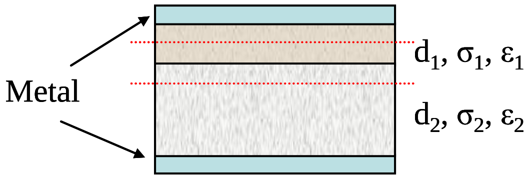
Example-0 (Still on conducting media)
In between the parallel metal plates of area A are two slabs of media (d the thickness, σ the conductivity, and ε the dielectric constant). Find the capacitance C and resistance R between the two plates.



Solution
The usual way to solve this kind of problems is to assume some charge Q on the top plate and $-Q$ on the bottom plate, then work out the electric field and potential difference V between the plates. Note that the electric displacement D is no longer constant, because there may be a surface charge between the two slabs. Instead, the electric current density J should be constant because under steady conditions the total current passing through any cross areas (the dashed lines) parallel to the plates must be the same. From J we get the electric field E in each slab, then voltage V across the plate, and finally C and R.

Static Magnetic field

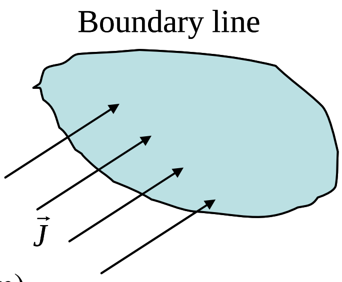
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\iiint \frac{\vec{J}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}dx'dy'dz' = \frac{\mu_0}{4\pi}\iiint \frac{\vec{J}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}dv' \qquad (1),$$

where $\mu_0 = 4\pi \times 10^{-7}$ N/A^2.

When current is confined in thin wires, the volume element $dv'=dx'dy'dz'$ becomes Adl', where A is the cross section area of the wire, and dl' is a small section of the wire. The current density must be in the same direction as the wire section, so $\vec{J}(\vec{r}\,')dx'dy'dz' = I(\vec{r}\,')d\vec{l}\,'$. Eq. (1) then becomes

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{I(\vec{r}\,')d\vec{l}\,'\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}\,'\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \qquad (2),$$

because for steady current it is the same everywhere along the wire.

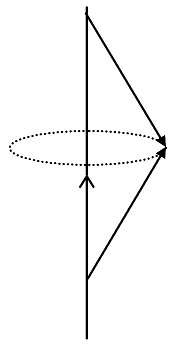


Ampere's Law

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \iint_S \vec{J}\cdot d\vec{S} = \mu_0 I_{enc} \qquad (3).$$

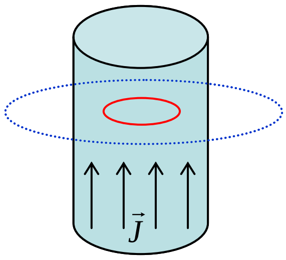
($\nabla\cdot\vec{B} = 0$, $\nabla\times\vec{B} = \mu_0\vec{J}$, which is Ampere's Law in differential form)

Example-1
Find the B-field of an infinitely long wire.



Solution:
Symmetry analysis shows that the B-field must be horizontal and circles around the wire, and depends on r (the distance of the position to the wire) only. Using Eq. (3),

$$\oint \vec{B}\cdot d\vec{l} = 2\pi rB = \mu_0 \iint_S \vec{J}\cdot d\vec{S} = \mu_0 I, \text{ so } B = \frac{\mu_0 I}{2\pi r}.$$ Ans.



Example-1A
Find the B-field of an infinitely long cylindrical conductor of radius R carrying uniform current density J.

Solution:
Take a loop of radius r, as shown (the red or blue circle).
For $r > R$, the answer is the same as above with $I = J\pi R^2$.
For $r < R$, the current within the loop is $J\pi r^2$. So

$$B = \frac{\mu_0 Jr}{2}.$$ Ans.

There are only a few cases where Eq. (1) leads to answers in analytical form.

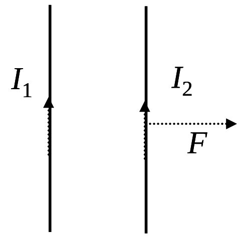
Force of B-field on current wire and charged particle (Lorentz force)

Lorentz force on a point charge q is $\vec{F} = q\vec{v}\times\vec{B}$, where $\vec{v}$ is the velocity of the charge.

Force on current density $\vec{F} = \iiint \vec{J}(\vec{r})\times\vec{B}(\vec{r})dxdydz$.

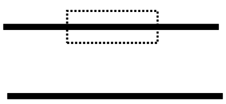
On a thin wire, it is $\vec{F} = I\int d\vec{l}\times\vec{B}(\vec{r})$

Example-2A
Force per unit length between two parallel wires carrying currents I_1 and I_2, and separated by a distance D.
Solution:



From Example 1A, the magnetic field due to I_1 at I_2 is $B = \frac{\mu_0 I_1}{2\pi D}$, and pointing into the paper plane. The force is then pointing to the right, and is the same along the wire. For any section of the wire of length L, the force is $F = \frac{\mu_0 I_1 I_2}{2\pi D}L$.

Example-2B
Pressure on two large parallel plates carrying opposite surface current density σ.
Solution:

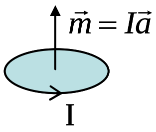


As shown in the figure, assuming that the current on the top plate is flowing out of the paper plane and on the bottom one is flowing into the paper plane. By symmetry we see that the B-field is parallel to the plates and perpendicular to the current. Take a small loop of length L as shown, and apply Ampere's law, we get for a single plate, the B-field is given by $2BL = \mu_0\sigma L$, or $B = 0.5\mu_0\sigma$. Now take a rectangular area of the bottom plate of dimension L by b, the total current being σL and the force is $F = \sigma LBb$. The pressure is then $\sigma B = 0.5\mu_0\sigma^2$

Magnetic dipole – a current loop with physical dimension << distance

Dipole moment $\vec{m} = I\vec{a}$, where *I* is the current and *a* is the area. The direction of the moment follows the right-hand rule.

B-field of a dipole $\vec{B} = \frac{\mu_0}{4\pi r^3}\left[\frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^2} - \vec{m}\right]$ (3)



Torque of B-field on a dipole $\vec{N} = \vec{m}\times\vec{B}$ (4)

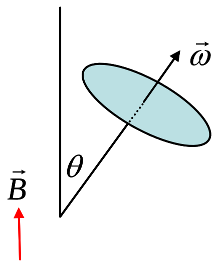
Energy of a dipole in B-field $E = \vec{m}\cdot\vec{B}$ (5)

Force on a dipole in B-field $\vec{F} = \nabla(\vec{m}\cdot\vec{B})$. (6)

So only non-uniform field exerts force on a dipole.
The above formulae also work for an electric dipole if B-field is replaced by E-field and the magnetic dipole is replaced by an electric one.

Example-2C

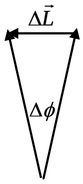
A disc of radius *R* is spinning with angular speed *ω* in a magnetic field *B* in weightless space. The edge of the disk carries a uniform total charge of *Q*. The moment of initia of the disc is I_0. The angle between the spin axis and the field is *θ*. Determine the motion of the disc.



Solution:

The angular momentum of the disc is $\vec{L} = I_0\vec{\omega}$.
The charge at the edge of the disc forms an electric current loop which can be viewed as a magnetic dipole with moment

$m = \pi R^2 \omega Q / 2\pi = \frac{1}{2}R^2\omega Q$.

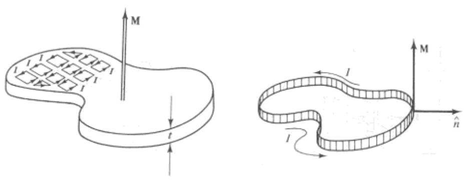


The torque on the disk is *mB*sin*θ* and perpendicular to the $\vec{L}\wedge\vec{B}$ plane.
The change of the angular momentum under the action of the torque is
$\Delta L = L\Delta\phi\sin\theta = mB\sin\theta\Delta t$.

So $\frac{\Delta\phi}{\Delta t} = \omega_p = \frac{mB}{L} = \frac{BQR^2}{2I_0}$.

The direction of the spin axis will rotate around the B-field direction with angular speed ω_p.
Such type of motion is called procession.
<u>Ans.</u>

Magnetization of media



$\vec{M}$ --magnetic dipole per unit volume.
Bound current density $\vec{J}_b = \nabla\times\vec{M}$
Bound current surface density $\vec{K}_b = \vec{M}\times\vec{n}$, where $\vec{n}$ is the surface normal pointing out of the medium.

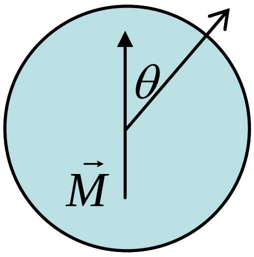
Ampere's law becomes $\nabla\times\vec{B}=\mu_0(\vec{J}_f+\vec{J}_b)=\mu_0(\vec{J}_f+\nabla\times\vec{M})$.

$$\text{So } \nabla\times\left(\frac{\vec{B}}{\mu_0}-\vec{M}\right)=\nabla\times\vec{H}=\vec{J}_f \qquad (3)$$

$$\text{And } \vec{H}\equiv\frac{\vec{B}}{\mu_0}-\vec{M} \qquad (4)$$

Example-3

Find the bound current of a uniformly magnetized sphere.



Solution

$\vec{M}$ is uniform inside the sphere so $\vec{J}_b=0$

$K_b=M\sin\theta$, and is similar to a spinning sphere of radius R carrying a uniform surface charge density σ, where the surface current density is $K_b=\sigma\omega R\sin\theta$

By the way, the B-field inside the sphere is $\vec{B}=\frac{2}{3}\mu_0\vec{M}$, outside the sphere it is the same as due to a pure dipole $\vec{m}=\frac{4}{3}\pi R^3\vec{M}$. (Eq. (3)) <u>ans.</u>

Boundary conditions

$\nabla\cdot\vec{B}=0$ leads to $B_1^{\perp}=B_2^{\perp}$ (5)

Equation (3) leads to $\vec{H}_1^{//}-\vec{H}_2^{//}=\vec{K}_f\times\vec{n}$, (6)

where $\vec{n}$ is the interface normal pointing from medium-1 to medium-2.