Transverse Waves

Reading: Chapter 16

Waves

1. Mechanical waves: e.g. water waves, sound waves, seismic waves

2. Electromagnetic waves: light, ultraviolet light, radio and television waves, microwaves, x rays, radar waves

3. Matter waves: electrons, protons, other fundamental particles, atoms and molecules

Transverse and Longitudinal Waves

See animation "Travelling Waves"

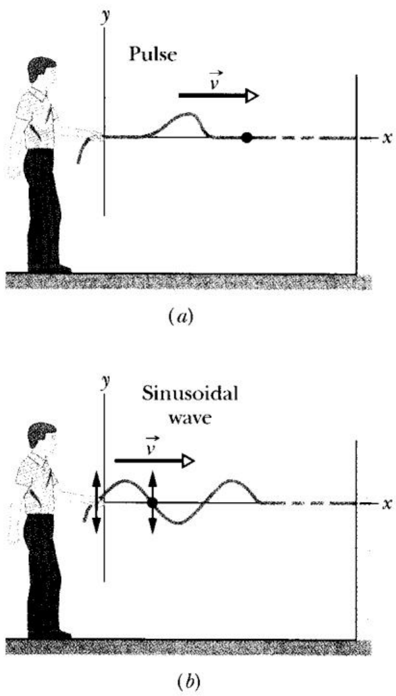


FIG. 16-1 (*a*) A single pulse is sent along a stretched string. A typical string element (marked with a dot) moves up once and then down as the pulse passes. The element's motion is perpendicular to the wave's direction of travel, so the pulse is a *transverse wave*. (*b*) A sinusoidal wave is sent along the string. A typical string element moves up and down continuously as the wave passes. This too is a transverse wave.

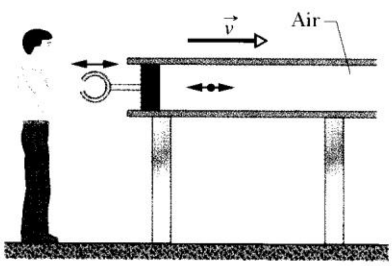


FIG. 16-2 A sound wave is set up in an air-filled pipe by moving a piston back and forth. Because the oscillations of an element of the air (represented by the dot) are parallel to the direction in which the wave travels, the wave is a *longitudinal wave*.

Transverse waves: the displacement of a point on the string is *perpendicular* to the direction of travel of the wave.

Longitudinal waves: the motion of a particle is *parallel* to the direction of travel of the wave.

Wavelength and Frequency

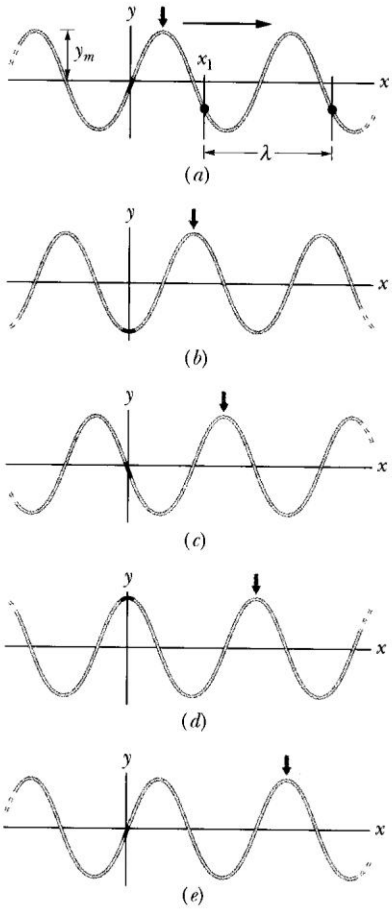


FIG. 16-5 Five "snapshots" of a string wave traveling in the positive direction of an x axis. The amplitude y_m is indicated. A typical wavelength λ, measured from an arbitrary position x_1, is also indicated.

Suppose that at $t = 0$, a travelling wave has the form

$$y(x,0) = y_m \sin kx.$$

At time t, the travelling wave will have the same form, except that it is displaced along the x direction by a displacement vt, where v = wave speed.

Hence the displacement at position x and time t is given by

$$y(x,t) = y_m \sin[k(x - vt)].$$

This is usually written as

$$\boxed{y(x,t) = y_m \sin(kx - \omega t),}$$

where $\omega = kv$, or $\boxed{v = \frac{\omega}{k}.}$

Amplitude y_m
Phase $kx - \omega t$

Wavenumber k:

At t = 0,

$$y(x,0) = y_m \sin kx.$$

Since the waveform repeats itself when displaced by one wavelength,

$$y_m \sin kx = y_m \sin[k(x+\lambda)] = y_m \sin(kx + k\lambda).$$

Thus, $k\lambda = 2\pi$,

$$\boxed{k = \frac{2\pi}{\lambda}.}$$

Angular frequency ω:

At x = 0,

$$y(0,t) = -y_m \sin \omega t.$$

Since the waveform repeats itself when delayed by one period,

$$-y_m \sin \omega t = -y_m \sin[\omega(t+T)] = -y_m \sin(\omega t + \omega T).$$

Thus, $\omega T = 2\pi$,

$$\boxed{\omega = \frac{2\pi}{T}.}$$

Frequency f:

$$\boxed{f = \frac{1}{T} = \frac{\omega}{2\pi}.}$$

Wave Speed

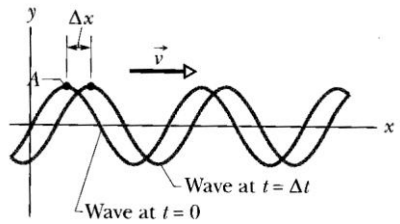


FIG. 16-8 Two snapshots of the wave of Fig. 16-5, at time $t = 0$ and then at time $t = \Delta t$. As the wave moves to the right at velocity $\vec{v}$, the entire curve shifts a distance Δx during Δt. Point A "rides" with the wave form, but the string elements move only up and down.

Since $k = 2\pi/\lambda$ and $\omega = 2\pi/T$,

$$\boxed{v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f.}$$

$v = \lambda/T$ means that the wave travels by a distance of one wavelength in one period.
Since $y(x, t) = y_m\sin(kx - \omega t)$, the peak of the travelling wave is described by:

$$kx - \omega t = \frac{\pi}{2}.$$

In general, any point on the waveform, as the wave moves in space and time, is described by:

$$kx - \omega t = \text{constant}.$$

Travelling wave in the opposite direction:

$$y(x, t) = y_m \sin[k(x + vt)] = y_m \sin(kx + \omega t).$$

A point on the waveform, as the wave moves in space and time, is described by $kx + \omega t$ = constant.

Example

16-2,3 A transverse wave travelling along a string is described by $y(x, t) = 0.00327\sin(72.1x - 2.72t)$, in which the numerical constants are in SI units.

(a) What is the amplitude of this wave?

(b) What are the wavelength, period, and frequency of this wave?

(c) What is the velocity of this wave?

(d) What is the displacement y at x = 22.5 cm and t = 18.9 s?

(e) What is the transverse velocity u of this element of the string, at that place and at that time?

(f) What is the transverse acceleration a_y at that position and at that time?

(a) $y_m = 0.00327 = 3.27$ mm (ans)

(b) $$\lambda = \frac{2\pi}{k} = \frac{2\pi}{72.1} = 0.0871 = 8.71\,\text{cm} \quad \text{(ans)}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2.72} = 2.31\,\text{s} \quad \text{(ans)}$$

$$f = \frac{1}{T} = \frac{1}{2.31} = 0.433\,\text{Hz} \quad \text{(ans)}$$

(c) $$v = \frac{\omega}{k} = \frac{2.72}{72.1} = 0.0377 = 3.77\,\text{cms}^{-1} \quad \text{(ans)}$$

(d) $$y(x,t) = 0.00327\sin(72.1\times 0.225 - 2.72\times 18.9)$$
$$= 0.00192 = 1.92\,\text{mm} \quad \text{(ans)}$$

(e) $$u = \frac{\partial y}{\partial t} = -\omega y_m \cos(kx - \omega t)$$
$$= -(2.72)(0.00327)\sin(72.1\times 0.225 - 2.72\times 18.9)$$
$$= 7.20\,\text{mms}^{-1} \quad \text{(ans)}$$

(f) $$a_y = \frac{\partial u}{\partial t} = -\omega^2 y_m \sin(kx - \omega t) = -\omega^2 y$$
$$= -(2.72)^2(0.00192) = -0.0142 = -14.2\,\text{mms}^{-2} \quad \text{(ans)}$$

Wave Speed on a Stretched String

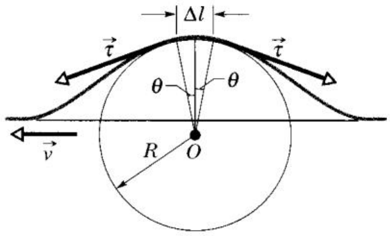


FIG. 16-10 A symmetrical pulse, viewed from a reference frame in which the pulse is stationary and the string appears to move right to left with speed v. We find speed v by applying Newton's second law to a string element of length Δl, located at the top of the pulse.

Consider the peak of a wave travelling from left to right on the stretched string.

If we observe the wave from a reference frame moving at the wave speed *v*, the peak becomes stationary, but the string moves from right to left with speed *v*.

Consider a small segment of length Δl at the peak.

Let τ be the tension in the string.

Vertical component of the force on the element:

$$F = 2\tau \sin\theta \approx \tau(2\theta) = \tau\frac{\Delta l}{R},$$

where *R* is the radius of curvature.

Mass of the segment: $\Delta m = \mu\Delta l$.

Its centripetal acceleration: $a = \dfrac{v^2}{R}$.

Using Newton's second law, $F = \Delta m \times a$,

$$\tau\frac{\Delta l}{R} = (\mu\Delta l)\frac{v^2}{R}.$$

This reduces to

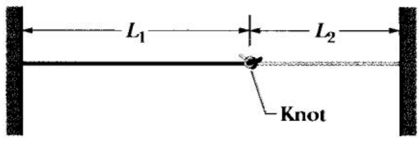
$$\boxed{v = \sqrt{\frac{\tau}{\mu}}.}$$

Note that τ represents the elastic property of the stretched string, and μ represents its inertial property.

Example

16-4 Two strings have been tied together with a knot and then stretched between two rigid supports. The strings have linear densities $\mu_1 = 1.4 \times 10^{-4}$ kgm^{-1} and $\mu_2 = 2.8 \times 10^{-4}$ kgm^{-1}. Their lengths are L_1 = 3 m and L_2 = 2 m, and string 1 is under a tension of 400 N. Simultaneously, on each string a pulse is sent from the rigid support end, towards the knot. Which pulse reaches the knot first?

FIG. 16-11 Two strings, of lengths L_1 and L_2, tied together with a knot and stretched between two rigid supports.



$$v_1 = \sqrt{\frac{\tau}{\mu_1}}$$

$$t_1 = \frac{L_1}{v_1} = L_1\sqrt{\frac{\mu_1}{\tau}} = 3\sqrt{\frac{1.4\times10^{-4}}{400}} = 1.77\times10^{-3}\ \text{s}$$

$$v_2 = \sqrt{\frac{\tau}{\mu_2}}$$

$$t_2 = \frac{L_2}{v_2} = L_2\sqrt{\frac{\mu_2}{\tau}} = 2\sqrt{\frac{2.8\times10^{-4}}{400}} = 1.67\times10^{-3}\ \text{s}$$

Thus, the pulse on string 2 reaches the knot first.

Energy and Power of a Travelling String Wave

Kinetic energy:

Consider a string element of mass dm. Kinetic energy:

$$dK = \frac{1}{2} dm u^2.$$

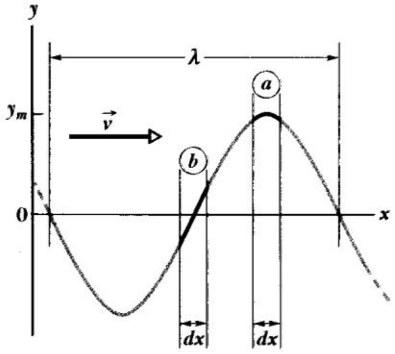


FIG. 16-12 A snapshot of a traveling wave on a string at time $t = 0$. String element a is at displacement $y = y_m$, and string element b is at displacement $y = 0$. The kinetic energy of the string element at each position depends on the transverse velocity of the element. The potential energy depends on the amount by which the string element is stretched as the wave passes through it.

Since $y(x, t) = y_m \sin(kx - \omega t)$,

$$u = \frac{\partial y}{\partial t} = -\omega y_m \cos(kx - \omega t).$$

Since $dm = \mu dx$,

$$dK = \frac{1}{2}(\mu dx)(-\omega y_m)^2 \cos^2(kx - \omega t).$$

Rate of kinetic energy transmission:

$$\frac{dK}{dt} = \frac{1}{2}\mu\omega^2 y_m^2 \cos^2(kx - \omega t)\frac{dx}{dt}.$$

Using $v = dx/dt$,

$$\frac{dK}{dt} = \frac{1}{2}\mu v\omega^2 y_m^2 \cos^2(kx - \omega t).$$

Kinetic energy is maximum at the $y = 0$ position.

Potential energy:

Potential energy is carried in the string when it is stretched.
Stretching is largest when the displacement has the largest gradient.
Hence, the potential energy is also maximum at the $y = 0$ position. This is different from the harmonic oscillator, in which case energy is conserved.
Detailed mathematics shows that:

$$\frac{dU}{dt} = \frac{1}{2}\mu v\omega^2 y_m^2 \cos^2(kx - \omega t) = \frac{dK}{dt}.$$

Mechanical energy:

$$\frac{dE}{dt} = \frac{dK}{dt} + \frac{dU}{dt} = \mu v\omega^2 y_m^2 \cos^2(kx - \omega t).$$

Average power of transmission:

$$\langle P \rangle = \left\langle \frac{dE}{dt} \right\rangle = \mu v\omega^2 y_m^2 \langle \cos^2(kx - \omega t) \rangle,$$

where $\langle \ldots \rangle$ represents averaging over time. Since $\langle \cos^2(kx - \omega t) \rangle = 1/2$, average power:

$$\boxed{\langle P \rangle = \frac{1}{2}\mu v\omega^2 y_m^2.}$$

Example

16-5 A string has a linear density μ of 525 g/m and is stretched with a tension τ of 45 N. A wave whose frequency f and amplitude y_m are 120 Hz and 8.5 mm, respectively, is travelling along the string. At what average rate is the wave transporting energy along the string?

$$\omega = 2\pi f = (2\pi)(120) = 754.0\,\text{rads}^{-1}$$

$$v = \sqrt{\frac{\tau}{\mu}} = \sqrt{\frac{45}{0.525}} = 9.258\,\text{ms}^{-1}$$

$$\langle P \rangle = \frac{1}{2}\mu v \omega^2 y_m^2$$

$$= \left(\frac{1}{2}\right)(0.525)(9.258)(754.0)^2(0.0085)^2$$

$$= 99.8\,\text{W} \quad \text{(ans)}$$

The Principle of Superposition for Waves

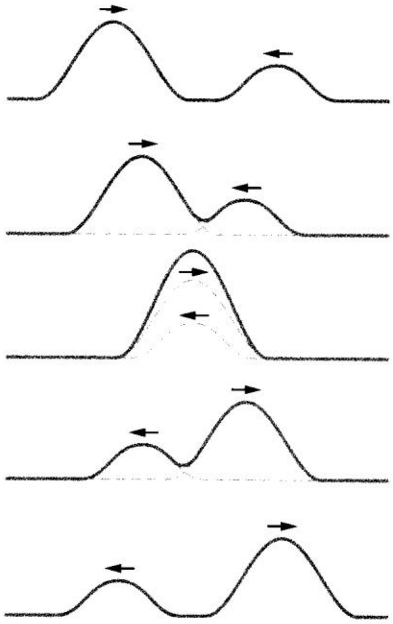


FIG. 16-14 A series of snapshots that show two pulses traveling in opposite directions along a stretched string. The superposition principle applies as the pulses move through each other.

Overlapping waves algebraically add to produce a resultant wave.

$$y'(x,t) = y_1(x,t) + y_2(x,t).$$

Overlapping waves do not in any way alter the travel of each other.

Interference of Waves

Suppose we send two sinusoidal waves of the same wavelength and amplitude in the same direction along a stretched string.

$$y_1(x,t) = y_m \sin(kx - \omega t),$$
$$y_2(x,t) = y_m \sin(kx - \omega t + \phi).$$

ϕ is called the *phase difference* or *phase shift* between the two waves. Combined displacement:

$$y'(x,t) = y_m \sin(kx - \omega t) + y_m \sin(kx - \omega t + \phi).$$

Using the trigonometric identity

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right),$$

we obtain

$$y'(x,t) = \left[2y_m \cos\frac{\phi}{2}\right]\sin\left(kx - \omega t + \frac{\phi}{2}\right).$$

The resultant wave:

(1) is also a travelling wave in the same direction,

(2) has a phase constant of $\phi/2$,

(3) has an amplitude of $y'_m = 2y_m\cos(\phi/2)$.

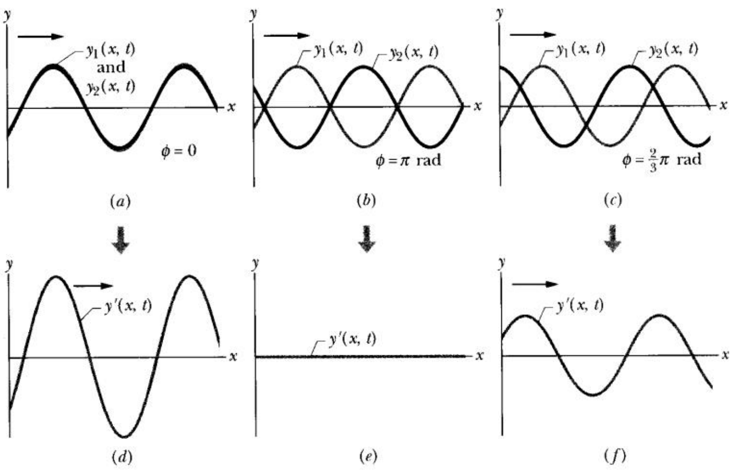


FIG. 16-16 Two identical sinusoidal waves, $y_1(x, t)$ and $y_2(x, t)$, travel along a string in the positive direction of an x axis. They interfere to give a resultant wave $y'(x, t)$. The resultant wave is what is actually seen on the string. The phase difference ϕ between the two interfering waves is (a) 0 rad or 0°, (b) π rad or 180°, and (c) $\frac{2}{3}\pi$ rad or 120°. The corresponding resultant waves are shown in (d), (e), and (f).

Fully constructive interference: If $\phi = 0$, the amplitude is maximum:

$$y'(x,t) = 2y_m \sin(kx - \omega t).$$

Fully destructive interference: If $\phi = \pi$,

$$y'(x,t) = 0.$$

Intermediate interference: If ϕ is between 0 and π, the amplitude is intermediate.

TABLE 16-1

Phase Difference and Resulting Interference Types[a]

Phase Difference, in			Amplitude of Resultant Wave	Type of Interference
Degrees	Radians	Wavelengths		
0	0	0	$2y_m$	Fully constructive
120	$\frac{2}{3}\pi$	0.33	y_m	Intermediate
180	π	0.50	0	Fully destructive
240	$\frac{4}{3}\pi$	0.67	y_m	Intermediate
360	2π	1.00	$2y_m$	Fully constructive
865	15.1	2.40	$0.60y_m$	Intermediate

[a]The phase difference is between two otherwise identical waves, with amplitude y_m, moving in the same direction.

Example

16-6 Two identical sinusoidal waves, moving in the same direction along a stretched string, interfere with each other. The amplitude y_m of each wave is 9.8 mm, and the phase difference ϕ between them is 100°.
(a) What is the amplitude y'_m of the resultant wave due to the interference, and what is the type of this interference?
(b) What phase difference, in radians and wavelengths, will give the resultant wave an amplitude of 4.9 mm?

(a) $y'_m = \left|2y_m \cos\frac{\phi}{2}\right| = \left|(2)(9.8\,\text{mm})\cos\left(\frac{100^{\circ}}{2}\right)\right|$

$= 13\,\text{mm} \quad \text{(ans)}$

(b) $y'_m = \left|2y_m \cos\frac{\phi}{2}\right|$

$4.9 = (2)(9.8)\cos\frac{\phi}{2}$

$\phi = 2\cos^{-1}\left(\frac{4.9}{(2)(9.8)}\right) = \pm 2.636\,\text{rad} \approx \pm 2.6\,\text{rad} \quad \text{(ans)}$

ϕ = +2.6 rad: The second wave *leads* (travels ahead of) the first wave.
ϕ = –2.6 rad: The second wave *lags* (travels behind) the first wave.
In wavelengths, the phase difference is

$\pm\frac{2.636}{2\pi} = \pm 0.42\ \text{wavelength} \quad \text{(ans)}$

Standing Waves

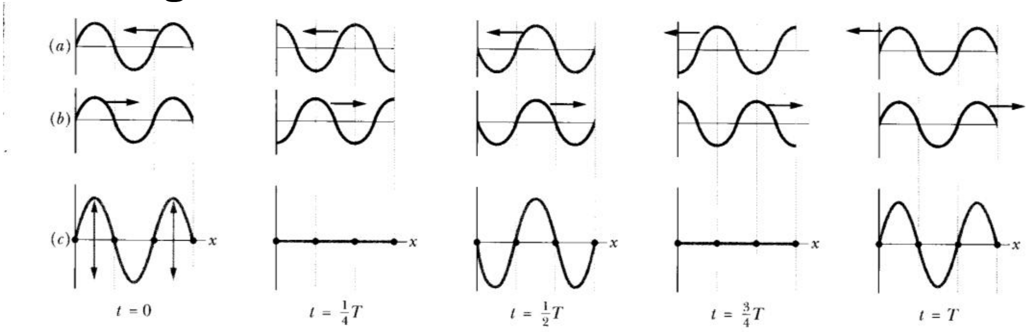


FIG. 16-19 (*a*) Five snapshots of a wave traveling to the left, at the times t indicated below part (*c*) (T is the period of oscillation). (*b*) Five snapshots of a wave identical to that in (*a*) but traveling to the right, at the same times t. (*c*) Corresponding snapshots for the superposition of the two waves on the same string. At $t = 0, \frac{1}{2}T$, and T, fully constructive interference occurs because of the alignment of peaks with peaks and valleys with valleys. At $t = \frac{1}{4}T$ and $\frac{3}{4}T$, fully destructive interference occurs because of the alignment of peaks with valleys. Some points (the nodes, marked with dots) never oscillate; some points (the antinodes) oscillate the most.

See animation "Two Waves on the Same String"
Consider two sinusoidal waves of the same wavelength and amplitude travelling in the *opposite* direction along a stretched string.

$$y_1(x,t) = y_m \sin(kx - \omega t),$$
$$y_2(x,t) = y_m \sin(kx + \omega t).$$

Combined displacement:

$$y'(x,t) = y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t).$$

Using the trigonometric identity

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right),$$

we obtain

$$y'(x,t) = [2y_m \sin kx]\cos\omega t.$$

Properties:

(1) The resultant wave is *not* a travelling wave, but is a standing wave, e.g. the locations of the maxima and minima do not change,

(2) There are positions where the string is permanently at rest. They are called **nodes**, and are located at

$$kx = n\pi, \quad \text{for} \quad n = 0, 1, 2, \cdots$$

$$x = \frac{n\pi}{k} = n\frac{\lambda}{2}, \quad \text{for} \quad n = 0, 1, 2, \cdots$$

They are separated by half wavelength.

(3) There are positions where the string has the maximum amplitude. They are called **antinodes**, and are located at

$$kx = \tfrac{1}{2}\pi, \tfrac{3}{2}\pi, \tfrac{5}{2}\pi, \cdots = (n + \tfrac{1}{2})\pi, \quad \text{for} \quad n = 0, 1, 2, \cdots$$

$$x = \left(n + \frac{1}{2}\right)\frac{\lambda}{2}, \quad \text{for} \quad n = 0, 1, 2, \cdots$$

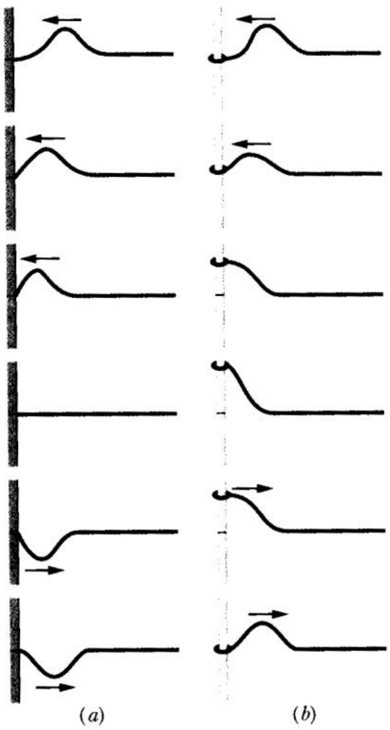
They are separated by half wavelength.

Reflections at a Boundary

Fixed end:
(1) The fixed end becomes a node.
(2) The reflected wave vibrates in the opposite transverse direction.

Free end:
(1) The free end becomes an antinode.
(2) The reflected wave vibrates in the same transverse direction.

At the fixed end:
Displacement = 0



At the free end:
Displacement gradient = 0

FIG. 16-21 (*a*) A pulse incident from the right is reflected at the left end of the string, which is tied to a wall. Note that the reflected pulse is inverted from the incident pulse. (*b*) Here the left end of the string is tied to a ring that can slide without friction up and down the rod. Now the pulse is not inverted by the reflection.

See animation "Constructive and Destructive Interference"

Standing Waves and Resonance

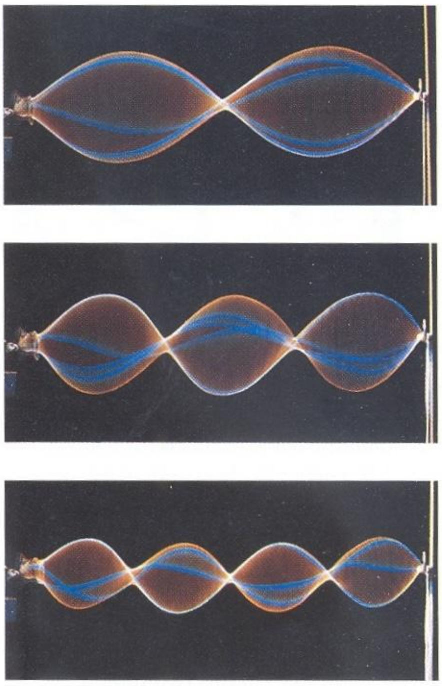


FIG. 16-22 Stroboscopic photographs reveal (imperfect) standing wave patterns on a string being made to oscillate by an oscillator at the left end. The patterns occur at certain frequencies of oscillation. *(Richard Megna/Fundamental Photographs)*

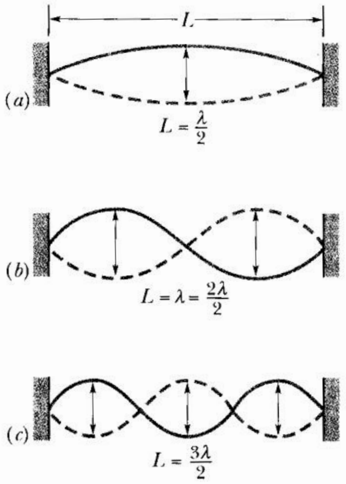


FIG. 16-23 A string, stretched between two clamps, is made to oscillate in standing wave patterns. (*a*) The simplest possible pattern consists of one *loop*, which refers to the composite shape formed by the string in its extreme displacements (the solid and dashed lines). (*b*) The next simplest pattern has two loops. (*c*) The next has three loops.

Consider a string with length *L* stretched between two fixed ends.
Boundary condition: nodes at each of the fixed ends.
When the string is driven by an external force, at a certain frequency the standing wave will fit this boundary condition.
Then this **oscillation mode** will be excited.
The frequency at which the oscillation mode is excited is called the **resonant frequency**.
See animation "Standing Waves"

FIG. 16-24 One of many possible standing wave patterns for a kettledrum head, made visible by dark powder sprinkled on the drumhead. As the head is set into oscillation at a single frequency by a mechanical oscillator at the upper left of the photograph, the powder collects at the nodes, which are circles and straight lines in this two-dimensional example. *(Courtesy Thomas D. Rossing, Northern Illinois University)*

Case (a): 2 nodes at the ends, 1 antinode in the middle

$$L = \frac{\lambda}{2}, \qquad \lambda = 2L.$$

Resonant frequency: $f = \dfrac{v}{\lambda} = \dfrac{v}{2L}.$

Case (b): 3 nodes and 2 antinodes

$$L = \lambda.$$

Resonant frequency: $f = \dfrac{v}{\lambda} = \dfrac{v}{L}.$

Case (c): 4 nodes and 3 antinodes

$$L = \frac{3\lambda}{2}, \qquad \lambda = \frac{2L}{3}.$$

Resonant frequency: $f = \dfrac{v}{\lambda} = \dfrac{3v}{2L}.$

In general,

$$L = \frac{n\lambda}{2}, \quad \lambda = \frac{2L}{n}, \quad \text{for } n = 1, 2, 3, \ldots$$

Resonant frequency:

$$\boxed{f = \frac{v}{\lambda} = n\frac{v}{2L}, \qquad \text{for } n = 1, 2, 3, \ldots}$$

$n = 1$ fundamental mode, or first harmonic
$n = 2$ second harmonic
$n = 3$ third harmonic

See Youtube “Millenium Bridge Opening”.

Example 16-8 A string of mass m = 2.5 g and length L = 0.8 m is under tension τ = 325 N.

(a) What is the wavelength λ of the transverse waves producing the standing-wave pattern in Fig. 16-25, and what is the harmonic number n?

(b) What is the frequency f of the transverse waves and of the oscillations of the moving string elements?

(c) What is the maximum magnitude of the transverse velocity u_m of the element oscillating at coordinate x = 0.18 m?

(d) At what point during the element's oscillation is the transverse velocity maximum?

(a) $\lambda = \frac{L}{2} = \frac{0.8}{2} = 0.4\,\text{m}$ (ans)

Since there are four loops, n = 4. (ans)

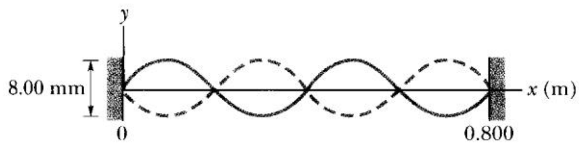


FIG. 16-25 Resonant oscillation of a string under tension.

(b) $v = \sqrt{\frac{\tau}{\mu}} = \sqrt{\frac{325}{0.0025/0.8}} = 322.5\,\text{ms}^{-1}$

$f = \frac{v}{\lambda} = \frac{322.5}{0.4} = 806.2 \approx 806\,\text{Hz}$ (ans)

(c) $y'(x,t) = [2y_m \sin kx]\cos\omega t$ where y_m = 0.002 m.

$u(x,t) = \frac{\partial y}{\partial t} = \frac{\partial}{\partial t}(2y_m \sin kx \cos\omega t)$

$= -2\omega y_m \sin kx \sin\omega t$

Magnitude: $u_m = |-2\omega y_m \sin kx|$

Here, $\omega = 2\pi f = (2\pi)(806.2)$, $k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.4}$. At x=0.18 m,

$$u_m = \left| -2(2\pi)(806.2)(0.002)\sin\left[\frac{2\pi}{0.4}(0.18)\right] \right| = 6.26\,\text{ms}^{-1}$$

(d) The transverse velocity is maximum when y = 0. (ans)

Summary of Equations:

Travelling wave:	$y(x,t) = y_m \sin[k(x - vt)],$
or	$y(x,t) = y_m \sin(kx - \omega t).$
Wavenumber:	$k = \frac{2\pi}{\lambda}.$
Angular frequency:	$\omega = \frac{2\pi}{T}.$
Frequency:	$f = \frac{1}{T} = \frac{\omega}{2\pi}.$
Wave velocity:	$v = \frac{\omega}{k} = f\lambda.$
Travelling wave (opposite):	$y(x,t) = y_m \sin(kx + \omega t).$
Stretched string:	$v = \sqrt{\frac{\tau}{\mu}}.$
Transmitted power:	$\langle P \rangle = \frac{1}{2}\mu v \omega^2 y_m^2.$
Interference:	$y'_m = 2y_m \cos\frac{\phi}{2}.$
Standing wave:	$y(x,t) = [2y_m \sin kx]\cos \omega t.$
Reflection at fixed end:	node, osc. opp. dir.
Reflection at free end:	antinode, osc. same dir.
Vibrating string (fixed ends):	$f = \frac{v}{\lambda} = n\frac{v}{2L}.$