# Simple Harmonic Motion

Reading: Chapter 15

## Simple Harmonic Motion

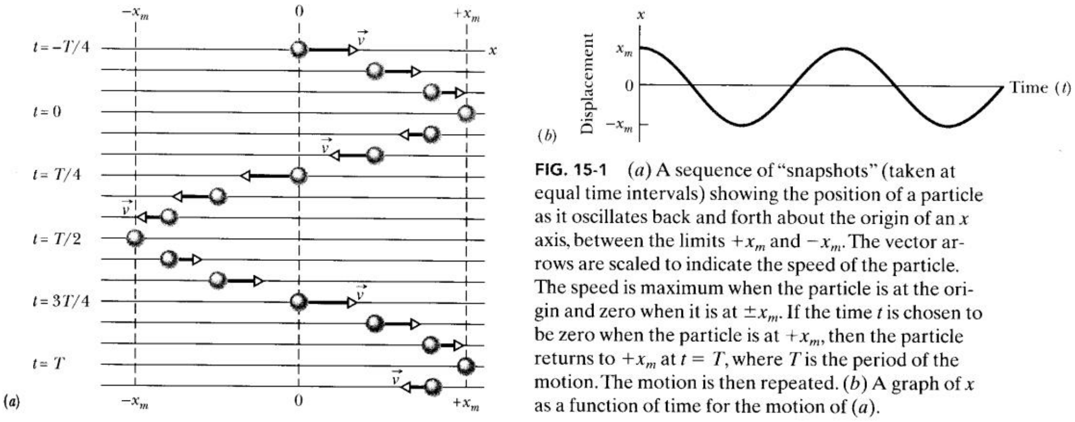


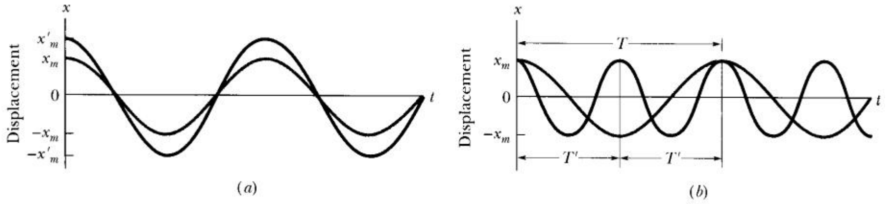

**FIG. 15-1** ($a$) A sequence of "snapshots" (taken at equal time intervals) showing the position of a particle as it oscillates back and forth about the origin of an $x$ axis, between the limits $+x_m$ and $-x_m$. The vector arrows are scaled to indicate the speed of the particle. The speed is maximum when the particle is at the origin and zero when it is at $\pm x_m$. If the time $t$ is chosen to be zero when the particle is at $+x_m$, then the particle returns to $+x_m$ at $t = T$, where $T$ is the period of the motion. The motion is then repeated. ($b$) A graph of $x$ as a function of time for the motion of ($a$).

Frequency $f$
Period $T$

$$\boxed{T = \frac{1}{f}.}$$

Simple harmonic motion

$$\boxed{x(t) = x_m \cos(\omega t + \phi).}$$



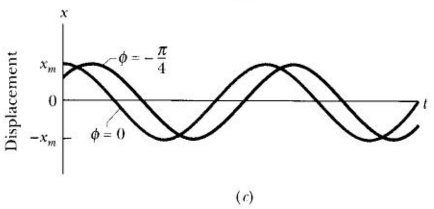




**FIG. 15-3** In all three cases, the blue curve is obtained from Eq. 15-3 with $\phi = 0$. (*a*) The red curve differs from the blue curve *only* in that the red-curve amplitude $x'_m$ is greater (the red-curve extremes of displacement are higher and lower). (*b*) The red curve differs from the blue curve *only* in that the red-curve period is $T' = T/2$ (the red curve is compressed horizontally). (*c*) The red curve differs from the blue curve *only* in that for the red curve $\phi = -\pi/4$ rad rather than zero (the negative value of $\phi$ shifts the red curve to the right).

Amplitude $x_m$
Phase $\phi$
Angular frequency $\omega$

Since the motion returns to its initial value after one period $T$,

$$x_m \cos(\omega t + \phi) = x_m \cos[\omega(t + T) + \phi)],$$
$$\omega t + \phi + 2\pi = \omega(t + T) + \phi,$$
$$\omega T = 2\pi.$$

Thus

$$\boxed{\omega = \frac{2\pi}{T} = 2\pi f\,.}$$

## Velocity

$$v(t) = \frac{dx}{dt} = \frac{d}{dt}[x_m \cos(\omega t + \phi)],$$
$$v(t) = -\omega x_m \sin(\omega t + \phi).$$

Velocity amplitude $v_m = \omega x_m$.

## Acceleration

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}[-\omega x_m \sin(\omega t + \phi)],$$
$$a(t) = -\omega^2 x_m \cos(\omega t + \phi).$$

Acceleration amplitude $a_m = \omega^2 x_m$.

Note that

$$a(t) = -\omega^2 x(t),$$
$$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0.}$$

This equation of motion will be very useful in identifying simple harmonic motion and its frequency.

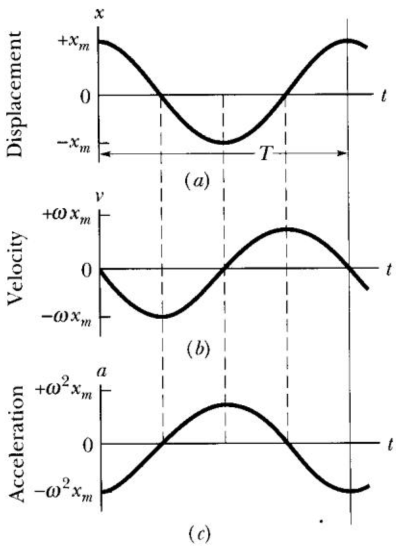



**FIG. 15-4** (*a*) The displacement $x(t)$ of a particle oscillating in SHM with phase angle $\phi$ equal to zero. The period $T$ marks one complete oscillation. (*b*) The velocity $v(t)$ of the particle. (*c*) The acceleration $a(t)$ of the particle.

## The Force Law for Simple Harmonic Motion

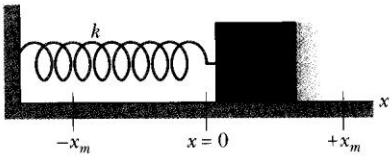



**FIG. 15-5** A linear simple harmonic oscillator. The surface is frictionless. Like the particle of Fig. 15-1, the block moves in simple harmonic motion once it has been either pulled or pushed away from the $x = 0$ position and released. Its displacement is then given by Eq. 15-3.

Consider the simple harmonic motion of a block of mass *m* subject to the elastic force of a spring. Newton's law:

$$F = -kx = ma.$$

$$m\frac{d^2x}{dt^2} + kx = 0.$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0.$$

Comparing with the equation of motion for simple harmonic motion,

$$\omega^2 = \frac{k}{m}.$$

| |
|---|
| Simple harmonic motion is the motion executed by a particle of mass *m* subject to a force that is proportional to the displacement of the particle but opposite in sign. |

Angular frequency:

$$\boxed{\omega = \sqrt{\frac{k}{m}}.}$$

Period: Since $T = \dfrac{2\pi}{\omega}$,

$$\boxed{T = 2\pi\sqrt{\frac{m}{k}}.}$$

## Examples

**15-1** A block whose mass $m$ is 680 g is fastened to a spring whose spring constant $k$ is 65 $\mathrm{Nm^{-1}}$. The block is pulled a distance $x$ = 11 cm from its equilibrium position at $x$ = 0 on a frictionless surface and released from rest at $t$ = 0.

(a) What are the angular frequency, the frequency, and the period of the resulting oscillation?
(b) What is the amplitude of the oscillation?
(c) What is the maximum speed of the oscillating block?
(d) What is the magnitude of the maximum acceleration of the block?
(e) What is the phase constant $\phi$ for the motion?
(f) What is the displacement function $x$(t)?

(a) $\omega = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{65}{0.68}} = 9.78\,\mathrm{rads^{-1}}$ (ans)

$f = \dfrac{\omega}{2\pi} = 1.56\,\mathrm{Hz}$ (ans)

$T = \dfrac{1}{f} = 0.643\,\mathrm{s}$ (ans)

(b) $x_m = 11\,\mathrm{cm}$ (ans)

(c) $v_m = \omega x_m = (9.78)(0.11) = 1.08\,\mathrm{ms^{-1}}$ (ans)

(d) $a_m = \omega^2 x_m = (9.78)^2(0.11) = 10.5\,\mathrm{ms^{-2}}$ (ans)

(e) At $t$ = 0,

$x(0) = x_m \cos\phi = 0.11$ (1)

$v(0) = -\omega x_m \sin\phi = 0$ (2)

(2): $\sin\phi = 0 \quad \Rightarrow \quad \phi = 0$ (ans)

(f) $x(t) = x_m \cos(\omega t + \phi) = 0.11\cos(9.78t)$ (ans)

**15-2** At $t$ = 0, the displacement of $x(0)$ of the block in a linear oscillator is –8.50 cm. Its velocity $v(0)$ then is –0.920 $ms^{-1}$, and its acceleration $a(0)$ is +47.0 $ms^{-2}$.
(a) What are the angular frequency $\omega$?
(b) What is the phase constant $\phi$ and amplitude $x_m$?

(a) $x(t) = x_m \cos(\omega t + \phi)$

$v(t) = -\omega x_m \sin(\omega t + \phi)$

$a(t) = -\omega^2 x_m \cos(\omega t + \phi)$

At $t$ = 0,

$$x(0) = x_m \cos\phi = -0.085 \qquad (1)$$

$$v(0) = -\omega x_m \sin\phi = -0.920 \qquad (2)$$

$$a(0) = -\omega^2 x_m \cos\phi = +47.0 \qquad (3)$$

(3) ÷ (1): $\dfrac{a(0)}{x(0)} = -\omega^2$

$$\omega = \sqrt{-\frac{a(0)}{x(0)}} = \sqrt{-\frac{47.0}{-0.0850}} = 23.5\,\text{rads}^{-1} \quad \text{(ans)}$$

(c) (2) ÷ (1): $\dfrac{v(0)}{x(0)} = -\omega\dfrac{\sin\phi}{\cos\phi} = -\omega\tan\phi$

$$\tan\phi = -\frac{v(0)}{\omega x(0)} = -\frac{-0.920}{(23.51)(-0.085)} = -0.4603$$

$$\phi = -24.7^{o} \quad \text{or} \quad \phi = 180^{o} - 24.7^{o} = 155^{o}$$

(1): $x_m = \dfrac{x(0)}{\cos\phi}$

If $\phi = -24.7^{o}$, $x_m = \dfrac{-0.085}{\cos 24.7^{o}} = -0.094\,\text{m} = -9.4\,\text{cm}$

If $\phi = 155^{o}$, $x_m = \dfrac{-0.085}{\cos 155^{o}} = 0.094\,\text{m} = 9.4\,\text{cm}$

Since $x_m$ is positive, $\phi = 155^{o}$ and $x_m$ = 9.4 cm. (ans)

**Energy in Simple Harmonic Motion**

Potential energy:

Since $x(t) = x_m \cos(\omega t + \phi)$,

$$U(t) = \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2 \cos^2(\omega t + \phi).$$

Kinetic energy:

Since $v(t) = -\omega x_m \sin(\omega t + \phi)$,

$$K(t) = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 x_m^2 \sin^2(\omega t + \phi).$$

Since $\omega^2 = k/m$,

$$K(t) = \frac{1}{2}kx_m^2 \sin^2(\omega t + \phi).$$

Mechanical energy:

$$\begin{aligned} E = U + K &= \frac{1}{2}kx_m^2 \cos^2(\omega t + \phi) + \frac{1}{2}kx_m^2 \sin^2(\omega t + \phi) \\ &= \frac{1}{2}kx_m^2[\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)]. \end{aligned}$$

Since $\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi) = 1$,

$$\boxed{E = U + K = \frac{1}{2}kx_m^2.}$$

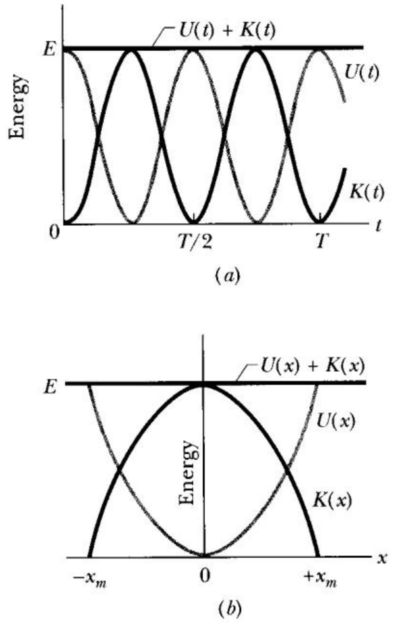



**FIG. 15-6** (*a*) Potential energy $U(t)$, kinetic energy $K(t)$, and mechanical energy $E$ as functions of time $t$ for a linear harmonic oscillator. Note that all energies are positive and that the potential energy and the kinetic energy peak twice during every period. (*b*) Potential energy $U(x)$, kinetic energy $K(x)$, and mechanical energy $E$ as functions of position $x$ for a linear harmonic oscillator with amplitude $x_m$. For $x = 0$ the energy is all kinetic, and for $x = \pm x_m$ it is all potential.

The mechanical energy is conserved.

**15-3** Suppose the damper of a tall building has mass $m$ = $2.72 \times 10^5$ kg and is designed to oscillate at frequency $f$ = 10 Hz and with amplitude $x_m$ = 20 cm.
(a) What is the total mechanical energy $E$ of the damper?
(b) What is the speed of the damper when it passes through the equilibrium point?

See Youtube "Discovery Channel Taipei 101 (3/5)" and "Taipei 101 Damper"

(a) $k = m\omega^2 = m(2\pi f)^2$

$= (2.72 \times 10^5)(20\pi)^2 = 1.073 \times 10^9$ N

The energy:

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$= 0 + \frac{1}{2}(1.073 \times 10^9)(0.2)^2$$

$= 2.147 \times 10^7$ J $\approx 21.5$ MJ (ans)

(b) Using the conservation of energy,

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$2.147 \times 10^7 = \frac{1}{2}(2.72 \times 10^5)v^2 + 0$$

$v = 12.6\ \mathrm{ms}^{-1}$ (ans)

## An Angular Simple Harmonic Oscillator

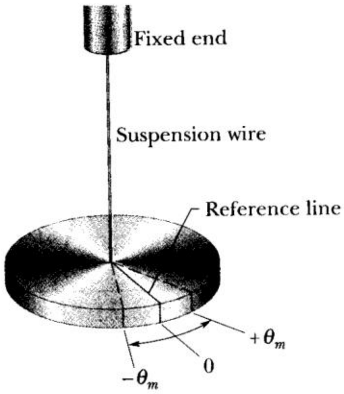



**FIG. 15-7** A torsion pendulum is an angular version of a linear simple harmonic oscillator. The disk oscillates in a horizontal plane; the reference line oscillates with angular amplitude $\theta_m$. The twist in the suspension wire stores potential energy as a spring does and provides the restoring torque.

When the suspension wire is twisted through an angle $\theta$, the **torsional pendulum** produces a restoring torque given by

$$\tau = -\kappa\theta.$$

$\kappa$ is called the torsion constant.
Using Newton's law for angular motion, $\tau = I\alpha$,

$$-\kappa\theta = I\alpha, \qquad \frac{d^2\theta}{dt^2} + \frac{\kappa}{I}\theta = 0.$$

Comparing with the equation of motion for simple harmonic motion,

$$\omega^2 = \frac{\kappa}{I}.$$

Period: Since $T = \dfrac{2\pi}{\omega}$,

$$T = 2\pi\sqrt{\frac{I}{\kappa}}.$$

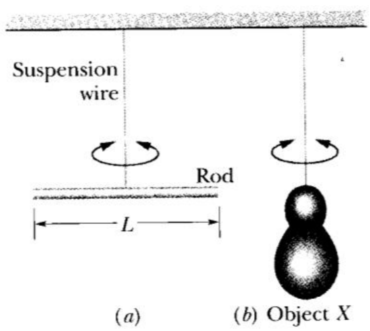
**Example**

**15-4** A thin rod whose length $L$ is 12.4 cm and whose mass $m$ is 135 g is suspended at its midpoint from a long wire. Its period $T_a$ of angular SHM is measured to be 2.53 s. An irregularly shaped object, which we call $X$, is then hung from the same wire, and its period $T_b$ is found to be 4.76 s. What is the rotational inertia of object $X$ about its suspension axis?

Rotational inertia of the rod about the center

$$= I_a = \frac{1}{12} ML^2$$

IG. 15-8 Two torsion endulums, consisting of a) a wire and a rod and b) the same wire and an rregularly shaped ob- ect.




$$= \left(\frac{1}{12}\right)(0.135)(0.124)^2$$

$$= 1.7298 \times 10^{-4} \text{ kgm}^2$$

Since $T_a = 2\pi\sqrt{\frac{I_a}{\kappa}}$ and $T_b = 2\pi\sqrt{\frac{I_b}{\kappa}}$, we have

$$\frac{T_a}{T_b} = \sqrt{\frac{I_a}{I_b}}$$

Therefore,

$$I_b = \left(\frac{T_b}{T_a}\right)^2 I_a$$

$$= \left(\frac{4.76}{2.53}\right)^2 (1.73 \times 10^{-4}) = 6.12 \times 10^{-4} \text{ kgm}^2 \quad \text{(ans)}$$

# The Simple Pendulum

The restoring torque about the point of suspension is $\tau = -mg \sin\theta L$.
Using Newton's law for angular motion, $\tau = I\alpha$,

$$-mg\sin\theta L = mL^2\alpha,$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0.$$

When the pendulum swings through a small angle, $\sin\theta \approx \theta$. Therefore

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0.$$

Comparing with the equation of motion for simple harmonic motion,

$$\omega^2 = \frac{g}{L}.$$

Period: Since $T = \dfrac{2\pi}{\omega}$,

$$T = 2\pi\sqrt{\frac{L}{g}}.$$

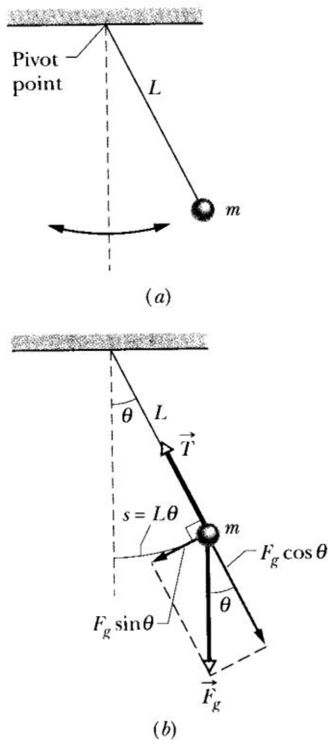



FIG. 15-9 (*a*) A simple pendulum. (*b*) The forces acting on the bob are the gravitational force $\vec{F}_g$ and the force $\vec{T}$ from the string. The tangential component $F_g \sin\theta$ of the gravitational force is a restoring force that tends to bring the pendulum back to its central position.

**The Physical Pendulum**

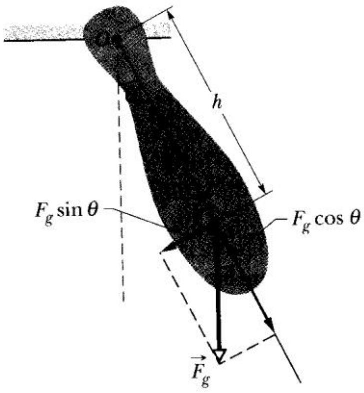



**FIG. 15-10** A physical pendulum. The restoring torque is $hF_g \sin\theta$. When $\theta = 0$, center of mass $C$ hangs directly below pivot point $O$.

The restoring torque about the point of suspension is $\tau = -mg \sin\theta\, h$.
Using Newton's law for angular motion, $\tau = I\alpha$,

$$-mg\sin\theta\, h = I\alpha, \qquad \frac{d^2\theta}{dt^2} + \frac{mgh}{I}\sin\theta = 0.$$

When the pendulum swings through a small angle, $\sin\theta \approx \theta$. Therefore

$$\frac{d^2\theta}{dt^2} + \frac{mgh}{I}\theta = 0.$$

Comparing with the equation of motion for simple harmonic motion,

$$\omega^2 = \frac{mgh}{I}.$$

Period: Since $T = \dfrac{2\pi}{\omega}$,

$$T = 2\pi\sqrt{\frac{I}{mgh}}.$$

If the mass is concentrated at the center of mass $C$, such as in the simple pendulum, then

$$T = 2\pi\sqrt{\frac{I}{mgh}} = 2\pi\sqrt{\frac{mL^2}{mgL}} = 2\pi\sqrt{\frac{L}{g}}.$$

We recover the result for the simple pendulum.

**Examples**

**15-5** A meter stick, suspended from one end, swings as a physical pendulum.
(a) What is its period of oscillation $T$?
(b) A simple pendulum oscillates with the same period as the stick. What is the length $L_0$ of the simple pendulum?

Rotational inertia of a rod about one end

$$= \frac{1}{3}ML^2$$

Period $T = 2\pi\sqrt{\frac{I}{mgh}}$

$$= 2\pi\sqrt{\frac{mL^2/3}{mgL/2}}$$

$$= 2\pi\sqrt{\frac{2L}{3g}} \quad \text{(ans)}$$

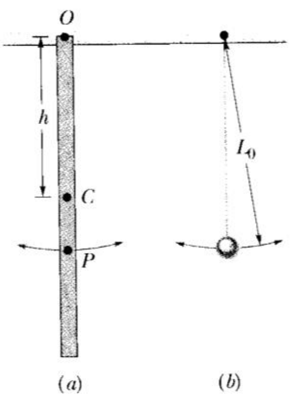


FIG. 15-11 (a) A meter stick suspended from one end as a physical pendulum. (b) A simple pendulum whose length $L_0$ is chosen so that the periods of the two pendulums are equal. Point $P$ on the pendulum of (a) marks the center of oscillation

(b) For a simple pendulum of length $L_0$,

$$T = 2\pi\sqrt{\frac{L_0}{g}}$$

$$2\pi\sqrt{\frac{L_0}{g}} = 2\pi\sqrt{\frac{2L}{3g}} \quad \Rightarrow \quad L_0 = \frac{2}{3}L = 66.7\ \text{cm} \quad \text{(ans)}$$

**15-6** A diver steps on the diving board and makes it move downwards. As the board rebounds back through the horizontal, she leaps upward and lands on the free end just as the board has completed 2.5 oscillations during the leap. (With such timing, the diver lands when the free end is moving downward with greatest speed. The landing then drives the free end down substantially, and the rebound catapults the diver high into the air.) Modeling the spring board as the rod-spring system (Fig. 15-12(d)), what is the required spring constant *k*? Given *m* = 20 kg, diver's leaping time $t_{fl}$ = 0.62 s.

See Youtube "Guo Jingjing".

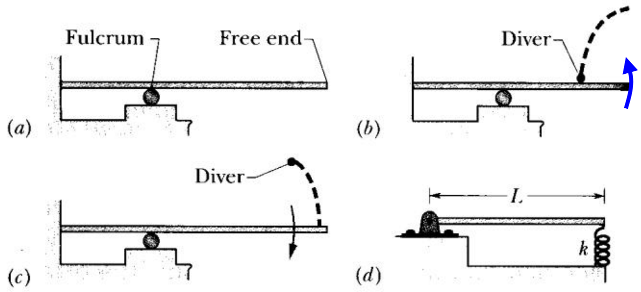



**FIG. 15-12** (*a*) A diving board. (*b*) The diver leaps upward and forward as the board moves through the horizontal. (*c*) The diver lands 2.5 oscillations later. (*d*) A spring-oscillator model of the oscillating board.

## Damped Simple Harmonic Motion

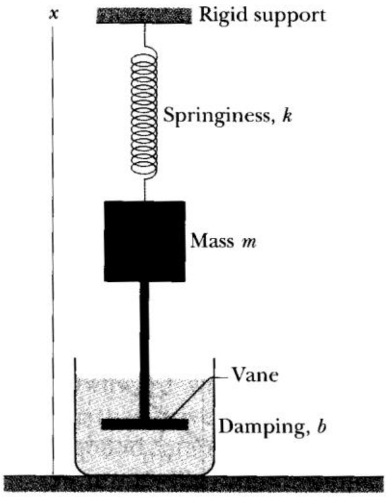


FIG. 15-15 An idealized damped simple harmonic oscillator. A vane immersed in a liquid exerts a damping force on the block as the block oscillates parallel to the x axis.

The liquid exerts a damping force proportional to the velocity. Then,

$$F_d = -bv,$$

$b$ = damping constant.
Using Newton's second law,

$$-bv - kx = ma.$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0.$$

Solution:
$x(t) = x_m e^{-bt/2m}\cos(\omega' t + \phi)$, where

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}.$$

If $b = 0$, $\omega'$ reduces to $\omega = \sqrt{k/m}$ of the undamped oscillator. If $b << \sqrt{km}$, then $\omega' \approx \omega$.
The amplitude, $x(t) = x_m e^{-bt/2m}$, gradually decreases with time.
The mechanical energy decreases exponentially with time.

$$E(t) = \frac{1}{2}kx_m^2 e^{-bt/m}.$$

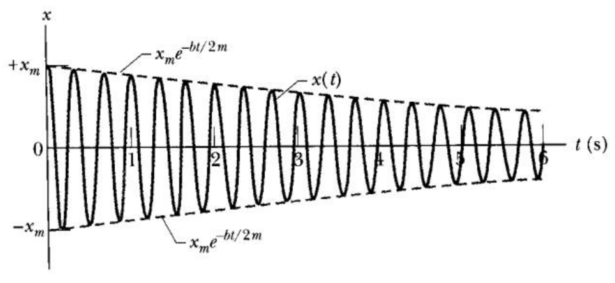


FIG. 15-16 The displacement function $x(t)$ for the damped oscillator of Fig. 15-15, with values given in Sample Problem 15-7. The amplitude, which is $x_m e^{-bt/2m}$, decreases exponentially with time.

**Example**

**15-7** For the damped oscillator with $m$ = 250 g, $k$ = 85 Nm$^{-1}$, and $b$ = 70 gs$^{-1}$.
(a) What is the period of the motion?
(b) How long does it take for the amplitude of the damped oscillations to drop to half its initial value?
(c) How long foes it take for the mechanical energy to drop to half its initial value?

(a) $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.25}{85}} = 0.34\,\text{s}$ (ans)

(b) When the amplitude drops by half,

$$x_m e^{-bt/2m} = \frac{1}{2}x_m$$

$$e^{-bt/2m} = \frac{1}{2}$$

Taking logarithm, $-\frac{bt}{2m} = \ln\frac{1}{2} = -\ln 2$

$$t = \frac{2m\ln 2}{b} = \frac{(2)(0.25)(\ln 2)}{0.07} = 4.95\,\text{s} \quad \text{(ans)}$$

(c) When the energy drops by half,

$$\frac{1}{2}kx_m^2 e^{-bt/m} = \frac{1}{2}\left(\frac{1}{2}kx_m^2\right)$$

$$e^{-bt/m} = \frac{1}{2}$$

Taking logarithm, $-\frac{bt}{m} = \ln\frac{1}{2} = -\ln 2$

$$t = \frac{m\ln 2}{b} = \frac{(0.25)(\ln 2)}{0.07} = 2.48\,\text{s} \quad \text{(ans)}$$

## Forced Oscillations and Resonance

When a simple harmonic oscillator is driven by a periodic external force, we have *forced oscillations* or *driven oscillations*.
Its behavior is determined by *two* angular frequencies:
(1) the *natural* angular frequency $\omega$
(2) the angular frequency $\omega_d$ of the external driving force.

The motion of the forced oscillator is given by

$$x(t) = x_m \cos(\omega_d t + \phi).$$

(1) It oscillates at the angular frequency $\omega_d$ of the external driving force.
(2) Its amplitude $x_m$ is greatest when

$$\omega_d = \omega.$$

This is called **resonance**.
See Youtube "Tacoma Bridge Disaster".

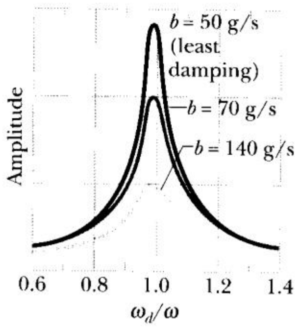



**FIG. 15-17** The displacement amplitude $x_m$ of a forced oscillator varies as the angular frequency $\omega_d$ of the driving force is varied. The curves here correspond to three values of the damping constant $b$.