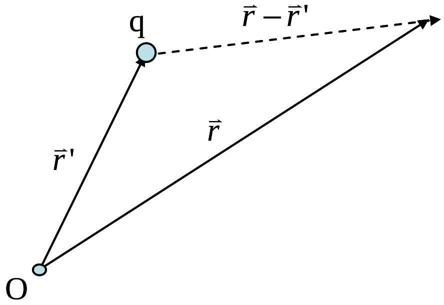
EM-2

1 Coulomb's law, electric field, potential field, superposition



Electric field of a point charge

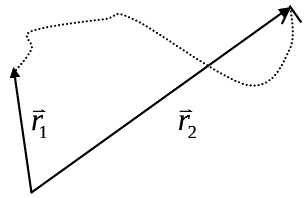
$\vec{E}(\vec{r}) = kq\dfrac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$, where $k \equiv 1/4\pi\varepsilon_0$ (1)

Force of q on a test charge e at position $\vec{r}$ is $e\vec{E}(\vec{r})$

Electric potential

$V(\vec{r}) = \dfrac{kq}{|\vec{r}-\vec{r}\,'|}$ (2).

The potential energy of a test charge e at position $\vec{r}$ is $eV(\vec{r})$



$V(\vec{r}_2) - V(\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E}\cdot d\vec{l}$ (3)

which is *independent* of the path.

Superposition of field and potential. Given charge distribution $\rho(\vec{r}\,')$, the potential at $\vec{r}$ is

$$V(\vec{r}) = k\iiint \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dx'dy'dz' = k\iiint \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dv' \qquad (4)$$

And the electric field is

$$\vec{E}(\vec{r}) = k\iiint \frac{\rho(\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}dx'dy'dz' = k\iiint \frac{\rho(\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}dv' \qquad (5)$$

Uniqueness theorem: It is quite obvious from Eqs. (4) and (5) that given $\rho(\vec{r}\,')$ then $V(\vec{r})$ and $\vec{E}(\vec{r})$ are uniquely determined. The reverse, which is not obvious from the equations, is also true. This is quite important in some cases where one can guess the charge distribution to produce the given potential field. If the guess is correct, then it is the one and the only correct answer.

2 Gauss's law

Gauss's Law

$$\frac{Q}{\varepsilon_0} = \oiint_S \vec{E}\cdot d\vec{S} \qquad (6)$$

Here $\vec{E}$ is the *total field*, but Q is the charge within the space enclosed by S.

Differential form: $\nabla\cdot\vec{E} = \rho/\varepsilon_0$ (6a)

For the electric potential, it is

$$\nabla^2 V = -\rho/\varepsilon_0 \qquad (6b)$$

This is called Poisson's Equation. In the space where the charge density is zero, we have the Laplace Equation

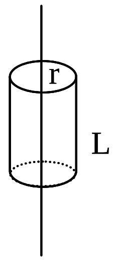
$$\nabla^2 V = 0 \qquad (6c)$$

Example-1: Verify Eq. (6) with a point charge.
Take a spherical surface of radius r centered on the point charge at $\vec{r}' = 0$. The electric field at any point on the surface is parallel to the surface normal and its amplitude is $\frac{kq}{r^2}$. The total surface of the sphere is $4\pi r^2$, so $\oiint_S \vec{E} \cdot d\vec{S} = 4\pi r^2 \frac{kq}{r^2} = 4\pi kq = \frac{q}{\varepsilon_0}$ <u>ans.</u>

Gauss's Law is useful in cases with high symmetry.

Example-2:
Find the E-field and potential of an infinitely long straight wire carrying uniform charge of λ per unit length.

Solution:
Symmetry analysis shows that the electric field is pointing outwards along the radial direction, and depends on r (the radial distance) only. Take a closed Gauss surface as shown, the top/bottom surfaces will have no flux. The flux through the size wall is $2\pi r L E = \frac{\lambda L}{\varepsilon_0}$. So $E = \frac{\lambda}{2\pi r \varepsilon_0}$.



The electric potential V(r) is likewise depending on r only. Taking a line integral along the radial direction from r_1 to r_2, we have

$$V(r_2) - V(r_1) = -\int_{r_1}^{r_2} E dr = -\int_{r_1}^{r_2} \frac{\lambda}{2\pi r \varepsilon_0} dr = \frac{\lambda}{2\pi \varepsilon_0} \ln(\frac{r_1}{r_2})$$

Or simply $V(r) = -\frac{\lambda}{2\pi\varepsilon_0} \ln(r)$

<u>ans.</u>

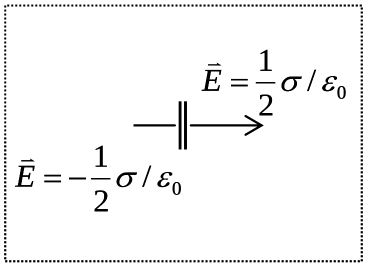
3 Conductors

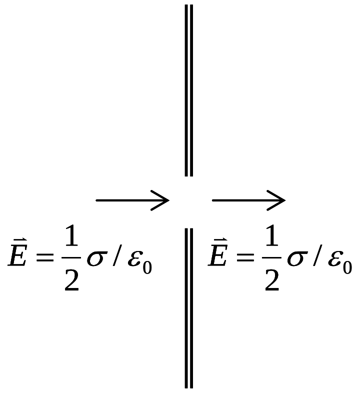
Conductors contain virtually infinite amount of free charges, usually electrons, and a fixed positive charge background. Therefore, in equilibrium, there is no electric field inside a conductor, and the electric potential of the whole body of a conductor is equal. There is charge neutrality everywhere inside a conductor, so the net charge must all be at the surface.
Example-3

Surface charge density and pressure on a charged conductor surface.

At a point very close to the surface, the surface can be taken as flat and infinitely large. Choose a small flat box with one of its large surface in the conductor and the other outside, use Gauss's law, it is straightforward to show that the E-field is $E = \sigma / \varepsilon_0$ outside the conductor and 0 inside. The direction is perpendicular to the surface because any parallel E-field component would drive the surface charge to move around. (The parallel component of the E-fields at both side of the boundary must be the same.)

Now let us find out the electric field force upon a small patch of surface charge on a conductor. The force is not σEA, where A is the area of the patch, because the E-field contains contribution from the charge on the patch, and that charge does not exert force on itself.





A patch of charge will produce E-field $E = \frac{1}{2}\sigma/\varepsilon_0$ on both sides with opposite directions.

The rest of the flat charge sheet produces a field $E = \frac{1}{2}\sigma/\varepsilon_0$ pointing to the right. The combined field of the patch and the flat sheet with the hole is σ/ε_0 pointing to the right. So the force of the flat sheet with hole on the patch is $F = \frac{1}{2\varepsilon_0}\sigma^2 A$. The pressure is $P = \frac{1}{2\varepsilon_0}\sigma^2$.

<u>ans.</u>

The main problem associated with conductors is that only the potential and/or total charge of each conductor is usually known. The exact charge distributions on the conductors are not known in advance, and have to be found as part of the solution of the Laplace Equation or Poisson's Equation. In mathematical terms, the potential and/or total charge of each conductor is called the *boundary conditions*, which must be given in prior. For example, the surface of a sphere of radius R centered at (x_0, y_0, z_0) is $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=R^2$.

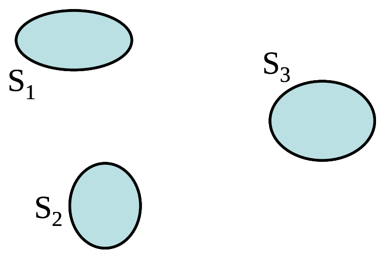
If it is a conductor held at potential V_0, then the potential field $V(\vec{r})$ must be equal to V_0 when $\vec{r}$ satisfy the above equation for the sphere surface.
The boundary conditions are of the same in math as the initial conditions in mechanics. Consider for example a particle in uniform acceleration a along the X-direction. The equation is then $\frac{dv_x}{dt}=a$. Solving it one gets $v_x = at + v_0$, and v_0 is a constant and determined by the initial condition of v_x at t = 0.

The potential or the total charge of all conductors must be known in order to determine the potential field $V(\vec{r})$.

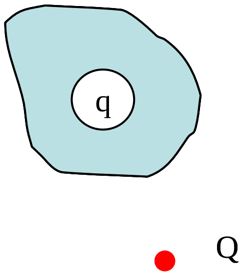
Uniqueness theorem: Given the potential on the surface of a closed space and the charge distribution in the space, the potential (and electric field) in the space is uniquely determined.

Here are some examples of *closed space*.

(1) The *closed space* is the rest of the rest of the space not occupied by the conductors, and the boundaries are the surfaces of the conductors.



(2) Here there are two *closed spaces*. One is the cavity, with its boundary being the inner spherical surface. The other is the space outside the conductor, and its boundary is the outer surface of the conductor. These two spaces are not connected.

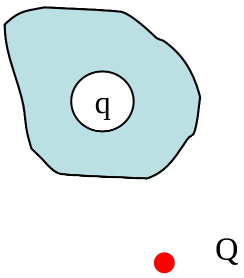


Proof:

Situation-1: The potential of each and every conductor is known.
Consider the case as (1) above, assume that there are two potential fields, $V_1(\vec{r})$ and $V_2(\vec{r})$, both satisfy the Poisson's Equation and the boundary conditions that their values on conductor-1, -2, -3 are U_1, U_2, and U_3, respectively. Then, let $V(\vec{r}) = V_1(\vec{r}) - V_2(\vec{r})$, then it is straightforward to show that $V(\vec{r})$ satisfy the Laplace Equation (no charge density outside the conductors), and its value on all conductors is 0. This is the case where there is no charge everywhere, so $V(\vec{r}) = 0$.

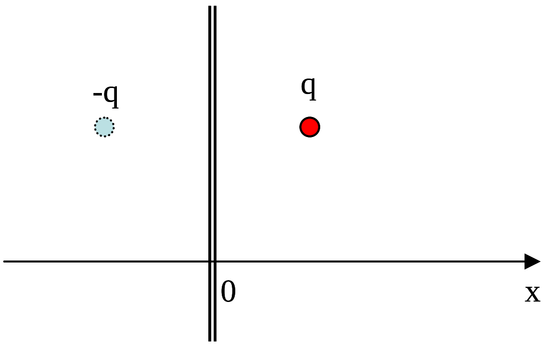
Situation-2: The total charge on each conductor is known. (HW exercise)

Example-4
A point charge is in a cave inside a conductor held at potential V. The potential and E-field in the cave remain the same as long as the conductor is at potential V, regardless of how this is achieved, e. g., by putting charge on the conductor or by placing some charge Q near the conductor.
Likewise, the potential and E-field outside the conductor is uniquely determined as long as the conductor remains at potential V, regardless of the interior of the conductor, which can even be carved out till only a thin shell remains, completely filled, or the charge q removed.
ans.



The technique of image charge: Use a charge distribution to create the same potential distribution on the surface of a closed space to facilitate the finding of potential (field) in the space.

Example-5 A point charge is placed at distance d from an infinitely large conductor plate at $V = 0$. Find the potential, E-field, the surface charge density on the plate, and the force on the charge.



Solution:

The E-field in the space of $x < 0$ is zero. So is V.
For the space $x > 0$, the boundary of it is the plate at $x = 0$. The boundary condition is $V(x = 0) = 0$. This is maintained by the induced charge on the plate which is so properly distributed and the point charge q at $x = d$.

Put an 'image charge' $-q$ at $x = -d$ will produce the same boundary condition. So for the E-field in $x > 0$ the effect of the induced charge on the plate is exactly the same as the image charge $-q$ while removing the plate. The force on q is the same as the force of point charge $-q$ on it, i. e., $F = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{4d^2}$, which is attractive. ans.

As an exercise, find the E-field and the charge density at the plate surface.

Note that the image charge $-q$ cannot be applied to the $x < 0$ region, because its presence changes the condition there (Originally there is no charge in that region).

4 Several conductors, capacitance, capacitors

Suppose there are three conductors (1, 2, 3) in a closed space. The electric potential of the conductors are V_i (i = 1, 2, 3) and the total charge on each is Q_i. Then it can be shown that

$$V_i = \sum_{j=1}^{3} P_{ij} Q_j \quad (7),$$

where $P_{ij} = P_{ji}$ depend on the shapes and the positions of the conductors only. If there are n conductors, then i and j run from 1 to n. Note that P_{12} for two conductors alone usually will change when the third conductor is added.

With only one conductor, $V = PQ$, and $C = 1/P$ is the *capacitance* of the conductor.

If conductor-1 is inside conductor-2, then P_{12} remains the same regardless of the presence of other conductors (use uniqueness theorem to proof). They form a *capacitor*. The potentials due to charges on other conductors will be the same for both conductor-1 and -2. So $P_{1j} = P_{2j}$, j > 2. Also, $P_{22} = P_{12}$. Let $Q_1 = -Q_2$, then $V_2 = 0$, and $V_1 = (P_{11} - P_{12})Q_1$. The *capacitance* of the capacitor is $C = 1/(P_{11} - P_{12})$. In general, to find the capacitance of a pair of conductors that **form** a capacitor, we put opposite charges on the pair of conductors and find their potential difference.

Total energy:
$$W = \frac{1}{2}\sum_{i=1}^{N} Q_i V_i \quad (8).$$

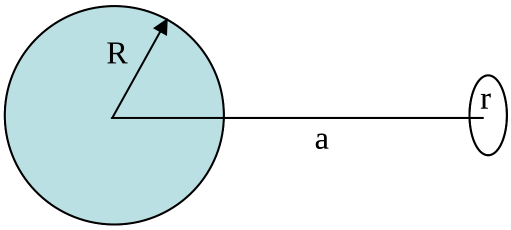
For a capacitor,
$$W = \frac{1}{2} C V^2 \quad (8a).$$

Example-6
A very small conductor is placed at distance d from the center of a conductor sphere with radius R ($< d$). Find the potential of the sphere when the small conductor carries charge q.

Solution.
Let the sphere be the conductor-1 and the other conductor-2, $V_1 = P_{12}q$. But $P_{12} = P_{21} = V_2/q_1$. So the problem is converted to finding the potential of the small conductor when the sphere carries charge q, the answer of which is readily available: $V = kq/d$ ans.

Example-7

As shown in the figure, a thin conducting circular ring is placed at distance a from the center of a conductor sphere of radius R. The line joining the center of the sphere and that of the ring is perpendicular to the ring plane. The ring carries total charge $q > 0$. (Let $k \equiv 1/4\pi\varepsilon_0$)



(1) When the sphere carries no net charge, what is its voltage?
(2) When the sphere is grounded, what is the total charge on it?
(3) When the voltage of the sphere is V_0, what is the total charge on it?
(4) Compare (3) with (2), what is the amount and direction of the change of force acting upon the ring by the sphere?
(5) Compare (1) with (2), what is the amount and direction of the change of force acting upon the ring by the sphere?

Solution:

According to $P_{12} = P_{21}$, the answer of (1) is equal to the voltage of the ring when the sphere is carrying charge q and the ring carries none. The distance between the edge of the ring to the sphere center is $\sqrt{r^2+a^2}$. Note that the space occupied by the ring is an equal-potential one when the ring is absent. Therefore the presence of the ring carrying no charge will not change the charge distribution of the sphere surface. This is only true when the ring plane is perpendicular to the line joining the two centers. So the answer to (1) is V=$kq/\sqrt{r^2+a^2}$. (A charge distribution $\sigma(\theta)$ on the sphere surface produces an E-field that cancels precisely that by the charged ring, so the interior of the sphere is zero-field everywhere.)

For (2), the charge q_1 on the sphere must produce $q_1/R = V = -kq/\sqrt{r^2+a^2}$, so $q_1 = -kRq/\sqrt{r^2+a^2}$. Note that the charge q_1 is uniformly distributed on the sphere surface.

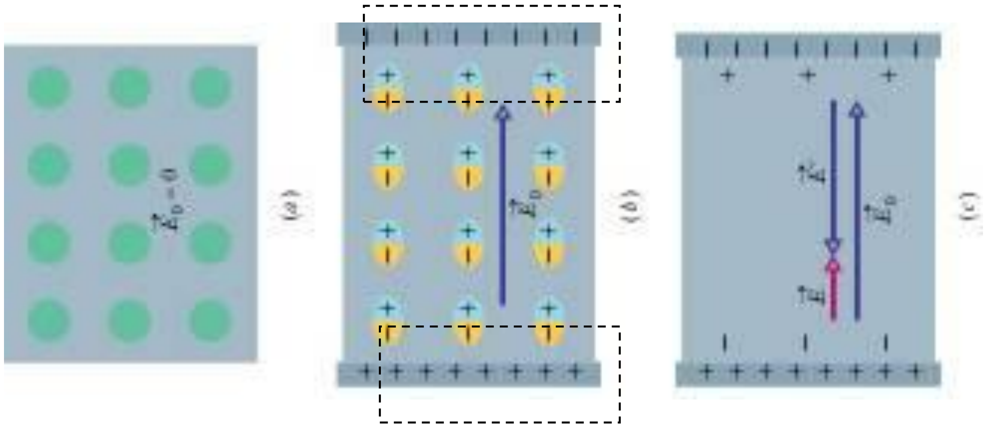
For (3), $V_0 = kq_2/R + kq/\sqrt{r^2+a^2}$, so q_2 can be readily found.

For (4), the difference in the amount of charge is $q_2 - q_1$, which is uniform on the surface, so the force is $F = \dfrac{kq(q_2-q_1)}{r^2+a^2}\dfrac{a}{\sqrt{r^2+a^2}} = \dfrac{kaq(q_2-q_1)}{(r^2+a^2)^{3/2}}$

For (5), the difference in the amount of charge is $-q_1$, which is uniform on the surface, so the force is $F = -\dfrac{kaqq_1}{(r^2+a^2)^{3/2}}$.   ans.

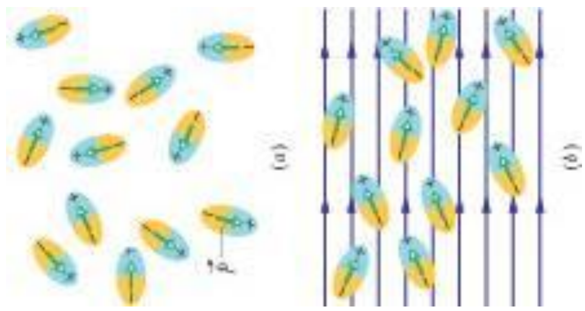
5 Dielectrics Media

The microscopic origin of this effect can be understood if we consider insulators as composed of closed packed, *non-polar* atoms, with each atom composed of negative and positive charges *centered at the same point*. When the insulator is put under an electric field, the atom will be polarized – the positive charge is pushed away a little bit by the electric field and the negative charge is pulled towards the electric field a little bit (see figure).



Notice that the insulator as a whole is neutral except at the surface, where a layer of induced *negative (positive)* charge *q'* is found next to the capacitor plate with *positive (negative)* charge *q*. The layers of induced charge produce an electric field $\vec{E}'$ that opposes the electric field $\vec{E}$ coming from the capacitor charge. The total electric field between the plates $\vec{E}^{tot} = \vec{E} - \vec{E}'$ is smaller than $\vec{E}$, resulting in an weaker voltage between the plates.

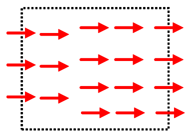
Some dielectrics (like water) have molecules with permanent electric dipole moments. In the absence of external electric field the dipoles are pointing in random directions. When they are put under an electric field, the dipoles are aligned with the external field (see figure below).



The net effect is similar to non-polar neutral atoms except that induced effective charge *q'* is usually larger, and the total electric field $\vec{E}^{tot} = \vec{E} - \vec{E}'$ is smaller.

E-field induces dipoles. Polarization $\vec{P}$ = dipole moment per unit volume.

5.1 Bound Charge



Consider the small volume of area *a* and length *dx*, the net charge within the volume is $-a(P_x(x+dx) - P_x(x)) = -a\frac{\partial P_x}{\partial x}dx$, so the bound charge density is $\rho_b = \frac{\partial P_x}{\partial x}$.

In general, consider any closed surface, the net charge within is $Q_b = -\oiint_S \vec{P} \bullet d\vec{s}$, its differential form is then (compare to Gauss's Law) $\rho_b = -\nabla \cdot \vec{P}$; surface charge density $\sigma_b = \vec{n} \cdot \vec{P}$, where $\vec{n}$ is the surface normal direction.

Define electric displacement $\vec{D} \equiv \varepsilon_0 \vec{E} + \vec{P}$

Gauss's law becomes: $Q_f = \oiint_S \vec{D}\cdot d\vec{S}$; $\nabla\cdot\vec{D}=\rho_f$

The force on a test charge is still $e\vec{E}(\vec{r})$, and the electric potential is still given by Eq. (3).

Linear media: $\vec{P}=\varepsilon_0\chi\vec{E}$. So

$$\vec{D}=\varepsilon_0(\chi+1)\vec{E}\equiv\varepsilon_0\varepsilon\vec{E} \qquad (9)$$

ε is permeability (dielectric constant)

5.2 Boundary conditions: $\vec{E}_1^{//}=\vec{E}_2^{//}$; (10a) $\vec{D}_1^{\perp}-\vec{D}_2^{\perp}=\sigma_f$ (10b).

Generalized uniqueness theorem:
In a space containing conductors (shapes, positions, and potentials given) and dielectrics (shapes, positions, and ε's all given), and free charge distribution, the potential $V(\vec{r})$ throughout the space is then uniquely determined.

Image charges can be used to produce the same boundary conditions.

Ohm's law: $\vec{J}=\sigma\vec{E}$ (11),
$\vec{J}=\rho\vec{v}$ is the electric current density, ρ the mobile charge density. σ is the conductivity of the medium. Both σ and ε are the material parameters of a medium.

Electric current: $I=\iint_S \vec{J}\cdot d\vec{S}$.
So in general one has to define the surface before talking about current.
Under steady conditions (nothing changes with time but things can move), $\nabla\cdot\vec{J}=0$. So

$$J_1^{\perp}=J_2^{\perp} \qquad (12)$$

at an interface (boundary) between two media.

Example-8
At the boundary between two media (ε_1, σ_1, ε_2, σ_2), the electric displacement $\vec{D}_1$ in medium-1 is known. Find $\vec{D}_2$ and the free surface charge density.

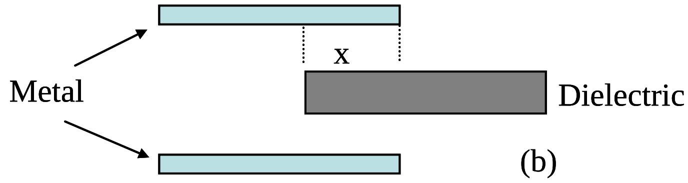
Solution:

Using Eqs. (9), (10a) and (11), we first have $\frac{\vec{D}_1^{//}}{\varepsilon_1}=\frac{\vec{D}_2^{//}}{\varepsilon_2}$. Using Eq. (12), we then have $\frac{\sigma_1 D_1^{\perp}}{\varepsilon_1}=\frac{\sigma_2 D_2^{\perp}}{\varepsilon_2}$. Then use Eq. (10b) to find the free surface charge density. The electric fields and the current densities in both media can be found using Eqs. (9) and (11). Ans.

Example-9

Consider a parallel capacitor made of two large metal plates of L by L separated by distance d ($<<L$) with a neutral dielectric slab (thickness a, same area as the metal plates). The potential difference between the two plates is V. Find the amount of charge on the plates and energy stored in (a) and (b).

(a)



(b)

Solution:

(a)

Since there is no free charge in the space between the plates, D is the same between the plates, as one can check that such D satisfies the boundary condition for D. The E-field inside the dielectric is $E_1 = D/\varepsilon\varepsilon_0$, and in the air gaps $E_2 = D/\varepsilon_0$. Let the upper air gap thickness be x_1, and that of the lower air gap be x_2, then

$V = E_1(x_1 + x_2) + aE_2 = D[(d-a)/\varepsilon + a]/\varepsilon_0$

Then the surface charge density is $\sigma = \pm D$ for upper/lower plates. Total charge $Q = \sigma L^2$. The capacitance $C = Q/V = \varepsilon_0 L^2/[(d-a)/\varepsilon + a]$

Energy $W = 0.5QV$

(b)

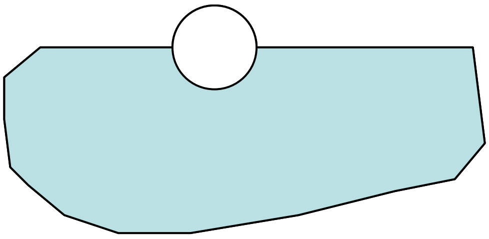
In the x portion the answer of (a) can be applied. For the rest part $\sigma_2 = \varepsilon_0 V/d$.

Total charge is $Q = \varepsilon_0 xL/[(d-a)/\varepsilon + a] + \varepsilon_0 L(L-x)/d$.

$W = QV/2$ depends on x. So the force of the plates on the dielectric is $F = -\dfrac{dW}{dx}$. F being positive means W increases with decreasing x, i. e., the force is pushing the slab out. <u>ans.</u>

Example-10

As shown, half the conductor sphere (radius R) is buried in a dielectric medium. The sphere is held at potential V_0. Find V, E, free and bound charge distributions.



Solution:

The potential is $V(r) = \dfrac{R}{r}V_0$, i. e., as if the dielectric was not there!

Let us examine whether all the boundary conditions are met, which are (i) $V(r=R) = V_0$; (ii) Eq. (10a) at the dielectric/air boundary, and (iii) $\vec{D}_1^{\perp} - \vec{D}_2^{\perp} = 0$ at the dielectric/air boundary. (i) is obvious. Note that with the potential given above, the E-field is parallel to the boundary, so (ii) and (iii) do hold.

The E-field is then $\vec{E}(r) = \dfrac{RV_0}{r^3}\vec{r}$.

The free charge density of the upper hemisphere is $\sigma_1 = \frac{\varepsilon_0 V_0}{R}$, which is also equal to the total charge density at the lower hemisphere surface.

The electric displacement $\vec{D}(r) = \frac{\varepsilon\varepsilon_0 R V_0}{r^3}\vec{r}$ in the lower half of the space, and the free charge density at the lower hemisphere surface is $\sigma_2 = \frac{\varepsilon\varepsilon_0 V_0}{R}$.

The bound charge density is $\sigma_3 = \frac{(1-\varepsilon)\varepsilon_0 V_0}{R}$ <u>ans.</u>

Example-11

A point charge *q* is placed at *x* = *d* in front of an infinitely large dielectric medium filling the space of *x* < 0. Find everything also.

Partial solution:

Again we use image charges. For E-field in *x* > 0, we put a charge *q*' at *x* = –*d*. For E-field in *x* < 0, put a charge *q* + *q*'' at *x* = *d*. The boundary conditions at *x* = 0 are again Eqs. (10a) and (10b) with zero free charge at the interface. There boundary conditions determine *q*' in terms of *q* and ε (permeability).