Rotational Motion

Reading: Chapter 10

Angular Displacement

$$\Delta\theta = \theta_2 - \theta_1.$$

Angular Velocity

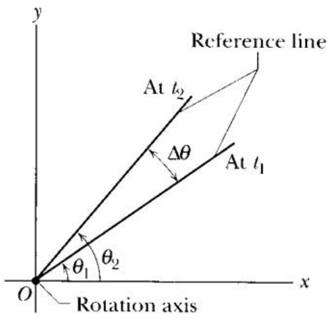


FIG. 10-4 The reference line of the rigid body of Figs. 10-2 and 10-3 is at angular position θ_1 at time t_1 and at angular position θ_2 at a later time t_2. The quantity $\Delta\theta$ ($= \theta_2 - \theta_1$) is the angular displacement that occurs during the interval Δt ($= t_2 - t_1$). The body itself is not shown.

Average angular velocity

$$\overline{\omega} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}.$$

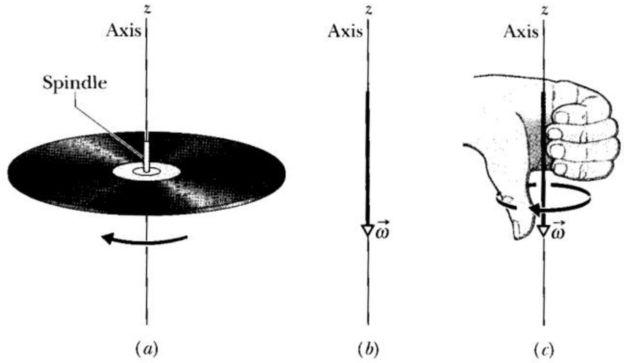
Instantaneous angular velocity

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}.$$

The magnitude is called the **angular speed**.

Angular Velocity as a Vector

FIG. 10-6 (*a*) A record rotating about a vertical axis that coincides with the axis of the spindle. (*b*) The angular velocity of the rotating record can be represented by the vector $\vec{\omega}$, lying along the axis and pointing down, as shown. (*c*) We establish the direction of the angular velocity vector as downward by using a right-hand rule. When the fingers of the right hand curl around the record and point the way it is moving, the extended thumb points in the direction of $\vec{\omega}$.



The direction of the vector $\vec{\omega}$ points along the axis of rotation, according to the **right-hand rule**.

Angular Acceleration

Average angular acceleration

$$\overline{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}.$$

Instantaneous angular acceleration

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}.$$

Rotation with Constant Acceleration

$$\omega = \omega_0 + \alpha t,$$
$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2,$$
$$\omega^2 = \omega_0^2 + 2\alpha\theta.$$

Example

10-3 A grindstone rotates at constant angular acceleration α = 0.35 rad s^{-2}. At time t = 0, it has angular velocity ω_0 = –4.6 rad s^{-1} and a reference line on it is horizontal, at the angular position θ_0 = 0.
(a) At what time after t = 0 is the reference line at the angular position θ = 5 rev?
(b) Describe the grindstone's rotation between t = 0 and t = 32 s.
(c) At what time t does the grindstone momentarily stop?

(a) Since $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$,

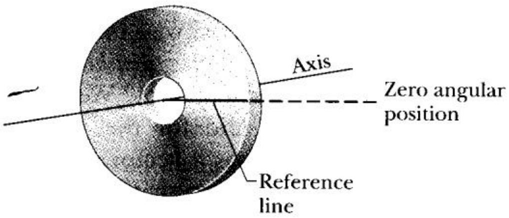


FIG. 10-8 A grindstone. At t = 0 the reference line (which we imagine to be marked on the stone) is horizontal.

$$10\pi = -4.6t + \left(\frac{1}{2}\right)(0.35)t^2$$

$$0.35t^2 - 9.2t - 20\pi = 0$$

$$t = \frac{9.2 \pm \sqrt{9.2^2 + (4)(0.35)(20\pi)}}{0.7} = 31.9 \text{ or } -5.63$$

Therefore, t = 32 s. (ans)

(b) The wheel is initially rotating in the clockwise direction. It slows down, stops, and then reverses to rotate in the anticlockwise direction. The reference line returns to its initial orientation of θ = 0, and turns an additional 5 rev by time t = 32 s. (ans)

(c) Since $\omega = \omega_0 + \alpha t$,

$$0 = -4.6 + 0.35t$$

$$t = \frac{0 - (-4.6)}{0.35} = 13\,\text{s} \qquad \text{(ans)}$$

Relating the Linear and Angular Variables

The position

$$\boxed{s = \theta r.}$$

The velocity

$$\boxed{v = \omega r.}$$

The acceleration

Tangential component:

$$\boxed{a_t = \alpha r.}$$

Radial component:

$$\boxed{a_r = \frac{v^2}{r} = \omega^2 r.}$$

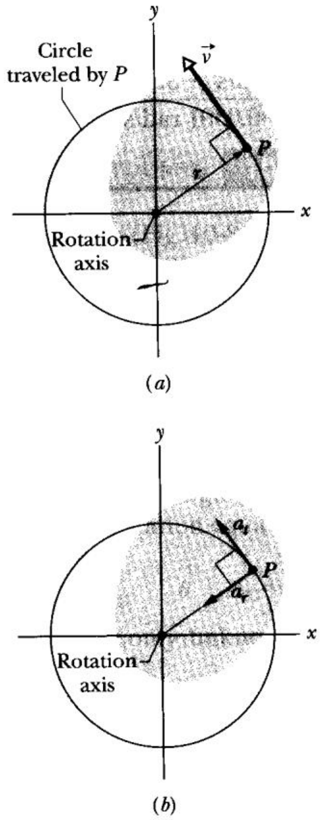
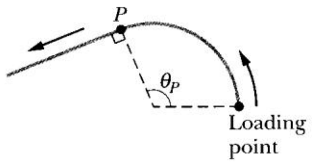


FIG. 10-9 The rotating rigid body of Fig. 10-2, shown in cross section viewed from above. Every point of the body (such as P) moves in a circle around the rotation axis. (a) The linear velocity $\vec{v}$ of every point is tangent to the circle in which the point moves. (b) The linear acceleration $\vec{a}$ of the point has (in general) two components: tangential a_t and radial a_r.

Example

10-5 A roller coaster track is designed as follows.
(1) The passenger leaves the loading point with acceleration *g* along the horizontal track.
(2) The first section of the track forms a circular arc, so that the passenger also experiences a centripetal acceleration.
(3) When the magnitude a of the net acceleration reaches 4*g* at some point *P* and angle θ_P along the arc, the passenger then moves in a straight line along the tangent of the arc.
(a) What is the angle θ_P?
(b) What is the magnitude *a* of the passenger's net acceleration at *P* and after *P*?



10-10 An overhead view of a horizontal track for a roller coaster. The track begins as a circular arc at the loading point and then, at point *P*, continues along a tangent to the arc.

(a) Tangential acceleration: $a_t = g$
Centripetal acceleration: $a_r = r\omega^2$
Since $\omega^2 = \omega_0^2 + 2\alpha\theta = 2\alpha\theta$ and

$\alpha = \frac{a_t}{r} = \frac{g}{r}$, we have $a_r = r(2\alpha\theta) = r\left(\frac{2g\theta}{r}\right) = 2g\theta$

Magnitude of acceleration: $a = \sqrt{a_t^2 + a_r^2} = \sqrt{g^2 + (2g\theta)^2}$
$= g\sqrt{1 + 4\theta^2}$

When *a* reaches 4*g* at *P*, $4g = g\sqrt{1 + 4\theta^2}$

$\Rightarrow \quad \theta = \sqrt{\frac{15}{4}} = 1.94\,\text{rad} = 111^\text{o}$ (ans)

(b) At *P*, *a* = 4*g*. (ans)
After *P*, $a_t = g$ and $a_r = 0$. Therefore, *a* = *g*. (ans)
Remark: This abrupt change in acceleration can cause roller-coaster headache.

Kinetic Energy of Rotation

Consider the kinetic energy of a rotating rigid body:

$$K = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \cdots = \sum_i \frac{1}{2}m_iv_i^2.$$

Since $v = \omega r$, and ω is the same for all particles, we have

$$K = \sum_i \frac{1}{2}m_i(\omega r_i)^2 = \frac{1}{2}\left(\sum_i m_ir_i^2\right)\omega^2.$$

$\sum_i m_ir_i^2$ is called the **rotational inertia**. It tells us how the mass of the rotating body is distributed about its axis of rotation. In summary,

$$\boxed{I = \sum_i m_ir_i^2} \quad \text{and} \quad \boxed{K = \frac{1}{2}I\omega^2.}$$

For continuous bodies,

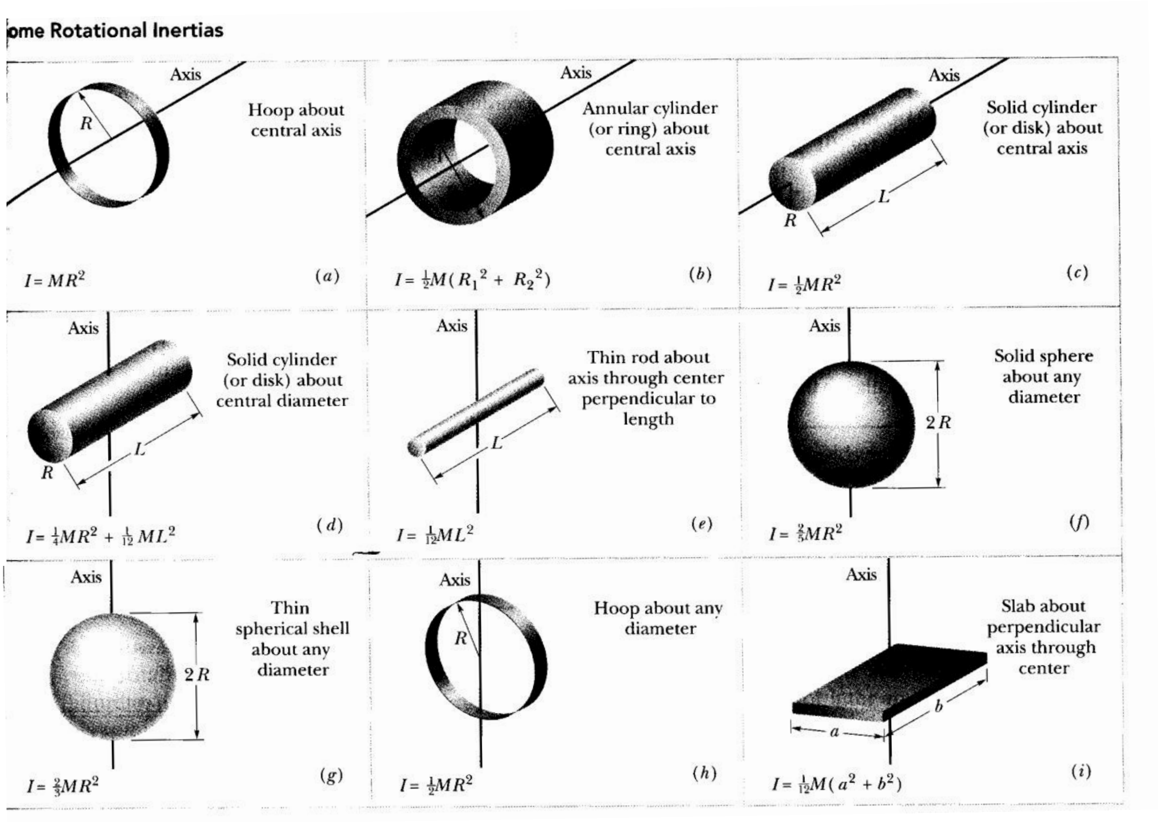
$$\boxed{I = \int r^2 dm.}$$

Ring $I = MR^2$ (axis) $I = \frac{1}{2}MR^2$ (diameter)

Cylinder $I = \frac{1}{2}MR^2$

Rod $I = \frac{1}{12}ML^2$ (centre) $I = \frac{1}{3}ML^2$ (end)

Sphere $I = \frac{2}{5}MR^2$



Parallel Axis Theorem

$$\boxed{I = I_{\text{cm}} + Mh^2.}$$

The rotational inertia of a body about any axis is equal to the rotational inertia (= Mh^2) it would have about that axis if all its mass were concentrated at its centre of mass, plus its rotational inertia (= I_{cm}) about a parallel axis through its centre of mass.

Proof

$$I = \int r^2 dm \; = \int \left[(x-a)^2 + (y-b)^2 \right] dm,$$

which can be written as

$$I = \int (x^2 + y^2) dm \; - 2a \int x dm \; - 2b \int y dm \; + \int (a^2 + b^2) dm.$$

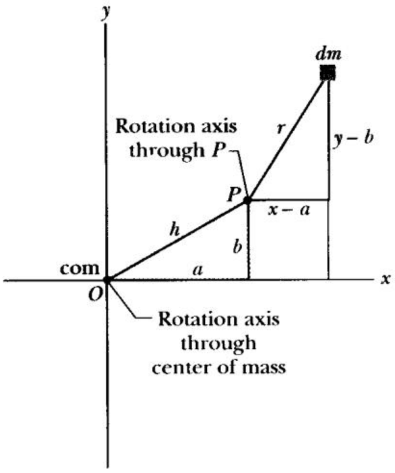


FIG. 10-12 A rigid body in cross section, with its center of mass at *O*. The parallel-axis theorem (Eq. 10-36) relates the rotational inertia of the body about an axis through *O* to that about a parallel axis through a point such as *P*, a distance *h* from the body's center of mass. Both axes are perpendicular to the plane of the figure.

In the first term, $x^2 + y^2 = R^2$. Hence the first term becomes

$$\int (x^2 + y^2)dm = \int R^2 dm = I_{cm}.$$

In the second and third terms, the position of the centre of mass gives

$$x_{\rm cm} = \frac{1}{M}\int xdm = 0 \text{ and } y_{\rm cm} = \frac{1}{M}\int ydm = 0.$$

Hence these terms vanish.

In the last term, $a^2 + b^2 = h^2$. Hence the last term becomes

$$\int (a^2 + b^2)dm = \int h^2 dm = Mh^2.$$

Examples

10-6 A rigid body consists of two particles of mass m connected by a rod of length L and negligible mass.
(a) What is the rotational inertia I_{cm} about an axis through the center of mass perpendicular to the rod?
(b) What is the rotational inertia I of the body about an axis through the left end of the rod and parallel to the first axis?

10-7 Consider a thin, uniform rod of mass *M* and length *L*.
(a) What is the rotational inertia about an axis perpendicular to the rod, through its center of mass?
(b) What is the rotational inertia of the rod about an axis perpendicular to the rod through one end?

(a) $I = \int r^2 dm$

$dm = \frac{m}{L} dx$

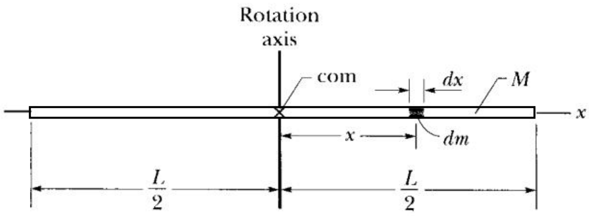


FIG. 10-14 A uniform rod of length L and mass M. An element of mass dm and length dx is represented.

$$I = \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \left(\frac{M}{L} dx \right) = \frac{M}{3L} \left[x^3 \right]_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{M}{3L} \left[\left(\frac{L}{2} \right)^3 - \left(-\frac{L}{2} \right)^3 \right] = \frac{ML^2}{12}$$

(b) Using the parallel axis theorem,

$$I = I_{\text{cm}} + Mh^2 = \frac{1}{12} ML^2 + M\left(\frac{L}{2} \right)^2 = \frac{1}{3} ML^2 \qquad \text{(ans)}$$

Torque

The ability of $\vec{F}$ to rotate the body depends on:

(1) the magnitude of the tangential component $F_t = F\sin\phi$,

(2) the distance between the point of application and the axis of rotation.

Define the **torque** as

$$\tau = rF\sin\phi.$$

It can be considered as either $rF_\perp$ or $r_\perp F$. Terms:

line of action
moment arm

τ is positive if it tends to rotate the body counterclockwise.
It is negative if it tends to rotate the body clockwise.

Considering the vector direction,

$$\vec{\tau} = \vec{r} \times \vec{F}.$$

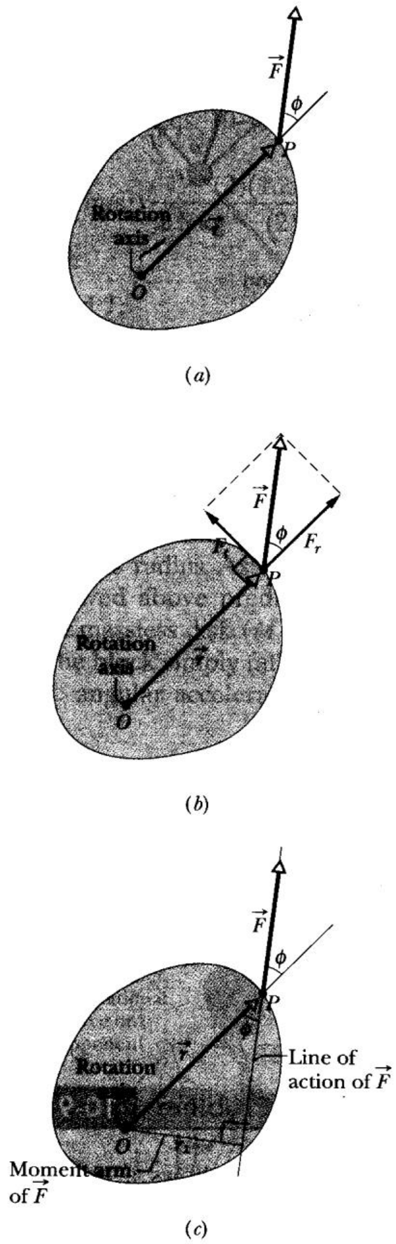


FIG. 10-16 (*a*) A force $\vec{F}$ acts at point *P* on a rigid body that is free to rotate about an axis through *O*; the axis is perpendicular to the plane of the cross section shown here. (*b*) The torque due to this force is $(r)(F\sin\phi)$. We can also write it as rF_t, where F_t is the tangential component of $\vec{F}$. (*c*) The torque can also be written as $r_\perp F$, where $r_\perp$ is the moment arm of $\vec{F}$.

Newton's Second Law for Rotation

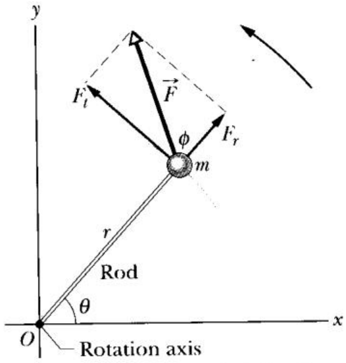


FIG. 10-17 A simple rigid body, free to rotate about an axis through O, consists of a particle of mass m fastened to the end of a rod of length r and negligible mass. An applied force $\vec{F}$ causes the body to rotate.

Newton's second law:

$$F_t = ma_t.$$

Torque:

$$\tau = F_t r = ma_t r.$$

Since $a_t = \alpha r$, we obtain

$$\tau = m(\alpha r)r = (mr^2)\alpha.$$

Conclusion:

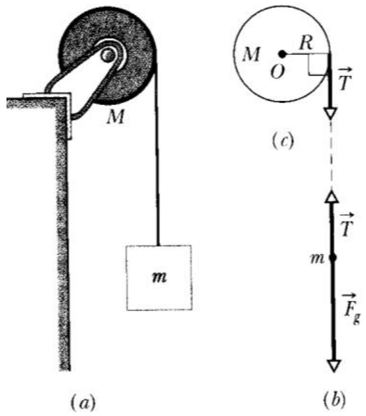
$$\tau = I\alpha.$$

If there are more than one forces,

$$\boxed{\sum \tau = I\alpha.}$$

Examples

10-9 A uniform disk of mass M = 2.5 kg and radius R = 20 cm is mounted on a fixed horizontal axle. A block whose mass m is 1.2 kg hangs from a massless cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the angular acceleration of the disk, and the tension in the cord. The cord does not slip, and there is no friction at the axle.



Newton's law for the hanging block (upward is positive):

$$T - mg = ma \qquad (1)$$

Newton's law for the rotating disk (anticlockwise is positive):

$$-TR = \frac{1}{2}MR^2\alpha \qquad (2)$$

Since $a = R\alpha$,

(2): $T = -\frac{1}{2}Ma$

(1): $-\frac{1}{2}Ma - mg = ma$

$$-mg = \left(m + \frac{M}{2}\right)a$$

$$a = -\frac{2mg}{M + 2m} = -\frac{(2)(1.2)(9.8)}{2.5 + (2)(1.2)} = -4.8\,\text{ms}^{-2} \qquad \text{(ans)}$$

$$\alpha = \frac{a}{R} = -\frac{4.8}{0.2} = -24\,\text{rad s}^{-2} \qquad \text{(ans)}$$

$$T = -\frac{1}{2}Ma = -\left(\frac{1}{2}\right)(2.5)(-4.8) = 6\,\text{N} \qquad \text{(ans)}$$

10-10 (Physics of judo) To throw an 80-kg opponent with a basic judo hip throw, you intend to pull his uniform with a force $\vec{F}$ and a moment arm d_1 = 0.30 m from a pivot point (rotation axis) on your right hip, about which you wish to rotate him with an angular acceleration of –6.0 rad s^{-2}, that is, with a clockwise acceleration. Assume that his rotational inertia I is 15 kg m^2.

(a) What must the magnitude of $\vec{F}$ be if you initially bend your opponent forward to bring his centre of mass to your hip?

(b) What must the magnitude of $\vec{F}$ be if he remains upright and his weight $m\vec{g}$ has a moment arm d_2 = 0.12 m from the pivot point?

See Youtube "Judo hip throw".

(a) Newton's law for the rotating opponent (anticlockwise is positive):

$$\tau = -d_1 F = I\alpha$$

$$F = -\frac{I\alpha}{d_1} = -\frac{(15)(-6)}{0.3}$$

= 300 N (ans)

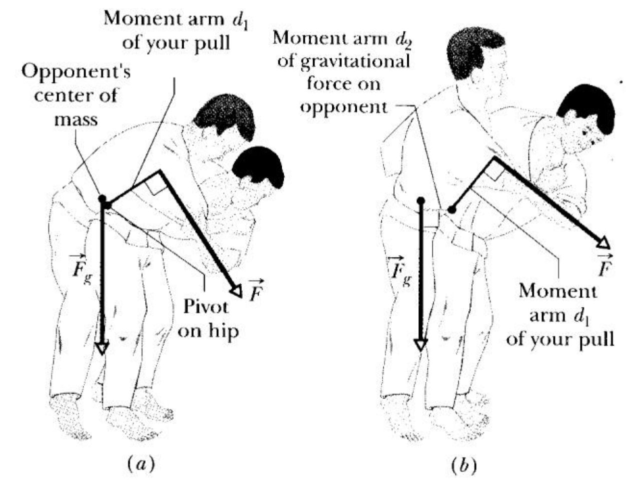


FIG. 10-19 A judo hip throw (a) correctly executed and (b) incorrectly executed.

(b)

$$\sum \tau = -d_1 F + d_2 mg = I\alpha$$

$$F = -\frac{I\alpha}{d_1} + \frac{d_2 mg}{d_1} = -\frac{(15)(-6)}{0.3} + \frac{(0.12)(80)(9.8)}{0.3} \approx 614 \text{ N}$$

(ans)

Remark: In the correct execution of the hip throw, you should bend your opponent to bring his center of mass to your hip.

Work and Rotational Kinetic Energy

Work done by the force:

$$dW = \vec{F} \cdot d\vec{s} = F_t ds = F_t r d\theta = \tau d\theta.$$

Total work done:

$$\boxed{W = \int_{\theta_i}^{\theta_f} \tau d\theta.}$$

Work-kinetic energy theorem:

$$\tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\omega\frac{d\omega}{d\theta} = \frac{d}{d\theta}\left(\frac{1}{2}I\omega^2\right).$$

Integrating over the angular displacement,

$$W = \int_{\theta_i}^{\theta_f} \tau d\theta \quad = \int_{\theta_i}^{\theta_f} \frac{d}{d\theta}\left(\frac{1}{2}I\omega^2\right)d\theta = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = \Delta K$$

$$\boxed{W = \Delta K.}$$

Power

$$P = \frac{dW}{dt} = \tau\frac{d\theta}{dt} = \tau\omega.$$

Some Corresponding Relations for Translational and Rotational Motion

Pure Translation (Fixed Direction)		Pure Rotation (Fixed Axis)	
Position	x	Angular position	θ
Velocity	$v = dx/dy$	Angular velocity	$\omega = d\theta/dt$
Acceleration	$a = dv/dt$	Angular acceleration	$\alpha = d\omega/dt$
Mass	m	Rotational inertia	I
Newton's second law	$F_{net} = ma$	Newton's second law	$\tau_{net} = I\alpha$
Work	$W = \int F\,dx$	Work	$W = \int \tau\,d\theta$
Kinetic energy	$K = \frac{1}{2}mv^2$	Kinetic energy	$K = \frac{1}{2}I\omega^2$
Power (constant force)	$P = Fv$	Power (constant torque)	$P = \tau\omega$
Work–kinetic energy theorem	$W = \Delta K$	Work–kinetic energy theorem	$W = \Delta K$

Example

10-11 As in Example 10-9, a uniform disk of mass M = 2.5 kg and radius R = 20 cm is mounted on a fixed horizontal axle. A block whose mass m is 1.2 kg hangs from a massless cord that is wrapped around the rim of the disk. What is the rotational kinetic energy K at t = 2.5 s?

10-12 A tall, cylindrical chimney will fall over when its base is ruptured. Treat the chimney as a thin rod of length L = 55 m. At the instant it makes an angle of $\theta = 35^{o}$ with the vertical, what is its angular speed ω_f?

FIG. 10-20 (*a*) A cylindrical chimney. (*b*) The height of its center of mass is determined with the right triangle.