Newtonian Gravitation

Reading: Chapter 13 (13-1 to 13-8)

Newton's Law of Gravitation

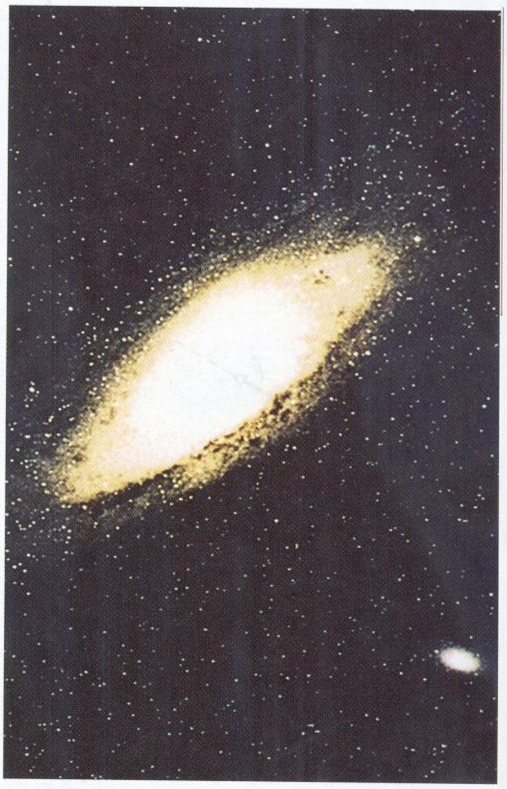


FIG. 13-1 The Andromeda Galaxy. Located 2.3×10^6 light-years from us, and faintly visible to the naked eye, it is very similar to our home galaxy, the Milky Way. *(Courtesy NASA)*

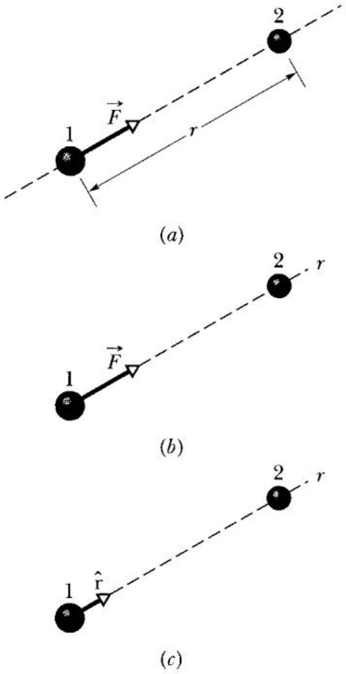


FIG. 13-2 (*a*) The gravitational force $\vec{F}$ on particle 1 due to particle 2 is an attractive force because particle 1 is attracted to particle 2. (*b*) Force $\vec{F}$ is directed along a radial coordinate axis *r* extending from particle 1 through particle 2. (*c*) $\vec{F}$ is in the direction of a unit vector $\hat{r}$ along the *r* axis.

$$\boxed{F = G\frac{m_1 m_2}{r^2},}$$

where *G* is the gravitational constant

$$G = 6.67 \times 10^{-11}\ \text{Nm}^2\text{kg}^{-2}.$$

A uniform spherical shell of matter attracts a particle that is outside the shell as if all the shell's mass were concentrated at its centre.

Gravitational Potential Energy

$$\boxed{U = -\frac{GMm}{r}.}$$

If there are more than two particles, the total gravitational potential energy is equal to the sum of the gravitational potential energy for each pair.

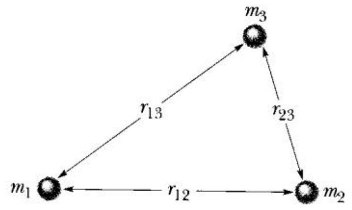


FIG. 13-9 A system consisting of three particles. The gravitational potential energy *of the system* is the sum of the gravitational potential energies of all three pairs of particles.

e.g. $U = -\left(\frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}}\right).$

Proof

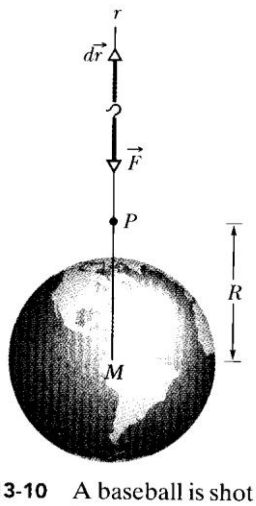


FIG. 13-10 A baseball is shot directly away from Earth, through point *P* at radial distance *R* from Earth's center. The gravitational force $\vec{F}$ on the ball and a differential displacement vector $d\vec{r}$ are shown, both directed along a radial *r* axis.

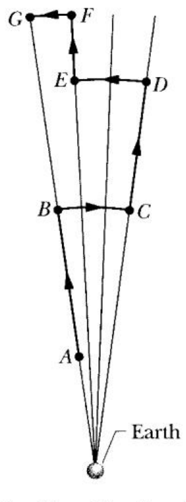


FIG. 13-11 Near Earth, a baseball is moved from point *A* to point *G* along a path consisting of radial lengths and circular arcs.

Work done by the gravitational force in moving the ball from infinity to a distance R from Earth:

$$W_g = \int_{\infty}^{R} \vec{F}(r) \cdot d\vec{r}.$$

Work done by the applied force:

$$W_a = -W_g = -\int_{\infty}^{R} \vec{F}(r) \cdot d\vec{r}.$$

This becomes the increase in gravitational potential energy:

$$U = -\int_{\infty}^{R} \vec{F}(r) \cdot d\vec{r}.$$

Since $F = -\frac{GMm}{r^2}$ (minus sign due to the inward direction),

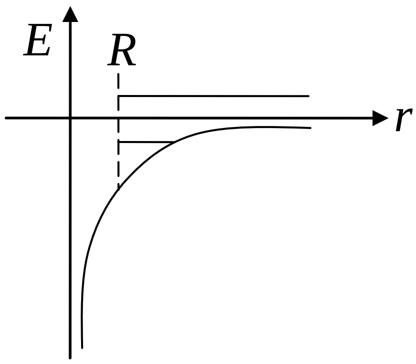
$$U = \int_{\infty}^{R} \left(\frac{GMm}{r^2} \right) dr = \left[-\frac{GMm}{r} \right]_{\infty}^{R} = -\frac{GMm}{R}.$$

Potential Energy and Force

$$F = -\frac{dU}{dr} = -\frac{d}{dr}\left(-\frac{GMm}{r} \right) = -\frac{GMm}{r^2}.$$

Thus we recover the Newton's law of gravitation.

Escape Speed



Consider a projectile fired from distance R.

If $E < 0$, the projectile is bounded.

If $E > 0$, the projectile is unbounded.

If $E = 0$, the projectile just have sufficient energy to escape from the gravitational attraction of Earth.

The initial speed just sufficient to escape from Earth is called the **escape speed**.

It can be obtained from the conservation of energy:

$$E = \frac{1}{2}mv^2 + \left(-\frac{GMm}{R}\right) = 0.$$

This yields the escape speed

$$v = \sqrt{\frac{2GM}{R}}.$$

Example

13-5 An asteroid, headed directly toward Earth, has a speed of 12 km s^{-1} relative to the planet when it is at a distance of 10 Earth radii from Earth's center. Ignoring the effects of the terrestrial atmosphere on the asteroid, find the asteroid's speed when it reaches Earth's surface. (M_E = 5.98 × 10^{24} kg, R_E = 6,370 km)

Using the conservation of energy,

$$K_f + U_f = K_i + U_i$$

$$\frac{1}{2}mv_f^2 - \frac{GM_E m}{R_E} = \frac{1}{2}mv_i^2 - \frac{GM_E m}{10R_E}$$

$$v_f^2 = v_i^2 + \frac{2GM_E}{R_E}\left(1 - \frac{1}{10}\right)$$

$$= (12\times10^3)^2 + \frac{2(6.67\times10^{-11})(5.98\times10^{24})}{6.37\times10^6}0.9$$

$$= 2.567\times10^8 \text{ m}^2\text{s}^{-2}$$

$$v_f = 1.602\times10^4 \text{ ms}^{-1} = 16.0 \text{ km s}^{-1} \quad \text{(ans)}$$

Remark: Even if the comet were only 5 m across, the energy released matched the Hiroshima nuclear explosion.

See Youtube "Comet Shoemaker Levy Colliding with Jupiter".

Planets and Satellites: Kepler’s Laws

1. The Law of Orbits: All planets move in elliptical orbits, with the Sun at one focus.

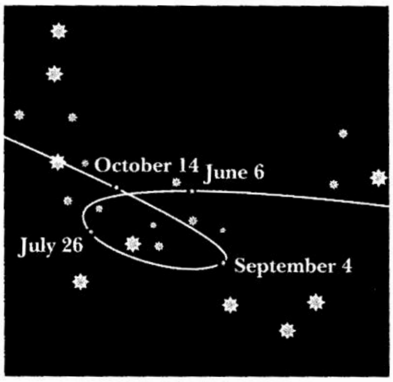


FIG. 13-12 The path seen from Earth for the planet Mars as it moved against a background of the constellation Capricorn during 1971. The planet’s position on four days is marked. Both Mars and Earth are moving in orbits around the Sun so that we see the position of Mars relative to us; this relative motion sometimes results in an apparent loop in the path of Mars.

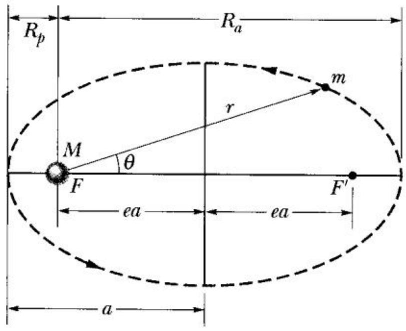


FIG. 13-13 A planet of mass m moving in an elliptical orbit around the Sun. The Sun, of mass M, is at one focus F of the ellipse. The other focus is F', which is located in empty space. Each focus is a distance ea from the ellipse’s center, with e being the eccentricity of the ellipse. The semimajor axis a of the ellipse, the perihelion (nearest the Sun) distance R_p, and the aphelion (farthest from the Sun) distance R_a are also shown.

a = semimajor axis
e = eccentricity
e = 0 for a circle.

At the point nearest to the sun,

$$r_{\min} = a(1-e).$$

At the point furthest from the sun,

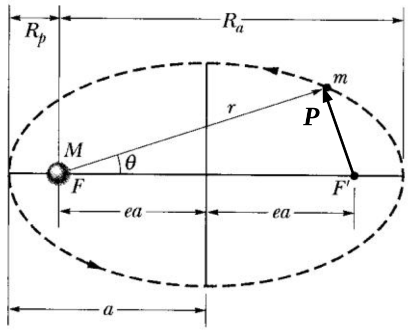
$$r_{\max} = a(1+e).$$

Eliminating a, we have

$$\boxed{e = \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}}.}$$

Polar equation of an ellipse:

An ellipse is the set of points such that whose sum of distances from the two foci is a constant.



In the figure,

$$PF + PF' = \text{constant}.$$

Furthermore, when P is nearest the Sun,

$$PF + PF' = a(1-e) + a(1+e) = 2a.$$

Using the cosine law,

$$PF' = \sqrt{r^2 + (2ea)^2 - 2r(2ae)\cos\theta}.$$

Hence, the polar equation of the ellipse is

$$r + \sqrt{r^2 + (2ea)^2 - 2r(2ae)\cos\theta} = 2a.$$

Collecting terms and squaring both sides,

$$r^2 + (2ea)^2 - 2r(2ae)\cos\theta = (2a-r)^2.$$

Simplifying,

$$\boxed{r = \frac{a(1-e^2)}{1-e\cos\theta}.}$$

This equation can be derived from the conservation of energy (see Appendix A).

2. The Law of Areas: A line that connects a planet to the Sun sweeps out equal areas in equal times.

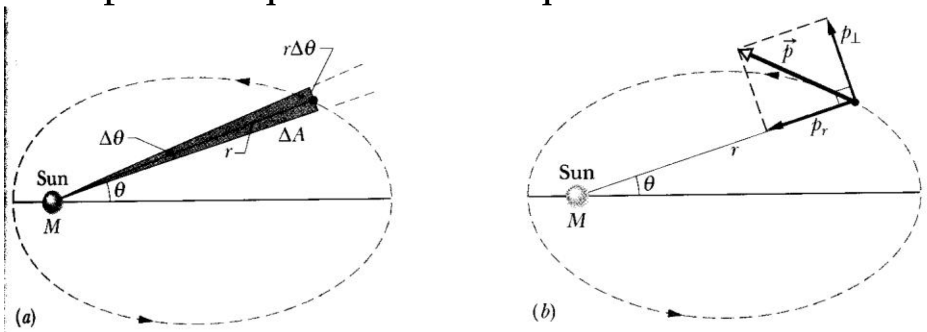


FIG. 13-14 (*a*) In time Δt, the line r connecting the planet to the Sun moves through an angle $\Delta\theta$, sweeping out an area ΔA (shaded). (*b*) The linear momentum $\vec{p}$ of the planet and the components of $\vec{p}$.

This is equivalent to the conservation of angular momentum.

Area ΔA of the triangle swept out in time Δt

$= \frac{1}{2}(r\Delta\theta)r = \frac{1}{2}r^2\Delta\theta$

The rate at which the area is swept:

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\omega.$$

Conservation of angular momentum:

$$L = mr^2\omega.$$

Therefore

$$\frac{dA}{dt} = \frac{L}{2m} = \text{constant}.$$

3. The Law of Periods: The square of the period of any planet is proportional to the cube of the semimajor axis of its orbit.

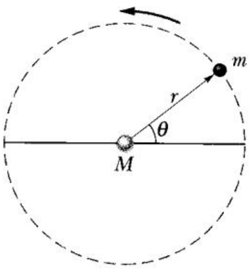


FIG. 13-15 A planet of mass m moving around the Sun in a circular orbit of radius r.

e.g. for circular orbit:
Using Newton's second law,

$$\frac{GMm}{r^2} = ma = m\omega^2 r.$$

$$\omega^2 = \frac{GM}{r^3}.$$

Since $T = \frac{2\pi}{\omega}$,

$$\boxed{T^2 = \left(\frac{4\pi^2}{GM}\right) r^3.}$$

For elliptical orbits, r is replaced by a, the semimajor axis of the ellipse. (See Appendix B.)

TABLE 13-3

Kepler's Law of Periods for the Solar System

Planet	Semimajor Axis a (10^{10} m)	Period T (y)	T^2/a^3 (10^{-34} y^2/m^3)
Mercury	5.79	0.241	2.99
Venus	10.8	0.615	3.00
Earth	15.0	1.00	2.96
Mars	22.8	1.88	2.98
Jupiter	77.8	11.9	3.01
Saturn	143	29.5	2.98
Uranus	287	84.0	2.98
Neptune	450	165	2.99
Pluto	590	248	2.99

Examples

13-6 Comet Halley orbits about the Sun with a period of 76 years and, in 1986, had a distance of closest approach to the Sun, its *perihelion distance* R_p, of 0.59 AU (between the orbits of Mercury and Venus).

(a) What is the comet's farthest distance from the Sun, its *aphelion distance* R_a (in AU)?

(b) What is the eccentricity of the orbit of comet Halley?

(1 AU = 1 Astronomical Unit = distance between Earth and Sun = 1.50×10^{11} m)

13-7 The star S2 moves around a mysterious and unobserved object called Sagittarius A*, which is at the center of the Milky Way galaxy. S2 orbits Sagittarius A* with a period of T = 15.2 y and a semimajor axis of a = 5.50 light days (=1.42 × 10^{14} m). What is the mass M of Sagittarius A* (in solar masses)?
($M_{\text{Sun}} = 1.99 \times 10^{30}$ kg)

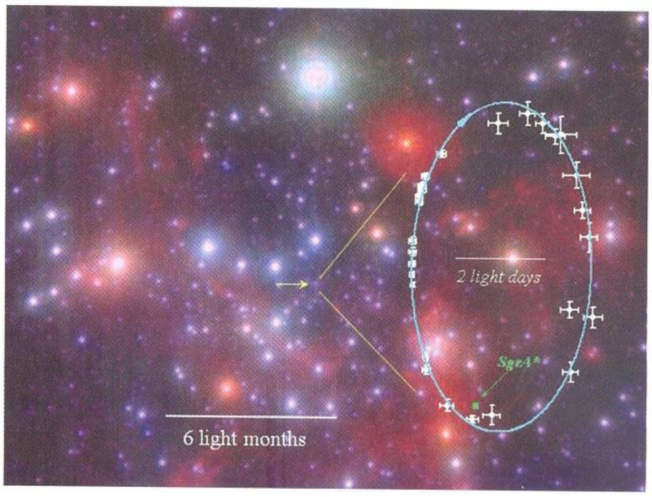


FIG. 13-16 The orbit of star S2 about Sagittarius A* (Sgr A*). The elliptical orbit appears skewed because we do not see it from directly above the orbital plane. Uncertainties in the location of S2 are indicated by the crossbars. *(Courtesy Reinhard Genzel)*

Using Kepler's law of periods,

$$T^2 = \left(\frac{4\pi^2}{GM}\right)a^3$$

$$M = \frac{4\pi^2 a^3}{GT^2}$$

$$= \frac{4\pi^2(1.42\times10^{14})^3}{(6.67\times10^{-11})[(15.2)(365)(24)(60)(60)]^2}$$

$$= 7.35\times10^{36}\text{ kg} = 3.7\times10^6 M_{\text{Sun}} \quad \text{(ans)}$$

Remark: It is believed that Sagittarius A* is a *supermassive* black hole, and most galaxies have supermassive black holes at their centers. See Youtube "Chandra X-ray Observatory images of Sagittarius A".

Satellites: Orbits and Energy

Potential energy:

$$U = -\frac{GMm}{r}$$

Kinetic energy for a circular orbit:
Using Newton's second law,

$$\frac{GMm}{r^2} = m\frac{v^2}{r}.$$

$$K = \frac{1}{2}mv^2 = \frac{GMm}{2r}.$$

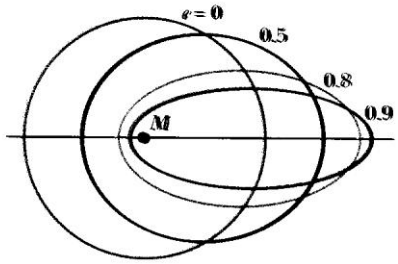


FIG. 13-17 **Four orbits with different eccentricities *e* about an object of mass *M*. All four orbits have the same semimajor axis *a* and thus correspond to the same total mechanical energy *E*.**

Therefore, for circular orbits,

$$K = -\frac{U}{2}.$$

Total mechanical energy:

$$E = K + U = -\frac{GMm}{2r}.$$

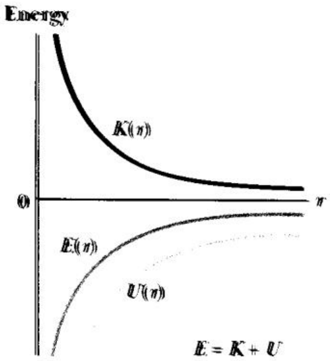


FIG. 13-18 **The variation of kinetic energy *K*, potential energy *U*, and total energy *E* with radius *r* for a satellite in a circular orbit. For any value of *r*, the values of *U* and *E* are negative, the value of *K* is positive, and $E = -K$. As $r \to \infty$, all three energy curves approach a value of zero.**

For an elliptical orbit with a semimajor axis *a*, analysis shows that

$$E = -\frac{GMm}{2a},$$

independent of the eccentricity *e*. (See Appendix C.)

Example

To launch a spacecraft from Earth to Mars, physicists suggested that the best way is to put it in a *transfer orbit* around the sun as shown in the figure, which is an elliptical orbit whose nearest point is tangent to Earth's orbit, and whose furthest point is tangent to Mars's orbit.

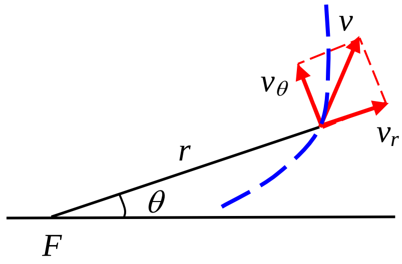
(a) Given that the period of Mars is 1.88 y, calculate the time it takes to arrive at Mars.

(b) What is the eccentricity of the transfer orbit?

(c) What is the fractional increase in the kinetic energy of the spacecraft when it transfers from Earth's orbit into the transfer orbit?

Appendix A: Deriving the Orbital Equation

Using the conservation of energy,



$$E = \frac{1}{2}m\left(v_r^2 + v_\theta^2\right) - \frac{GMm}{r}.$$

Using the conservation of angular momentum,

$$L = mrv_\theta.$$

Eliminating v_θ in the energy expression,

$$E = \frac{1}{2}mv_r^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r}.$$

This is equivalent to the motion of a single particle moving in an effective potential energy

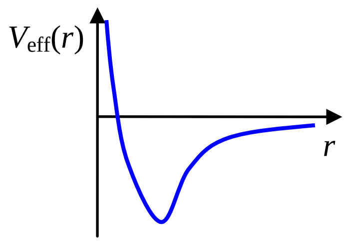
$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r}.$$

The term $L^2/2mr^2$ is sometimes called the *centrifugal energy* in the literature.

Properties of $V_{\text{eff}}(r)$: The first term dominates at small distance, and the second term dominates at large distance. There is a minimum at an intermediate distance.
Position of the minimum:

$$\frac{dV_{\text{eff}}(r)}{dr} = -\frac{L^2}{mr^3} + \frac{GMm}{r^2} = 0 \quad \Rightarrow \quad r_0 = \frac{L^2}{GMm^2}.$$

Minimum energy: $E_{\min} = -\dfrac{GMm}{2r_0}$.



There are several cases:

Case 1) $E = E_{min}$: circular orbit
Case 2) $0 < E < E_{min}$: elliptical orbit
Case 3) $E = 0$: parabolic orbit
Case 4) $E > 0$: hyperbolic orbit

To derive the orbital equation in Case 2, we note that $v_r = \frac{dr}{dt} = \left(\frac{dr}{d\theta}\right)\left(\frac{d\theta}{dt}\right)$. Since $L = mr^2 \frac{d\theta}{dt}$, we have $v_r = \frac{L}{mr^2}\left(\frac{dr}{d\theta}\right)$, and the energy equation becomes

$$E = \frac{L^2}{2mr^4}\left(\frac{dr}{d\theta}\right)^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r}.$$

Change of variable: Let $u = 1/r$. Then,

$$E = \frac{L^2}{2m}\left(\frac{du}{d\theta}\right)^2 + \frac{L^2}{2m}u^2 - GMmu.$$

Differentiating with respect to θ,

$$\frac{d^2u}{d\theta^2} + u - \frac{GMm^2}{L^2} = 0.$$

The solution to this differential equation is

$$u = \frac{GMm^2}{L^2} + C\cos\theta.$$

Substituting into the energy expression,

$$E = \frac{L^2}{2m}C^2 - \frac{G^2M^2m^3}{2L^2}.$$

After some algebra, we have

$$r = \frac{a(1-e^2)}{1-e\cos\theta},$$

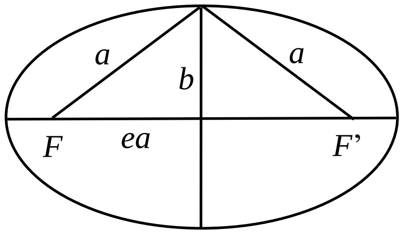
where

$$e = \sqrt{1+\frac{2L^2E}{G^2M^2m^3}},$$
$$a = -\frac{GMm}{2E}.$$

The orbital equation for Case 3 is the same, except that $E = 0$, implying that $e = 1$ (parabola).
The orbital equation for Case 4 is the same, except that $E > 0$, implying that $e > 1$ (hyperbola).

Appendix B: Deriving Kepler's Third Law for Elliptical Orbits

From Kepler's second law, we have $\frac{L}{2m} = \frac{dA}{dt} = \frac{\pi ab}{T}$, where πab is the area of an ellipse, b is the semiminor axis. Using Pythagorean theorem,



$$b = a\sqrt{1-e^2}.$$

Using the conservation of angular momentum,

$$L = mr_{\min}v_{\max} = mr_{\max}v_{\min}, \quad \Rightarrow \quad \frac{v_{\min}}{v_{\max}} = \frac{r_{\min}}{r_{\max}}.$$

Using the conservation of energy,

$$E = \frac{1}{2}mv_{\max}^2 - \frac{GMm}{r_{\min}} = \frac{1}{2}mv_{\min}^2 - \frac{GMm}{r_{\max}},$$

$$\frac{1}{2}mv_{\max}^2\left(1-\frac{v_{\min}^2}{v_{\min}^2}\right) = \frac{GMm}{r_{\min}}\left(1-\frac{r_{\min}}{r_{\max}}\right),$$

$$v_{\max}^2 = \frac{2GM}{r_{\min}}\left(1+\frac{r_{\min}}{r_{\max}}\right)^{-1} = \frac{2GM}{r_{\max}+r_{\min}}\left(\frac{r_{\max}}{r_{\min}}\right) = \frac{GM}{a}\left(\frac{r_{\max}}{r_{\min}}\right).$$

Hence, $L = mr_{\min}v_{\max} = m\sqrt{\frac{GM}{a}}\sqrt{r_{\min}r_{\max}}$. Since $r_{\min}$ = a(1 – e) and $r_{\max}$ = a(1 + e), we have $L = m\sqrt{GMa(1-e^2)}$. Finally,

$$T^2 = \left(\frac{2m}{L}\right)^2(\pi ab)^2 = \frac{4\pi^2a^4(1-e^2)}{GMa(1-e^2)} = \left(\frac{4\pi^2}{GM}\right)a^3.$$

Appendix C: Deriving the Orbital Energy for Elliptical Orbits

Using the conservation of angular momentum,

$$L = mr_{\min}v_{\max} = mr_{\max}v_{\min}, \quad \Rightarrow \quad \frac{v_{\min}}{v_{\max}} = \frac{r_{\min}}{r_{\max}}.$$

Using the conservation of energy,

$$E = \frac{1}{2}mv_{\max}^2 - \frac{GMm}{r_{\min}} = \frac{1}{2}mv_{\min}^2 - \frac{GMm}{r_{\max}},$$

$$\frac{1}{2}mv_{\max}^2\left(1 - \frac{v_{\min}^2}{v_{\min}^2}\right) = \frac{GMm}{r_{\min}}\left(1 - \frac{r_{\min}}{r_{\max}}\right),$$

$$v_{\max}^2 = \frac{2GM}{r_{\min}}\left(1 + \frac{r_{\min}}{r_{\max}}\right)^{-1} = \frac{2GM}{r_{\max} + r_{\min}}\left(\frac{r_{\max}}{r_{\min}}\right) = \frac{GM}{a}\left(\frac{r_{\max}}{r_{\min}}\right).$$

Hence,

$$E = \frac{GMm}{2a}\left(\frac{r_{\max}}{r_{\min}}\right) - \frac{GMm}{r_{\min}} = -\frac{GMm}{2ar_{\min}}(2a - r_{\max}).$$

Since $2a - r_{\max} = r_{\min}$, we have

$$E = -\frac{GMm}{2a}.$$