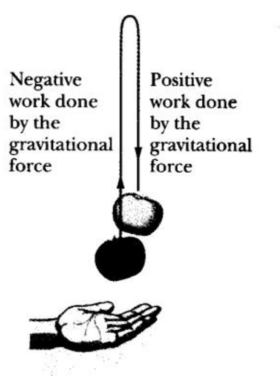


FIG. 8-2 A tomato is thrown upward. As it rises, the gravitational force does negative work on it, decreasing its kinetic energy. As the tomato descends, the gravitational force does positive work on it, increasing its kinetic energy.

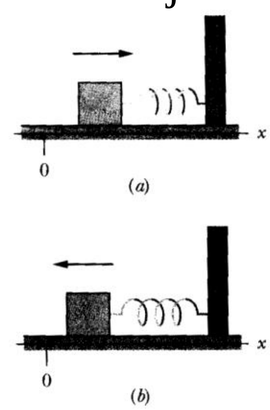


FIG. 8-3 A block, attached to a spring and initially at rest at $x = 0$, is set in motion toward the right. (a) As the block moves rightward (as indicated by the arrow), the spring force does negative work on it. (b) Then, as the block moves back toward $x = 0$, the spring force does positive work on it.

The change ΔU in the gravitational potential energy
= the work done by the applied force
= the negative of the work done by the gravitational force.

$$\boxed{\Delta U = -W.}$$

Conservative and Nonconservative Forces

The net work done by a conservative force on a particle moving around any closed path is zero.

e.g. of conservative force: gravitational force, spring force
e.g. of nonconservative force: frictional force

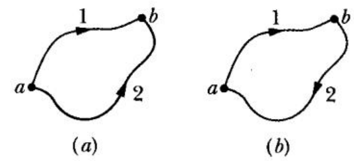


FIG. 8-4 (*a*) As a conservative force acts on it, a particle can move from point *a* to point *b* along either path 1 or path 2. (*b*) The particle moves in a round trip, from point *a* to point *b* along path 1 and then back to point *a* along path 2.

Since

$$W_{ab,1} + W_{ba,2} = 0,$$

we have

$$W_{ab,1} = -W_{ba,2} = W_{ab,2}.$$

The work done by a conservative force on a particle moving between two points *does not depend on the path* taken by the particle.

Determining Potential Energy

Work done by the force:

$$W = \int_{x_i}^{x_f} F(x)dx.$$

Hence the change in potential energy is:

$$\Delta U = -W = -\int_{x_i}^{x_f} F(x)dx.$$

Gravitational Potential Energy

$$\Delta U = -\int_{y_i}^{y_f} (-mg)dy \quad = \Big[\quad mgy \quad \Big]_{y_i}^{y_f},$$

which yields

$$\Delta U = mg(y_f - y_i) = mg\Delta y.$$

Choosing the gravitational potential energy to be $U_i = 0$ at the reference point y_i, we obtain

$$\boxed{U = mgy.}$$

Elastic Potential Energy

$$\Delta U = -\int_{x_i}^{x_f} (-kx)dx = \left[\tfrac{1}{2}kx^2\right]_{x_i}^{x_f},$$

which yields

$$\Delta U = \tfrac{1}{2}kx_f^2 - \tfrac{1}{2}kx_i^2.$$

Choosing the spring potential energy to be $U_i = 0$ at the reference point $x_i = 0$, which is the equilibrium position of the system, we obtain

$$\boxed{U(x) = \tfrac{1}{2}kx^2.}$$

Conservation of Mechanical Energy

Mechanical energy

$$E_{\mathrm{mec}} = K + U.$$

When a conservative force does work W on an object, it transfers kinetic energy to the object:

$$\Delta K = W.$$

The change in potential energy is:

$$\Delta U = -W.$$

Combining,

$$\Delta K = -\Delta U,$$

or

$$\Delta E = \Delta K + \Delta U = 0.$$

Principle of conservation of mechanical energy – When only conservative forces act within a system, the kinetic energy and potential energy can change. However, their sum, the mechanical energy E of the system, does not change.

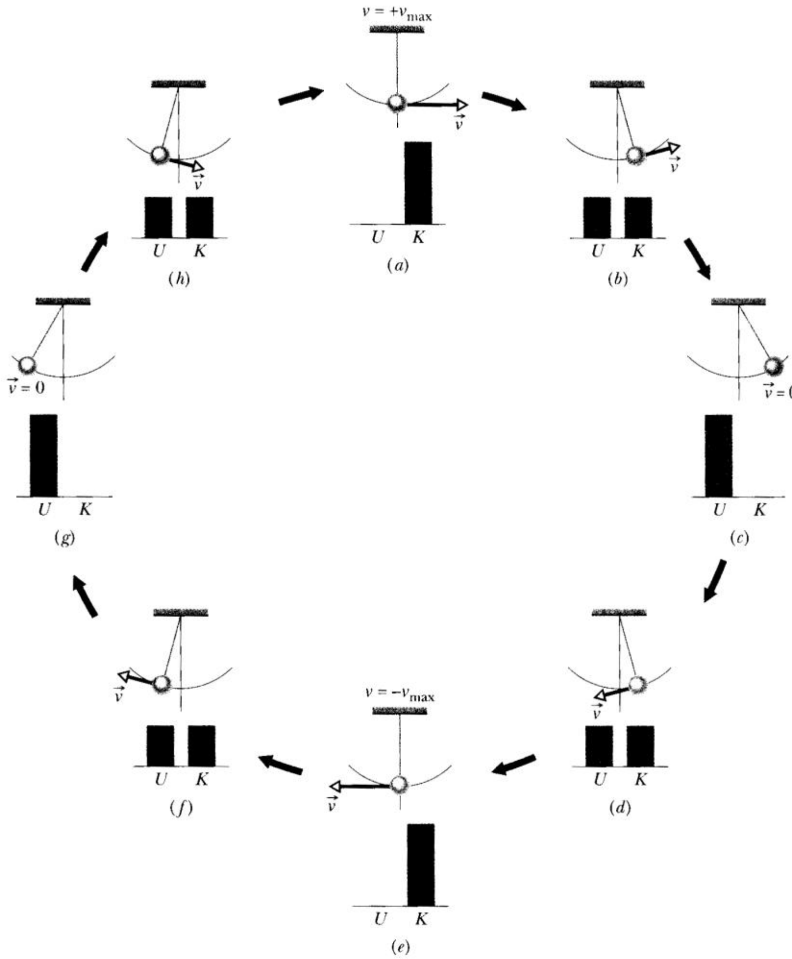


FIG. 8-7 A pendulum, with its mass concentrated in a bob at the lower end, swings back and forth. One full cycle of the motion is shown. During the cycle the values of the potential and kinetic energies of the pendulum–Earth system vary as the bob rises and falls, but the mechanical energy E_{mec} of the system remains constant. The energy E_{mec} can be described as continuously shifting between the kinetic and potential forms. In stages (a) and (e), all the energy is kinetic energy. The bob then has its greatest speed and is at its lowest point. In stages (c) and (g), all the energy is potential energy. The bob then has zero speed and is at its highest point. In stages (b), (d), (f), and (h), half the energy is kinetic energy and half is potential energy. If the swinging involved a frictional force at the point where the pendulum is attached to the ceiling, or a drag force due to the air, then E_{mec} would not be conserved, and eventually the pendulum would stop.

See demonstration “The Interrupted Pendulum”.
See Youtube “Conceptual Physics Conservation of Energy”

Application – When the mechanical energy of a system is conserved, we can relate the total of kinetic energy and potential energy at one instant to that at another instant *without considering the intermediate motion and without finding the work done by the forces involved*.

Example

8-3 A child of mass m is released from rest at the top of a water slide, at height h = 8.5 m above the bottom of the slide. Assuming that the slide is frictionless because of the water on it, find the child's speed at the bottom of the slide.

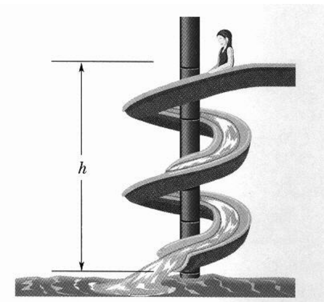


FIG. 8-8 A child slides down a water slide as she descends a height h.

Since the normal force does not do work on the child, energy is conserved.

$$K_b + U_b = K_t + U_t$$

$$\frac{1}{2}mv_b^2 + mgy_b = \frac{1}{2}mv_t^2 + mgy_t$$

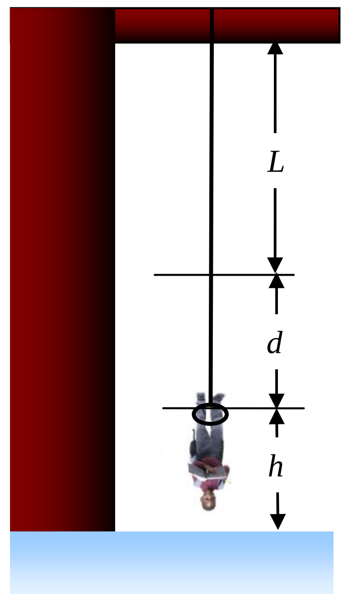
$$v_b^2 = v_t^2 + 2g(y_t - y_b)$$

Since $v_t = 0$, $y_t - y_b = h$, we have

$$v_b^2 = 2gh$$

$$v_b = \sqrt{2gh} = \sqrt{2(9.8)(8.5)} = 13\,\mathrm{ms^{-1}} \qquad \text{(ans)}$$

A 61.0 kg bungee-cord jumper is on a bridge 45.0 m above a river. In its relaxed state, the elastic bungee cord has length L = 25.0 m. Assume that the cord obeys Hooke's law, with a spring constant of 160 Nm^{-1}.
(a) If the jumper stops before reaching the water, what is the height h of his feet above the water at his lowest point?
(b) What is the net force on him at his lowest point (in particular, is it zero)?



(a) Using the conservation of energy,

$$\Delta K + \Delta U_e + \Delta U_g = 0$$

$$\Delta K = 0$$

$$\Delta U_e = \frac{1}{2}kd^2$$

$$\Delta U_g = -mg(L+d)$$

Hence

$$0 + \frac{1}{2}kd^2 - mg(L+d) = 0$$

$$\frac{1}{2}kd^2 - mgd - mgL = 0$$

$$d = \frac{mg \pm \sqrt{m^2g^2 + 2kmgL}}{k} = \frac{mg}{k} \pm \sqrt{\left(\frac{mg}{k}\right)^2 + 2L\left(\frac{mg}{k}\right)}$$

$$\frac{mg}{k} = \frac{(61)(9.8)}{160} = 3.7363$$

$$d = 3.7363 \pm \sqrt{3.7363^2 + 2(25)(3.7363)} = 17.9 \text{ or } -10.4$$

$$d = 17.9\,\text{m} \qquad \text{(ans)}$$

(b) Force = $-k(-d) - mg$ = (160)(17.9) – (61)(9.8)
= 2270 N (ans)

Reading a Potential Energy Curve

From force to potential energy: $\Delta U = -\int_{x_i}^{x_f} F(x)dx.$

From potential energy to force: $F(x) = -\dfrac{dU(x)}{dx}.$

e.g. spring: $U(x) = \dfrac{1}{2}kx^2$ yields $F(x) = -kx.$

e.g. gravitation: $U(x) = mgx$ yields $F(x) = -mg.$

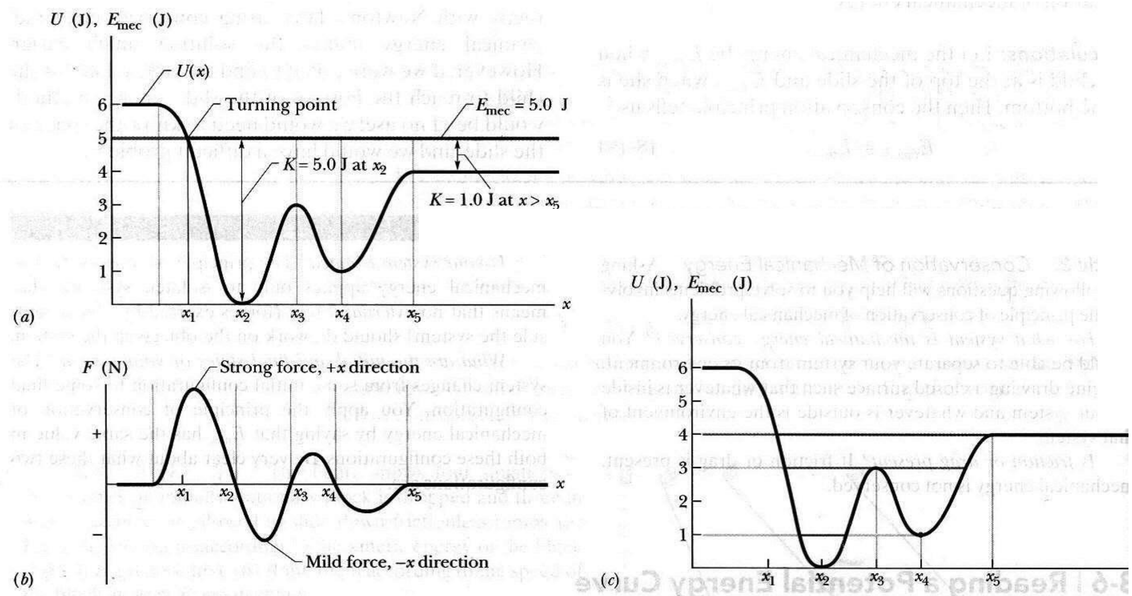


FIG. 8-9 (*a*) A plot of $U(x)$, the potential energy function of a system containing a particle confined to move along an x axis. There is no friction, so mechanical energy is conserved. (*b*) A plot of the force $F(x)$ acting on the particle, derived from the potential energy plot by taking its slope at various points. (*c*) The $U(x)$ plot of (*a*) with three possible values of E_{mec} shown.

Turning Points

In the potential energy curve, since $U(x) + K(x) = E$,

$$K(x) = E - U(x).$$

Since $K(x) = \frac{1}{2}mv^2$, it can never be negative. Hence the particle can never move to the left of x_1.
At x_1, dU/dx is negative, hence the force on the particle is positive, and the particle will turn back and move to the right. x_1 is called a **turning point**.

Equilibrium Points – Positions where no forces act on the particle, i.e. $U(x)$ has zero slope.

Types of Equilibrium:

Stable equilibrium – If slightly displaced, a restoring force appears and the particle returns to the original position. They correspond to the minima in $U(x)$.
e.g. when E = 1 J and $x = x_4$.
Unstable equilibrium – If slightly displaced, a force pushes it further away from the original position. They correspond to the maxima in $U(x)$.
e.g. when E = 3 J and $x = x_3$.
Neutral equilibrium – If slightly displaced, no forces act on the particle and it remains there.
e.g. when E = 4 J and x is beyond x_5.

Types of Motion:

Equilibrium e.g. when E = 0 J.
Bounded motion e.g. when E = 1 J.
e.g. when E = 2 J, the motion may be bounded in the left or the right valley, depending on the initial condition.
Unbounded motion e.g. when E = 5 J.

~~~~~~~~~~~~~~~

Work Done by an External Force

Case 1: No Friction Involved

Consider the work done in pushing a ball vertically upward.

$$W_a + W_g = \Delta K.$$

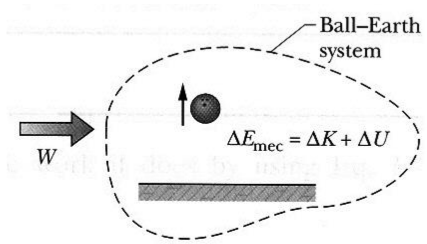


FIG. 8-12 Positive work W is done on a system of a bowling ball and Earth, causing a change ΔE_{mec} in the mechanical energy of the system, a change ΔK in the ball's kinetic energy, and a change ΔU in the system's gravitational potential energy.

Since $W_g = -\Delta U,$ we have

$$W_a = \Delta K + \Delta U.$$

Hence the work-energy theorem becomes

$$\boxed{W_a = \Delta E_{\text{mec}}.}$$

The work done on a system is equal to the change in the mechanical energy.
~~~~~~~~~~~~~~~

Case 2: Friction Involved

Consider the sliding motion of the block pulled by an external force. Using Newton's law of motion,

$$F - f_k = ma.$$

Since a is constant,

$$v^2 = v_0^2 + 2ad.$$

Eliminating a, we have

$$Fd = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 + f_k d.$$
$$Fd = \Delta E_{\text{mec}} + f_k d.$$

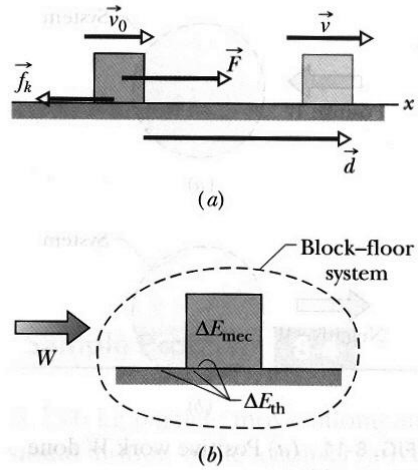


FIG. 8-13 (*a*) A block is pulled across a floor by force $\vec{F}$ while a kinetic frictional force $\vec{f}_k$ opposes the motion. The block has velocity $\vec{v}_0$ at the start of a displacement $\vec{d}$ and velocity $\vec{v}$ at the end of the displacement. (*b*) Positive work W is done on the block–floor system by force $\vec{F}$, resulting in a change ΔE_{mec} in the block's mechanical energy and a change ΔE_{th} in the thermal energy of the block and floor.

The work done against friction is $f_k d$. Usually it is converted to the thermal energy of the object and its environment. The change in the thermal energy is

$$\Delta E_{\text{th}} = f_k d.$$

Then we can write

$$Fd = \Delta E_{\text{mec}} + \Delta E_{\text{th}}.$$
$$\boxed{W = \Delta E_{\text{mec}} + \Delta E_{\text{th}}.}$$

Example

8-6 Statues of Easter Island were most likely moved by cradling them in a wooden sled and pulling them over a "runway" of roller logs. In a modern reenactment of this technique, 25 men were able to move a 9000 kg statue 45 m over level ground in 2 min. Suppose each men pulled with a force of 1400 N.

(a) Estimate the work done by the men.

(b) What is the increase ΔE_{th} in the thermal energy of the system during the 45 m displacement?

(a) $W = Fd\cos\phi$

$= (25)(1400)(45)\cos 0^{o}$

$= 1.575\times 10^{6}\ \text{J} \approx 1.6\ \text{MJ}$ (ans)

(b) $W = \Delta E_{\text{mec}} + \Delta E_{\text{th}}$

Since $\Delta E_{\text{mec}} = 0$,

$\Delta E_{\text{th}} = W = 1.575\times 10^{6}\ \text{J} \approx 1.6\ \text{MJ}$ (ans)

Conservation of Energy

Isolated System

The total energy E of an isolated system cannot change.

$$\boxed{\Delta E_{\text{tot}} = \Delta E_{\text{mec}} + \Delta E_{\text{th}} + \Delta E_{\text{int}} = 0.}$$

Here, $\Delta E_{\text{mec}} = \Delta K + \Delta U$ is any change in the mechanical energy of the system,
ΔE_{th} is any change in the thermal energy of the system,
ΔE_{int} is any change in any other type of the internal energy of the system.

Summary: In an isolated system, energy can be transferred from one type to another, but the total energy of the system remains constant.

Empowerment: In an isolated system, we can relate the total energy at one instant to the total energy at another instant, *without considering the energies at intermediate times*.

If the system is not isolated, external forces are present to transfer energy to or from the system, then

$$\boxed{W = \Delta E_{\text{tot}} = \Delta E_{\text{mec}} + \Delta E_{\text{th}} + \Delta E_{\text{int}}.}$$

Examples

8-7 A 2.0 kg package of tamales slides along a floor with speed v_1 = 4.0 ms^{-1}. It then runs into and compresses a spring, until the package momentarily stops. Its path to the initially relaxed spring is frictionless, but as it compresses the spring, a kinetic frictional force from the floor, of magnitude 15 N, acts on the package. If k = 10,000 Nm^{-1}, by what distance d is the spring compressed when the package stops?

Using the conservation of energy,

$$\Delta E_{\text{mec}} + \Delta E_{\text{th}} = 0$$

$$\Delta E_{\text{mec}} = \Delta K + \Delta U$$

$$\Delta K = 0 - \frac{1}{2}mv^2$$

$$\Delta U = \frac{1}{2}kd^2 - 0$$

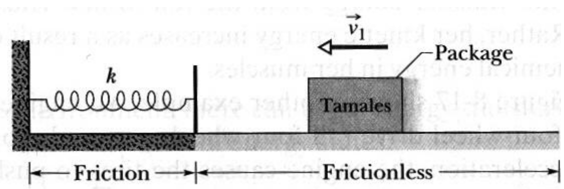


FIG. 8-18 A package slides across a frictionless floor with velocity $\vec{v}_1$ toward a spring of spring constant k. When the package reaches the spring, a frictional force from the floor acts on the package.

Since the change in the thermal energy comes from the work done by the moving package against friction,

$$\Delta E_{\text{th}} = f_k d$$

Therefore,

$$\frac{1}{2}kd^2 - \frac{1}{2}mv^2 + f_k d = 0$$

$$\left(\frac{1}{2}\right)(10{,}000)d^2 - \left(\frac{1}{2}\right)(2)(4)^2 + 15d = 0$$

$$5000d^2 + 15d - 16 = 0$$

$$d = \frac{-15 \pm \sqrt{15^2 + 320{,}000}}{10{,}000} = 0.055 \text{ or} -0.058$$

$$d = 0.055\,\text{m} = 5.5\,\text{cm} \qquad \text{(ans)}$$

8-8 During a rock avalanche on a mountain slope, the rocks, of total mass m, fall from a height $y = H$, move a distance d_1 along a slope of angle $\theta = 45^{\circ}$, and then move a distance d_2 along a flat valley. What is the ratio d_2/H of the runout to the fall height if the coefficient of kinetic friction has the reasonable value of 0.60?

Using the conservation of energy,

$$\Delta E_{\text{mec}} + \Delta E_{\text{th}} = 0$$

$$\Delta E_{\text{mec}} = \Delta K + \Delta U$$

$$\Delta K = 0$$

$$\Delta U = 0 - mgH$$

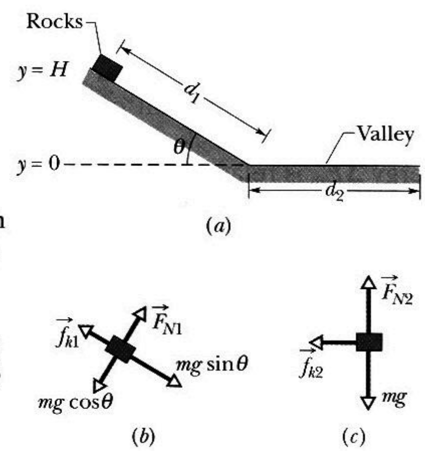


FIG. 8-19 (*a*) The path a rock avalanche takes down a mountainside and across a valley floor. The forces on the rock material along (*b*) the mountainside and (*c*) the valley floor.

Since the change in the thermal energy comes from the work done by the rocks against friction,

$$\Delta E_{\text{th}} = f_{k1} d_1 + f_{k2} d_2$$

where $f_{k1} = \mu_k mg \cos\theta$, $f_{k2} = \mu_k mg$, and $d_1 = H / \sin\theta$.

Therefore,

$$-mgH + \mu_k mg \cos\theta \frac{H}{\sin\theta} + \mu_k mg d_2 = 0$$

$$\mu_k d_2 = H - \mu_k \cos\theta \frac{H}{\sin\theta}$$

$$\frac{d_2}{H} = \frac{1}{\mu_k} - \frac{1}{\tan\theta}$$

$$= \frac{1}{0.6} - \frac{1}{\tan 45^{\circ}} = 0.67 \quad \text{(ans)}$$

Remark: For a large avalanche, d_2/H may be as large as 20, corresponding to $\mu_k = 0.05$! This remains an open question.

8-9 A 20 kg block is about to collide with a spring at its relaxed length. As the block compresses the spring, a kinetic frictional force between the block and the floor acts on the block. Using Fig. 8-20*b*, find the coefficient of kinetic friction μ_k between the block and the floor.

Using the conservation of energy,

$$\Delta E_{\text{mec}} + \Delta E_{\text{th}} = 0$$

From Fig. 8-20*b*,

$$\Delta E_{\text{mec}} = 14 - 30 = -16 \text{ J}$$

$$\Delta E_{\text{th}} = -\Delta E_{\text{mec}} = 16 \text{ J}$$

Since the change in the thermal energy comes from the work done by the moving block against friction,

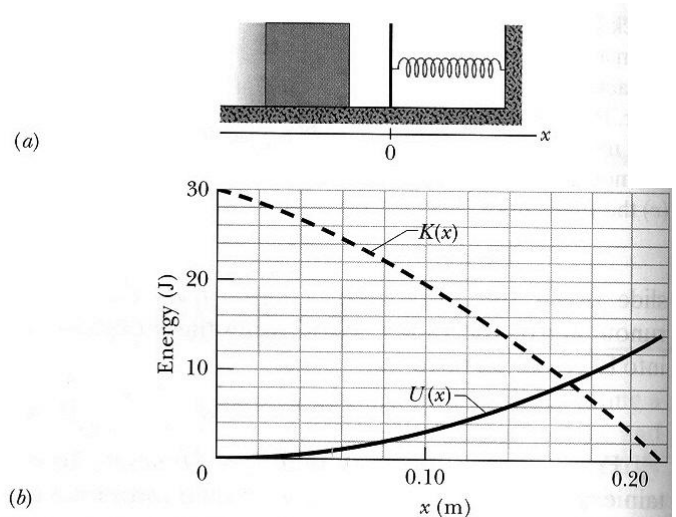


FIG. 8-20 (*a*) Block on the verge of colliding with a spring. (*b*) The change of kinetic energy K and potential energy U as the spring is compressed and the block slows to a stop.

$$\Delta E_{\text{th}} = f_k d = \mu_k N d = \mu_k mgd$$

From Fig. 8-20b, d = 0.215 m. Therefore,

$$16 = \mu_k (20)(9.8)(0.215)$$

$$\mu_k = \frac{16}{(20)(9.8)(0.215)} = 0.38 \qquad \text{(ans)}$$