Angular Momentum

Reading: Chapter 11 (11-7 to 11-12)

Angular Momentum

$$\boxed{\vec{l} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v}).}$$

$$l = mrv\sin\phi.$$

Alternatively, $l = rp_{\perp} = rmv_{\perp}$ or $l = r_{\perp}p = r_{\perp}mv$.

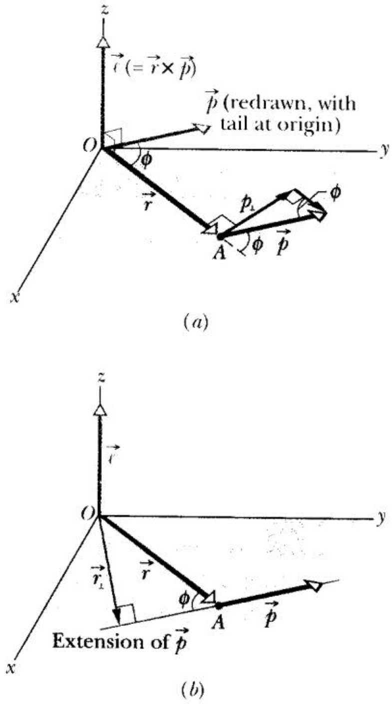

FIG. 11-12 Defining angular momentum. A particle passing through point A has linear momentum $\vec{p}\,(=m\vec{v})$, with the vector $\vec{p}$ lying in an xy plane. The particle has angular momentum $\vec{\ell}\,(=\vec{r}\times\vec{p})$ with respect to the origin O. By the right-hand rule, the angular momentum vector points in the positive direction of z. (a) The magnitude of $\vec{\ell}$ is given by $\ell = rp_{\perp} = rmv_{\perp}$. ($b$) The magnitude of $\vec{\ell}$ is also given by $\ell = r_{\perp}p = r_{\perp}mv$.

Newton's Second Law

$$\boxed{\Sigma\vec{\tau} = \frac{d\vec{l}}{dt}.}$$

The vector sum of all the torques acting on a particle is equal to the time rate of change of the angular momentum of that particle.

Proof

$$\vec{l} = m(\vec{r} \times \vec{v}).$$

Differentiating with respect to time,

$$\frac{d\vec{l}}{dt} = m\left(\vec{r} \times \frac{d\vec{v}}{dt} + \frac{d\vec{r}}{dt} \times \vec{v}\right).$$

$$\frac{d\vec{l}}{dt} = m(\vec{r} \times \vec{a} + \vec{v} \times \vec{v}).$$

$\vec{v} \times \vec{v} = 0$ because the angle between $\vec{v}$ and itself is zero.

$$\frac{d\vec{l}}{dt} = m(\vec{r} \times \vec{a}) = \vec{r} \times m\vec{a}.$$

Using Newton's law, $\Sigma\vec{F} = m\vec{a}$. Hence

$$\frac{d\vec{l}}{dt} = \vec{r} \times \left(\sum \vec{F}\right) = \sum \left(\vec{r} \times \vec{F}\right).$$

Since $\vec{\tau} = \vec{r} \times \vec{F}$, we arrive at

$$\sum \vec{\tau} = \frac{d\vec{l}}{dt}.$$

Example

11-5 A penguin of mass m falls from rest at point A, a horizontal distance D from the origin O of an xyz coordinate system.
(a) What is the angular momentum $\vec{l}$ of the falling penguin about O?
(b) About the origin O, what is the torque $\vec{\tau}$ on the penguin due to the gravitational force $\vec{F}_g$?

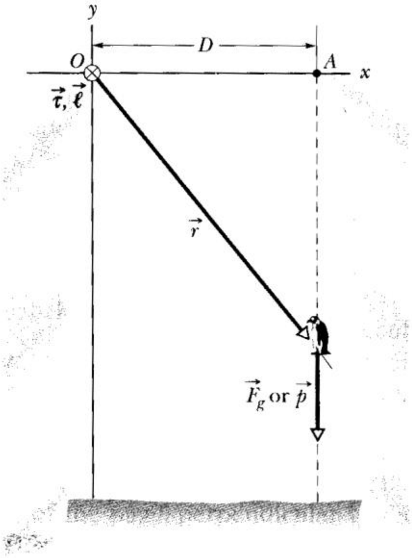

FIG. 11-14 A penguin falls vertically from point A. The torque $\vec{\tau}$ and the angular momentum $\vec{\ell}$ of the falling penguin with respect to the origin O are directed into the plane of the figure at O.

(a) Angular momentum: $l = r_{\perp}mv = Dmgt$
Direction: Using the right-hand rule, $\vec{l}$ is directed into the plane of the figure.
(b) Torque: $\tau = DF_g = Dmg$
Direction: Using the right-hand rule, $\vec{\tau}$ is directed into the plane of the figure.
Remark: The results agree with Newton's law for angular motion:

$$\tau = \frac{dl}{dt} = \frac{d(Dmgt)}{dt} = Dmg$$

The Angular Momentum of a System of Particles

Total angular momentum for *n* particles:

$$\vec{L} = \vec{l}_1 + \vec{l}_2 + \cdots = \Sigma \vec{l}_i.$$

Newtons' law for angular motion:

$$\Sigma \vec{\tau}_i = \Sigma \frac{d\vec{l}_i}{dt} = \frac{d}{dt} \Sigma \vec{l}_i = \frac{d\vec{L}}{dt}.$$

$\Sigma \vec{\tau}_i$ includes torques acting on all the *n* particles. Both internal torques and external torques are considered. Using Newton's law of action and reaction, the internal forces cancel in pairs. Hence

$$\Sigma \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}.$$

TABLE 11-1

More Corresponding Variables and Relations for Translational and Rotational Motion[a]

Translational		Rotational	
Force	$\vec{F}$	Torque	$\vec{\tau}\ (= \vec{r} \times \vec{F})$
Linear momentum	$\vec{p}$	Angular momentum	$\vec{\ell}\ (= \vec{r} \times \vec{p})$
Linear momentum[b]	$\vec{P}\ (= \Sigma \vec{p}_i)$	Angular momentum[b]	$\vec{L}\ (= \Sigma \vec{\ell}_i)$
Linear momentum[b]	$\vec{P} = M\vec{v}_{com}$	Angular momentum[c]	$L = I\omega$
Newton's second law[b]	$\vec{F}_{net} = \dfrac{d\vec{P}}{dt}$	Newton's second law[b]	$\vec{\tau}_{net} = \dfrac{d\vec{L}}{dt}$
Conservation law[d]	$\vec{P}$ = a constant	Conservation law[d]	$\vec{L}$ = a constant

[a]See also Table 10-3.

[b]For systems of particles, including rigid bodies.

[c]For a rigid body about a fixed axis, with L being the component along that axis.

[d]For a closed, isolated system.

The Angular Momentum of a Rigid Body

For the ith particle, angular momentum:

$$l_i = r_i p_i \sin 90^{\circ} = r_i \Delta m_i \, v_i .$$

The component of angular momentum parallel to the rotation axis (the z component):

$$l_{iz} = l_i \sin\theta = (r_i \sin\theta)(\Delta m_i v_i)$$
$$= r_{i\perp} \Delta m_i v_i .$$

The total angular momentum for the rotating body

$$L_z = \sum_i l_{iz} = \sum_i \Delta m_i v_i r_{i\perp}$$
$$= \sum_i \Delta m_i (\omega r_{i\perp}) r_{i\perp} = \omega\left(\sum_i m_i r_{i\perp}^2\right).$$

This reduces to

$$\boxed{L = I\omega.}$$

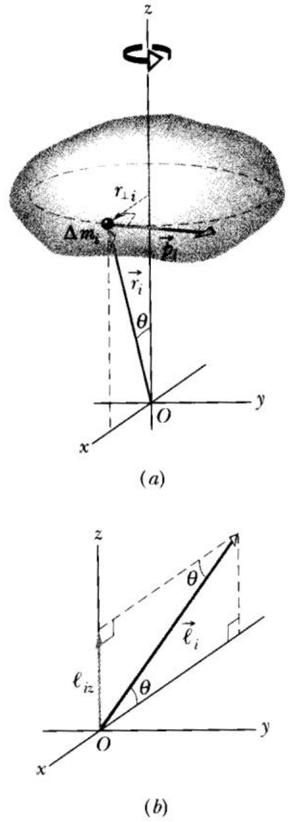

FIG. 11-15 (a) A rigid body rotates about a z axis with angular speed ω. A mass element of mass Δm_i within the body moves about the z axis in a circle with radius $r_{\perp i}$. The mass element has linear momentum $\vec{p}_i$, and it is located relative to the origin O by position vector $\vec{r}_i$. Here the mass element is shown when $r_{\perp i}$ is parallel to the x axis. (b) The angular momentum $\vec{\ell}_i$, with respect to O, of the mass element in (a). The z component ℓ_{iz} is also shown.

Conservation of Angular Momentum

$$\Sigma \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}.$$

If no external torque acts on the system,

$$\frac{d\vec{L}}{dt} = 0.$$

$$\boxed{\vec{L} = \text{constant.}}$$

$$\boxed{\vec{L}_i = \vec{L}_f.}$$

Law of conservation of angular momentum.

Examples

1. The spinning volunteer: When the student pulls in the dumbbells, the rotational inertia I decreases. Since $I_i\omega_i = I_f\omega_f$, the angular velocity increases.

2. The springboard diver: When the diver is in the tuck position, the rotational inertia decreases, and the angular velocity increases.

When the diver is in the layout position, the rotational inertia increases, and the angular velocity decreases.

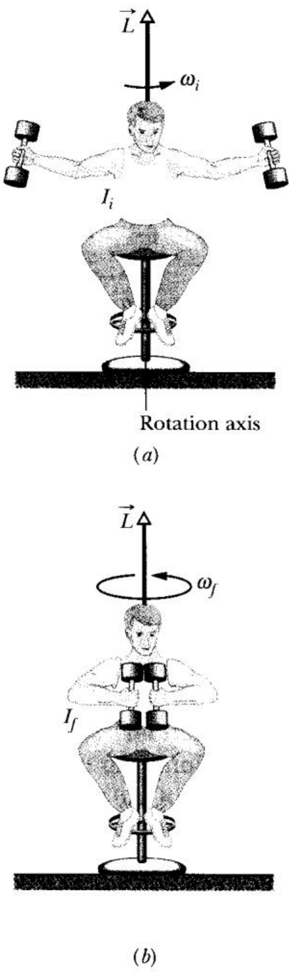

FIG. 11-17 (*a*) The student has a relatively large rotational inertia about the rotation axis and a relatively small angular speed. (*b*) By decreasing his rotational inertia, the student automatically increases his angular speed. The angular momentum $\vec{L}$ of the rotating system remains unchanged.

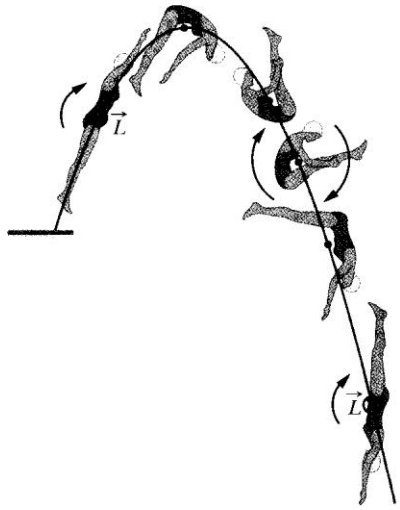

FIG. 11-18 The diver's angular momentum $\vec{L}$ is constant throughout the dive, being represented by the tail $\otimes$ of an arrow that is perpendicular to the plane of the figure. Note also that her center of mass (see the dots) follows a parabolic path.

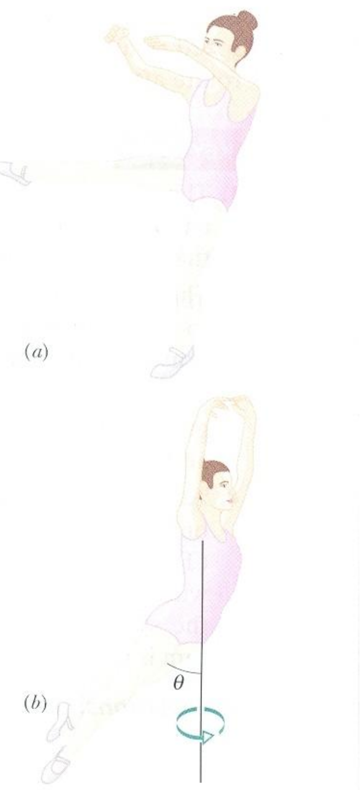

FIG. 11-20 (*a*) Initial phase of a tour jeté: large rotational inertia and small angular speed. (*b*) Later phase: smaller rotational inertia and larger angular speed.

FIG. 11-19 Windmill motion of the arms during a long jump helps maintain body orientation for a proper landing.

3. Long jump: When an athlete takes off, her angular momentum gives her a forward rotation around a horizontal axis.

In the air, the jumper shifts the angular momentum to her arms by rotating them in a windmill fashion. Then the body carries little angular momentum, keeping her body upright. She can then land with her legs extended forward.

4. Tour jeté: The dancer/gymnast leaps with one leg perpendicular to the body. In the air, the outstretched leg is brought down and the other leg is brought up, with both ending up at an angle θ to the body. The rotational inertia decreases and the angular speed increases.

On landing, a leg is again outstretched and the rotation seems to vanish.

See Youtube "Chen Ruolin Wang Xin", "Irving Saladino" and "tour jete".
See demonstration "Bicycle wheel gyroscope" and "swinging Atwood machine".

Examples

11-7 A student sits on a stool that can rotate freely about a vertical axis. The student, initially at rest, is holding a bicycle wheel whose rim is loaded with lead and whose rotational inertia I about its central axis is 1.2 kg m^2. The wheel is rotating at an angular speed ω_{wh} of 3.9 rev/s; as seen from overhead, the rotation is counterclockwise. The axis of the wheel is vertical, and the angular momentum L_{wh} of the wheel points vertically upward. The student now inverts the wheel; as a result, the student and stool rotate about the stool axis. The rotational inertia I_b of the student + stool + wheel system about the stool axis is 6.8 kg m^2. With what angular speed ω_b and in what direction does the composite body rotate after the inversion of the wheel?

Using the conservation of angular momentum,

$$L_{\mathrm{wh}} = L + (-L_{\mathrm{wh}})$$

$$L = 2L_{\mathrm{wh}}$$

$$I_b\omega_b = 2I\omega_{\mathrm{wh}}$$

$$\omega_b = \frac{2I\omega_{\mathrm{wh}}}{I_b} = \frac{(2)(1.2)(3.9)}{6.8}$$

$$= 1.38\,\mathrm{rev\,s^{-1}}$$

(ans)

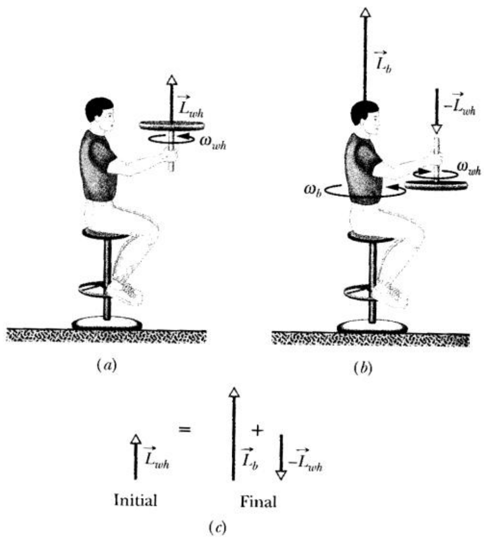

FIG. 11-21 (*a*) A student holds a bicycle wheel rotating around a vertical axis. (*b*) The student inverts the wheel, setting himself into rotation. (*c*) The net angular momentum of the system must remain the same in spite of the inversion.

11-8 A cockroach with mass m rides on a disk of mass $6m$ and radius R. The disk rotates like a merry-go-round around its central axis at angular speed $\omega_i = 1.5$ rad s^{-1}. The cockroach is initially at radius $r = 0.8R$, but then it crawls out to the rim of the disk. Treat the cockroach as a particle. What then is the angular speed?

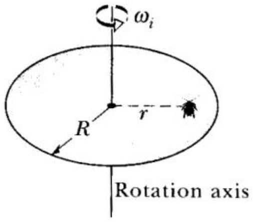

FIG. 11-22 A cockroach rides at radius r on a disk rotating like a merry-go-round.

Precession of a Gyroscope

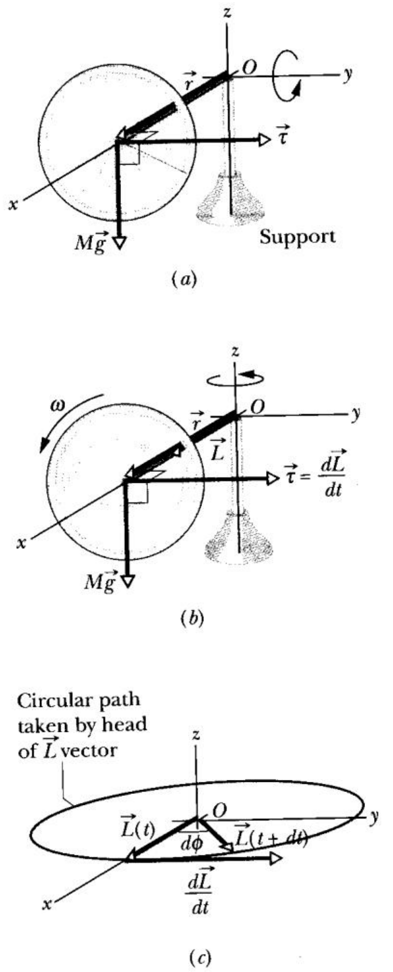

FIG. 11-23 (*a*) A nonspinning gyroscope falls by rotating in an xz plane because of torque $\vec{\tau}$. (*b*) A rapidly spinning gyroscope, with angular momentum $\vec{L}$, precesses around the z axis. Its precessional motion is in the xy plane. (*c*) The change $d\vec{L}/dt$ in angular momentum leads to a rotation of $\vec{L}$ about O.

Torque due to the gravitational force

$$\tau = Mgr\sin 90^{\circ} = Mgr$$

Angular momentum

$$L = I\omega$$

For a rapidly spinning gyroscope, the magnitude of $\vec{L}$ is not affected by the precession,

$$dL = Ld\phi$$
$$\frac{dL}{dt} = L\frac{d\phi}{dt}$$

Using Newton's second law for rotation,

$$\tau = \frac{dL}{dt}$$

$Mgr = L\dfrac{d\phi}{dt} = L\Omega$ where $\Omega = \dfrac{d\phi}{dt}$ is the precession rate

$$\Omega = \frac{Mgr}{L} = \frac{Mgr}{I\omega}$$