Electrostatics

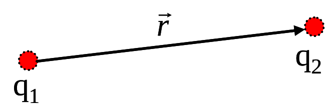
1 Introduction to electric charge

The crucial point is that some physical quantity which we call "charges" is discovered. The charges are associated with a special force we called electric force.

1.1 Coulomb's Law

This is the first Law of Physics on electromagnetism discovered by human beings. It says that the *electrostatic force* between two charges q_1 and q_2 separated by distance r is given by

$$\vec{F}_{12} = k\frac{q_1 q_2}{r^2}\hat{r}$$



where k is a constant, and $\hat{r}$ is a unit vector pointing in the direction from charge 1 to charge 2. $\vec{F}_{12}$ is the force acting on charge 2 from charge 1.

SI Unit of charge: Coulomb(C)

One coulomb is the amount of charge that is transferred through the cross section of a wire in 1 second when there is a current of 1 ampere in the wire.

Notice that the relationship between electric current and electric charges is already assumed in this definition. In SI Unit k is given by $k = \dfrac{1}{4\pi\varepsilon_0} = 8.99\times10^9\, N.m^2/C^2$,

or $\varepsilon_0 = 8.85\times10^{-12} C^2/N.m^2$.

For many charges, the forces satisfy the law of superposition,

$$\vec{F}_{i,net} = \sum_{j\neq i}\vec{F}_{ij} = \frac{1}{4\pi\varepsilon_o}\sum_{j\neq 1}\frac{q_i q_j}{|\vec{r}_{ij}|^3}\vec{r}_{ij}\,.$$

Notice the similarity of Coulomb's Law with Law of Gravitation, $\vec{F}_{12} = -G\dfrac{m_1 m_2}{r^2}\hat{r}$. The main difference is that charges can be both positive and negative, whereas masses are always positive.

The Coulomb's Law summarizes many experimental observations that were usually not discussed today. However, it is still interesting and useful to ask the question:

Can you design experiments that can verify Coulomb's Law?

Notice that in secondary schools (and here) you are often asked to perform experiments that can verify the inverse square dependence of force, i.e. $|\vec{F}_{12}| \propto 1/r_{12}^2$. How about (i) the direction of force, (ii) the law of superposition, and (iii) $|\vec{F}_{12}| \propto q_1 q_2$. In the last relation, why is it not $|\vec{F}_{12}| \propto q_1^2 q_2^2$ or $|\vec{F}_{12}| \propto q_1^3 q_2^3$?

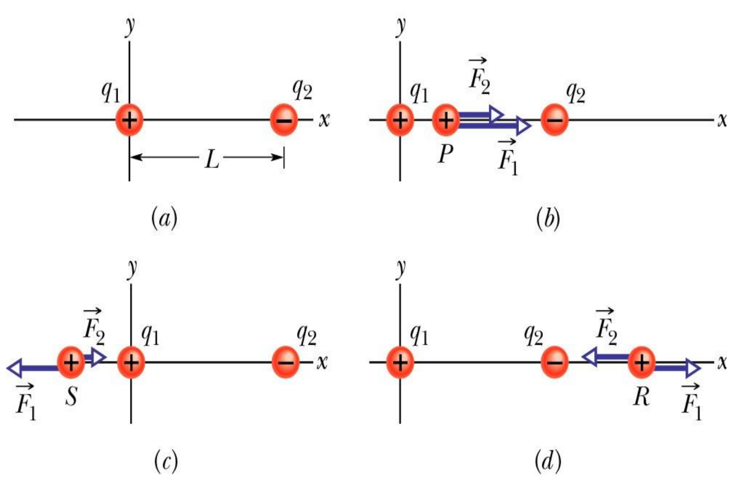
The similarity between Coulomb Law and Law of Gravitation also enable us to draw some conclusion about Coulomb forces easily. For example,

A shell of uniform charge attracts or repels a charged particle that is outside the shell as if all the shell's charge were concentrated at its center.

If a charged particle is located inside a shell of uniform charge, there is no net electrostatic force on the particle from the shell.

Do you know how to prove these statements? We shall come back to them after we learn Gauss Law.

Example:



The figure shows two particles fixed in place: a particle of charge $q_1 = 8q$ at the origin and a particle of charge $q_2 = -2q$ at $x = L$. At what point (other than infinitely far away) can a proton be placed so that it is in equilibrium? Is that equilibrium stable or unstable? ($-q$ = charge of electron)

1.2 Spherical Conductors

If excess charges Q are placed on a piece of metal, the charge will move under each other's repulsion to stay as far away from each other as possible. That means they prefer to stay at the *surface of metal*. For spherical conductors with radius R the final distribution of charges is simple. Because of symmetry, the charges Q will be spread uniformly on the surface of the spherical conductor, the surface charge density being $\sigma = \dfrac{Q}{4\pi R^2}$.

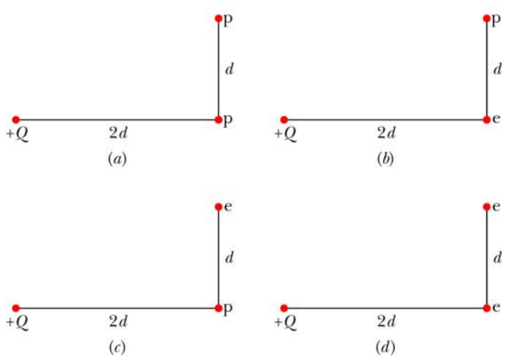
1.3 Charge is Quantized

We now know that like other materials in nature electric charges have a smallest unit, $e = 1.60 \times 10^{-19} C$, and all charges are multiple of the smallest unit, i.e. q=ne, $n = 0, \pm 1, \pm 2, ...$ etc. When a physical quantity such as charge can have only discrete values, we say that the quantity is *quantized*. It is in fact not obvious at all why matters in our universe all seem to be "quantized" somehow.

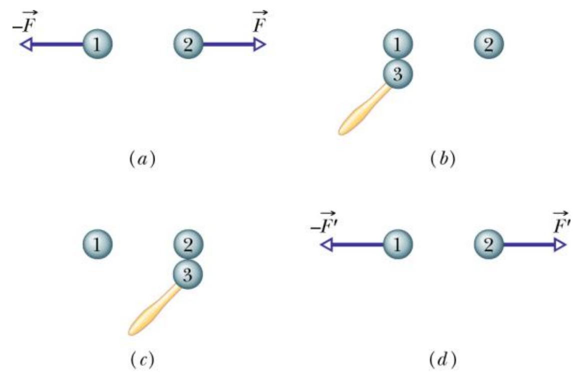
1.4 Charge is Conserved

We now know also that charges cannot be created or destroyed singly. When we "create" charges (by rubbing a glass rod with a piece of cloth, for example) we are separating positive and negative charges that are originally bound together. We can destroy pair of opposite charges if they are so-call particle and anti-particle pair, for example, $e + e^{+} \rightarrow 2\gamma$, where e^{+} called positron is the anti-particle of electron (e), γ represents photons (~ EM waves).

1) We show below your arrangements of charged particles. Rank the arrangements according to the magnitude of the net electrostatic force on the particle with charge +Q, greatest first.



2) Identical isolated conducting spheres 1 and 2 have equal charges and are separated by a distance that is large compared with their diameters (fig.(a)). The electrostatic force acting on sphere 2 due to sphere 1 is $\vec{F}$. Suppose now that a third identical sphere 3, having an insulating handle and initially neutral, is touched first to sphere 1 (fig.(b)), then to sphere 2 (fig.(c)), and finally removed (fig.(d)). In terms of magnitude *F*, what is the magnitude of the electrostatic force $\vec{F}'$ that now acts on sphere 2? (1/2,1/4,3/4,3/8)



2 Electric Fields

2.1 Introducing electric field

The concept of "Field" was initially introduced to describe forces between 2 objects that are separated by a distance (action at a distance). It is convenient to have a way of viewing how the force coming from object 1 felt by another object is "distributed" around object 1. This is particularly easy for charges obeying Coulomb's Law, since the force felt by charge 2 coming from object 1 is proportional to charge of object 2, i.e. we can write $\vec{F}_{12} = k\frac{q_1 q_2}{r^2}\hat{r} = \vec{E}_{12} q_2$, where $\vec{E}_{12}$ is the *force per unit charge* acting on q_2 due to charge 1. Because of linearity of force, this concept can be generalized to the force felt by a test charge *q* at position $\vec{r}$ due to a distribution of other charges. In that case, we may write

$$\vec{F}(\vec{r}) = \sum_j \vec{F}_j = \frac{q}{4\pi\varepsilon_o}\sum_j \frac{q_j}{|\vec{r}-\vec{r}_j|^3}(\vec{r}-\vec{r}_j) = q\vec{E}(\vec{r}),$$

where $\vec{E}(\vec{r})$ is the force per unit charge of a charge q felt at position $\vec{r}$. $\vec{E}(\vec{r}) = \vec{F}(\vec{r})/q$ was given a name called *electric field*. The SI unit for electric field is obviously Newton per coulomb (N/C).

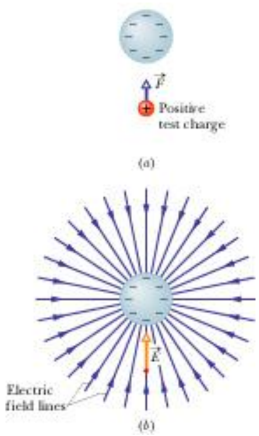
2.2 Electric Field Lines

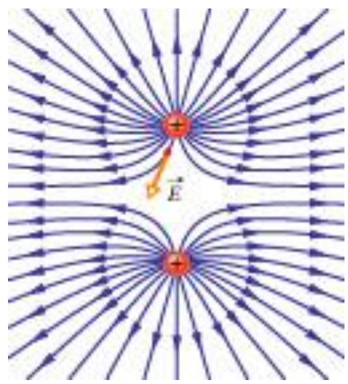
To make it easier to visualize, Michael Faraday introduced the idea of *lines of force*, or electric field lines. Electric field lines are diagrams that represent electric fields. They are drawn with the following rules: (1) At any point, the direction of a straight field line or the direction of the tangent to a curved field line gives the direction of $\vec{E}$ at that point, and (2) the field lines are drawn so that the number of lines per unit area, measured in a plane that is perpendicular to the lines, is proportional to the magnitude of $\vec{E}$.

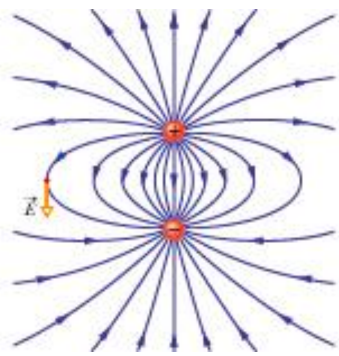
Some examples:

a) field lines from a (-) spherical charge distribution

b) Field lines from 2 point charges of equal magnitude (i) charges are same (ii) charges are opposite.







2.3 Electric field due to different charge distributions

1) Point charge at origin ($\vec{r}=0$)

$$\vec{E}=\frac{q}{4\pi\varepsilon_o r^2}\hat{r}$$

Exercise: what is the electric field at position $\vec{r}=(x,y,z)$ from 2 point charges with magnitude q, and q', respectively located at $\vec{r}=\pm z_0\hat{z}$?

We use the formula

$$\vec{E}(\vec{r})=\frac{1}{4\pi\varepsilon_o}\sum_j\frac{q_j}{|\vec{r}-\vec{r}_j|^3}(\vec{r}-\vec{r}_j)=\frac{1}{4\pi\varepsilon_o}\left[q\frac{(\vec{r}-z_0\hat{z})}{|\vec{r}-z_0\hat{z}|^3}+q'\frac{(\vec{r}+z_0\hat{z})}{|\vec{r}+z_0\hat{z}|^3}\right]$$

The last equality is valid for our 2-charges problem.

Therefore

$$E_x(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left[q\frac{x}{|\vec{r}-z_0\hat{z}|^3} + q'\frac{x}{|\vec{r}+z_0\hat{z}|^3}\right],$$

$$E_y(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left[q\frac{y}{|\vec{r}-z_0\hat{z}|^3} + q'\frac{y}{|\vec{r}+z_0\hat{z}|^3}\right],$$

$$E_z(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left[q\frac{z-z_0}{|\vec{r}-z_0\hat{z}|^3} + q'\frac{z+z_0}{|\vec{r}+z_0\hat{z}|^3}\right]$$

where $|\vec{r} \pm z_0\hat{z}| = \sqrt{x^2+y^2+(z\pm z_0)^2}$.

2) *Electric dipole*

This is the electric field due to 2 point charges with magnitude q, and $-q$, respectively located at $\vec{r}' = \pm d/2\,\hat{z}$, and at distances $|\vec{r}| >> d$ from the origin.

Using the above result, we obtain for the dipole electric field;

$$E_x(\vec{r}) = \frac{q}{4\pi\varepsilon_o}\left[\frac{x}{|\vec{r}-d/2\,\hat{z}|^3} - \frac{x}{|\vec{r}+d/2\,\hat{z}|^3}\right] \underset{r>>d}{\sim} \frac{3qd}{4\pi\varepsilon_o}\frac{zx}{r^5}$$

$$E_y(\vec{r}) = \frac{q}{4\pi\varepsilon_o}\left[\frac{y}{|\vec{r}-d/2\,\hat{z}|^3} - \frac{y}{|\vec{r}+d/2\,\hat{z}|^3}\right] \underset{r>>d}{\sim} \frac{3qd}{4\pi\varepsilon_o}\frac{zy}{r^5}$$

$$E_z(\vec{r}) = \frac{q}{4\pi\varepsilon_o}\left[\frac{z-d/2}{|\vec{r}-d/2\,\hat{z}|^3} - \frac{z+d/2}{|\vec{r}+d/2\,\hat{z}|^3}\right] \underset{r>>d}{\sim} \frac{qd}{4\pi\varepsilon_o}\frac{3z^2-r^2}{r^5}\,.$$

The electric field lines coming from a pair of opposite charges are shown in previous section (b). Notice that at distances far away from origin, the electric field is only proportional to the product *qd*. The quantity $\vec{p} = (qd)\hat{z}$ is called an electric dipole moment. The direction of the dipole is taken to be along the direction from the negative to the positive charge of a dipole. To derive the last result, we have used Taylor expansion,

$$f(x_0+\Delta x) \sim f(x_0) + \Delta x\left.\frac{df}{dx}\right|_{x=x_0} + \frac{1}{2}(\Delta x)^2\left.\frac{d^2f}{dx^2}\right|_{x=x_0} + \ldots$$

The mathematics will be explained in the tutorials.

You probably feel that it's not very easy to calculate the electric field when there is more than one charge in the system. We shall now make the situation worse but more realistic by considering electric field coming from a continuous charge distribution.

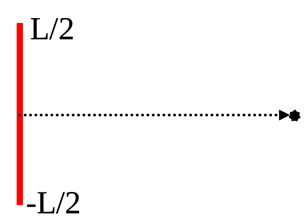
3) *Electric field from a line of charge*.

Imagine a very narrow rod which is charged. The rod may not be straight. The rod is so narrow that we can neglect the cross section of the rod and describe the charge distribution on the rod as $\lambda(\vec{x})$, where $0 < |\vec{x}| < L$, L is the length of the rod. The total charge on the rod is $Q = \int_0^L \lambda(x)dx$. The electric field produced by this rod is, by generalizing the sum over charges into integral, $\left[\sum_i f(\vec{r} - \vec{r}_i)q_i \rightarrow \int_0^L f(\vec{r} - \vec{r}\,')\lambda(\vec{r}\,')dr'\right]$

So $\vec{E}(\vec{r}) = \dfrac{1}{4\pi\varepsilon_o}\displaystyle\int_0^L \frac{\lambda(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}(\vec{r} - \vec{r}\,')d\vec{r}\,'$.

This integral is often impossible to evaluate exactly except in some simple situations. We shall consider one example here, the electric field coming from a straight line of charge at a symmetric point P at distance *r* from the line. The charge density λ is uniform along the line.

To calculate the electric field we employ a coordinate system where the origin *o* is placed at the mid-point of the straight line. By symmetry, we expect that the electric field at point P will be pointing at *x*-direction, with magnitude



$$E_x(r) = \frac{1}{4\pi\varepsilon_o}\int_{-L/2}^{L/2} \frac{\lambda r}{(r^2 + y^2)^{3/2}}dy .$$

This integral can be evaluated easily if we introduce $y = r\tan\phi$. We obtain $r^2 + y^2 = r^2\sec^2\phi$ and $dy = r\sec^2\phi d\phi$. Inserting these into the integral, we obtain

$$E_x(r) = \frac{\lambda}{4\pi\varepsilon_o r}\int_{-\tan^{-1}\frac{L}{2r}}^{\tan^{-1}\frac{L}{2r}} \cos\phi d\phi = \frac{\lambda}{4\pi\varepsilon_o r}\frac{L}{\sqrt{\left(L/2\right)^2 + r^2}} .$$

We have use $\sin(\tan^{-1} a/b) = a/\sqrt{a^2 + b^2}$ in the last equality. Notice that as $L \rightarrow \infty$ (infinite wire), we obtain $E_x(r) = \dfrac{\lambda}{2\pi\varepsilon_o r}$.

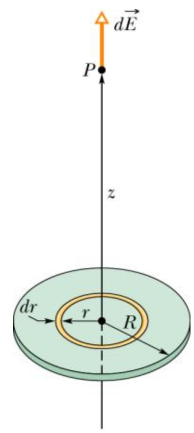
Another example: *electric field from a circular ring of charge*. Notice that in these examples we are calculating electric field at special points of symmetry only. The integrals at other points are often too complicated to be evaluated exactly.

4) Electric field from a surface of charge

It is obvious that we have physical situations where charges $\sigma(x, y)$ are distributed over a surface of very small thickness instead of along a line. The formula for the electric field becomes $\left[\sum_i f(\vec{r}-\vec{r}_i)q_i \rightarrow \int_0^L f(\vec{r}-\vec{r}\,')\sigma(\vec{r}\,')d^2r'\right]$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int_0^L \frac{\sigma(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}(\vec{r}-\vec{r}\,')d^2r'.$$

Now let us apply it to calculate the electric field at a symmetry point *P*, at a distance *z* above a plastic circular disc with uniform charge density $\sigma(x, y) = \sigma$.



From symmetry, we expect that the electric field be pointing towards *z*-direction. The formula for electric field is, in *cylindrical coordinate*

$$E_z(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int d\vartheta \int_0^R \frac{\sigma z}{(z^2+r^2)^{3/2}} r dr = \frac{\sigma}{2\varepsilon_o}\left(1 - \frac{z}{(z^2+R^2)^{1/2}}\right).$$

5) Electric field from a volume of charge

It is natural also to generalize to consider electric field from charge distributions $\rho(x, y, z)$ in a volume. In this case, we have $\left[\sum_i f(\vec{r}-\vec{r}_i)q_i \rightarrow \int_0^L f(\vec{r}-\vec{r}\,')\rho(\vec{r}\,')d^3r'\right]$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int_0^L \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}(\vec{r}-\vec{r}\,')d^3r'.$$

We expect that we can perform the integral for some simple cases like electric field from a sphere with uniform charge density. In fact, we know that the electric field outside the sphere should be $\vec{E} = \frac{Q}{4\pi\varepsilon_o r^2}\hat{r}$, where r is the distance from the center of the sphere. *Q* = total charge carried by the sphere. However, if you try to do the integral, you will find that it is rather difficult even for this simple case. We shall introduce in next chapter a new angle of looking at Coulomb’s Law called the *Gauss Law*. One advantage of this new angle is that many of the difficult integrals in evaluating electric fields can be avoided and can be replaced by much simpler mathematical expressions in situations with high symmetry.

2.4 Motion of Charges in electric field

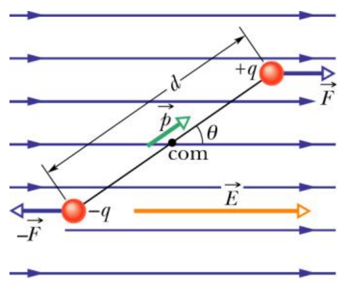
a) Point charge

A particle with charge q satisfies Newton's Equation of motion $\vec{F} = m\vec{a}$ under electric field, where m is the mass of the particle. The force the particle feels is $\vec{F} = q\vec{E}$, followed from the definition of electric field.

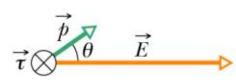
b) Dipole

Since a dipole is consist of 2 opposite charges of equal magnitude, the force acting on a dipole will be zero if electric is uniform. However, the forces on the charged ends do produce a net torque $\vec{\tau}$ on the dipole in general. The torque about its center of mass is

$$\tau = -(qE)(d/2)\sin\vartheta - (qE)(d/2)\sin\vartheta = -pE\sin\vartheta$$



(a)



(b)

Notice that the torque is trying to rotate the dipole clockwise to decrease θ in the figure, which is why there is a (-) sign.

In vector form, $\vec{\tau} = \vec{p} \times \vec{E}$. Notice that associated with the torque is also a potential energy

$$U = -\int_{\pi/2}^{\vartheta} \tau d\vartheta = \int_{\pi/2}^{\vartheta} pE\sin\vartheta d\vartheta = -pE\cos\vartheta = -\vec{p} \bullet \vec{E}\,.$$

Questions

What is the x-component of electric field at position $\vec{r} = (x, y, z)$ from 3 point charges with magnitude q , q' and q'', respectively located at $\vec{r} = x_0\hat{x},\ y_0\hat{y},\ z_0\hat{z}$, respectively? Recall: $\left(\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\sum_j \frac{q_j}{|\vec{r}-\vec{r}_j|^3}(\vec{r}-\vec{r}_j)\right)$

(a) $E_x(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left[q\frac{x-x_0}{|\vec{r}-x_0\hat{x}|^3} + q'\frac{x+x_0}{|\vec{r}+x_0\hat{x}|^3} + q''\frac{x}{|\vec{r}|^3}\right]$,

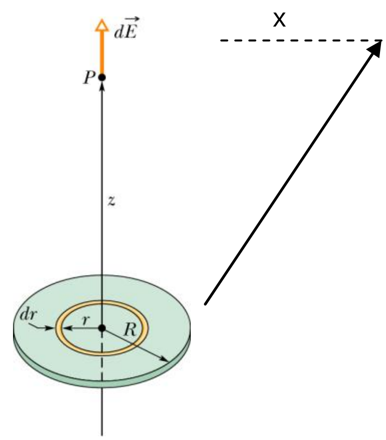
(b) $E_x(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left[q\frac{x-x_0}{|\vec{r}-x_0\hat{x}|^3} + q'\frac{y-y_0}{|\vec{r}-y_0\hat{y}|^3} + q''\frac{z-z_0}{|\vec{r}-z_0\hat{z}|^3}\right]$,

(c) $E_x(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left[q\frac{x-x_0}{|\vec{r}-x_0\hat{x}|^3} + q'\frac{x}{|\vec{r}-y_0\hat{y}|^3} + q''\frac{x}{|\vec{r}-z_0\hat{z}|^3} \right]$

(d) $E_x(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left[q\frac{x}{|\vec{r}-x_0\hat{x}|^3} + q'\frac{y}{|\vec{r}-y_0\hat{y}|^3} + q''\frac{z}{|\vec{r}-z_0\hat{z}|^3} \right]$

(e) $E_x(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left[q\frac{x}{|\vec{r}-x_0\hat{x}|^3} + q'\frac{x}{|\vec{r}-y_0\hat{y}|^3} + q''\frac{x}{|\vec{r}-z_0\hat{z}|^3} \right]$

What is the x-component of electric field at position $\vec{r} = (x, 0, z)$ from a surface charge distribution $\sigma(x', y') = ar'$ for $r' = \sqrt{x'^2 + y'^2} \le R$, and $\rho(x', y') = 0$ for $r' = \sqrt{x'^2 + y'^2} > R$. z is the vertical distance from the disc. $\left(\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int \frac{\sigma(x', y')}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}')d^2r' \right)$



(a) $E_x(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int_0^R r'dr' \int_0^{2\pi} d\vartheta \frac{(ar')(x - r'\cos\vartheta)}{\left(z^2 + r'^2\right)^{3/2}}$

(b) $E_x(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int_0^R r'dr' \int_0^{2\pi} d\vartheta \frac{(ar')(z - r'\cos\vartheta)}{\left(z^2 + r'^2\right)^{3/2}}$

(c) $E_x(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int_0^\infty dr' \int_0^{2\pi} d\vartheta \frac{(ar')(x - r'\cos\vartheta)}{\left(z^2 + (r'\cos\vartheta - x)^2 + (r'\sin\vartheta)^2\right)^{3/2}}$

(d) $E_x(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int_0^R r'dr' \int_0^{2\pi} d\vartheta \frac{(ar')x}{\left(z^2 + (r'\cos\vartheta - x)^2 + (r'\sin\vartheta)^2\right)^{3/2}}$

(e) $E_x(\vec{r}) = \frac{-1}{4\pi\varepsilon_o}\int_0^R r'dr' \int_0^{2\pi} d\vartheta \frac{(ar')(x - r'\cos\vartheta)}{\left(z^2 + (r'\cos\vartheta - x)^2 + (r'\sin\vartheta)^2\right)^{3/2}}$

3 Gauss' Law

3.1 What is Gauss' Law

Gauss' Law is discovered by the great mathematician Carl Friedrich Gauss. It's an alternative way to express Coulomb's Law. What is interesting about Gauss' Law is that at first sight it does not look related to Coulomb's Law in any way at all. The equivalence between Gauss' Law and Coulomb's Law relies on some very deep mathematics which gradually evolves into the field of *differential geometry*. It also begins an evolution in physics – that physical laws have profound geometrical meanings.

3.2 Language – closed surface and flux

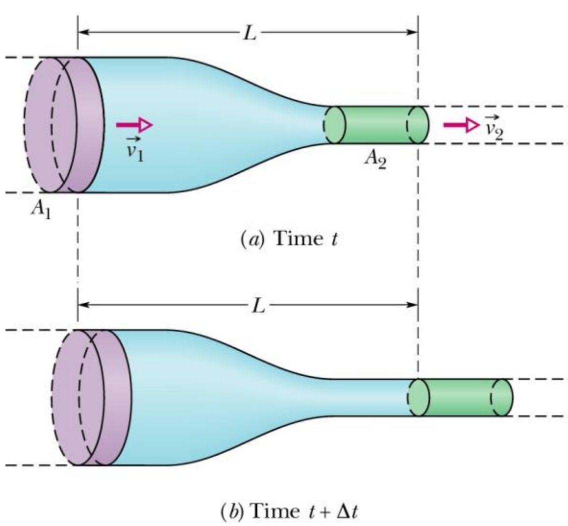
The Gauss' Law looks so different from Coulomb's Law that in order to describe it we first have to introduce a few new mathematical language or concept.

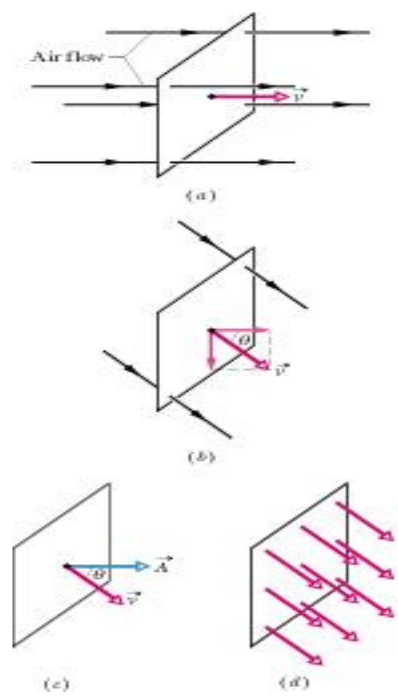
First we introduce the concept of (close) surface.

A close surface is a surface that closes on itself and has no boundary. In 3D it can be considered as the surface covering a solid object. The close surface (or the solid object) can have any shape or size.
Some examples of simple close surfaces include surface of a sphere, a table, or a rectangular box.

Next we introduce the concept of flux.

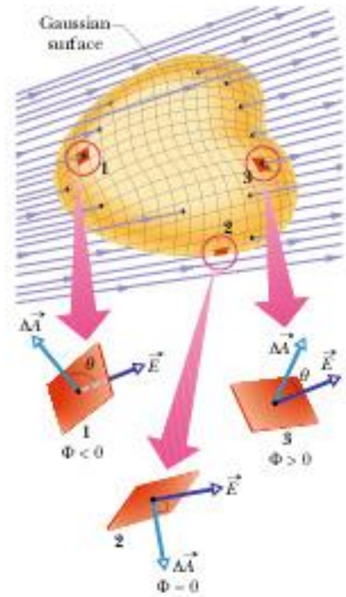
The concept of flux was first introduced in fluids. It describes how much fluid passes through a certain piece of (open) surface per unit time.





Let's consider the flow of fluid through a small square loop of area *A* as shown in the right figure. The velocity of fluid is $\vec{v}$. The rate of flow of fluid through the loop depends on the angle between $\vec{v}$ and the normal to the plane of the loop $\hat{n}$. If $\vec{v}$ is perpendicular to the plane (or parallel to $\hat{n}$), the rate of flow is *vA*. On the other hand, if $\vec{v}$ is parallel to the plane (perpendicular to $\hat{n}$), there is no fluid flowing *through* the surface. The rate of flow is zero. Mathematically, the rate of mass flow through a small piece of flat surface can be written as $\Phi = \rho \vec{v}.\hat{n} A$, where ρ is the density of fluid.

This construction can be generalized to surfaces S of arbitrary shape and for fluid flow where velocity is not constant in space. The idea is to divide the surface into small pieces S_i, each of area *dS*. The pieces are small enough such that each small surface can be considered as flat, and the fluid flow velocity is constant on this small surface (see figure below). Notice that the flow velocity can vary between different small surfaces.



The total flow rate through the surface S is thus

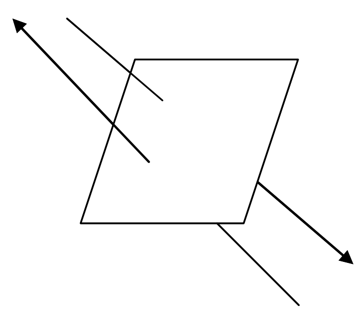
$$\Phi = \sum_i \rho(\vec{x}_i)\vec{v}(\vec{x}_i).\hat{n}_i ds = \sum_i \rho(\vec{x}_i)\vec{v}(\vec{x}_i).d\vec{s}_i \ ,$$

where $d\vec{s}_i$ is a vector with magnitude *ds*, and with direction along the normal of the surface element $\hat{n}_i$. In the limit when *ds* goes to zero, we obtain

$$\Phi = \iint \rho(\vec{r})\vec{v}(\vec{x}) \cdot d\vec{s} \ .$$

Sign of normal $d\vec{s}$

Notice that the sign of $d\vec{s}$ is arbitrary, depending on how we assign the sign to a surface. Different sign of $\vec{v} \cdot d\vec{s}$ reflects different direction of net current flow through the surface.



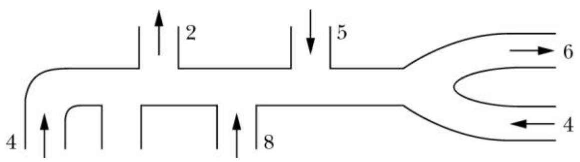
For close surface enclosing a volume, the usual sign convention is that current flowing *outward* from the volume is assign a positive (+) sign. In this case the total flux flowing outside from a close volume is usually written as

$$\Phi = \oiint \rho(\vec{x})\vec{v}(\vec{x}) \cdot d\vec{s} \ ,$$

where the circle on the integral sign indicates that the integration is to be taken over the entire (closed) surface. For example, a close surface covering a sphere with water coming out from all directions has $\Phi > 0$, and $\Phi < 0$ if water is flowing into the volume at all directions.

Example:

Imagine water flowing through a pipe system as shown in the figure below,



Can you construct closed surface where the total flux flowing through the closed surface is (i) > 0, (ii) = 0, (iii) < 0?

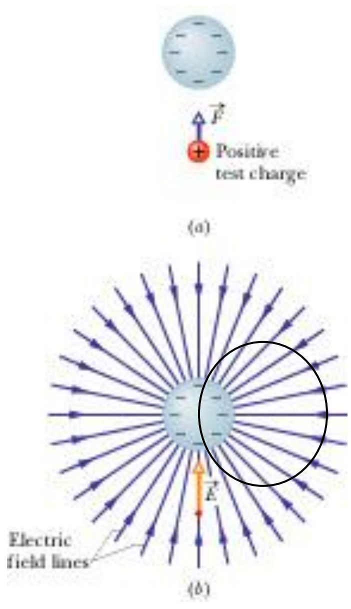
3.3 Electric flux

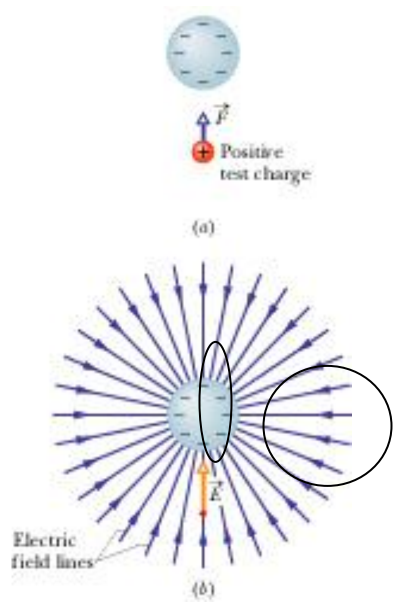
After introducing the language of close surface and flux, we introduce the idea of electric flux. The idea is simply to imagine that electric field represents something "flowing" in space like fluids and replace $\rho(\vec{x})\vec{v}(\vec{x})$ by $\vec{E}(\vec{x})$. Thus the electric flux passing through a surface is

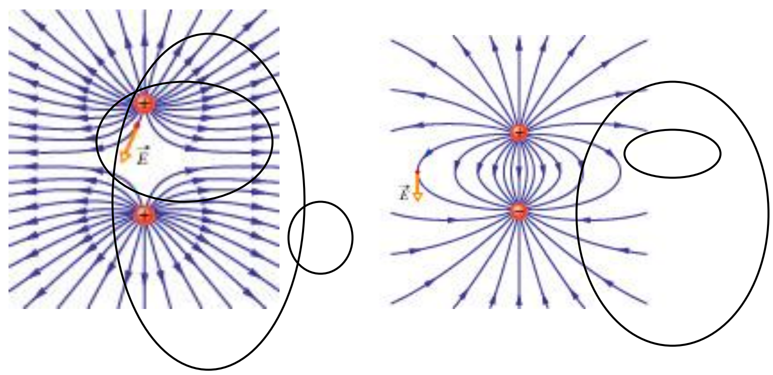
$$\Phi = \sum_i \vec{E}(\vec{x}_i).\hat{n}_i ds = \sum_i \vec{E}(\vec{x}_i)\cdot d\vec{s}_i$$

and $\Phi = \oiint_S \vec{E}(\vec{x})\cdot d\vec{s}$ for a close surface. *Pictorially, the electric flux Φ through a closed surface measures the net number of electric field lines passing through that surface.(see textbook p.546)*

Example: What is the flux Φ passing through each of the closed (ellipsoid) surfaces below?







3.4 Gauss' Law

Gauss's Law states that the net flux Φ of electric field passing through any closed surface is proportional to the *net charge* Q_{en} that is enclosed by the same surface. The proportionally constant is $1/\varepsilon_o$, i.e.

$$\Phi = \oiint_S \vec{E}(\vec{r})\cdot d\vec{s} = \frac{Q_{en}}{\varepsilon_o}.$$

Exercise:

Apply Gauss' Law to the examples we have discussed above. What are the Φ*'s*?

(Gauss' Law seems rather 'magical', but it is in fact not to hard to prove. One only needs to prove it for a point charge, of which the electric field is known, and then generalize the law to an arbitrary charge distribution using the superposition principle.)

3.5 Examples of application of Gauss' Law

Gauss' Law is most useful in calculations when we can construct simple closed surfaces where the electric field $\vec{E}$ is either (i) always perpendicular to and have the same value E on the surface or (ii) always parallel to the surface.

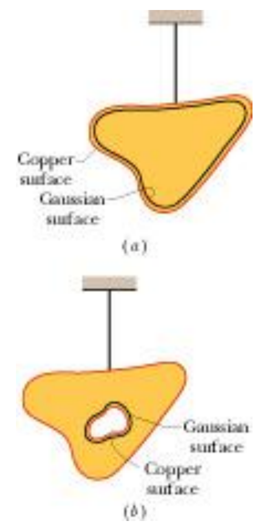
In case (i) we have $\oint \vec{E}(\vec{x})\cdot d\vec{s} = EA$, where $A = \oint ds$ is the surface area. In case two, we have $\oint \vec{E}(\vec{x})\cdot d\vec{s} = 0$.

1) A charged Isolated Conductor

Using Gauss' Law we can prove an important result about conductors:

If an excess charge is placed on an isolated conductor, the charge will move entirely to the surface of the conductor. None of the excess charge will stay within the body of the conductor.

Actually we have argued that this must be true using Coulomb's Law. To prove that this is true we note that the electric field inside a conductor must be zero. If this were not so, the electric force will force charges to move, and there will be electric current flowing in the conductor. This cannot happen to an isolated conductor in steady state. Now we can apply Gauss' Law to a close surface inside, but following the surface of the conductor (see figure). Since $\vec{E}=0$ everywhere inside the conductor, $\Phi=0$ and consequently, the net (excess) charges enclosed by the surface must be zero. The excess charges can only stay on the surface of the conductor.

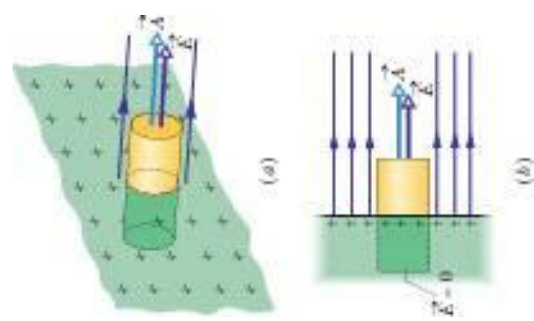


Notice that using the same argument; we can also prove that for an isolated conductor with an inside cavity, there is no charge accumulated on the surface of the cavity. The charges are all accumulated on the outer surface.

Electric field and charge distribution for a non-spherical conductor

For non-spherical conductors, the charges will not be uniformly distributed on the surface on the conductor. This variation makes the determination of the electric field set up by the surface charges very difficult.

However, Gauss' Law implies that there is a simple relation between the electric field just outside the surface of a conductor and the corresponding surface charge distribution $\sigma(\vec{x})$.



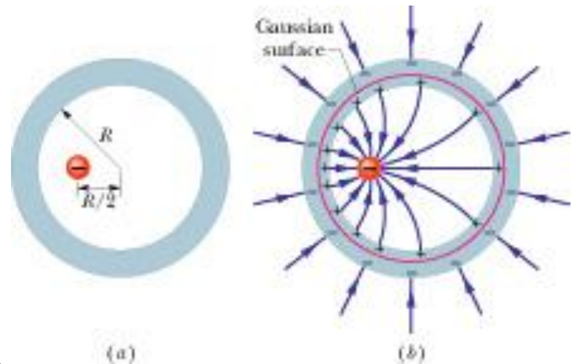
We apply Gauss' Law to a section of the surface *ds* that is small enough to be considered "flat". We then imagine a tiny cylindrical close surface to be embedded in the section as shown in the figure below. The cylinder is perpendicular to the surface and the length of the cylinder *dh* is small.

The electric field $\vec{E}$ at and just outside the conductor's surface must also be perpendicular to the surface. If it were not, there will be a parallel component to the surface and would cause a *surface current*. Again, this cannot happen for an isolated conductor in steady state.

We now apply Gauss' Law to the close surface. There is no electric field through the surface inside the conductor. There is also no electric field passing through the sides of the cylinder in the limit $dh \to 0$. In this limit, the only flux through the close surface is that through the external end cap. In the limit $ds \to 0$, the total flux is given by $\Phi = Eds = \frac{\sigma ds}{\varepsilon_o}$, or $\vec{E}(\vec{x}) = \frac{\sigma(\vec{x})}{\varepsilon_o}\hat{n}(\vec{x})$.

The equation relates the electric field just outside the surface of a conductor at position $\vec{x}$, to the surface charge density just below. $\hat{n}(\vec{x})$ is a unit vector pointing away from the surface at the same position.

Example



The figure shows a cross section of a spherical metal shell of inner radius R. A point charge of -5.0μC is located at a distance $R/2$ from the center of the shell. If the shell is electrically neutral, what are the (induced) charges on its inner and outer surfaces?

Solution:

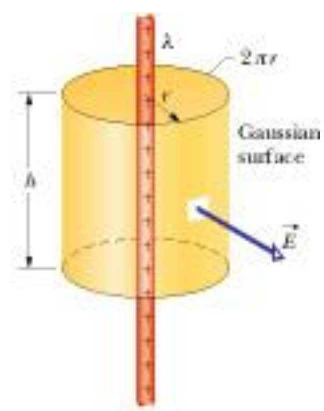
(1) Notice electric field = 0 inside conductor $\Rightarrow$ (a) induced charges can only be at inner and outer surfaces and (b) total induced charge on inner surface = 5.0 μC.
(2) charge neutrality of conductor $\Rightarrow$ total induce charge on outer surface = -5.0μC.

(3) Question for you: *how is the charge distributed outside, and why?*

2) Cylindrical Symmetry: Electric field from an infinite line of charge.

Recall that we have worked out this problem before. The electric field we obtain is given by a complicated integral. The final result is

$$E(r) = \frac{\lambda}{2\pi\varepsilon_o r}, \ \lambda \text{ is the line charge density.}$$



The direction of electric field is pointing away from the wire.
We now apply Gauss' Law to solve the problem. We construct a cylindrical closed surface of radius r and height h as shown in the figure. Notice that the two end caps are part of the surface.

By symmetry, we expect that the electric fields are pointing radially out from the wire, with the same magnitude for all points that are at the same distance from the wire, i.e.

$$\vec{E}(r,\vartheta,z) = E(r)\hat{r} \ ,$$

in cylindrical coordinate.

Applying Gauss' Law to the close surface, we obtain

$$\Phi = 2\pi r h E(r) = \frac{\lambda h}{\varepsilon_o}, \text{ or } \quad E(r) = \frac{\lambda}{2\pi\varepsilon_o r},$$

the same result as we obtained before.

3) Planar Symmetry: Electrical field from two conducting plates.

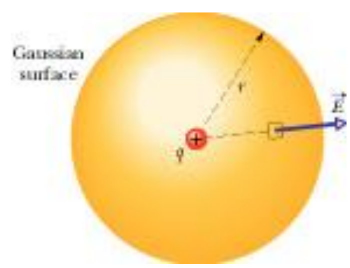
The result here will be used later in capacitors.

4) Spherical Symmetry

1) Point charge and Gauss' Law

At the beginning of this chapter we emphasize that Gauss' Law is actually equivalent to Coulomb's Law. That is, we can (a) derive Gauss' Law from Coulomb's Law, and we can also do the reverse, (b) derive Coulomb's Law from Gauss' Law. The proof of the equivalence of the two laws is mathematically quite subtle and requires familiarity of multiple variable calculus. Here, we shall consider a simple case. We shall show that Coulomb's Law for a point charge can be derived from Gauss' Law.

We apply Gauss' Law to a spherical surface with radius r surrounding a point charge q that sits at the center.



By symmetry we expect that the resulting electric field will be pointing outward from the center and has the same magnitude in all directions, i.e. $\vec{E}(\vec{r}) = E(r)\hat{r}$.

Therefore, the flux that passes through the surface is

$$\oint \vec{E}(\vec{x}) \bullet d\vec{s} = (4\pi r^2)E(r) = \frac{q}{\varepsilon_o},$$

or $E(r) = \frac{q}{4\pi\varepsilon_o r^2}$, which is Coulomb's Law for point charge.

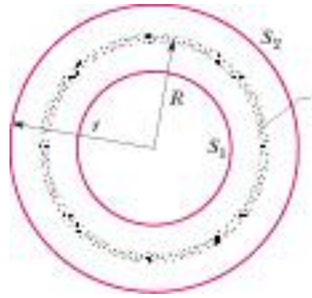
2) A spherical thin shell with uniform charge density σ

We consider positions both outside and inside the shell (see figure below). The radius of the shell is R.

We consider 2 concentric spherical surfaces, S_1 and S_2. S_2 is outside the charge shell, and S_1 is inside. Again we expect $\vec{E}(\vec{r}) = E(r)\hat{r}$ because of spherical symmetry.

Applying Gauss' Law, we obtain

$$\oint \vec{E}(\vec{x}).d\vec{s} = (4\pi r^2)E(r) = \frac{q}{\varepsilon_o} \text{ for } S_2,$$



where q = total charge on the shell, and

$$\oint \vec{E}(\vec{x}).d\vec{s} = (4\pi r^2)E(r) = 0 \text{ for } S_1, \text{ or}$$

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_o r^2}\hat{r} \text{ for } r > R, \; \vec{E}(\vec{r}) = 0 \text{ for } r < R.$$

Thus, a charge particle will feel no electrostatic force inside the shell, and will feel electrostatic force outside as if all the shell's charge is concentrated at the center of the shell.

The same analysis can also be applied to a sphere of radius *R* and with uniform charge densityρ.

For positions outside the sphere, it is easy to show that

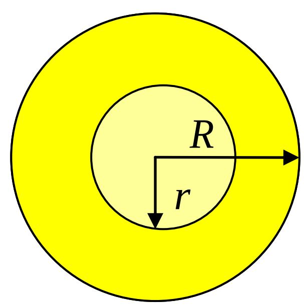
$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_o r^2}\hat{r},$$

where $q = \frac{4\pi}{3}R^3\rho$ is the charge density.

For positions inside the sphere at distance $r < R$ from the center, Gauss' Law implies

$$\vec{E}(\vec{r}) = \frac{q'(r)}{4\pi\varepsilon_o r^2}\hat{r},$$

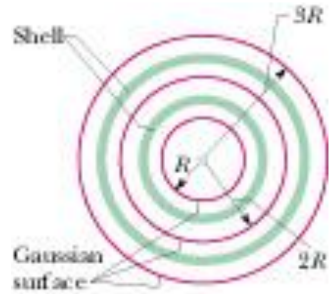
where $q'(r) = \frac{4\pi}{3}r^3\rho$ is the total charge enclosed in the sphere with radius r (see figure).



Therefore, $\vec{E}(\vec{r}) = \frac{\rho r}{3\varepsilon_o}\hat{r}$.

Questions:

The figure on the right shows, in cross section, a central metal ball, two spherical metal shells, and three spherical closed surfaces of radius R, $2R$ and $3R$, all with the same center. The uniform charges on the 3 objects are: ball, Q, smaller shell, $3Q$; larger shell, $5Q$. Rank the closed surfaces according to the magnitude of the electric field at any point on the surface, Greatest first.



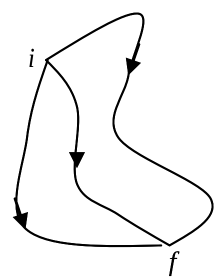
4 Electric Potential

4.1 Conservative force and potential energy

Recall that in Newtonian mechanics, a potential energy can be defined for a particle that obeys Newton's Law $\vec{F} = ma$ if the force is *conservative*. In this chapter, we shall see that the electric force felt by charges (we assume implicitly that charges have mass and obey Newton's Law) are conservative. Therefore, we can define potential energy for charges moving under electric force. We can also define the *electric potential – the potential energy per unit charge*, following the introduction of electric field from electric force. First we review what a conservative force is.

Conservative force

A conservative force is a force where the work done by the force in moving a particle (that experience the force) is path independent. It depends only on the initial and final positions of the particle.



In this case, the work done by the force on the particle can be expressed as minus the difference between the *potential energies* on the two end points, $-\Delta W = U_f - U_i$, the potential energy $U(\vec{x})$ is a function of the position of the particle only. The potential energy can be related to the force by noting that the work done is related to the force by $dW = \vec{F} \cdot d\vec{l}$. Consequently,

$$W = \int_i^f \vec{F} \bullet d\vec{l} = -(U_f - U_i).$$

The integral is path independent for conservative force. Using multi-variable calculus, it can be shown that the above equation can be inverted to obtain $\vec{F} = -\nabla U(\vec{r})$. This is a simplified notation for 3 equations

$$F_x = -\frac{\partial U(x,y,z)}{\partial x}, \quad F_y = -\frac{\partial U(x,y,z)}{\partial y}, \quad F_z = -\frac{\partial U(x,y,z)}{\partial z}.$$

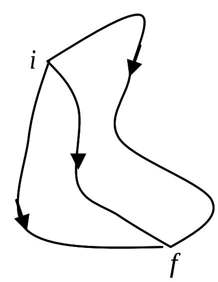
4.2 Electric Potential energy and electric potential

We can define electric potential energy for a point charge particle if electric force is conservative. Fortunately we know that electric force is conservative because of its similarity to gravitational force, at least if the electric force is coming from another point charge, $\vec{F}_{12} = \dfrac{q_1 q_2}{4\pi\varepsilon_o r_{12}^2}\hat{r}_{12}$. We know from analogy with gravitational force that the electric potential energy for a charge q_1 moving under the influence of another charge q_2 should be $U_{12} = \dfrac{q_1 q_2}{4\pi\varepsilon_o r_{12}}$, and the corresponding electric potential at a distance r from a charge q_2 is $V_2(r) = \dfrac{U_{12}}{q_2} = \dfrac{q_2}{4\pi\varepsilon_o r}$.

Direct Evaluation of V_2.

The potential energy expression U_{12} can be derived using the formula

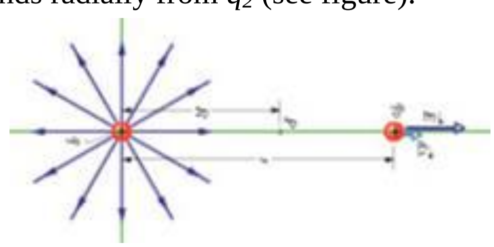
$$W = \int_i^f \vec{F}\cdot d\vec{l} = -(U_f - U_i),$$



where we shall set the initial point i at infinity and the final point f at a distance r from a fixed charge q_2. Notice that the electric potential is given by a similar integral

$$V_f - V_i = -\int_i^f \vec{E}\cdot d\vec{l},$$

since electric field is the force per unit charge by definition. We shall for simplicity chooses the initial and final points to be on a straight line that extends radially from q_2 (see figure).



In this case the integral becomes

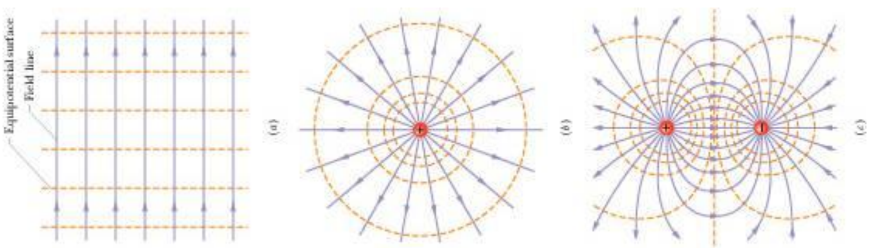
$$V_f - V_i = -\frac{q_2}{4\pi\varepsilon_o}\int_\infty^r \frac{1}{r^2}dr = \frac{q_2}{4\pi\varepsilon_o r}.$$

Choosing the reference electric potential to be $V_i = V(r=\infty) = 0$, we obtain the result $V_2(r) = \dfrac{q_2}{4\pi\varepsilon_o r}$.

Equipotential Surfaces

You may wonder what happens if in evaluating the electric potential, the initial and final points do not rest on a straight line that extends radially from q_2. How can we evaluate the integral $\int_i^f \vec{E}\cdot d\vec{l}$ in this case?

The construction of equipotential surfaces is useful in this case. Equipotential surfaces are points in space that have the same electric potential. These points usually form closed surfaces in space. For the electric potential from a single charge q, $V(r)=\dfrac{q}{4\pi\varepsilon_o r}$. The equipotential surfaces are surfaces consist of points at same distance r from the charge, i.e. they are *spherical surfaces* surrounding the charge.



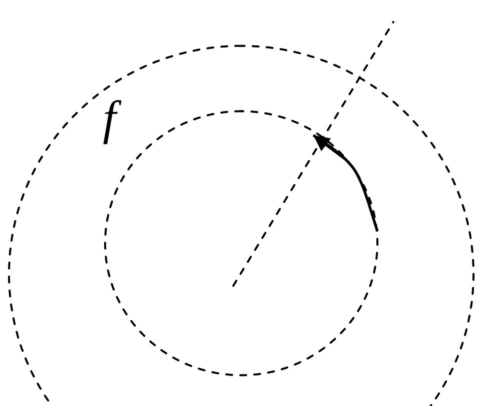
Notice that no (net) work W is done on a charged particle when the particle moves between any two points i and f on the same equipotential surface. This follows from

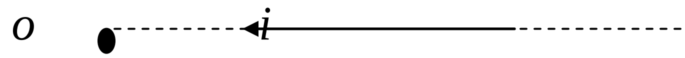
$$W=\int_i^f \vec{F}\cdot d\vec{l}=-(U_f-U_i)=0 \text{ if } U_f=U_i.$$

As a result the electric force (field) must be everywhere *perpendicular* to the equipotential surface. To see this we consider an arbitrary infinitesimal displacement $d\vec{l}$ on the equipotential surface. The work done is $dW=\vec{F}\cdot d\vec{l}=0$. This can be satisfied for arbitrary displacement $d\vec{l}$ on the equipotential surface only if the force $\vec{F}$ is everywhere perpendicular to the equipotential surface.

The figure above shows the equipotential surfaces for several different situations. The surfaces can be constructed easily from the electric field lines since they are surfaces perpendicular to the electric field (force) lines.

With the equipotential surfaces we can evaluate the integral $\int_i^f \vec{E}\cdot d\vec{l}$ rather easily for end points not on a straight line that extends radially from the charge q by choosing a line as shown below. Notice that the work done is zero between *o*-*f*.





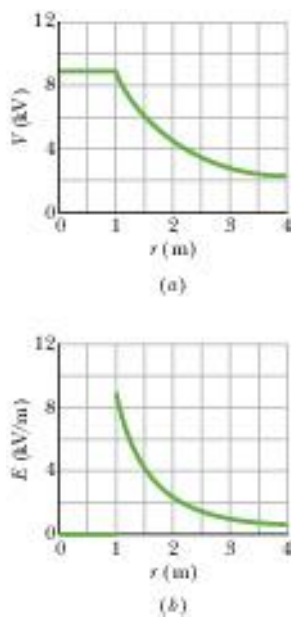
Potential of a charged Isolated Conductor

Recall that for excess charges distributed on an isolated conductor, all charges must be distributed on the surface such that the electric field on the surface cannot have a component parallel to the surface. (Otherwise there will be a current on the conductor surface)

Therefore, for any points *i* and *f* on the surface of conductor,

$$V_f - V_i = -\int_i^f \vec{E}\cdot d\vec{l} = 0,$$

i.e. the surface of an isolated conductor forms a equipotential surface. We show in the figure below the electric potential *V(r)* and electric field *E(r)* as a function of distance *r* from center for a spherical conductor with radius 1.



Notice that $E(r) \sim \dfrac{\partial V(r)}{\partial r}$.

Units of electric potential energy and electric potential:

The SI unit for energy is Joule (J) (1 Joule = 1 Newton × 1 meter). The SI unit for electric potential is therefore

1 *Volt (V) = 1 Joule per Coulomb (J/C).*

This new unit allows us to adopt a more conventional unit for electric field $\vec{E}$, which we have measured up to now in Newton per Coulomb (N/C). We note that

$$1 \quad N/C = \left(1\frac{N}{C}\right)\left(\frac{1V.C}{1J}\right)\left(\frac{1J}{1N.m}\right) = 1 \quad V/m.$$

We also notice that an energy unit that is often more convenient to use in atomic and subatomic domain is electron-volt (eV). 1 eV = energy equal to the work required to move a single elementary charge e through a potential difference of one volt, or

1 eV = $e \times$ (1V) = 1.6×10^{-19}C(1J/C) = 1.6×10^{-19} J.

It is interesting to compare this with the fundamental unit of thermal energy kT. With $k = 1.38\times10^{-23} J/K$, we find that 1eV of energy is equal to thermal energy with temperature $T = (1.6\times10^{-19}/1.38\times10^{-23})K$, or

1 eV ~ thermal energy at 1.16×10^{4} *K.*

4.3 Electric Potential due to a group of point charges

For many charges, the forces and electric field satisfy the law of superposition,

$$\vec{E}(\vec{r}) = \sum_j E_j(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\sum_j \frac{q_j}{|\vec{r}-\vec{r}_j|^3}(\vec{r}-\vec{r}_j)\,.$$

Therefore, the electric potential $V(\vec{r}) - V_\infty = -\int_\infty^{\vec{r}} \vec{E}\cdot d\vec{l}$ is given by (setting $V_\infty = 0$)

$$V(\vec{r}) = -\frac{1}{4\pi\varepsilon_o}\sum_j q_j \int_\infty^{\vec{r}} \frac{\vec{r}\,'-\vec{r}_j}{|\vec{r}\,'-\vec{r}_j|^3}\cdot d\vec{s}\,' = \frac{1}{4\pi\varepsilon_o}\sum_j \frac{q_j}{|\vec{r}-\vec{r}_j|}\,.$$

The last equality comes because the final potential is a sum of potentials from single point charges. Notice that electric potential is a *scalar* and the last sum is a simple algebraic sum of numbers (scalars). This is considerably simpler to evaluate than the sum over electric fields which are *vectors*.

Example: electric potential at position $\vec{r} = (x, y, z)$ from 2 point charges with magnitude q, and q', respectively located at $\vec{r} = \pm z_0\hat{z}$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\sum_j \frac{q_j}{|\vec{r}-\vec{r}_j|} = \frac{1}{4\pi\varepsilon_o}\left[\frac{q}{|\vec{r}-z_0\hat{z}|} + \frac{q'}{|\vec{r}+z_0\hat{z}|}\right]$$

where $|\vec{r} \pm z_0\hat{z}| = \sqrt{x^2 + y^2 + (z \pm z_0)^2}$

Evaluating the electric field from V(r).

Recall that for a conservative force, the force can be derived from the potential energy using

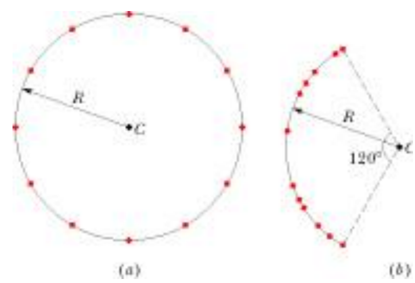
$$F_x = -\frac{\partial U(x,y,z)}{\partial x},\quad F_y = -\frac{\partial U(x,y,z)}{\partial y},\quad F_z = -\frac{\partial U(x,y,z)}{\partial z}\,.$$

Since $q\vec{E}=\vec{F}$ and $qV=U$, we also have

$$E_x=-\frac{\partial V(x,y,z)}{\partial x},\quad E_y=-\frac{\partial V(x,y,z)}{\partial y},\quad E_z=-\frac{\partial V(x,y,z)}{\partial z}.$$

Example:

What are the electric potential and electric field at the center of the circle due to electrons in the figure below?



(Notice that there are 12 electrons around the perimeter in both cases)

Exercise: evaluate the electric field for the 2-charge problem from *V(r)*. Show that it gives the same result as what we have obtained previously by directly adding up the electric fields from 2 separate charges.

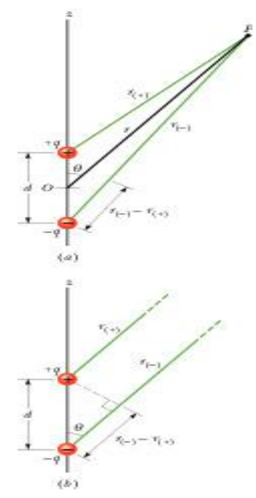
Electric dipole potential

This is the electric potential due to 2 point charges with magnitude q, and $-q$, respectively located at $\vec{r}=\pm{}^{d}\!/_{2}\,\hat{z}$, and at distances $|\vec{r}|>>d$ from the origin.

Using the above result, we obtain for the dipole electric potential;

$$V(\vec{r})=\frac{q}{4\pi\varepsilon_o}\left[\frac{1}{|\vec{r}-{}^{d}\!/_{2}\,\hat{z}|}-\frac{1}{|\vec{r}+{}^{d}\!/_{2}\,\hat{z}|}\right]\underset{r>>d}{\sim}\frac{qd(\vec{r}.\hat{z})}{4\pi\varepsilon_o r^3}=\frac{p\cos\vartheta}{4\pi\varepsilon_o r^2}$$

where p = qd, and θ is the angle between $\vec{r}$ and z-axis (see figure).



Notice that in Cartesian coordinate,

$$V(\vec{r})\underset{r>>d}{\sim}\frac{qd(\vec{r}.\hat{z})}{4\pi\varepsilon_o r^3}=\frac{pz}{4\pi\varepsilon_o(x^2+y^2+z^2)^{3/2}}.$$

Exercise: evaluate the electric field for the dipole problem from *V(r)*. Show that it gives the same result as what we have obtained previously for the dipole electric field. See also textbook p.574-575 for electric dipoles that occur in nature.

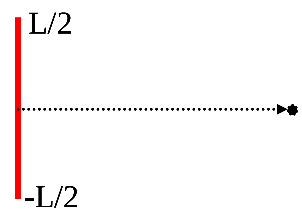
4.4 Electric Potential from a continuous charge distribution

6) *Electric field from a line of charge*.

We imagine a very narrow rod which may not be straight. The rod has charge distribution $\lambda(\vec{x})$, where $0 < |\vec{x}| < L$, L is the length of the rod. The total charge on the rod is $Q = \int_0^L \lambda(x)dx$. The electric potential produced by this rod is, by generalizing the formula for discrete charges,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int_0^L \frac{\lambda(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dr'.$$

We shall consider the example of electric potential coming from a straight line of charge at a point P at distance *r* from the line.



The charge density λ is uniform along the line. Applying the above formula, we obtain

$$V(r) = \frac{1}{4\pi\varepsilon_o}\int_{-L/2}^{L/2} \frac{\lambda}{(r^2+(y+z)^2)^{1/2}}dy = \frac{1}{4\pi\varepsilon_o}\int_{-L/2+z}^{L/2+z} \frac{\lambda}{(r^2+y^2)^{1/2}}dy.$$

This integral can be evaluated using standard integral formula (Integral 17 in Appendix E of textbook). We obtain

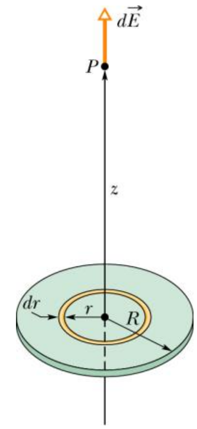
$$V(r) = \frac{\lambda}{4\pi\varepsilon_o}\left[\ln(y+(y^2+r^2)^{1/2}\right]_{-L/2+z}^{L/2+z}$$
$$= \frac{\lambda}{4\pi\varepsilon_o}\ln\left[\frac{L/2+z+((L/2+z)^2+r^2)^{1/2}}{L/2-z+((L/2-z)^2+r^2)^{1/2}}\right]$$

7) *Electric potential from a surface of charge*

For the situation where charges $\sigma(x,y)$ are distributed over a surface of very small thickness instead of along a line, the formula for the electric potential becomes

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int \frac{\sigma(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d^2r'.$$

Now let us apply it to calculate the electric potential at a symmetry point *P*, at a distance *z* above a plastic circular disc of radius *R* with uniform charge density $\sigma(x,y) = \sigma$.



with $d^2r' \to 2\pi r'dr'$, we obtain

$$V(z) = \frac{1}{2\varepsilon_o}\int_0^R \frac{\sigma}{\sqrt{z^2+r'^2}}r'dr' = \frac{\sigma}{2\varepsilon_o}\left(\sqrt{z^2+R^2}-z\right)$$

4.5 Electric potential energy of a system of point charges

We have shown that for a pair of charges q_1, q_2, the electric potential energy is $U_{12} = \frac{q_1 q_2}{4\pi\varepsilon_o r_{12}}$, which is minus the work done by the electrostatic force to move one of the charges from infinity (with the other charge fixed) to its position at r_{12}. For a collection of charges, we may define the electric potential energy similarly. We define:

The electric potential of a system of fixed point charges is equal to the work done by an external agent to assemble the system, bring each charge in from an infinite distance.

Let's see how we can derive the potential energy expression for 3 charges, q_1, q_2, q_3, located at r_1, r_2, r_3, respectively, using the expression for U_{12} and the principle of superposition.

Our strategy is to bring in the charges one by one, and sum up the energy we needed to bring in all the charges together.

First we bring in charge 1 to position r_1. The electric potential energy U_1 we needed is zero, since there is no charge around.

Next we bring in the second charge to position r_2. The electric potential energy we need is $U_2=U_{12}$ from definition.

We then bring in the third charge. From the principle of superposition, the force experienced by the third charge is the sum of the forces from the first two charges. Consequently the work done needed to bring in the third charge is equal to sum of the work done against the force of the first 2 charges, i.e. $U_3=U_{13}+U_{23}$.

Therefore the total electric potential energy needed to build up the system is

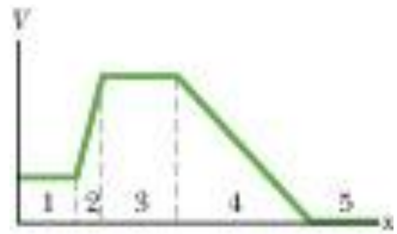
$$U = U_1+U_2+U_3 = (0)+(U_{12})+(U_{13}+U_{23}) = U_{12}+U_{13}+U_{23}.$$

In fact, we can continue this construction to show that the electric potential of a system of N charges is

$$U = \sum_{i<j} U_{ij} = \sum_{i<j}\frac{q_i q_j}{4\pi\varepsilon_o r_{ij}} = \frac{1}{2}\sum_{i\neq j}\frac{q_i q_j}{4\pi\varepsilon_o r_{ij}} .$$

Question

The figure below shows the electric potential V as a function of x. (a) Rank the five regions according to the magnitude of the x component of the electric field within them, greatest first. What is the direction of the field along the x-axis in (b) region 2 and (c) region 4? (d) Indicate points where charges may be present in the system.



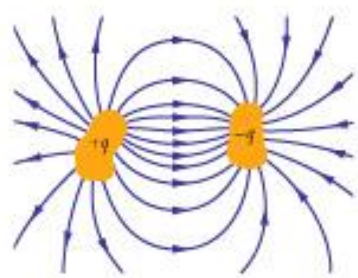
5 Capacitor

5.1 Introduction

In the previous few chapters we have learnt about the basic physics in electrostatics. In this and the next few chapters we shall discuss how these physics can be applied in daily life to build circuits. We start with learning a basic circuit-device element - capacitor. Crudely speaking, capacitors are devices for storing electric charges. Because of electrostatic forces energy is also stored in capacitor at the same time.

Capacitance

Generally speaking, capacitors are composed of two isolated conductors of any shape. We shall call them capacitor plates. The capacitor is "charged" when equal and opposite amount of charges is put on the two plates.



Because the plates are conductors, they are equipotential surfaces. Moreover, there is a potential difference V between the two plates when the capacitor is charged.

The potential difference between the two plates can be expressed as

$$V = -\int_{+}^{-} \vec{E}\cdot d\vec{l} ,$$

where the integral is performed a line joining the plate with positive charge to the plate with negative charge. Since the electric field from a charge distribution is directly proportional to the magnitude of charge, i.e., $\vec{E} \propto q$, we must have $V \propto q$, or

$$q = CV.$$

The proportionality constant C is called the capacitance of the capacitor. We shall see that C depends only on the *geometry of the plates* and not on their charge or potential difference. This is a direct consequence of the linearity of the Coulomb's Law.

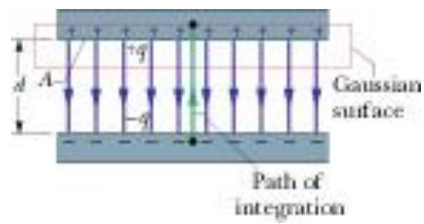
The SI unit for capacitance is *farad*.

1 *farad* = 1 *F* = 1 *Coulomb per volt* = 1*C/V*.

5.2 Calculating the Capacitance

The idea is to apply formula $V = -\int_{+}^{-} \vec{E} \cdot d\vec{l} = \int_{-}^{+} \vec{E} \cdot d\vec{l}$, for a given charge q on the capacitor, where + and – are end points of the conductor. The capacitance is deduced from the relation between V and q. We shall consider situations with high symmetry and will often apply Gauss' Law to calculate the electric field.

A Parallel-Plate capacitor



We assume that the plates of our parallel-plate capacitor are so large and so close together that we can "forget" that the plates has a boundary, i.e. we treat them as two infinite parallel plates put close together.

We draw a closed surface as shown in figure, Notice that one side of the surface is inside the conductor where $\vec{E} = 0$. In this case, all electric field come out perpendicular to the surface. Applied Gauss' Law, we have

$$q = \varepsilon_o EA, \quad A = \text{area of capacitor plate.}$$

We also have

$$V = \int_{=}^{+} \vec{E} \cdot d\vec{l} = Ed, \text{ where } d = \text{distance between two plates.} \quad \text{Therefore, } q = \varepsilon_o \left(\frac{V}{d}\right) A = CV.$$

The capacitance is $C = \dfrac{\varepsilon_o A}{d}$. Notice that the capacitance does depend only on geometrical factors, the plate area A and plate separate d. We shall see that this remains true in later examples.

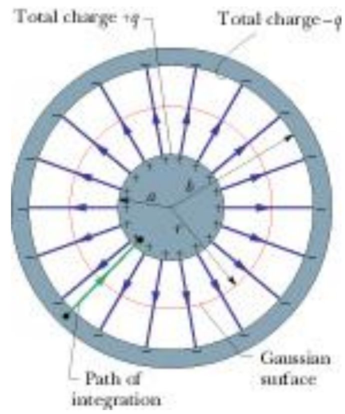
A Cylindrical Capacitor

The figure above shows, in cross section, a cylindrical capacitor of length L, formed by two coaxial cylinders of radii a and b. We assume $L >> b$ so that we can neglect end effects. Each plate contains a charge of magnitude q.

To apply Gauss' Law, we choose a closed surface = a cylinder of length L and radius r, closed by caps at the end of the cylindrical capacitor. Notice that by symmetry the electric field goes radially outward from the inside cylinder. Applying Gauss' Law, we obtain

$$q = \varepsilon_o EA = \varepsilon_o E(2\pi rL).$$

$(2\pi rL)$ is the area of the closed surface without the caps. There is no flux through the end caps. Solving for E we obtain



$$E(r) = \frac{q}{2\pi\varepsilon_o Lr}.$$

Therefore, $V = \int_{-}^{+} \vec{E}.d\vec{s} = -\frac{q}{2\pi\varepsilon_o L}\int_b^a \frac{dr}{r} = \frac{q}{2\pi\varepsilon_o L}\ln\left(\frac{b}{a}\right).$

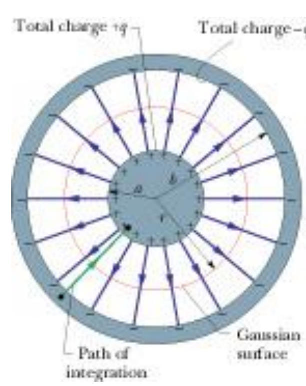
From the relation $q = CV$, we obtain $C = \dfrac{2\pi\varepsilon_o L}{\ln\left(b/a\right)}$.

Notice that C depends on geometrical factors characterizing the cylindrical capacitor only.

A Spherical Capacitor

A spherical capacitor consists of two concentric spherical shells, of radii a and b. It can be visualized by the same figure above, if you view it as a central cross section of the spherical capacitor

To apply Gauss' Law we consider a spherical surface between the two shells. In this case,



$$q = \varepsilon_o EA = \varepsilon_o E(4\pi r^2)$$

and $E(r) = \dfrac{q}{4\pi\varepsilon_o r^2}$, a result we have obtained before.

The potential difference between the shells is thus

$$V = \int_{-}^{+} \vec{E}\cdot d\vec{l} = -\frac{q}{4\pi\varepsilon_o}\int_b^a \frac{dr}{r^2} = \frac{q}{4\pi\varepsilon_o}\left(\frac{1}{a} - \frac{1}{b}\right)$$

and $\quad C = 4\pi\varepsilon_o \dfrac{ab}{b-a}$.

Isolated conductors

We can define a capacitance to a single isolated conductor by taking ("removing") one piece of the conductor to infinity. For an isolated spheres taking $b \to \infty$ we obtain $C = 4\pi\varepsilon_o a$.

To understand the meaning of this result we look at the equation $q/C = V$. In the previous cases V is the potential difference between the two pieces of conductors. What is the meaning V when $b \to \infty$?

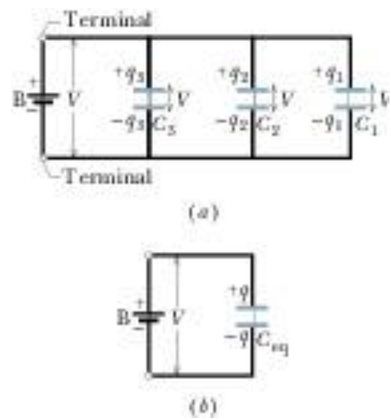
(ans: V is the potential difference between the charged conductor and *ground (or infinity)*.)

5.3 Capacitors in parallel and in series

When there is a combination of capacitors in a circuit, we can sometimes describe its behavior as an *equivalent capacitor* – a single capacitor that has the same capacitance as the actual combination of capacitors. In this way, we can simplify the circuit and make it easier to analyze. There are two basic configurations of capacitors that allow such a replacement.

Capacitors in parallel

The configuration is shown in the figure.



The equivalent capacitance can be derived by analyzing the applied voltage and charge on each capacitor.

The applied voltage is the same for the (3) capacitors, i.e., we have

$$q_1 = C_1V, \quad q_2 = C_2V \quad q_3 = C_3V, ...$$

The total charge stored is $q = q_1 + q_2 + q_3 + ...$. Therefore the effective capacitance is

$$C_{eq} = \frac{q}{V} = C_1 + C_2 + C_3 + ...$$

for capacitors in parallel.

Capacitors in Series

The configuration is shown in the figure below.

Notice that because of overall charge neutrality, the charge q stored on each capacitor must be the same = *charge stored in the equivalence capacitor*. The voltage across each capacitor is given by

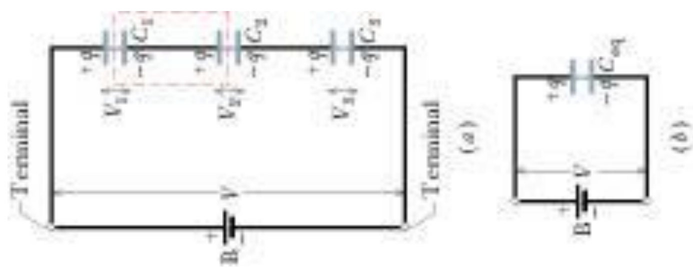
$$V_1 = q/C_1,\;\; V_2 = q/C_2 \;\; V_3 = q/C_3, ...$$

The total voltage across the equivalence capacitor is

$$V = V_1 + V_2 + V_3 + ... = q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + ..\right) = \frac{q}{C_{eq}} .$$

Therefore the equivalent capacitance is given by

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots$$



These results will be used later in circuit analysis.

5.4 **Energy Stored in Capacitors (electric field)**

To charge up a capacitor, we need work done. You may imagine that we need to move the electrons (-*Ve* charges) from neutral atoms in one capacitor plate to another. Energy is needed in this process because we have to separate positive and negative charges from an originally charge neutral configuration. In practice the work done is supplied by a battery.

The total energy needed to charge up a capacitor can be evaluated by noting that the energy needed to move unit charge dq against a potential V is, by definition

$$dW = Vdq = \frac{q}{C}dq .$$

Therefore the work required to bring the total capacitor charge up to final value Q is

$$W = \int dW = \frac{1}{C}\int_0^Q qdq = \frac{Q^2}{2C}$$

= potential energy (U) stored in a capacitor with charge Q.

Alternatively, using the relation $Q = CV$, we may also write $U = \frac{1}{2}CV^2$.

Electric field Energy density

The energy stored in a capacitor can be written in an alternative way. Let's consider a parallel capacitor. In this case $C = \varepsilon_o A/d$ and we may write

$$U = \frac{\varepsilon_o A}{2d} V^2 = \frac{\varepsilon_o Ad}{2}\left(\frac{V}{d}\right)^2 = \frac{\varepsilon_o Ad}{2} E^2,$$

where E is the electric field strength between the plates. The expression can be written as

$U = (Ad) \times u$,

where (AD)= volume between the capacitor plates and

$u = \frac{\varepsilon_o}{2} E^2$ = energy density.

What is interesting is that this result, which is derived for the special case of a parallel-plate capacitor, is valid for *any* capacitors, i.e. the energy stored in any capacitor can be written as

$$U = \int d^3 x u(\vec{x}), \quad \text{where } u(\vec{x}) = \frac{\varepsilon_o}{2} E(\vec{x})^2.$$

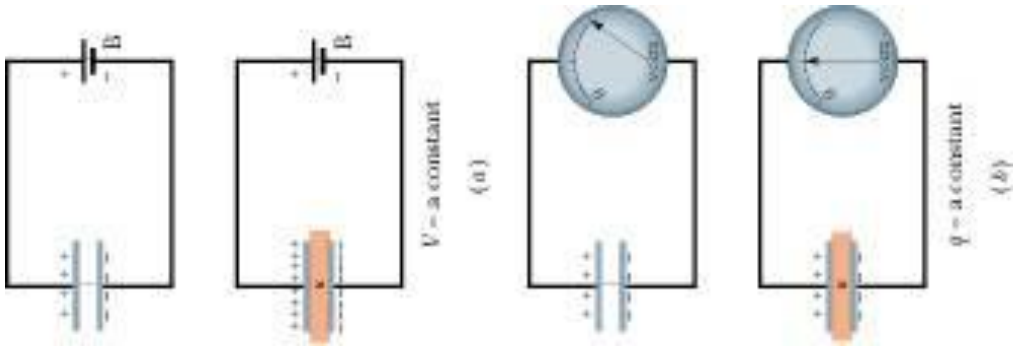
This expression suggests that the potential energy U, which is needed in separating the charges can be viewed as the energy needed to create the electric field associated with the charge separation!

This is a viewpoint which is very different from the conventional viewpoint of Newtonian mechanics, where potential energy is associated with the *position of a particle* (*matter*), but not associated with a *force field*. The viewpoint that energy is associated with a force field, if true, suggests that electric field has a physical significance that is more than representing force between charges. It is also a kind of matter – if you believe that *energy and matter are equivalent*. We shall address this question again later, when we discuss Maxwell equation and electromagnetic wave.

5.5 Dielectric and capacitors (More in EM-2)

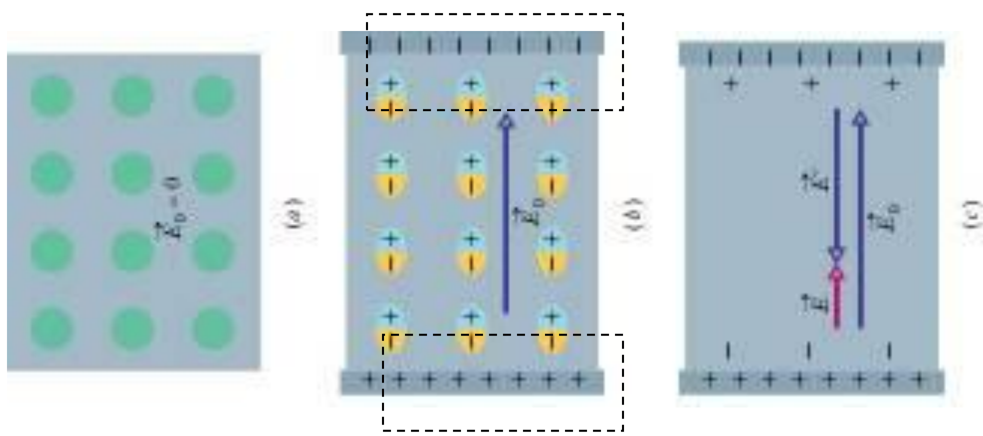
It was first observed by M. Faraday that if the space between capacitor plates is filled with *dielectric*, which is an insulating material such as mineral oil or plastic, the capacitance will increase by a numerical factor $\kappa > 1$ $(C \to \kappa C)$. κ is called the *dielectric constant* of the insulating material. The dielectric constant of a vacuum (unfilled capacitor) is unity by definition.

The effect of inserting the dielectric can be seen easily from the definition $Q = CV$. For the same voltage, a large C implies more charges will be accumulated (see figure).



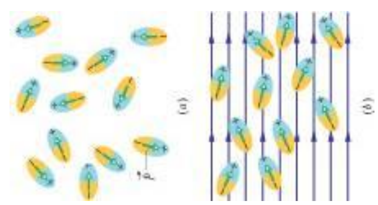
Microscopic Physics behind dielectrics

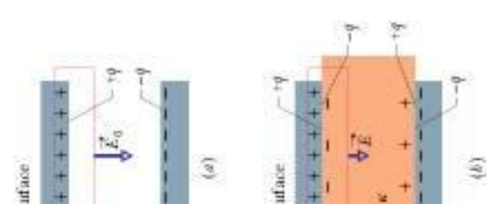
For the same amount of charge, a larger C implies a smaller voltage difference between the capacitors. Using the result $V = \int_i^f \vec{E} \cdot d\vec{l}$. We see that the effect of dielectric ($C \to \kappa C > C$) is to *reduce* the electric field ($E \to E/\kappa < E$) that comes out from the charge. The microscopic origin of this effect can be understood if we consider insulators as composed of closed packed, *non-polar* atoms, with each atom composed of negative and positive charges *centered at the same point*. When the insulator is put under an electric field, the atom will be polarized – the positive charge is pushed away a little bit by the electric field and the negative charge is pulled towards the electric field a little bit (see figure).



Notice that the insulator as a whole is neutral except at the surface, where a layer of induced *negative (positive)* charge *q'* is found next to the capacitor plate with *positive (negative)* charge *q*. The layers of induced charge produce an electric field $\vec{E}'$ that opposes the electric field $\vec{E}$ coming from the capacitor charge. The total electric field between the plates $\vec{E}^{tot} = \vec{E} - \vec{E}'$ is smaller than $\vec{E}$, resulting in an weaker voltage between the plates.

Some dielectrics (like water) have molecules with permanent electric dipole moments. In the absence of external electric field the dipoles are pointing in random directions. When they are put under an electric field, the dipoles are aligned with the external field (see figure below).





The net effect is similar to non-polar neutral atoms except that induced effective charge *q'* is usually larger, and the total electric field $\vec{E}^{tot} = \vec{E} - \vec{E}'$ is smaller.

The magnitude of the induced charges *q'* is related to the dielectric constant κ of the insulating medium. To derive this relation we consider the electric field between a parallel plate capacitor with a dielectric inserted in between.

The electric field between the plates, is using Gauss' Law (see previous notes on parallel plate capacitor), $E = \frac{q - q'}{\varepsilon_o A}$.

On the other hand, we notice that the effect of dielectric is to weaken the original field $E_0 = q/(\varepsilon_o A)$ by a factor κ, i.e. we expect

$$E = \frac{q - q'}{\varepsilon_o A} = \frac{E_0}{\kappa} = \frac{q}{\kappa \varepsilon_o A}.$$

Therefore, $q - q' = q/\kappa$, or $q' = q(1 - \frac{1}{\kappa})$.

Notice that $\kappa > 1$ for the above picture to be valid.

The introduction of dielectric is not restricted to the discussion of capacitors. A more fundamental issue is how the electric force between 2 charges is modified if the charges are put into an insulating medium. What is found is that the Coulomb force is in general weakened, with

$$\vec{F}_{12} \to \frac{q_1 q_2}{4\pi\varepsilon r^2}\hat{r}, \quad \text{where } \varepsilon = \kappa\varepsilon_o .$$

Correspondingly, Gauss' Law is modified to $\oiint \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon}$.