0. Measurement and Units

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International System of Units (SI units)

- A coherent system of units of measurement built on seven base units (called base standards)
- Based on the meter-kilogram-second (MKS) rather than centimeter-gram-second(CGS)
- All physical quantities can be expressed/derived in term of these seven base quantities

Quantity name	Unit name	Unit symbol
Length	Meter	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	Kelvin	K
Amount of substance	mole	mol
Luminous intensity	candela	cd

The fundamental base unit can be defined only in term of the procedure used to measure them

Distance and Time

- Time: second (s)

  The duration of 9,192,631,770 periods of the radiation corresponding to the transition between two energy levels of the caesium-133 atom
- Length: meter (m)

  The distance traveled by light in absolute vacuum in 1/299,792,458 of a second

All other units can be derived based on the 7 base units

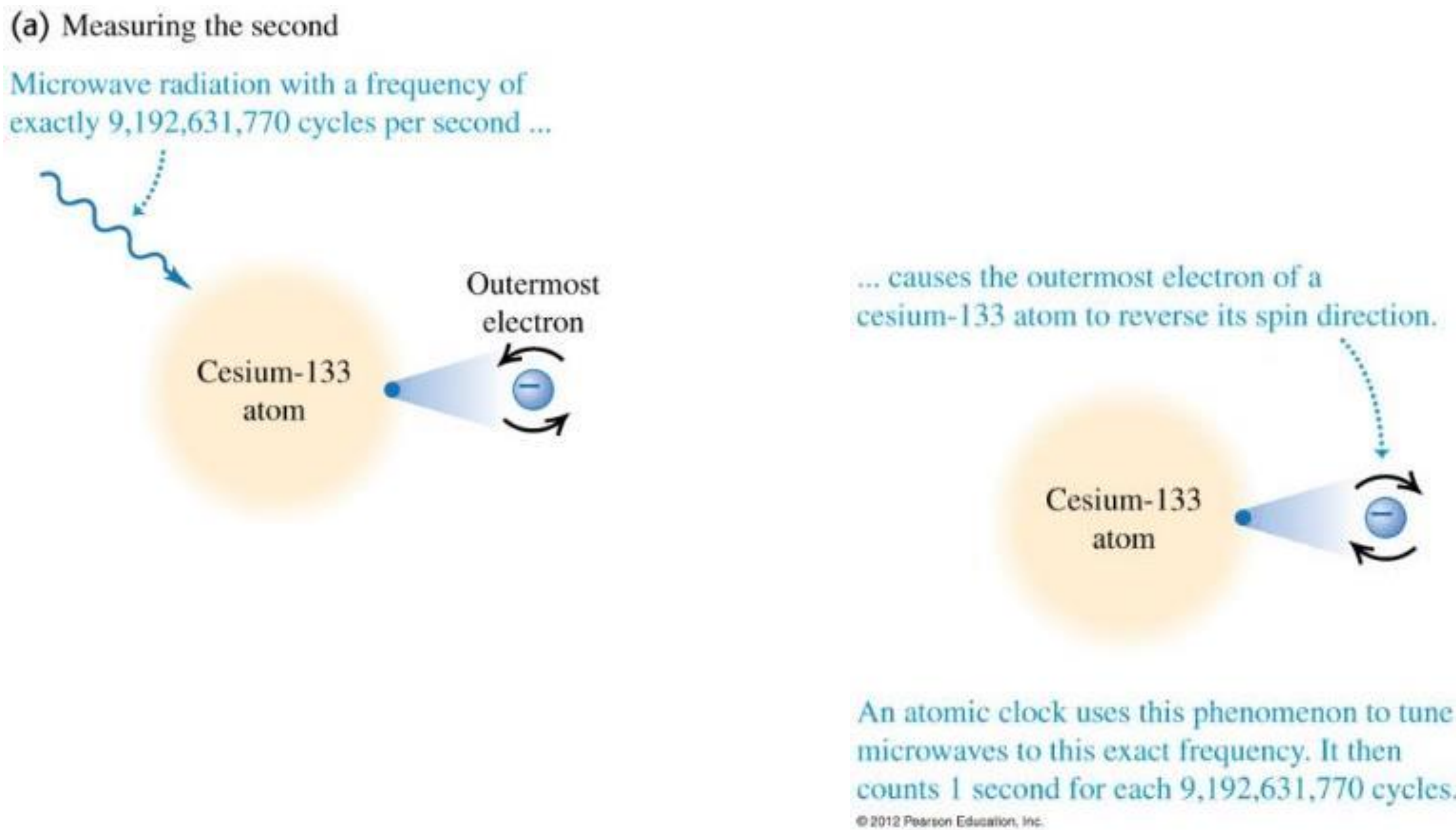
Examples:

Force = Newton = kg m s^{-2}

Speed = distance travelled/time = ms^{-1}

Energy = Joule = N m = kg m^2s^{-2}

Time (Second)


(a) Measuring the second
Microwave radiation with a frequency of exactly 9,192,631,770 cycles per second ...
Outermost electron
Cesium-133 atom
... causes the outermost electron of a cesium-133 atom to reverse its spin direction.
Cesium-133 atom
An atomic clock uses this phenomenon to tune microwaves to this exact frequency. It then counts 1 second for each 9,192,631,770 cycles.
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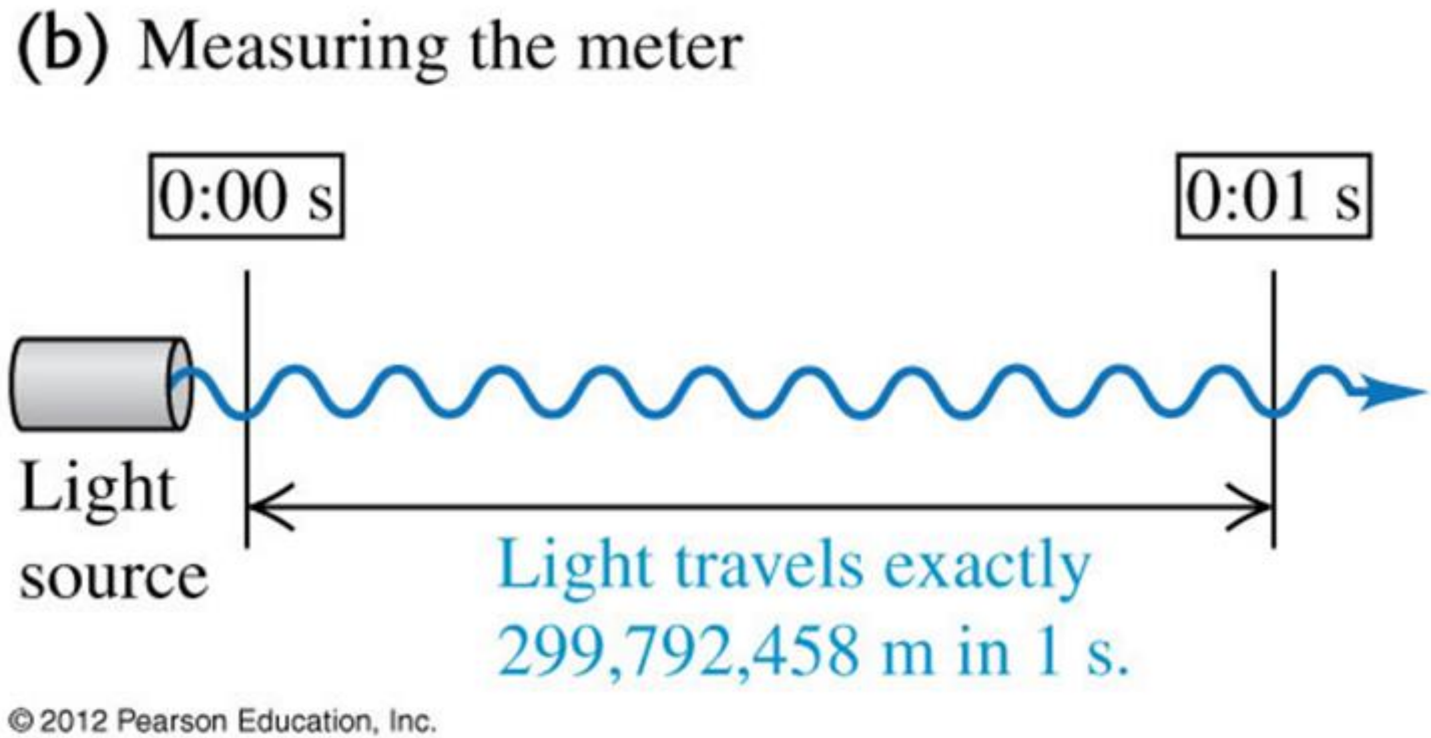
Length (Meter)



Mass (Kilogram)

Changing Units

- Change the units in which a physical quantity is expressed
- Conversion factor
- Example: 1 minute and 60 second are identical time intervals, we have

$$\frac{1\ \text{min}}{60\ \text{s}} = 1 \text{ and } \frac{60\ \text{s}}{1\ \text{min}} = 1$$

- We can use the conversion factor to cancel unwanted units.
- Example:

$$2\,\text{min} = (2\,\text{min})(1) = (2\,\cancel{\text{min}})\frac{60\text{s}}{1\cancel{\text{min}}} = 120s$$

Example:

Light-year (ly) – The distance the light travelled in one Julian year (365.25 days)

Speed of light c = 299 792 458 ms^{-1}

$$1\text{ ly} = (299792458 ms^{-1})(1 year) = (299792458 ms^{-1})(1 year)\frac{(365.25\times 24\times 60\times 60 s)}{(1 year)}$$

$$1\text{ ly} = 9460730472580800\text{ m}$$

Conversion factor

Dimension of physical quantities

- Physical quantities are derived from the base quantities (eg. **length (L), time (T), mass (M)….**) by a set of algebraic relations defining the physical relation between these quantities

- Example
  - Length (x) has the dimension L
  - Velocity (v) = distance travelled (x) divided by the time (t).
  - The dimension of v is:

$$[v] = [\frac{x}{t}] = \frac{[\text{length}]}{[\text{time}]} = LT^{-1}$$

More examples

Acceleration

$$a = \frac{dv}{dt}$$

$$[a] = \frac{[\text{velocity}]}{[\text{time}]} = \frac{[\text{length}]}{[\text{time}]^2} = LT^{-2}$$

Force

$$F = ma$$

$$[F] = \frac{[\text{mass}][\text{length}]}{[\text{time}]^2} = MLT^{-2}$$

Kinetic energy

$$K = \frac{1}{2}mv^2$$

$$[K] = [m][v]^2 = \frac{[\text{mass}][\text{length}]^2}{[\text{time}]^2} = ML^2T^{-2}$$

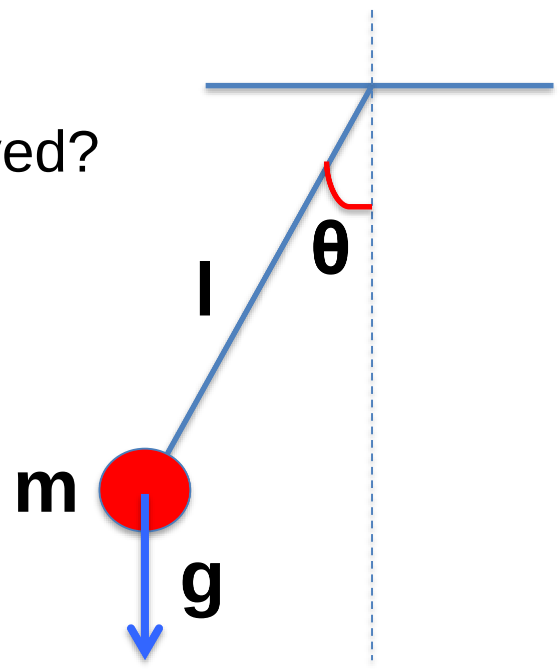
Dimensional analysis

Any physical meaningful equation must have the same dimensions on the left and right sides (dimensional homogeneity)

Example: Consider a simple pendulum consisting of a massive ball suspended from a fixed point by a string. Let t denote the time (period of pendulum) that it takes the bob to complete one cycle of oscillation. What is t?

What are the possible physical quantities involved?

- Length of pendulum, l
- Mass of pendulum bob, m
- Gravitational acceleration, g
- Initial angular amplitude, θ



We want to find a relationship such that

$$t = bl^X m^Y g^Z$$

where b is a dimensionless constant. Since angle θ is dimensionless, b=b(θ) can be a function of the angle θ.

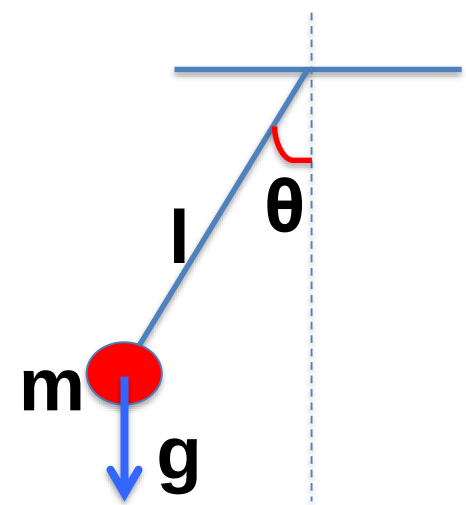
$$[t] = [b(q)][l]^X [m]^Y [g]^Z \text{ Þ } T = L^X M^Y L^Z T^{-2Z}$$

$$\backslash \quad X = -Z = 1/2, Y = 0$$

$$\text{Þ } t = b(q)\sqrt{\frac{l}{g}}$$

(We will solve the problem exactly and show b(q)=2p)

Name of Quantity	Symbol	Dimensional Formula
Time of swing	t	T
Length of pendulum	l	L
Mass of pendulum	m	M
Gravitational acceleration	g	$L \cdot T^{-2}$
Angular amplitude of swing	θ_0	No dimension



1. Vector

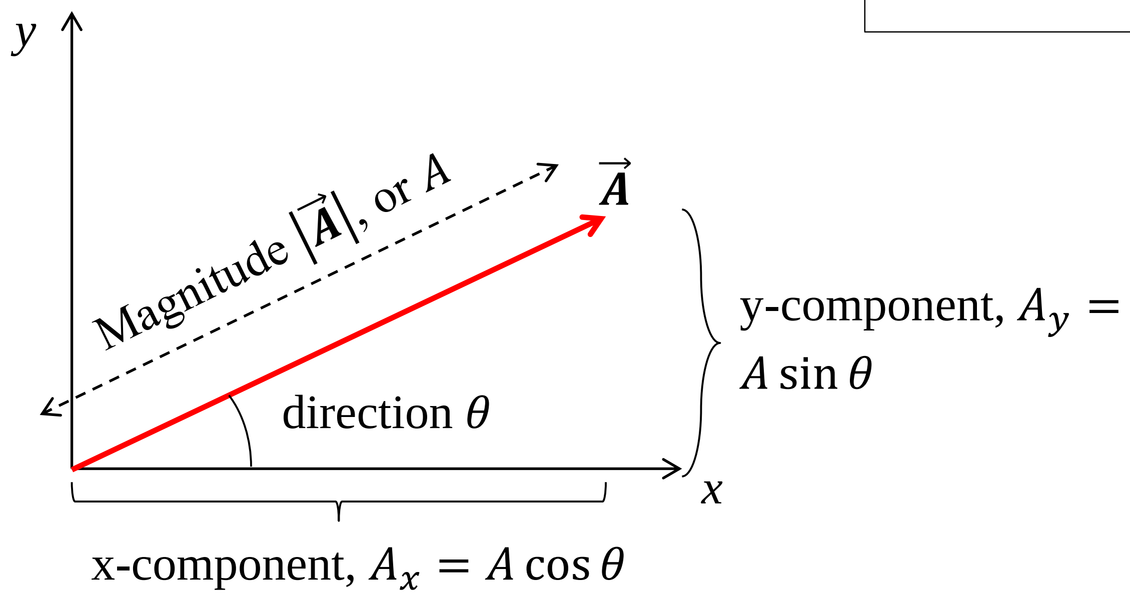
Vector

An "arrow" in space, has magnitude (length) and direction

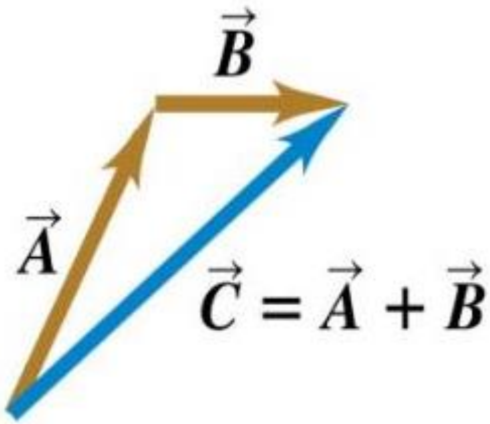
e.g. in 2D Cartesian coordinates (due to Renè Descartes)

Note: $A = \sqrt{A_x^2 + A_y^2}$ (Pythagoras thm)

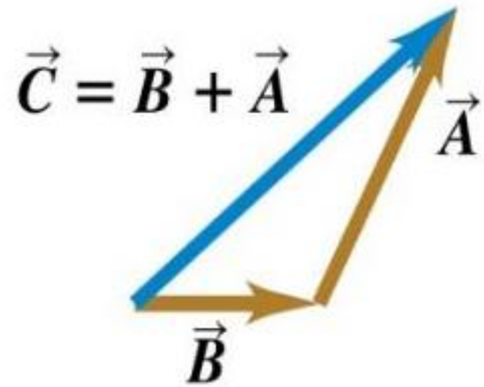
$\tan\theta = \frac{A_y}{A_x}$



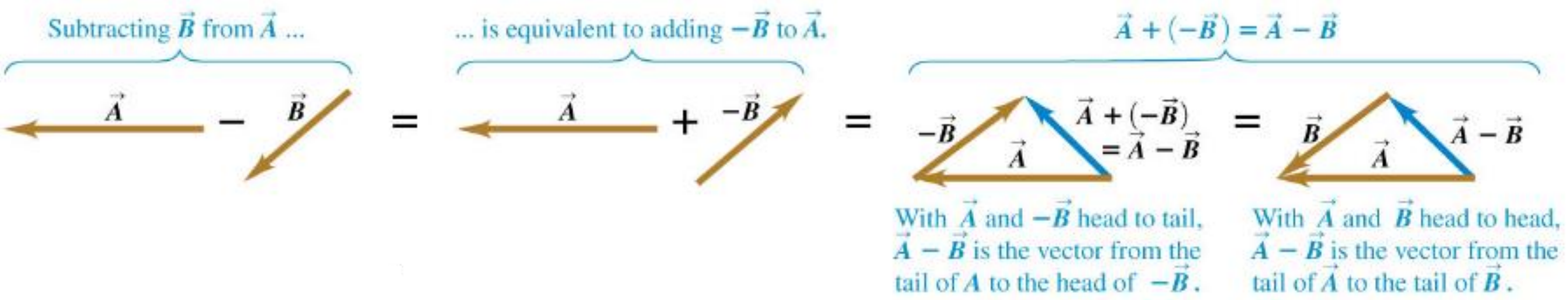
Addition



or



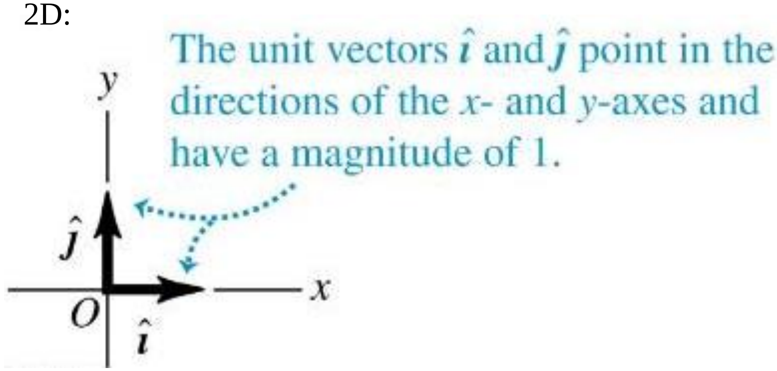
Subtraction



Unit Vectors

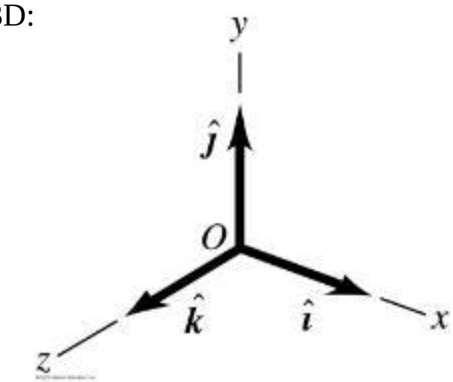
Vectors of unit magnitude are called <u>unit vectors</u>. Most commonly used unit vectors are $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, along x, y, and z directions in Cartesian coordinates

2D:



$$\boxed{\vec{\boldsymbol{A}} = A_x\hat{\imath} + A_y\hat{\jmath}}$$

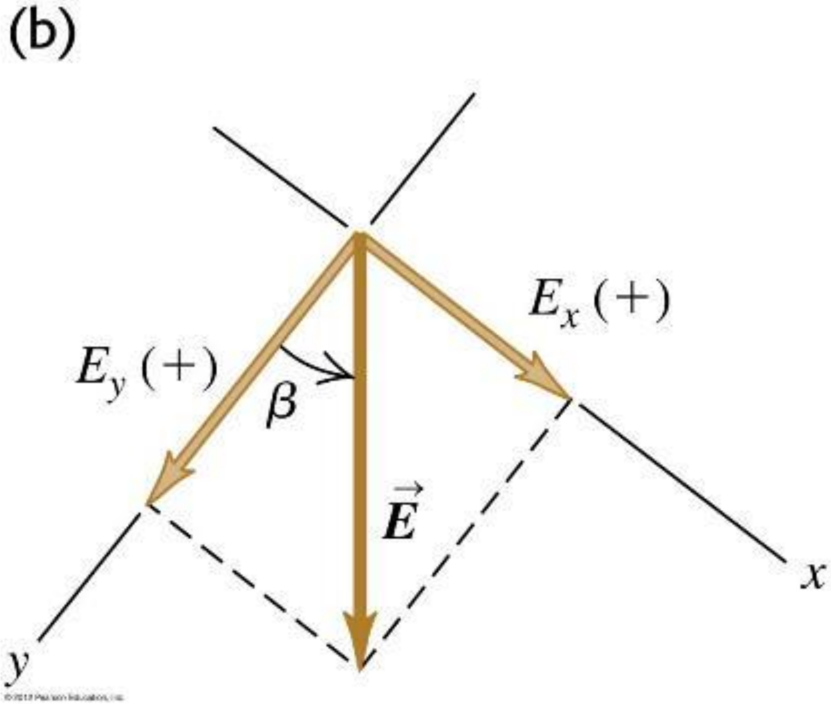
3D:



$$\boxed{\vec{\boldsymbol{A}} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}}$$

Q1.1

What are the *x*- and *y*-components of the vector $\vec{E}$?



A. $E_x = E \cos \beta$, $E_y = E \sin \beta$

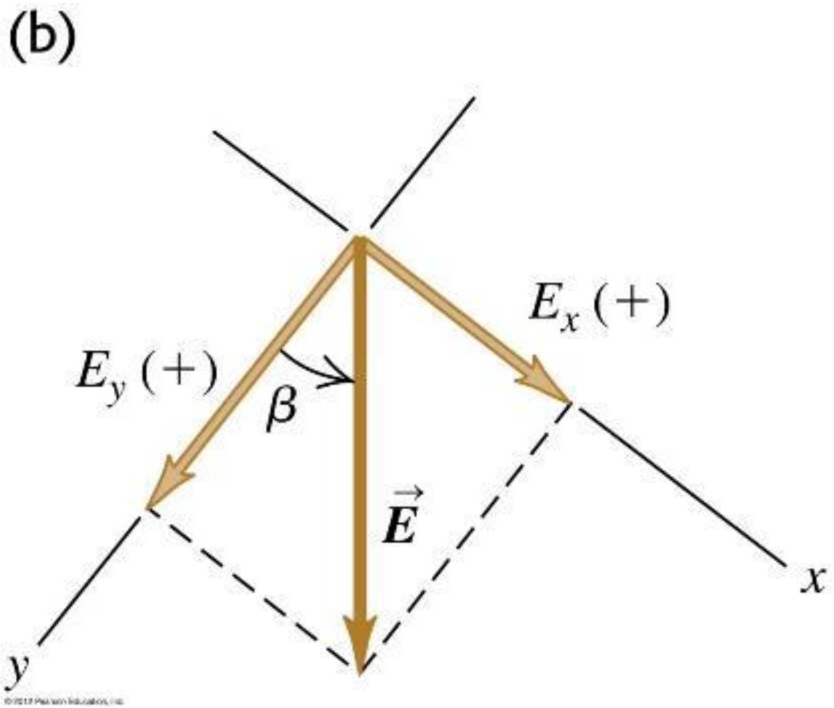
B. $E_x = E \sin \beta$, $E_y = E \cos \beta$

C. $E_x = -E \cos \beta$, $E_y = -E \sin \beta$

D. $E_x = -E \sin \beta$, $E_y = -E \cos \beta$

E. $E_x = -E \cos \beta$, $E_y = E \sin \beta$

Q1.1



What are the *x*- and *y*-components of the vector $\vec{E}$?

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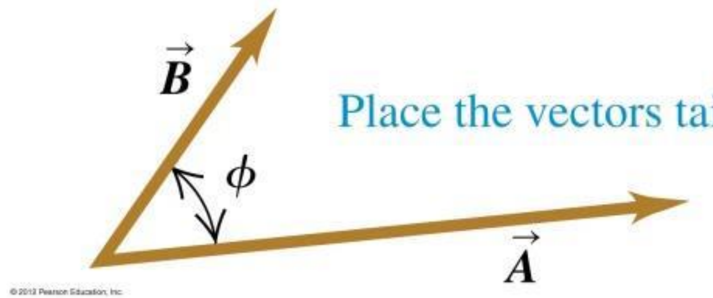
D. $E_x = -E \sin \beta$, $E_y = -E \cos \beta$

E. $E_x = -E \cos \beta$, $E_y = E \sin \beta$

Scalar/Dot product

$$\vec{A} \cdot \vec{B} = AB\cos\phi$$



Special cases:

(i) if $\vec{A} \parallel \vec{B}$, $\vec{A} \cdot \vec{B} = AB$, in particular, $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$

(ii) if $\vec{A} \perp \vec{B}$, $\vec{A} \cdot \vec{B} = 0$, in particular, $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

In analytical form, $\boxed{\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z}$

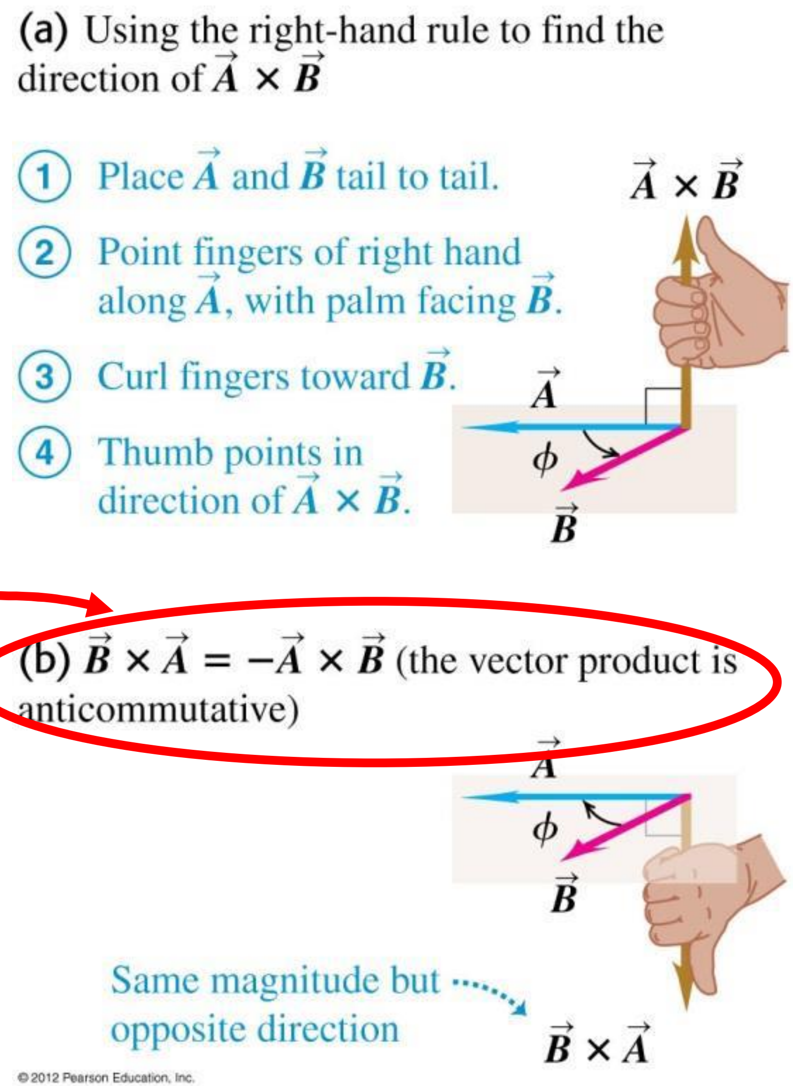
Vector/Cross product

$\vec{C} = \vec{A} \times \vec{B}$

Magnitude: $C = AB \sin \phi$

direction determined by *Right Hand Rule*

Important!



Example:

Consider the unit vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$

What is $\hat{\imath} \cdot \hat{\imath}$ and $\hat{\imath} \cdot \hat{\jmath}$?

What is $\hat{\imath} \times \hat{\imath}$ and $\hat{\imath} \times \hat{\jmath}$?

For the unit vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$, we have

$$\hat{i} \times \hat{j} = \hat{k}$$
$$\hat{j} \times \hat{k} = \hat{i}$$
$$\hat{k} \times \hat{I} = \hat{j}$$

For any two vectors,

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =$$
$$= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Exercise:

Consider two vectors

$$\vec{A} = \hat{i} + 2\hat{j} - \hat{z}$$

$$\vec{B} = \hat{i} + \hat{j} + \hat{j}$$

What is

$$\vec{A} + \vec{B}$$

$$\vec{A} - \vec{B}$$

$$\vec{A} \cdot \vec{B}$$

$$\vec{A} \times \vec{B}$$

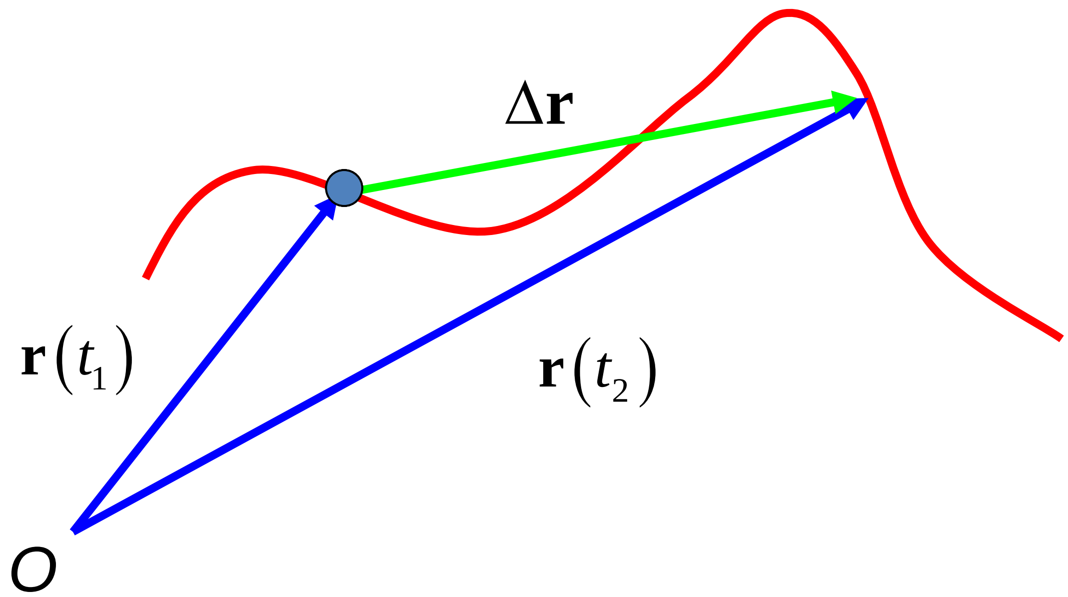
Can you construct a unit vector which is perpendicular (orthogonal) to both vector $\vec{A}$ and $\vec{B}$? Does the vector unique?

2. Trajectory and Motion
Kinematics

We first study the motion of point particle without considering its causes

How to describe the motion of a point particle?

Position Vector and Displacement

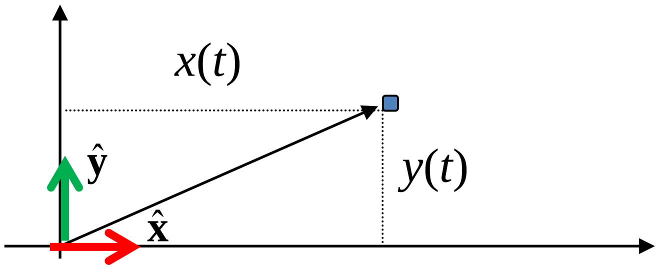


- We can use position vectors $\mathbf{r}(t)$ to specify the position of the particle on a plane at time t
- The above figure shows the position of the particle at two different moments, t_1 and t_2
- The displacement is given by

$$\Delta\mathbf{r} = \mathbf{r}(t_2) - \mathbf{r}(t_1)$$

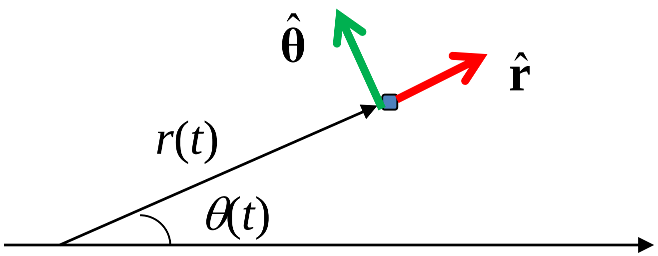
Coordinates in 2D

Cartesian



$$\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}}$$

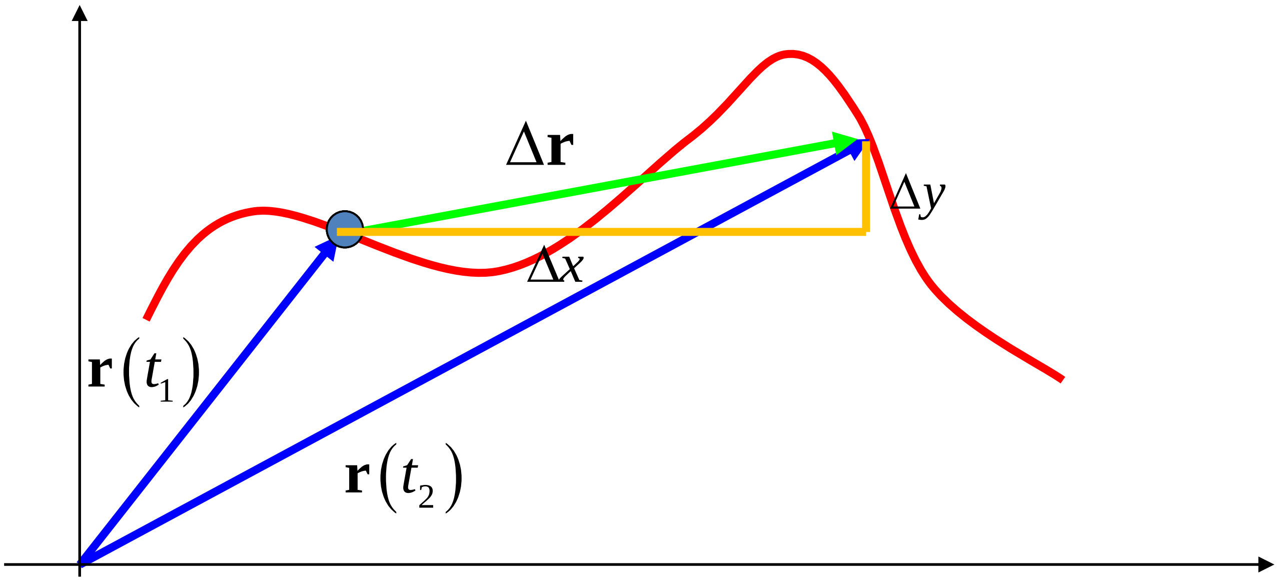
Polar



$$\mathbf{r}(t) = r(t)\hat{\mathbf{r}}(t)$$

We shall use Cartesian coordinates

Displacement in Cartesian Coordinates



$$\mathbf{r}(t_1) = x(t_1)\hat{\mathbf{x}} + y(t_1)\hat{\mathbf{y}} \qquad \mathbf{r}(t_2) = x(t_2)\hat{\mathbf{x}} + y(t_2)\hat{\mathbf{y}}$$

$$\begin{aligned}
\Delta \mathbf{r} &= \mathbf{r}(t_2) - \mathbf{r}(t_1) \\
&= \left(x(t_2)\hat{\mathbf{x}} + y(t_2)\hat{\mathbf{y}}\right) - \left(x(t_1)\hat{\mathbf{x}} + y(t_1)\hat{\mathbf{y}}\right) \\
&= \left(x(t_2) - x(t_1)\right)\hat{\mathbf{x}} + \left(y(t_2) - y(t_1)\right)\hat{\mathbf{y}} \\
&= \Delta x \hat{\mathbf{x}} + \Delta y \hat{\mathbf{y}}
\end{aligned}$$

where

$$\Delta x = x(t_2) - x(t_1)$$

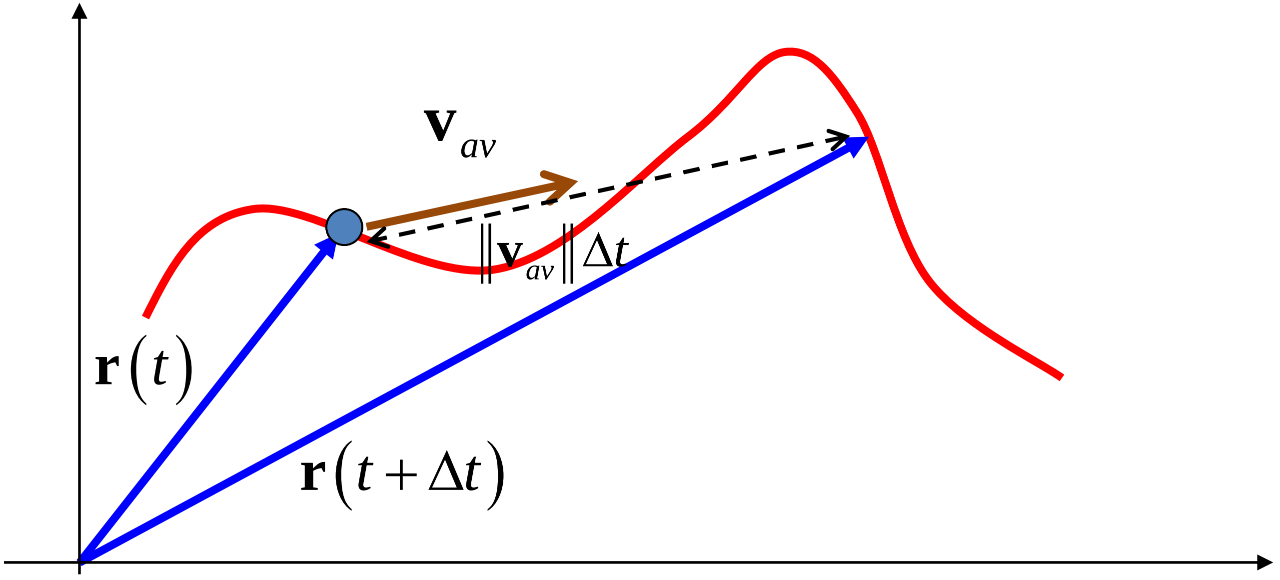
$$\Delta y = y(t_2) - y(t_1)$$

Average Velocity

- The average velocity during the interval Δt is a *vector* defined by

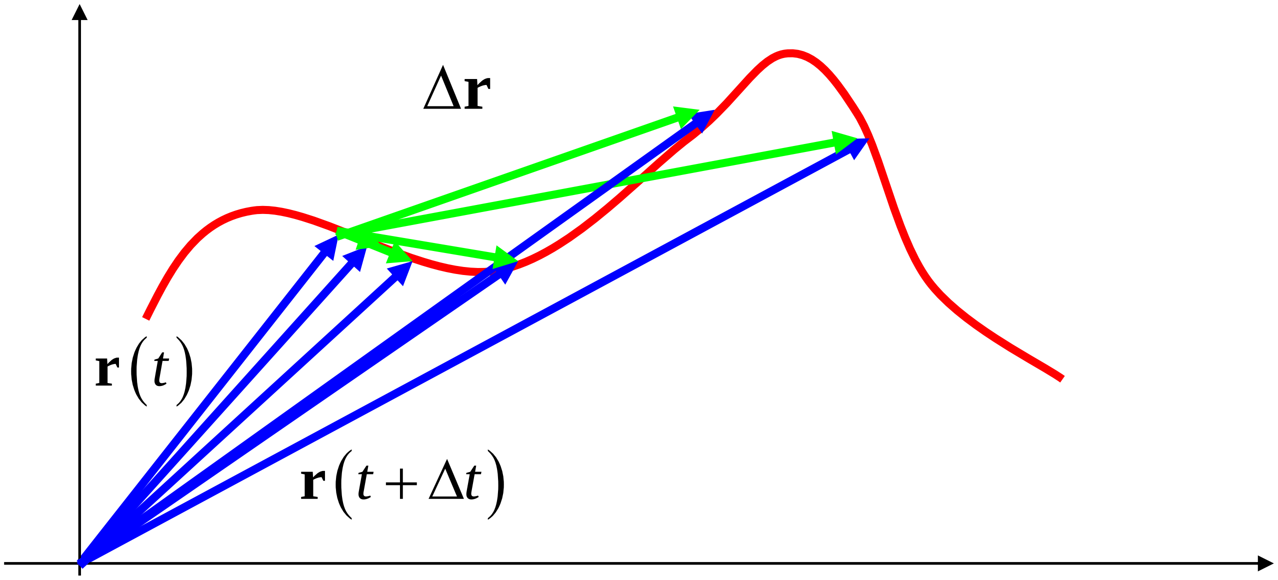
$$\mathbf{v}_{av} = \frac{\Delta \mathbf{r}}{\Delta t}$$

- Meaning: If you start from the initial position and move with constant velocity $\mathbf{v}_{av}$, you will reach the final position after Δt



Infinitesimal Displacement

What happens when the time interval tends to zero?

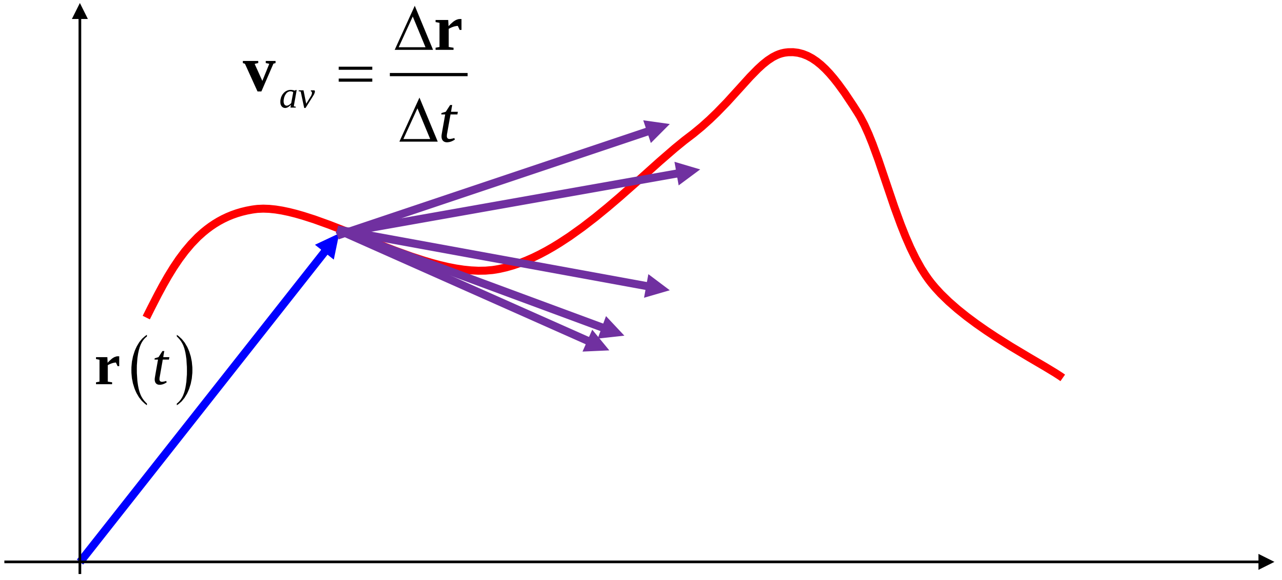


$$\|\Delta\mathbf{r}\| \to 0$$

Direction of $\Delta\mathbf{r} \to$ Direction of tangent of the trajectory

Limit of Average Velocity

However, in general the limit of the average velocity exists



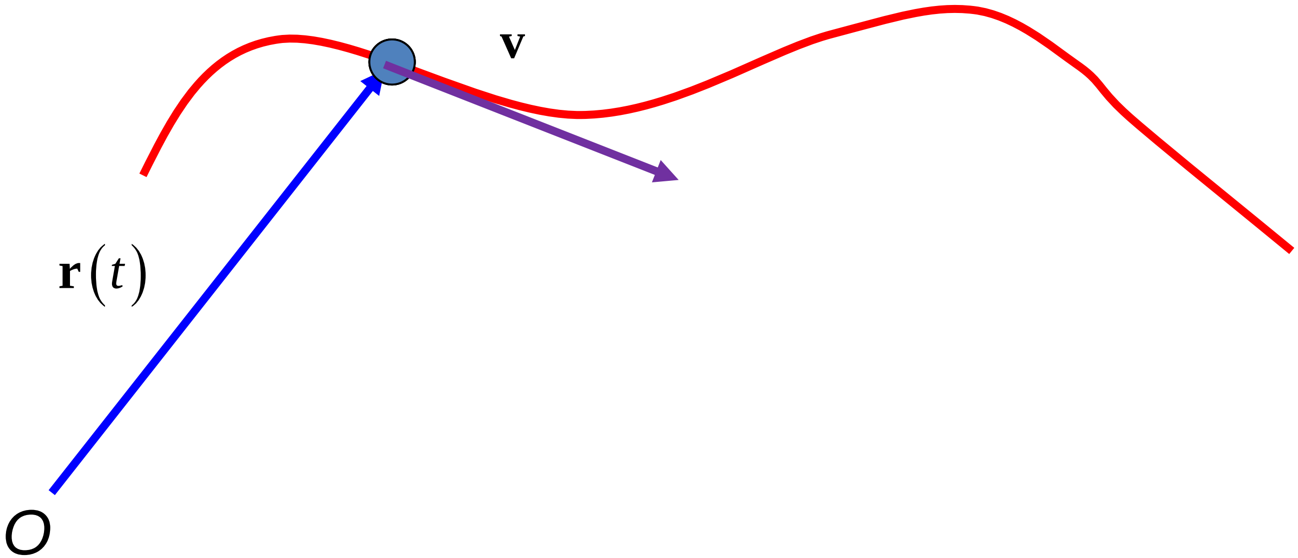
The instantaneous velocity is defined to be the limit of the average velocity

Instantaneous Velocity

- The instantaneous velocity at time t is defined as the limit of the average velocity when Δt tends to zero:

$$\mathbf{v}(t) = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{d\mathbf{r}}{dt}$$

- Its direction is along the tangent to the trajectory
- Its magnitude is the instantaneous speed $v = \|\mathbf{v}\|$



With respect to any function f(x), the operation

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

is called differentiation.

We say we are taking derivative of function f(x) with respect to x

(ref. to tutorial 1)

Derivative of function

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\sec x = \sec x \cdot \tan x$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\csc x = -\csc x \cdot \cot x$$

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Velocity in Cartesian Coordinates

Since $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are constant vectors

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta x \hat{\mathbf{x}} + \Delta y \hat{\mathbf{y}}}{\Delta t} = \lim_{\Delta t \to 0} \left(\frac{\Delta x}{\Delta t} \hat{\mathbf{x}} + \frac{\Delta y}{\Delta t} \hat{\mathbf{y}} \right)$$

$$= \left(\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \right) \hat{\mathbf{x}} + \left(\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} \right) \hat{\mathbf{y}} = \frac{dx}{dt} \hat{x} + \frac{dy}{dt} \hat{y}$$

$$v_x = \frac{dx}{dt}, \ v_y = \frac{dy}{dt}$$

$$\mathbf{v} = v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}}$$

$$v = \|\mathbf{v}\| = \sqrt{v_x^2 + v_y^2}$$

Example:

The trajectory of a particle is given by

$$\mathbf{r}(t) = 3t\hat{\mathbf{x}} + \left(t - 5t^2\right)\hat{\mathbf{y}}$$

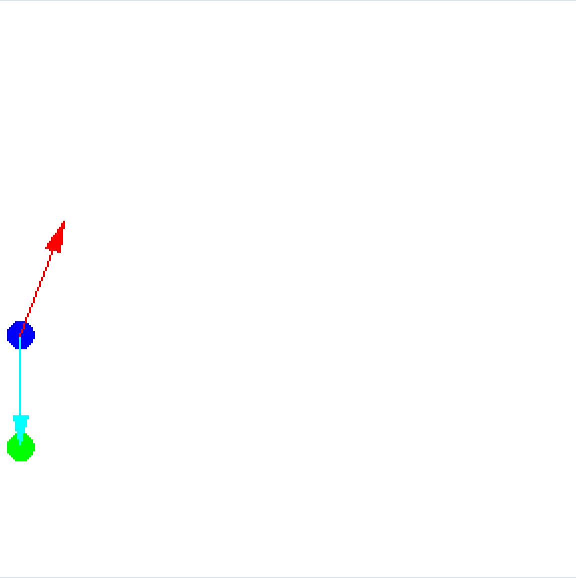
Find its velocity and speed.

Solution:

$$\begin{cases} x(t) = 3t \\ y(t) = t - 5t^2 \end{cases} \Rightarrow \begin{cases} dx/dt = 3 \\ dy/dt = 1 - 10t \end{cases}$$

$$\mathbf{v}(t) = 3\hat{\mathbf{x}} + (1 - 10t)\hat{\mathbf{y}}$$

$$v(t) = \sqrt{3^2 + (1 - 10t)^2}$$



Example:

The trajectory of a particle is given by

$$\mathbf{r}(t) = \cos\omega t\,\hat{\mathbf{x}} + \sin\omega t\,\hat{\mathbf{y}}$$

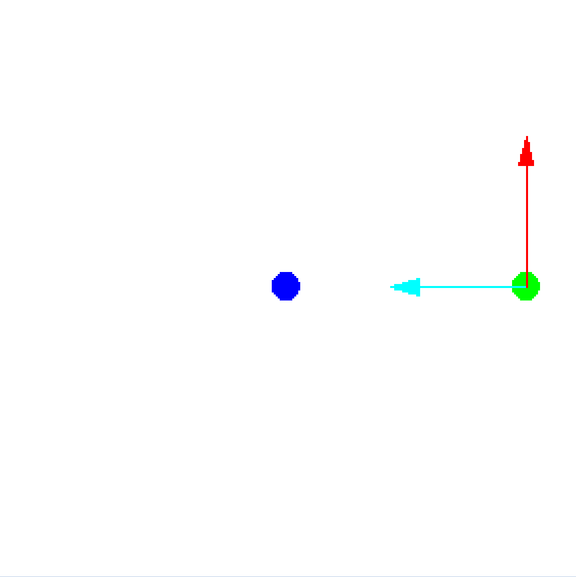
where $\omega > 0$ is a constant. Find its velocity and speed.

Solution:

$$\begin{cases} x(t) = \cos\omega t \\ y(t) = \sin\omega t \end{cases} \Rightarrow \begin{cases} v_x = dx/dt = -\omega\sin\omega t \\ v_y = dy/dt = \omega\cos\omega t \end{cases}$$

$$\mathbf{v}(t) = -\omega\sin\omega t\,\hat{\mathbf{x}} + \omega\cos\omega t\,\hat{\mathbf{y}}$$

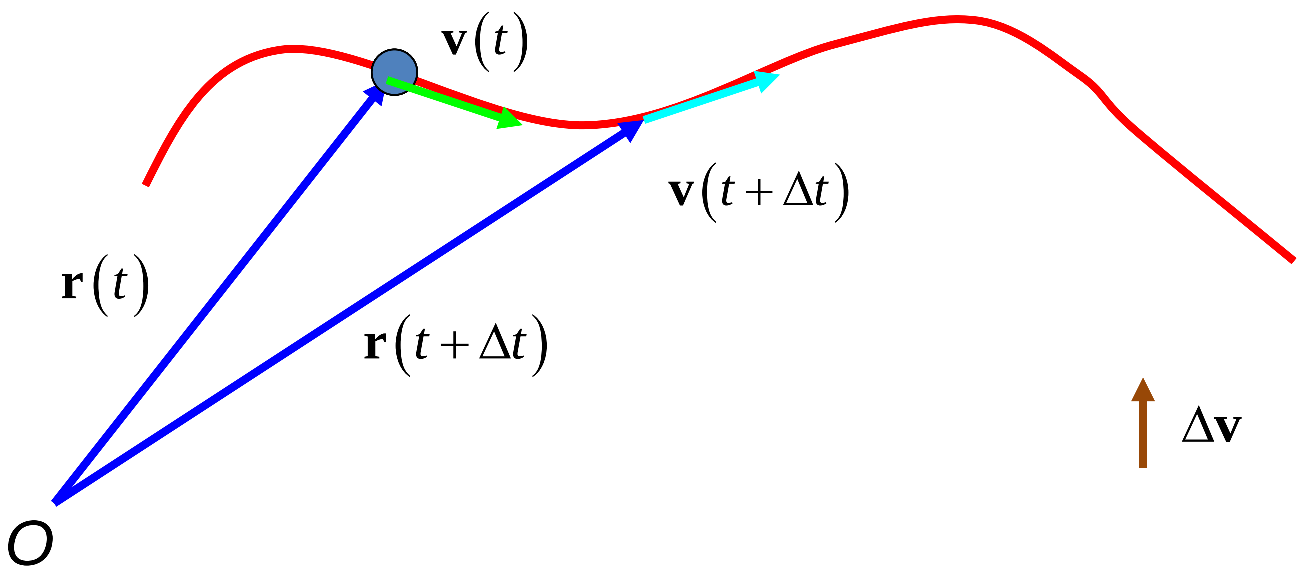
$$v(t) = \sqrt{(-\omega\sin\omega t)^2 + (\omega\cos\omega t)^2} = \omega$$



Change in Velocity

- After evaluating the instantaneous velocity at every moment t, we can then consider the rate of change of velocity w.r.t. time
- During a time interval Δt, the change in velocity is

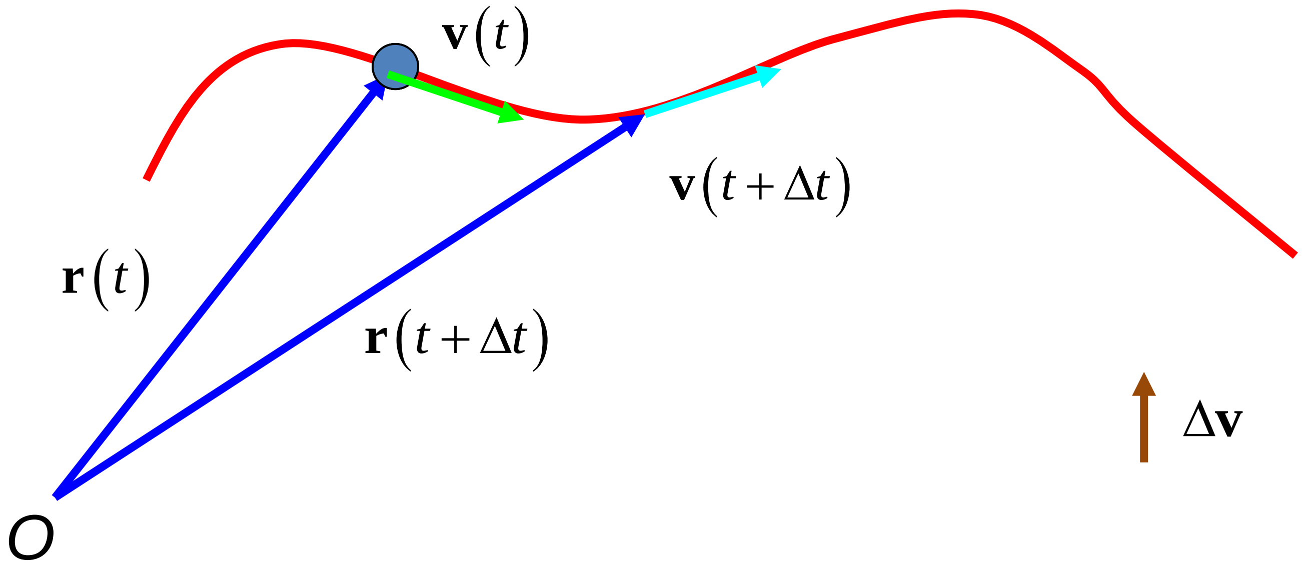
$$\Delta \mathbf{v} = \mathbf{v}(t+\Delta t) - \mathbf{v}(t)$$



Average Acceleration

- Acceleration is the rate of change of velocity
- The average acceleration during a time interval Δt is a *vector* defined by

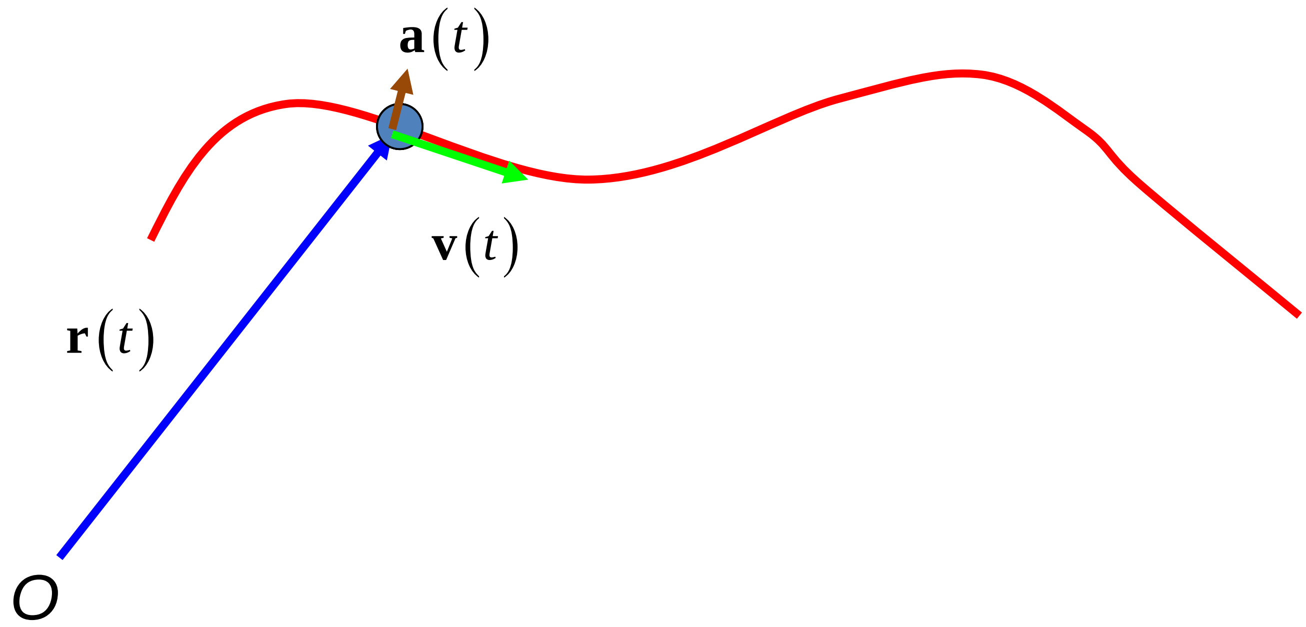
$$\mathbf{a}_{av} = \frac{\Delta \mathbf{v}}{\Delta t}$$



Instantaneous Acceleration

The instantaneous acceleration at time t is defined as the limit of the average acceleration when Δt tends to zero:

$$\mathbf{a}(t) = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t+\Delta t) - \mathbf{v}(t)}{\Delta t} = \frac{d\mathbf{v}}{dt}$$



Acceleration in Cartesian Coordinates

In Cartesian coordinates

$$\mathbf{v}(t) = v_x(t)\hat{\mathbf{x}} + v_y(t)\hat{\mathbf{y}} \qquad \mathbf{v}(t+\Delta t) = v_x(t+\Delta t)\hat{\mathbf{x}} + v_y(t+\Delta t)\hat{\mathbf{y}}$$

$$\begin{aligned}\Delta\mathbf{v} &= \mathbf{v}(t+\Delta t) - \mathbf{v}(t) \\ &= \left(v_x(t+\Delta t) - v_x(t)\right)\hat{\mathbf{x}} + \left(v_y(t+\Delta t) - v_y(t)\right)\hat{\mathbf{y}} \\ &= \Delta v_x\hat{\mathbf{x}} + \Delta v_y\hat{\mathbf{y}}\end{aligned}$$

Since $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are constant vectors

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \lim_{\Delta t\to 0}\frac{\Delta\mathbf{v}}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta v_x\hat{\mathbf{x}} + \Delta v_y\hat{\mathbf{y}}}{\Delta t} = \left(\lim_{\Delta t\to 0}\frac{\Delta v_x}{\Delta t}\right)\hat{\mathbf{x}} + \left(\lim_{\Delta t\to 0}\frac{\Delta v_y}{\Delta t}\right)\hat{\mathbf{y}}$$

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}, \; a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}$$

$$\mathbf{a} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}}$$

$$a = \|\mathbf{a}\| = \sqrt{a_x^2 + a_y^2}$$

Example:

The trajectory of a particle is given by

$$\mathbf{r}(t) = 3t\hat{\mathbf{x}} + \left(t - 5t^2\right)\hat{\mathbf{y}}$$

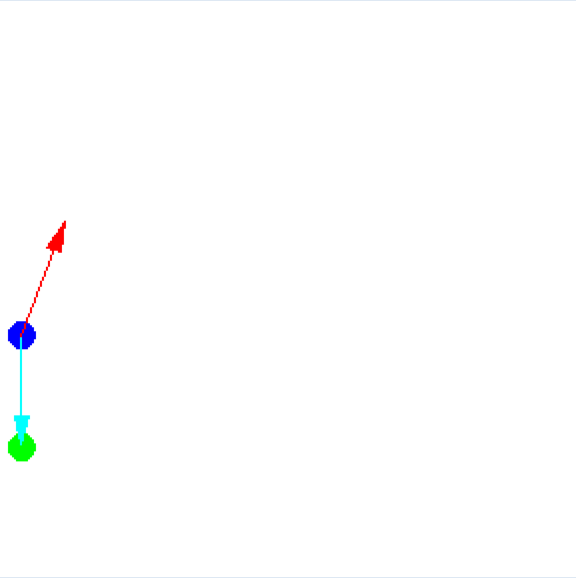
Find its acceleration.

Solution:

$$\begin{cases} v_x = 3 \\ v_y = 1 - 10t \end{cases} \Rightarrow \begin{cases} a_x = dv_x / dt = 0 \\ a_y = dv_y / dt = -10 \end{cases}$$

$$\mathbf{a}(t) = -10\hat{\mathbf{y}}$$

$$a(t) = 10$$



Example:

The trajectory of a particle is given by

$$\mathbf{r}(t) = \cos\omega t\hat{\mathbf{x}} + \sin\omega t\hat{\mathbf{y}}$$

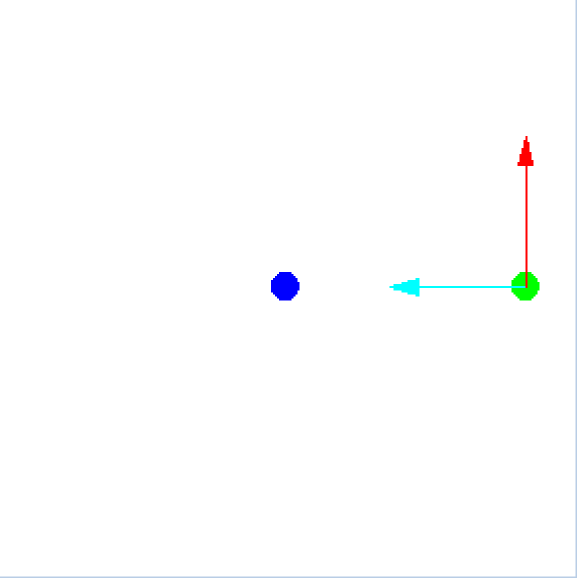
where $\omega > 0$ is a constant. Find its acceleration.

Solution:

$$\begin{cases} v_x = -\omega\sin\omega t \\ v_y = \omega\cos\omega t \end{cases} \Rightarrow \begin{cases} a_x = dv_x/dt = -\omega^2\cos\omega t \\ a_y = dv_y/dt = -\omega^2\sin\omega t \end{cases}$$

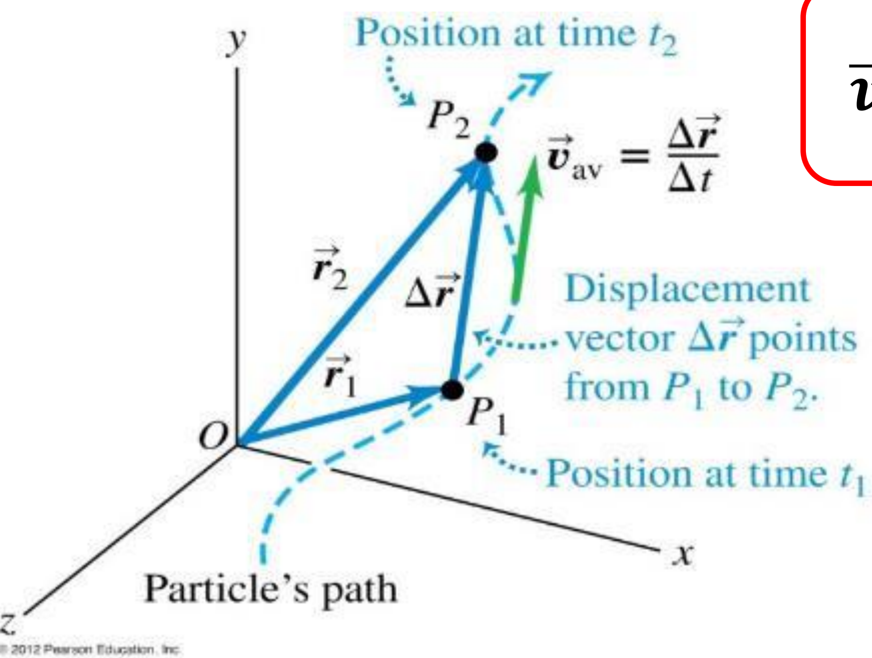
$$\mathbf{a}(t) = -\omega^2\cos\omega t\hat{\mathbf{x}} - \omega^2\sin\omega t\hat{\mathbf{y}} = -\omega^2\mathbf{r}$$

$$a(t) = \sqrt{\left(-\omega^2\sin\omega t\right)^2 + \left(-\omega^2\cos\omega t\right)^2} = \omega^2$$



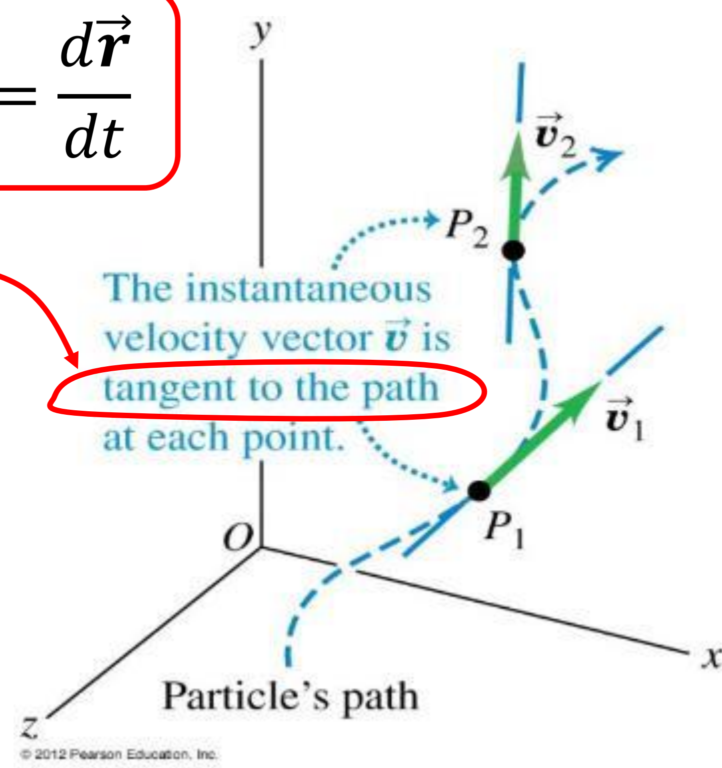
Displacement and velocity vectors

Distance and speed – scalars
Displacement and velocity – vectors

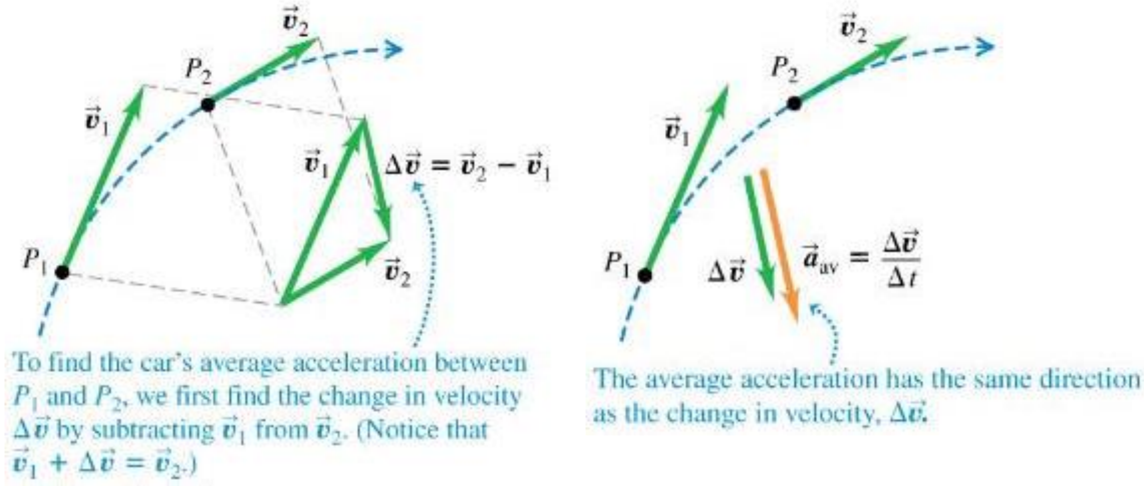


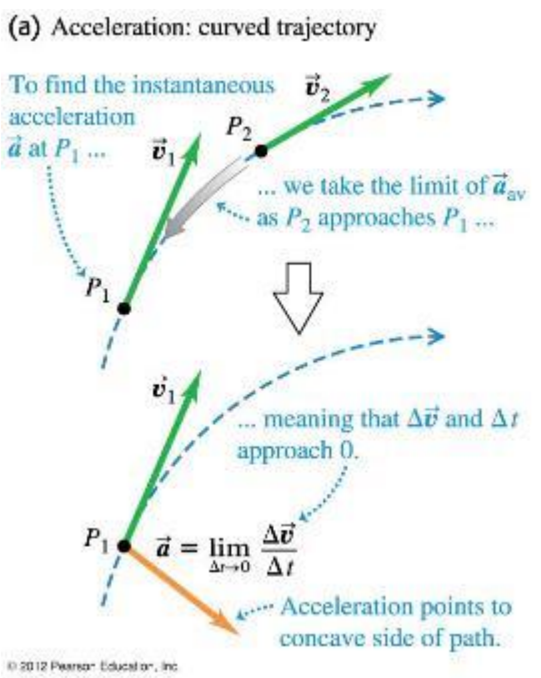
$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

Note:



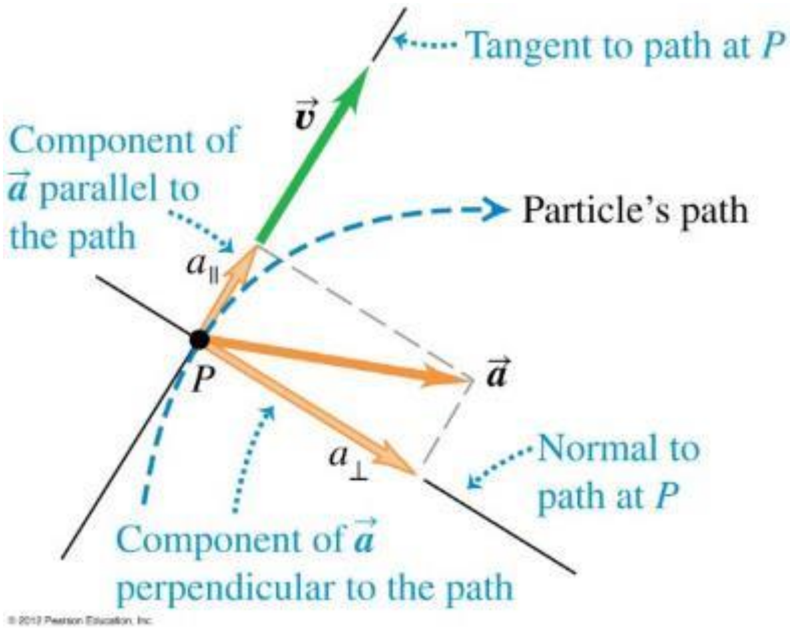
Acceleration vector

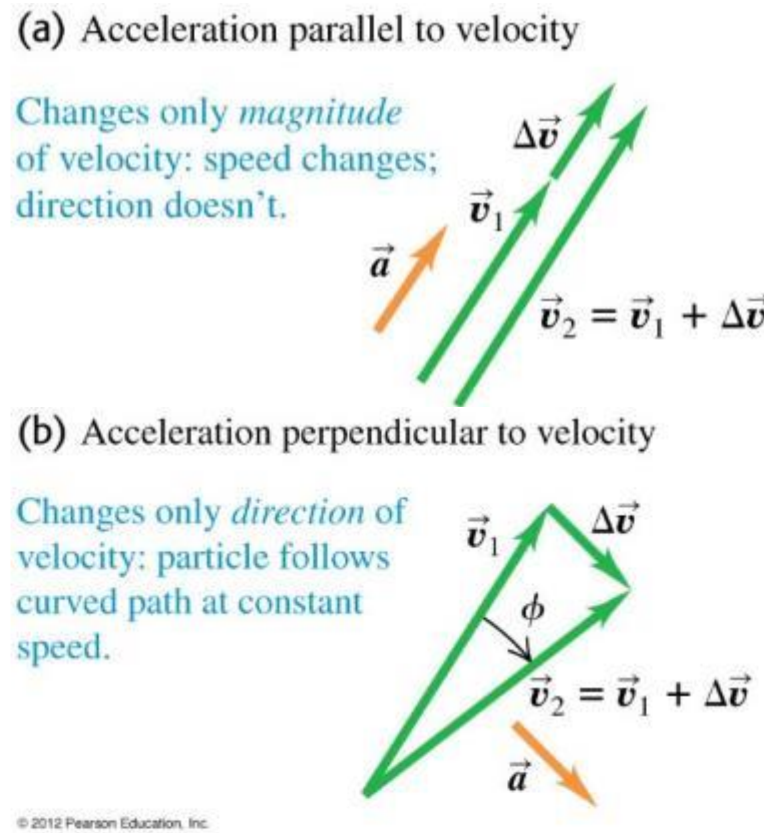




$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

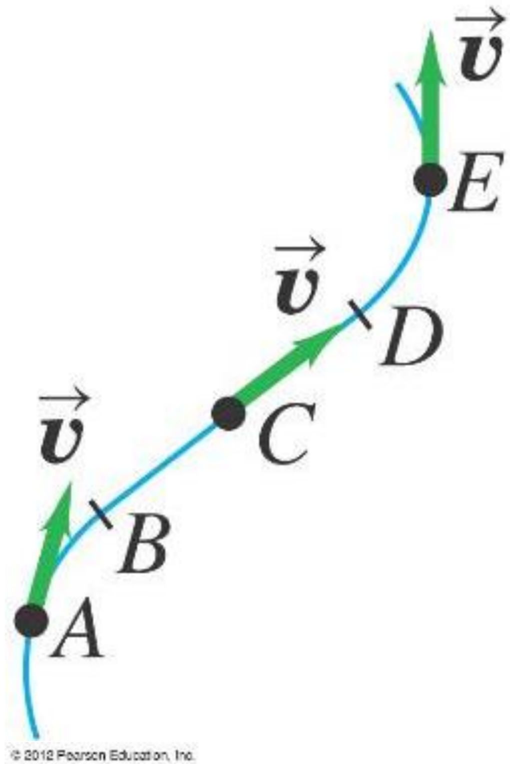
Resolve into parallel (or tangential) $a_{\parallel}$, and perpendicular (or radial) $a_{\perp}$ components





Q3.3

The motion diagram shows an object moving along a curved path at constant speed. At which of the points *A, C,* and *E* does the object have *zero* acceleration?



A. point *A* only

B. point *C* only

C. point *E* only

D. points *A* and *C* only

E. points *A, C,* and *E*

Q3.3

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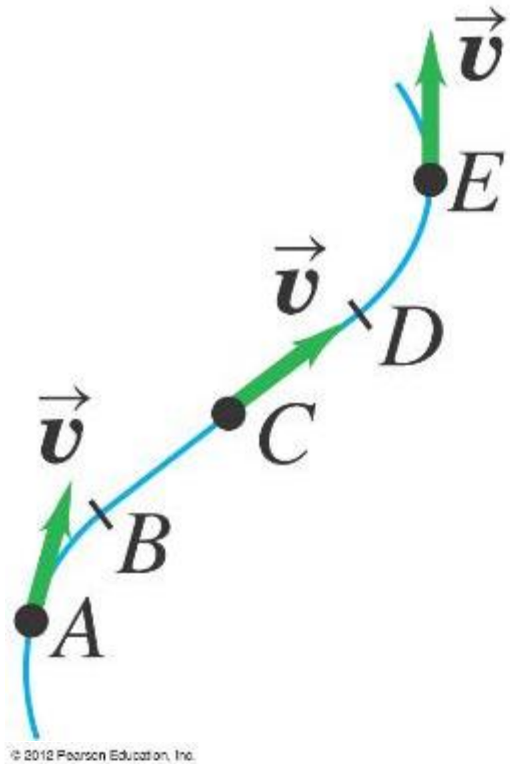

A. point *A* only

B. point *C* only

C. point *E* only

D. points *A* and *C* only

E. points *A, C,* and *E*

Equations of Constant-Acceleration Motions

Constant acceleration = a

At $t = t_0$, $v = v_0$, $x = x_0$

General equations

$$x = x_0 + v_0(t - t_0) + \frac{1}{2}a(t - t_0)^2$$

$$v = v_0 + a(t - t_0)$$

$$v^2 - v_0^2 = 2a(x - x_0)$$

$t_0 = x_0 = 0$

$$x = v_0 t + \frac{1}{2}at^2$$

$$v = v_0 + at$$

$$v^2 - v_0^2 = 2ax$$

Example:

An object initially at rest falls from the roof of a 100-m-high building. Suppose air resistance and all other forces except gravity can be ignored. Find the elapsed time and the speed of the object when it reaches the ground.

Take downward as positive in your calculation.

Solution:

Let the moment the object starts to fall be $t = 0$, and take the roof as the reference height. In other words, $t_0 = x_0 = 0$.

Hence

$$v^2 - v_0^2 = 2gx$$

It is also given that $v_0 = 0$. So $v^2 = 2gx$

When it reaches the ground $x = 100$. Therefore

$$v = \pm\sqrt{200g} = \pm\sqrt{200 \times 9.8} \approx \pm 44.3 \text{ m/s}$$

To obtain the time elapsed, use

$$v = v_0 + gt = gt \quad \Rightarrow \quad t = v / g = \pm\sqrt{200 / g}$$

Obviously the answer $t < 0$ is not physical and should be rejected.

Hence

$$t = \sqrt{200 / g} \approx 4.5 \text{ s} \qquad v = \sqrt{200g} \approx 44.3 \text{ m/s}$$

The object hits the ground at a velocity of 44.3 m/s after 4.5 s

Example:

Redo the last question. This time take upward as positive.

Solution: Still set $t_0 = x_0 = 0$. Now

$$v^2 - v_0^2 = -2gx \quad \Rightarrow \quad v = \pm\sqrt{-2gx}$$

This time when the object reaches the ground, $x = -100$.

Therefore

$$v = \pm\sqrt{-2gx} = \pm\sqrt{-2g(-100)} = \pm\sqrt{200g}$$

$$v = v_0 - gt = -gt \quad \Rightarrow \quad t = -v / g$$

$$v = \sqrt{200g} \Rightarrow t = -\sqrt{200 / g} < 0$$

which should be rejected. Hence the solution is

$$t = \sqrt{200 / g} \approx 4.5 \text{ s} \qquad v = -\sqrt{200g} \approx -44.3 \text{ m/s}$$

Exercise 1: A rocket is launched at t = 0 and has a constant upward acceleration of $2g$. The engine breaks down 20 s after the launch. Assume one can ignore air resistance.

(a) Find the maximum height H above ground reached by the rocket.

(b) Find the time T when the rocket hits the ground.

Exercise 2:

To estimate the height of a bridge, a stone (initially at rest) is dropped from the bridge to the water below. If the splash is heard after 4 s, find the height of the bridge. Assume that the speed of sound is 344 ms^{-1} and air resistance can be neglected.

Exercise 1: A rocket is launched at t = 0 and has a constant upward acceleration of $2g$. The engine breaks down 20 s after the launch. Assume one can ignore air resistance.

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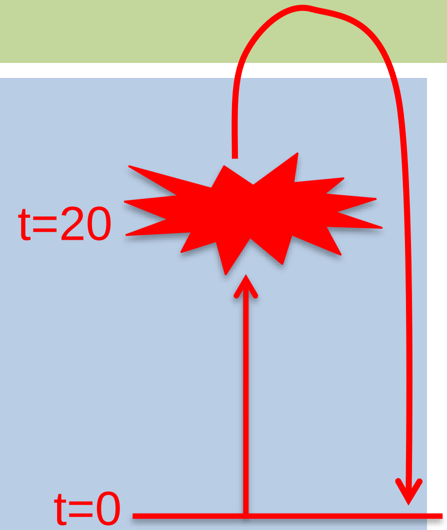


Solution:

Take the ground as the reference and upward as positive.

For $t < 20$, $a = 2g$. So at t = 20,

$$v = 2g \times 20 = 40g \qquad x = \frac{1}{2} \times 2g \times (20)^2 = 400g$$

For $t > 20$, $a = -g$.

(a)The object reaches maximum height H when its velocity is 0.

$$0^2 - (40g)^2 = -2g(H - 400g) \Rightarrow H = 400g + 1600g^2/2g = 1200g$$

(b) To find T, solve

$$x = x_0 + v_0(t - t_0) - \frac{1}{2}g(t - t_0)^2 \Rightarrow 0 = 400g + 40g(T - 20) - \frac{1}{2}g(T - 20)^2$$

$$T - 20 = \frac{-40 \pm \sqrt{(40)^2 + 800}}{-1} = 40 \mp \sqrt{2400} = 40 \mp 20\sqrt{6}$$

Rejecting the negative root → $T = 60 + 20\sqrt{6}$

Exercise 2:

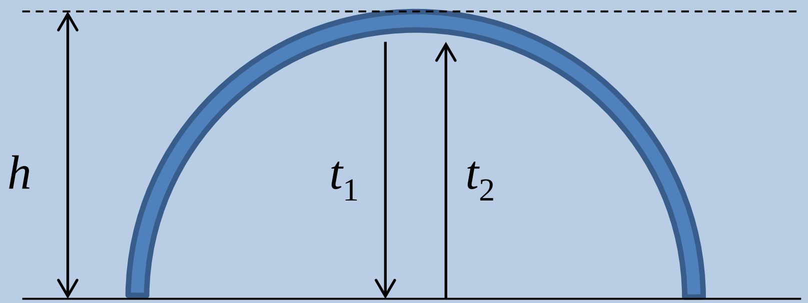
To estimate the height of a bridge, a stone (initially at rest) is dropped from the bridge to the water below. If the splash is heard after 4 s, find the height of the bridge. Assume that the speed of sound is 344 ms^{-1} and air resistance can be neglected.

Solution:

Let the height of the bridge be h, the time taken for the stone to reach the water be t_1, and the time taken for the splash to reach the bridge be t_2. Take downward as positive, and the ground to be $x = 0$. Let $t = 0$ at the moment when the stone starts to fall. In other words, $t_0 = 0$, $x_0 = -h$. Besides, $v_0 = 0$. Hence

$$x = x_0 + v_0\left(t - t_0\right) + \frac{1}{2}g\left(t - t_0\right)^2$$

$$= -h + \frac{1}{2}gt^2$$



Solution:

At the moment when the stone hits the water

$$0 = -h + \frac{1}{2} g t_1^2 \quad \Rightarrow \quad t_1 = \pm\sqrt{2h/g}$$

The solution of negative time should be rejected. Hence

$$t_1 = \sqrt{2h/g}$$

It is easy to see that $t_2 = h/344$

Therefore

$$t_1 + t_2 = \sqrt{2h/g} + h/344 = 4$$

$$\Rightarrow 344\sqrt{2h/g} + h = 1376$$

$$\Rightarrow h + 344\sqrt{2/g}\sqrt{h} - 1376 = 0$$

$$\Rightarrow \sqrt{h} = \frac{-344\sqrt{2/g} \pm \sqrt{344^2 \times 2/g + 4 \times 1376}}{2}$$

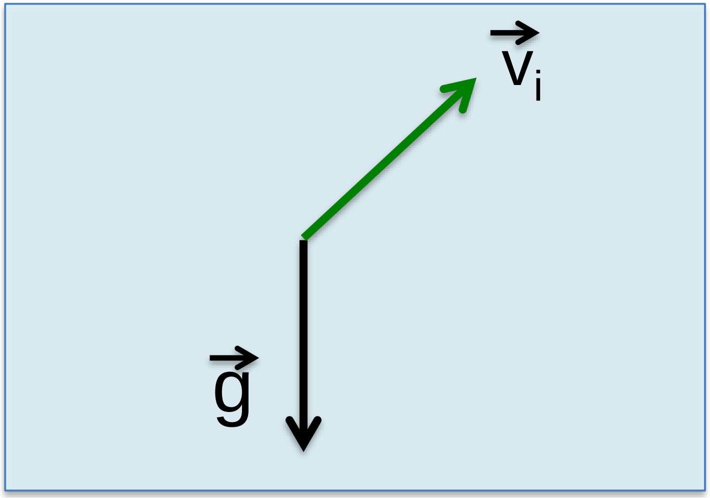
The smaller root is negative and should be rejected. Thus

$$\sqrt{h} = \sqrt{344^2/2g + 1376} - 344\Big/\sqrt{2g} \approx 8.400$$

$$h \approx 70.6 \text{ m}$$

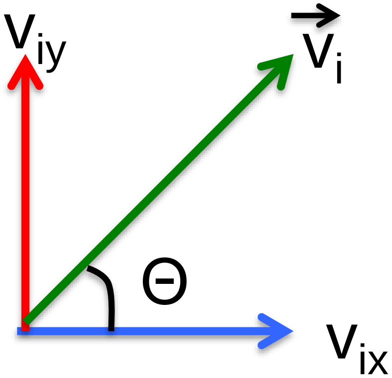
Projectile motion

Since a projectile moves in 2-dimensions, it therefore has two components.



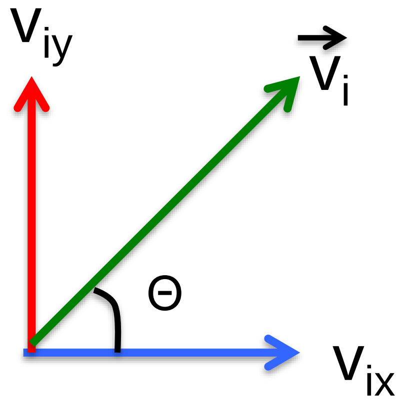
Two-dimensional motion of an object

- **Vertical**
- **Horizontal**



*Since the perpendicular components of motion are **independent** of each other.*

Key idea:
The horizontal and the vertical motion can be considered **independently**!!

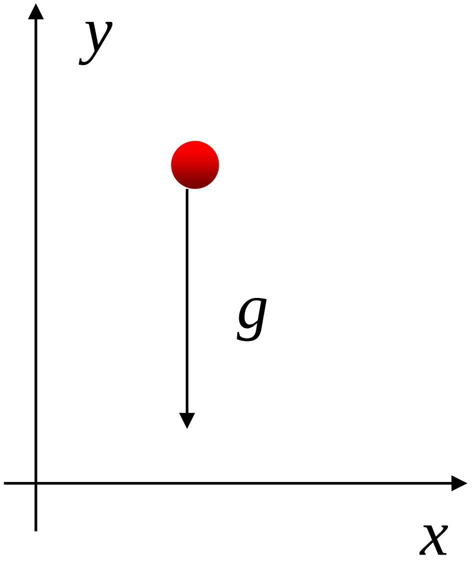


Acceleration due to Gravity

Near the Earth's surface, all objects are subject to a constant downward acceleration $g \approx 9.8\ \text{ms}^{-2}$

Set up the coordinate system so that:
x: horizontal
y: vertical (upward taken as positive)

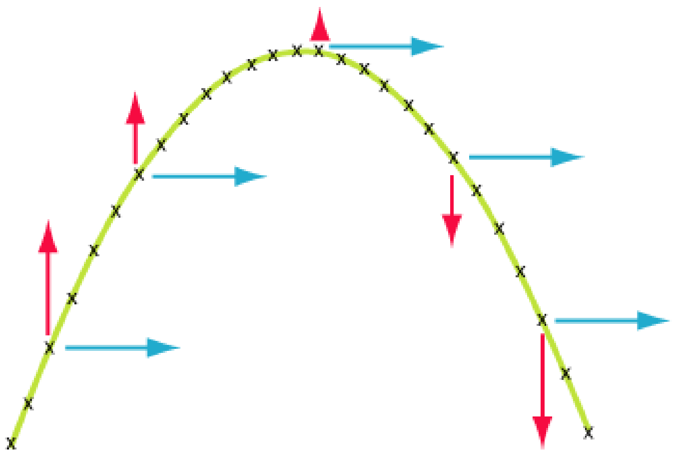
$$\mathbf{a} = -g\hat{\mathbf{y}} \quad \Rightarrow \quad \begin{cases} a_x = 0 \\ a_y = -g \end{cases}$$



Projectile motion:

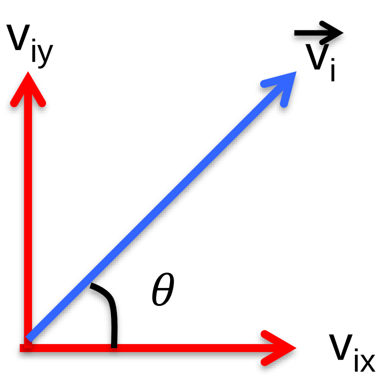
	Horizontal Motion	Vertical Motion
Acceleration	No	Yes g is downward at 9.8 m/s^2
Velocity	Constant	Changing By 9.8m/s per second

By combining two components, it will produce a trajectory/path, which is **parabolic**.



Equation of motion:

	Horizontal (x) Uniform motion	Vertical (y) Accelerating motion
acceleration	$a_x = 0$	$a_y = -g = -9.8 m/s^2$
velocity	$v_x = v_{ix} = v_x \cos\theta$	$v_y = v_{iy} - gt$ $v_y = v_i \sin\theta - gt$
displacement	$x = v_{ix}t = v_i t \cos\theta$	$y = v_{iy}t - \frac{1}{2}gt^2$ $y = v_i t \sin\theta - \frac{1}{2}gt^2$

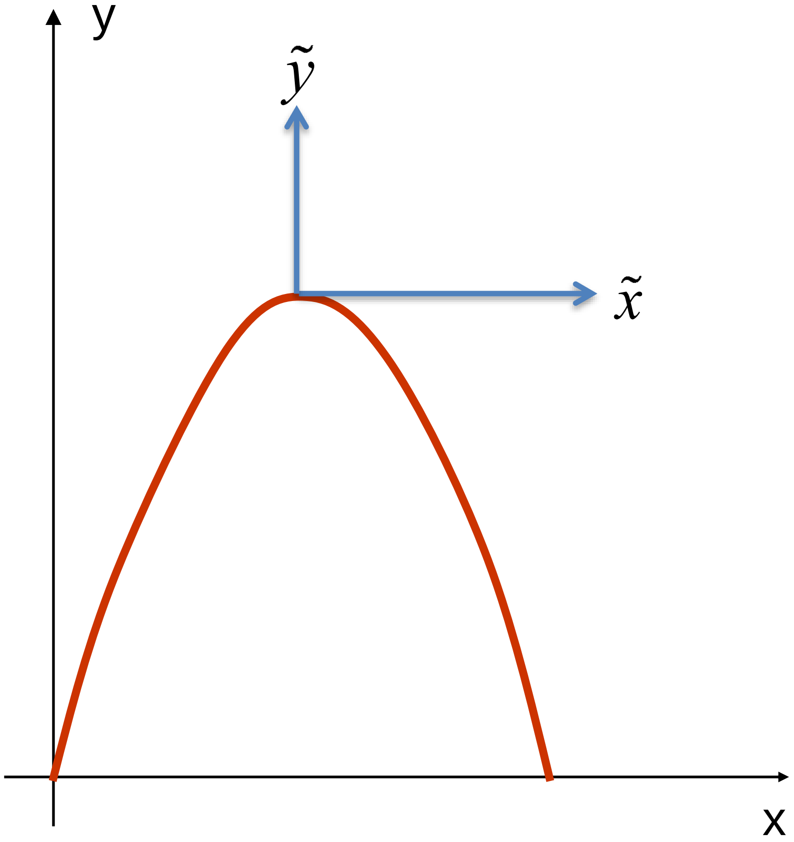


v_{ix},v_{iy}: initial horizontal and vertical velocity components
v_i: magnitude of the vector $\vec{v}_i$
a_x,a_y: acceleration along the horizontal and vertical direction
θ: angle between the initial velocity v and the horizontal direction

Trajectory of a projectile motion

$$x = v_i t \cos\theta$$
$$y = v_i t \sin\theta - \frac{1}{2} g t^2$$

Parabola, open *down*



Eliminate time, t

$$t = \frac{x}{v_i \cos\theta}$$

$$y = x\tan\theta - \frac{g}{2v_i^2\cos^2\theta}x^2$$

$$y = bx + ax^2 \Rightarrow \left(y + \frac{b^2}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2 \Rightarrow \tilde{y} = a\tilde{x}^2$$

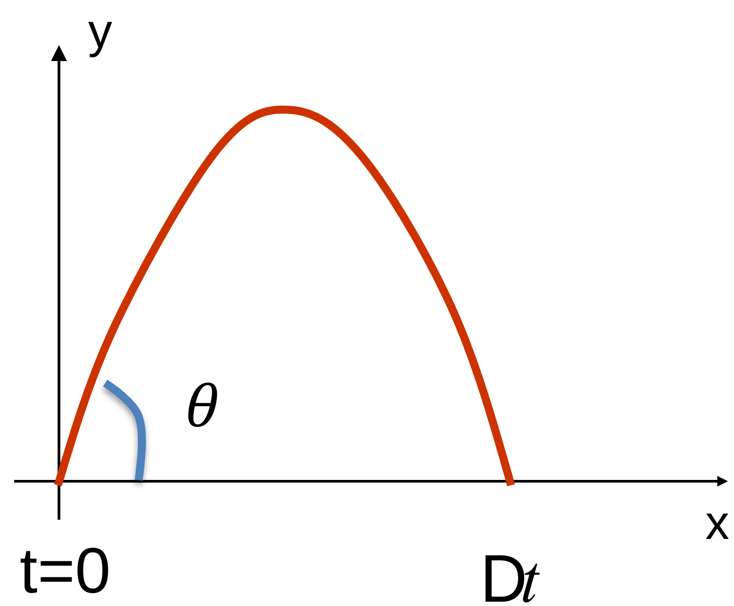
Parabola!!
Remember a<0 is negative!!

Total time travelled by the ball

Final height y=0 after time interval Δt

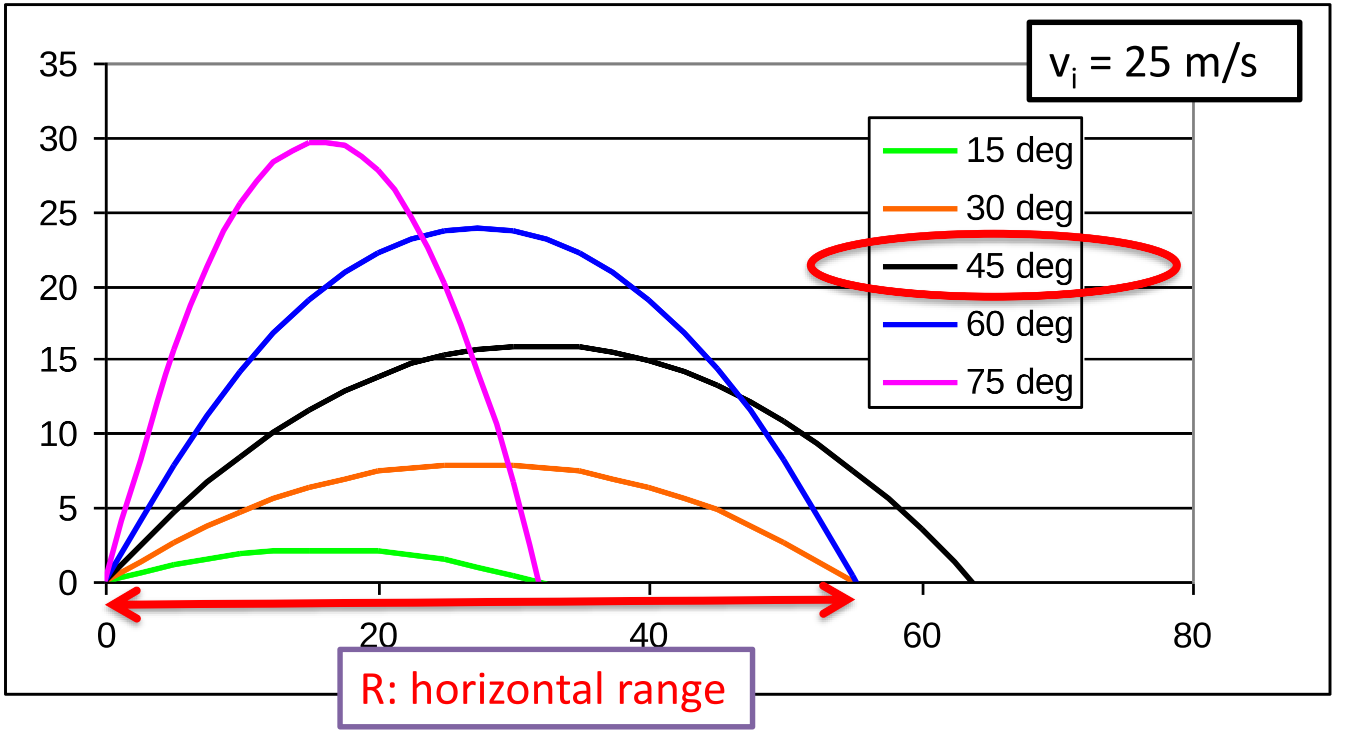
$$0 = v_i \Delta t \sin\Theta - \frac{1}{2} g (\Delta t)^2$$

$$\Delta t = \frac{2 v_i \sin\Theta}{g}$$



Trajectories at different angle θ

$$y = x\tan\theta - \frac{g}{2v_i^2\cos^2\theta}x^2$$



- Horizontal range are the same for angles θ and $(90-\theta)$
- Horizontal range is the greatest at $\theta = 45$ - **WHY?**

$$\sin(2\theta) = \sin\left(2(90^o - \theta)\right)$$

$$2\sin\theta\cos\theta = \sin(2\theta)$$

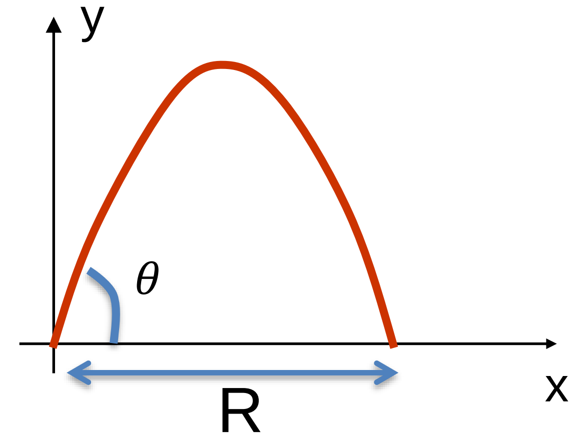
Horizontal range (R):

$$0 = R\tan\theta - \frac{g}{2v_i^2\cos^2\theta}R^2 \Rightarrow \quad R(\theta) = 0 \quad \text{OR}$$

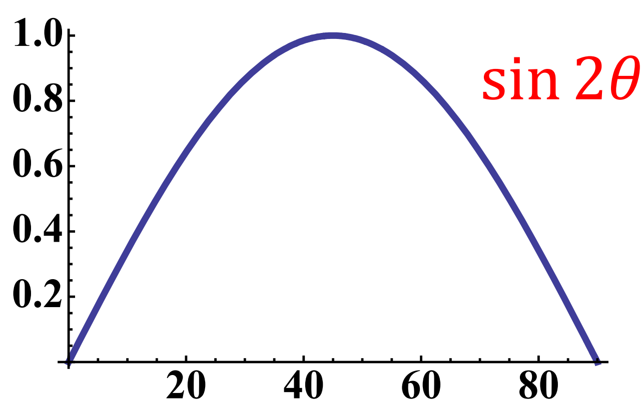
$$R(\theta) = \frac{2v_i^2\cos^2\theta\tan\theta}{g} = \frac{2v_i^2\cos\theta\sin\theta}{g} = \frac{v_i^2\sin 2\theta}{g}$$

Check!!

$$R(\theta) = R(90^o - \theta)$$



Why 45°?

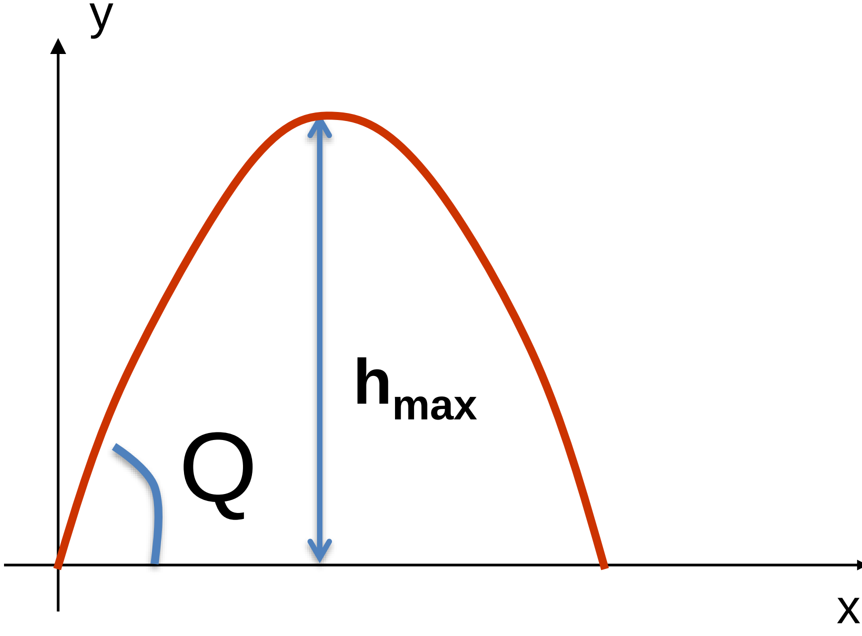


Maximum height

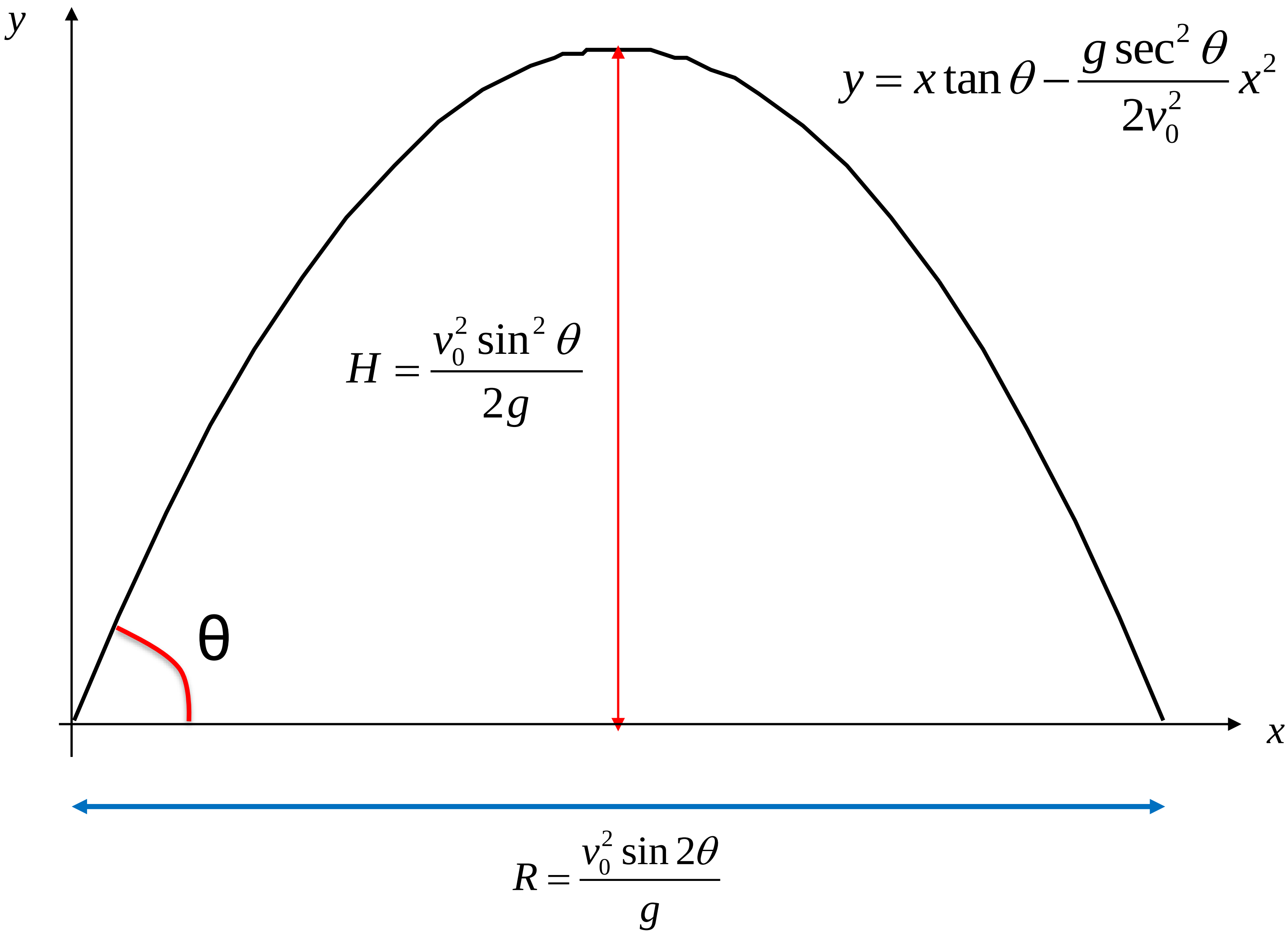
$$\left(y+\frac{b^2}{4a}\right)=a\left(x+\frac{b}{2a}\right)^2 \Rightarrow \left(y+\frac{b^2}{4a}\right)=-|a|\left(x+\frac{b}{2a}\right)^2 \le 0$$

$$\Rightarrow\ y \le -\frac{b^2}{4a}=\frac{2v_i^2\tan^2\Theta\cos^2\Theta}{4g}=\frac{v_i^2\sin^2\Theta}{2g}$$

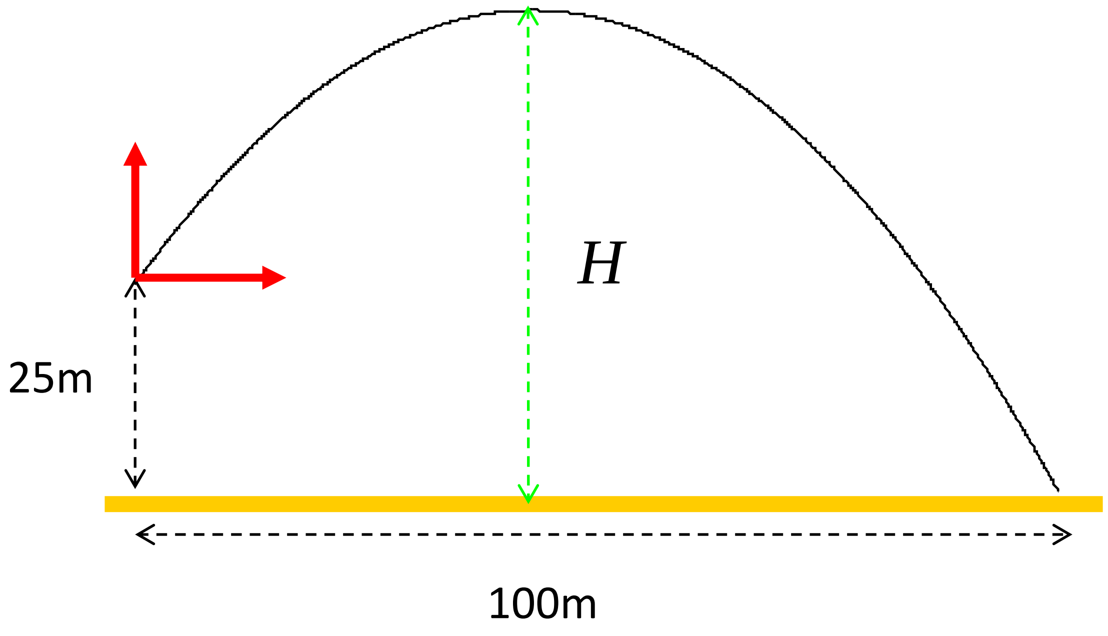
$$\Rightarrow\ h_{\max}=\frac{v_i^2\sin^2\Theta}{2g}$$



Path of a Particle Launched at Ground Level



Example: A particle is projected from a point *O* 25m above a horizontal plane. After 5 seconds it hits the plane at a point whose horizontal distance from *O* is 100m. Find the horizontal and vertical components of the initial velocity of the particle, and hence the initial speed. Also find the greatest height the particle reaches above the plane.



Solution: Let $t_0 = 0$. Set up the coordinate system as shown in the figure below so that $x_0 = 0$, $y_0 = 25$.

$$x(t) = v_{x0}t \quad \Rightarrow \quad v_{x0} = 100/5 = 20\text{ms}^{-1}$$

$$y(t) = 25 + v_{y0}t - \frac{1}{2}gt^2 \quad \Rightarrow \quad 0 = 25 + 5v_{y0} - \frac{25}{2}g \quad \Rightarrow \quad v_{y0} = 19.5\text{ms}^{-1}$$

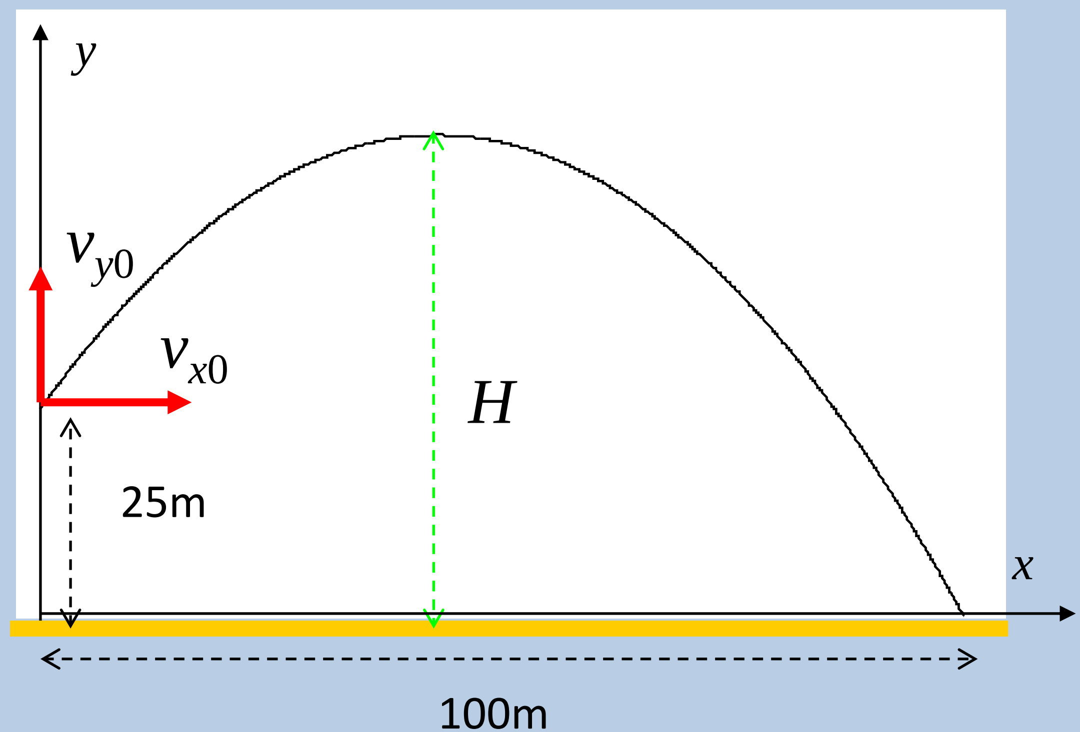
The initial speed is

$$v = \sqrt{20^2 + 19.5^2} \approx 27.9\text{ ms}^{-1}$$

To find the maximum height, use

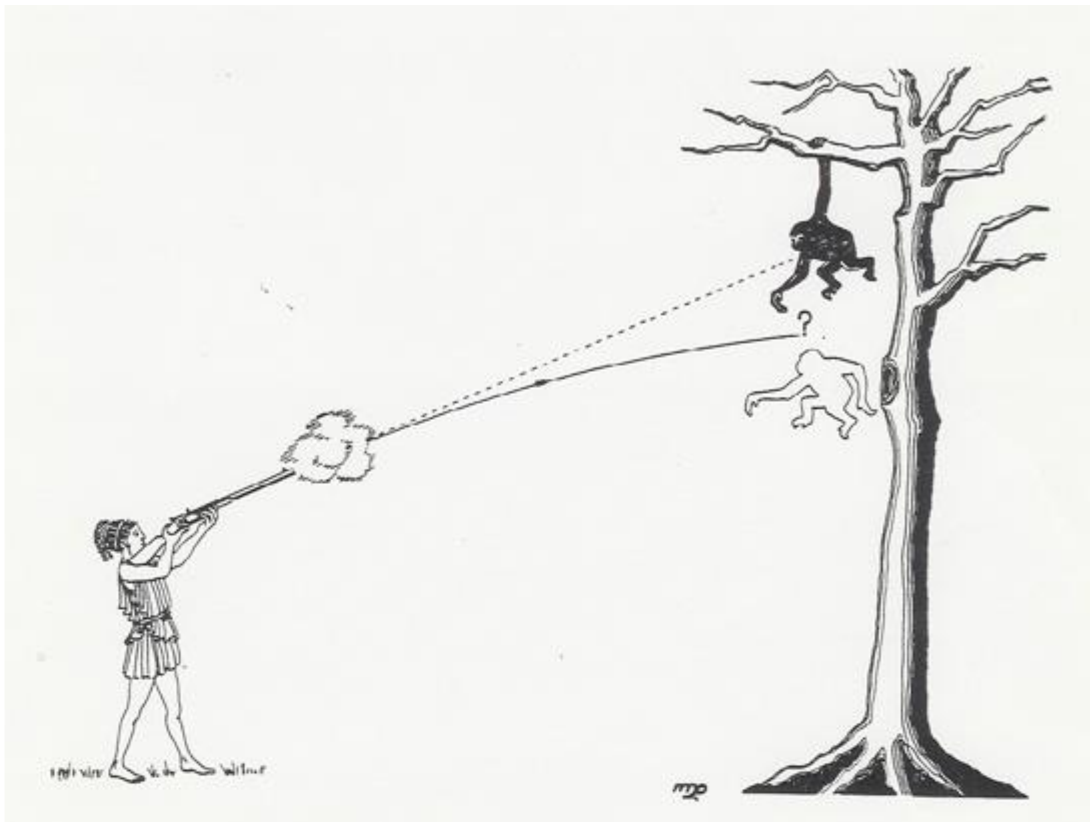
$$0^2 - v_{y0}^2 = -2g(H - 25)$$

$$H = 25 + \frac{v_{y0}^2}{2g} = 44.4\text{m}$$



Example: The Monkey and the Hunter

The monkey sees that the barrel of a hunter's gun is pointed directly at him. If he lets go of the branch at the instant the bullet is fired, will the bullet pass over his head?



Solution:

The bullet will always hit the target.

Time taken for the bullet to travel a horizontal distance d:

$$\tau = \frac{d}{v_0 \cos\theta}$$

Height of the target at τ:

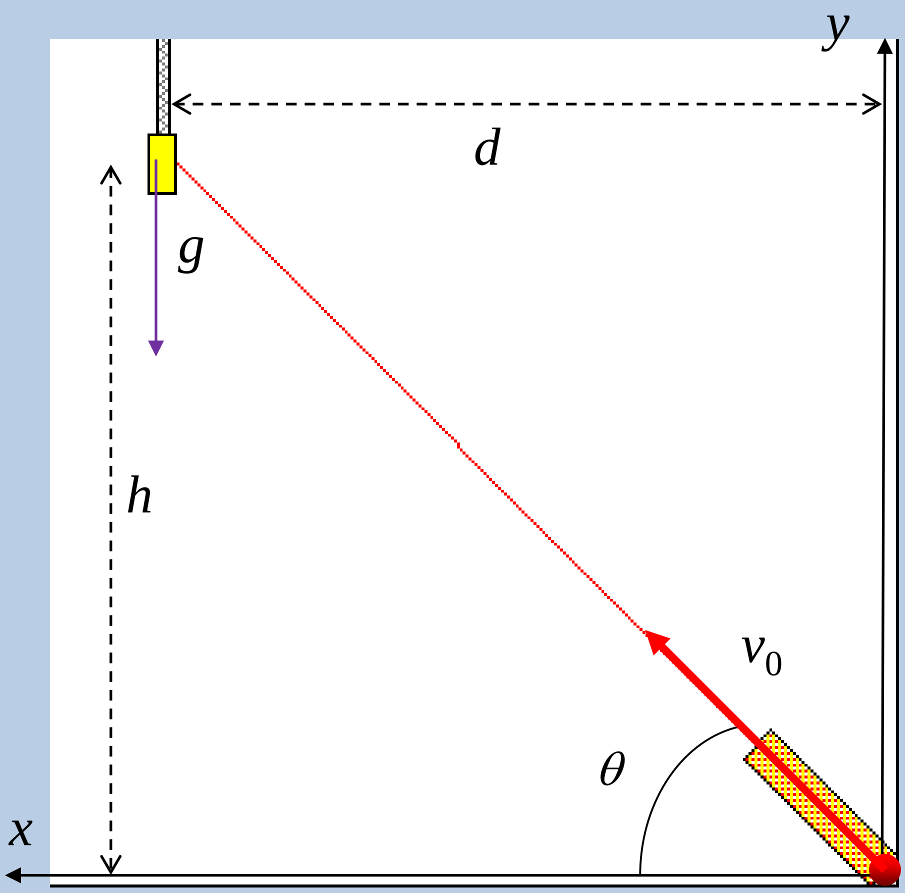
$$h - \frac{1}{2} g\tau^2$$

Height of bullet at τ:

$$v_0 \tau \sin\theta - \frac{1}{2} g\tau^2$$

Difference in height:

$$h - v_0 \tau \sin\theta = h - v_0 \frac{d}{v_0 \cos\theta} \sin\theta = h - d\tan\theta = 0$$



3. NEWTON’S LAWS OF MOTION

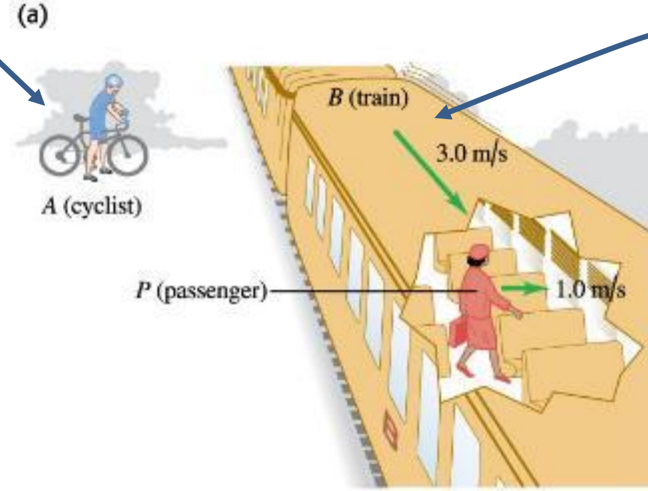
Relative velocity – concerning more than one observer

An observer making observation/measurement forms a reference frame

e.g.

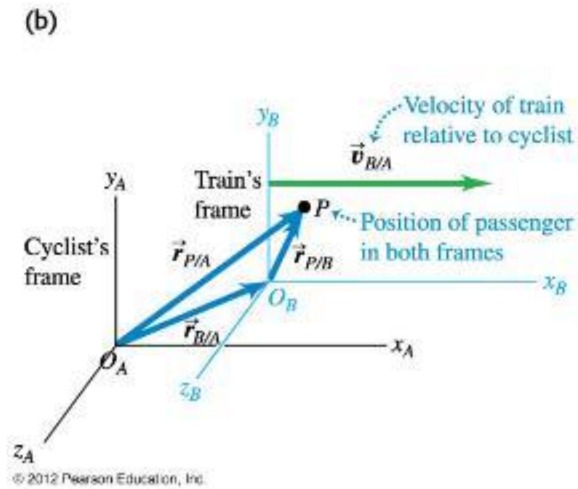
Reference Frame A

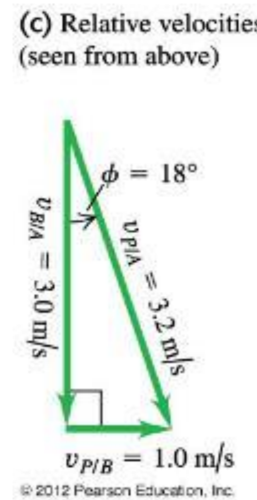
1st observer A, standing still on the platform



Reference Frame B

2nd observer B, sitting still inside the train





$$\vec{r}_{P/A} = \vec{r}_{B/A} + \vec{r}_{P/B}$$

$$\Rightarrow \boxed{\vec{v}_{P/A} = \vec{v}_{B/A} + \vec{v}_{P/B}}$$

velocity of P measured in frame A

velocity of frame B measured in frame A

velocity of P measured in frame B

Q3.12

The pilot of a light airplane with an airspeed of 200 km/h wants to fly due west. There is a strong wind of 120 km/h blowing from the north.

If the pilot points the nose of the airplane north of west so that her ground track is due west, what will be her ground speed?

A. 80 km/h

B. 120 km/h

C. 160 km/h

D. 180 km/h

E. It would impossible to fly due west in this situation.

Q3.12

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A. 80 km/h

B. 120 km/h

C. 160 km/h

D. 180 km/h

E. It would impossible to fly due west in this situation.

Exercise:

There are two boats across a river with width l and their connection line make an angle α with the river.
Assume that the maximum speed for two boats are u_A and u_B respectively and the speed of the water current is v.
If two boats depart at the same time, which direction should they move such that they will meet in the shortest time.
The velocity of boat A and B (relative to shore) are

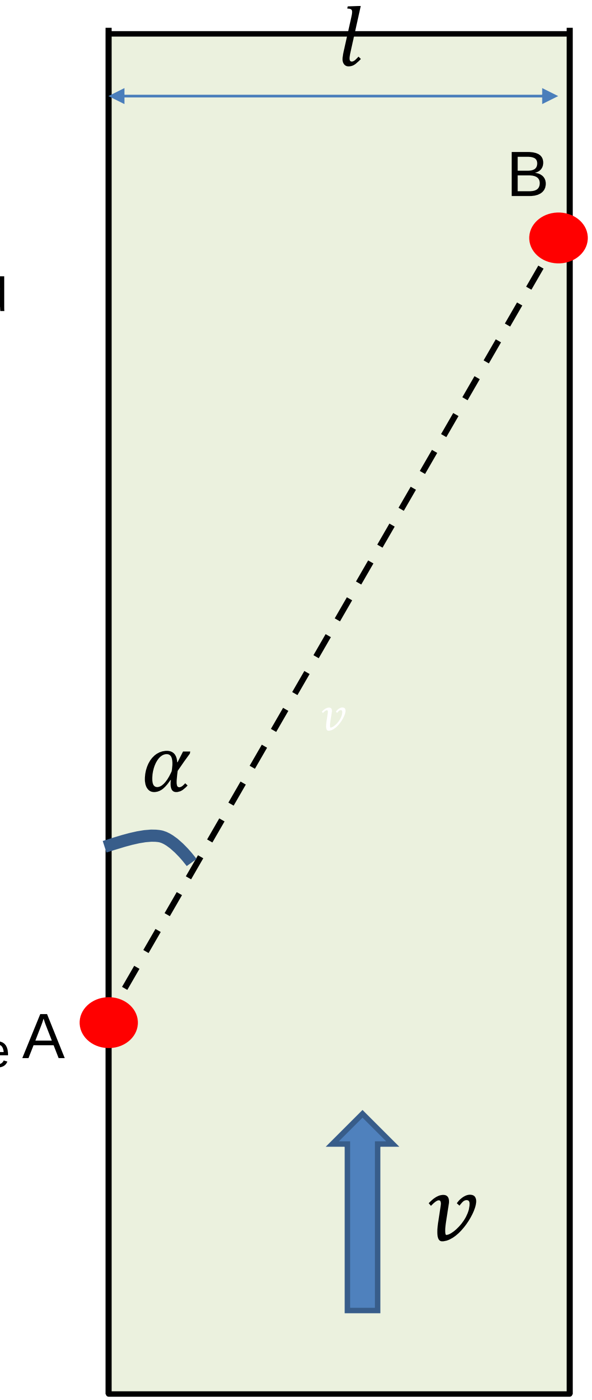
$$\vec{V}_A = \vec{u}_A + \vec{v}, \quad \vec{V}_B = \vec{u}_B + \vec{v}$$

The velocity of boat B relative to A is

$$\vec{V}_{B/A} = \vec{V}_B - \vec{V}_A = \vec{u}_B - \vec{u}_A$$

Two boats will meet if $\vec{V}_{B/A}$ is parallel to the connection line and the maximum speed if $\vec{u}_A$ and $\vec{u}_B$ are anti-parallel.
The time taken is

$$T = \frac{l/\sin\alpha}{u_1 + u_2}$$



Newtons’ first law of motion

A body acted on by no net force moves with constant velocity

$$\sum \vec{\boldsymbol{F}} = 0$$

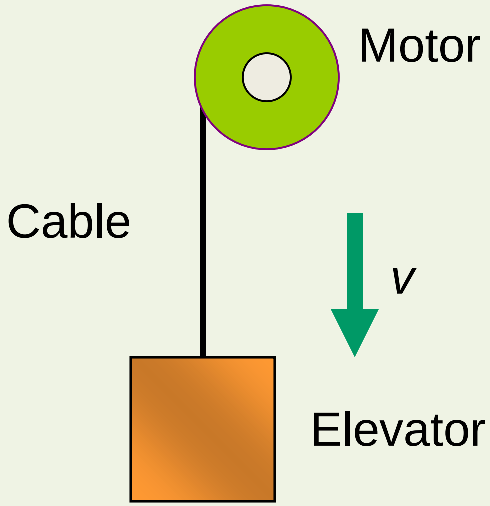
body in equilibrium

may be moving

Q4.6

An elevator is being lowered at constant speed by a steel cable attached to an electric motor. There is no air resistance, nor is there any friction between the elevator and the walls of the elevator shaft.

The upward force exerted on the elevator by the cable has the same magnitude as the force of gravity on the elevator, but points in the opposite direction. Why?



A. Newton’s first law

B. Newton’s second law

C. Newton’s third law

A4.6

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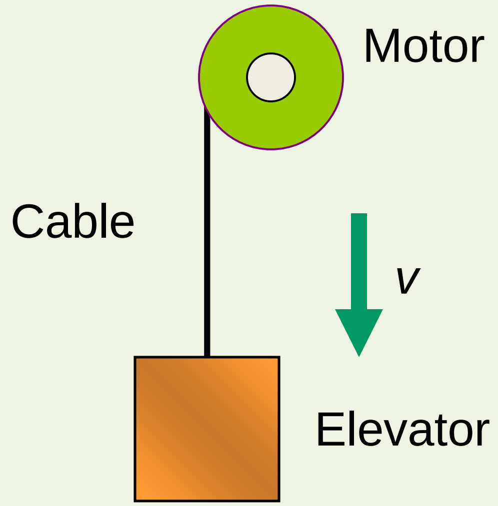
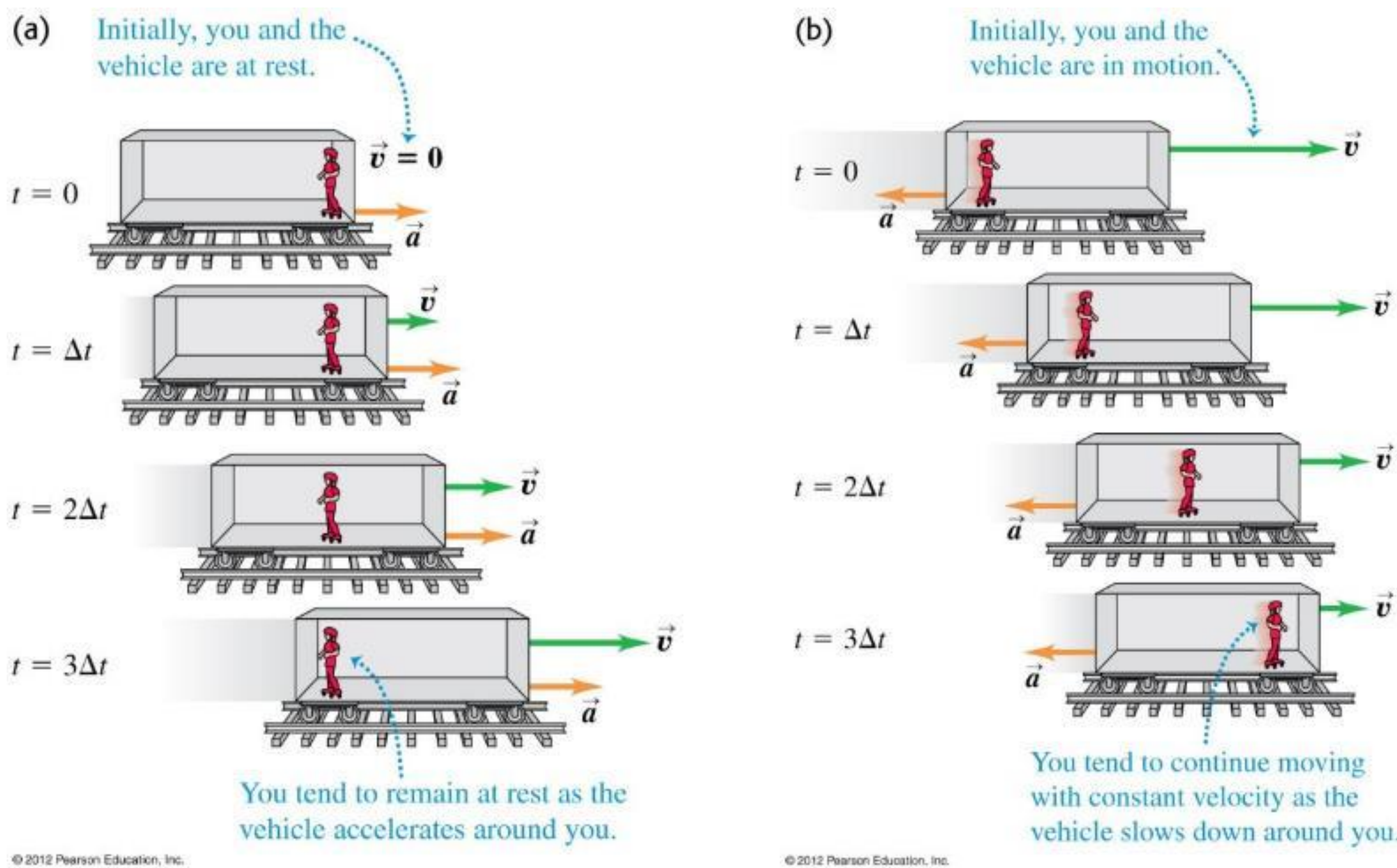


A. Newton’s first law ✓

B. Newton’s second law

C. Newton’s third law

Inertial Frame of Reference



Passenger (in roller skate) accelerates inside the train, but net force is zero.
Violate Newton's first law??
The train is not an inertial frame.

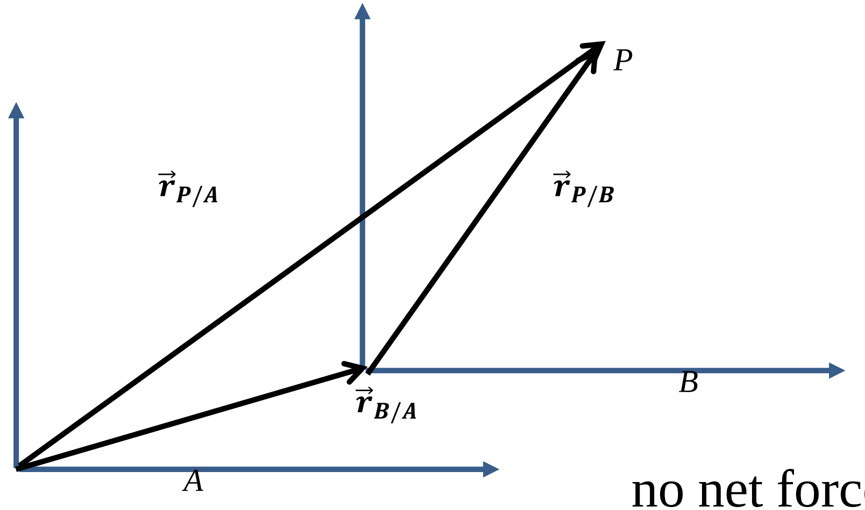
Inertial Frame

Definition:
A frame of reference in which Newton's first law is valid is called an inertial frame

Note: 1. Is the earth an inertial frame?

Only approximately

2. Given an inertial frame *A*,



$$\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A}$$

no net force

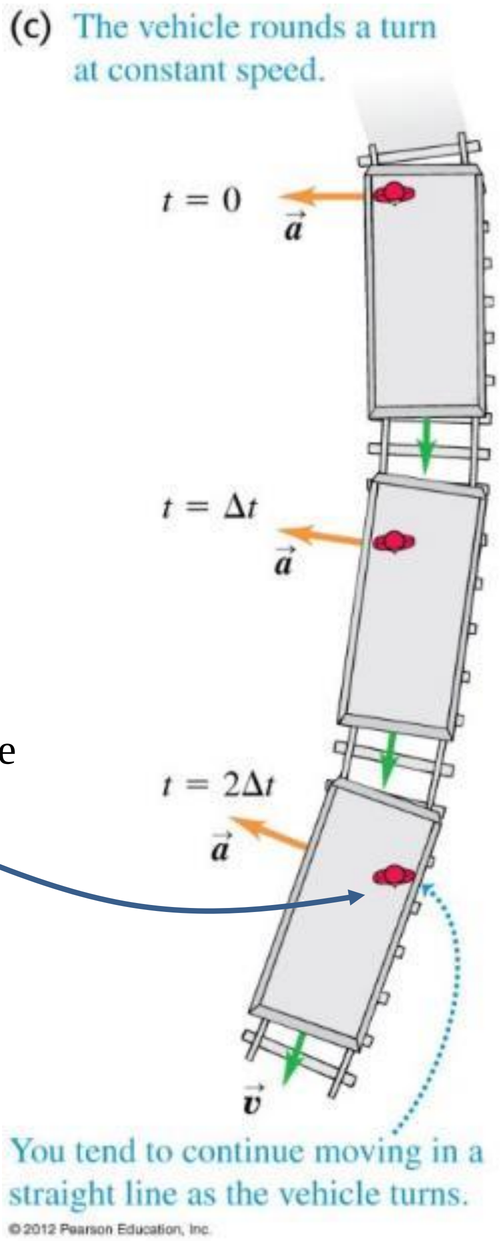
$\Rightarrow$ $\vec{v}_{P/A}$ constant

$\Rightarrow$ $\vec{v}_{P/B}$ constant provided $\vec{v}_{B/A}$ constant

Any frame *B* moving with constant $\boldsymbol{v}_{B/A}$ (can be zero) is also an inertial frame

In a non-inertial frame of reference, may feel like being acted on by a (non-existing) force.

feel like a force pushing you aside



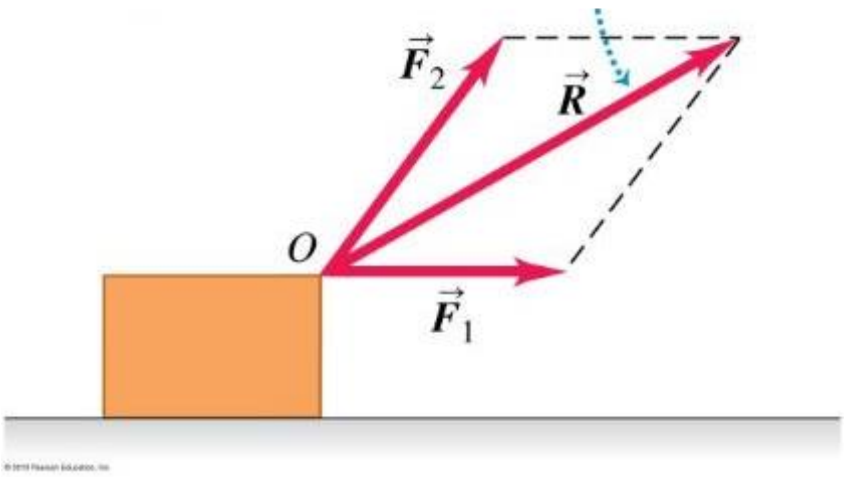
https://www.youtube.com/watch?v=49JwbrXcPjc

Question

- In which of the following situations is there zero net force on the body?
  a) an airplane flying due north at a steady speed and at a constant altitude, assuming that the earth is flat and is an inertial frame;
  b) a car driving straight up a hill at constant speed;
  c) a hawk circling at constant speed and constant height above an open field;
  d) a box with slick, frictionless surfaces in the back of a truck as the truck accelerates forward on a level road at constant acceleration.

Forces are vectors and can be added up (superposition of forces)

$\vec{R}$ is called the net or resultant force



The SI unit of force is newton,
1 N = 1 kg m/s^2

Newton's second law

If a *net* external force acts on a body, the body accelerates according to

$$\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$$

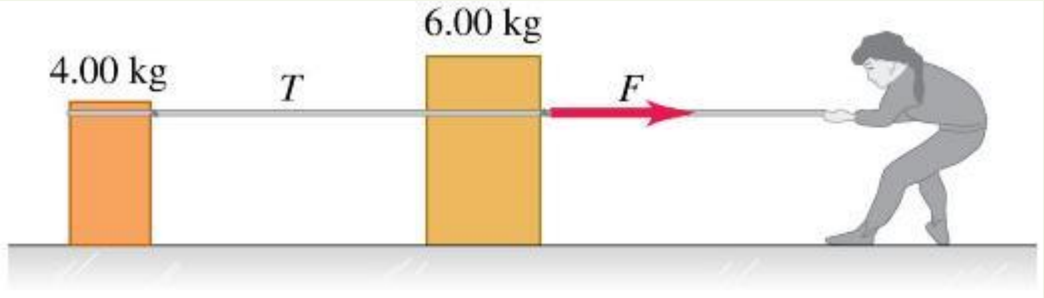
<u>inertial mass</u>

– how reluctant the body is to change its velocity

Make sure $\boldsymbol{F}$ is the net force, see Demonstration: fan car

Q4.12

A woman pulls on a 6.00-kg crate, which in turn is connected to a 4.00-kg crate by a light rope. It is given that both crates have non-zero accelerations and the light rope remains taut.



Compared to the 6.00-kg crate, the lighter 4.00-kg crate

A. is subjected to the same net force and has the same acceleration.

B. is subjected to a smaller net force and has the same acceleration.

C. is subjected to the same net force and has a smaller acceleration.

D. is subjected to a smaller net force and has a smaller acceleration.

E. none of the above

A4.12

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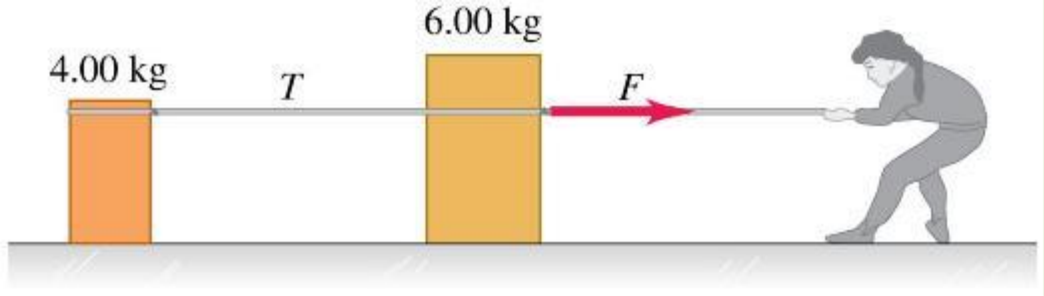

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B. is subjected to a smaller net force and has the same acceleration.

C. is subjected to the same net force and has a smaller acceleration.

D. is subjected to a smaller net force and has a smaller acceleration.

E. none of the above

Equation of motion:

$$\mathbf{F}(\mathbf{r},\mathbf{v}) = \sum_i \mathbf{F}_i(\mathbf{r},\mathbf{v}) = m\frac{d^2\mathbf{r}}{dt^2}$$

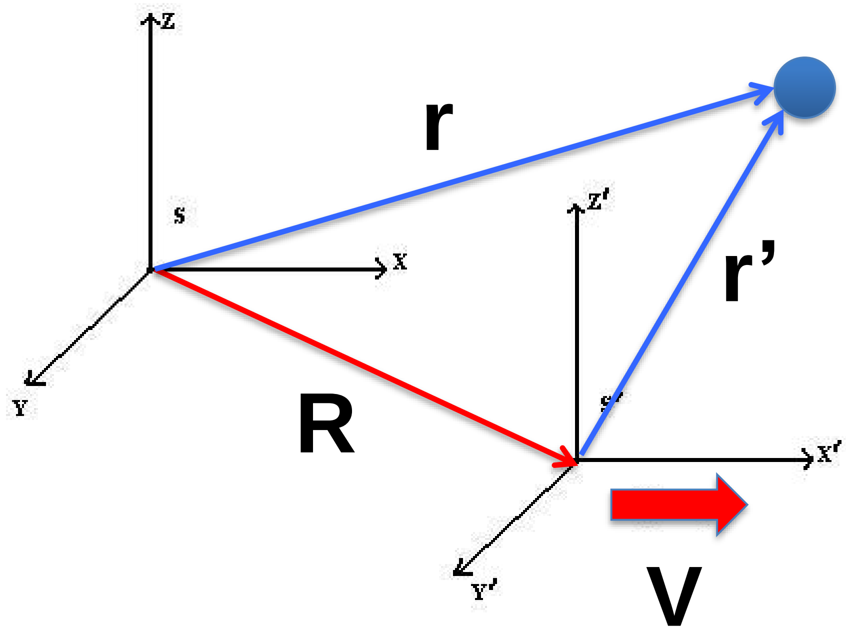
In particualr, if the net force is 0 (i.e. $\mathbf{F}(\mathbf{r},\mathbf{v}) = 0$), $\frac{d\mathbf{r}}{dt} = \mathbf{v} = \text{constant}$ (First law!)

Newton's second law, together with the force laws, becomes a law governing the accelerations of objects

Our goal is to find the laws for the force…..

Examples of force laws: Newton's law of universal gravitation, Coulomb's law, Hooke's law, law of friction….

One comment on the initial reference frame



$$r' = r - R$$

$$t' = t$$

$$v' = v - V$$

$$\mathbf{F}(\mathbf{r}, \mathbf{v}) = m\frac{d^2\mathbf{r}}{dt^2}$$

$$\mathbf{F}'(\mathbf{r}', \mathbf{v}') = m\frac{d^2\mathbf{r}'}{dt'^2}$$

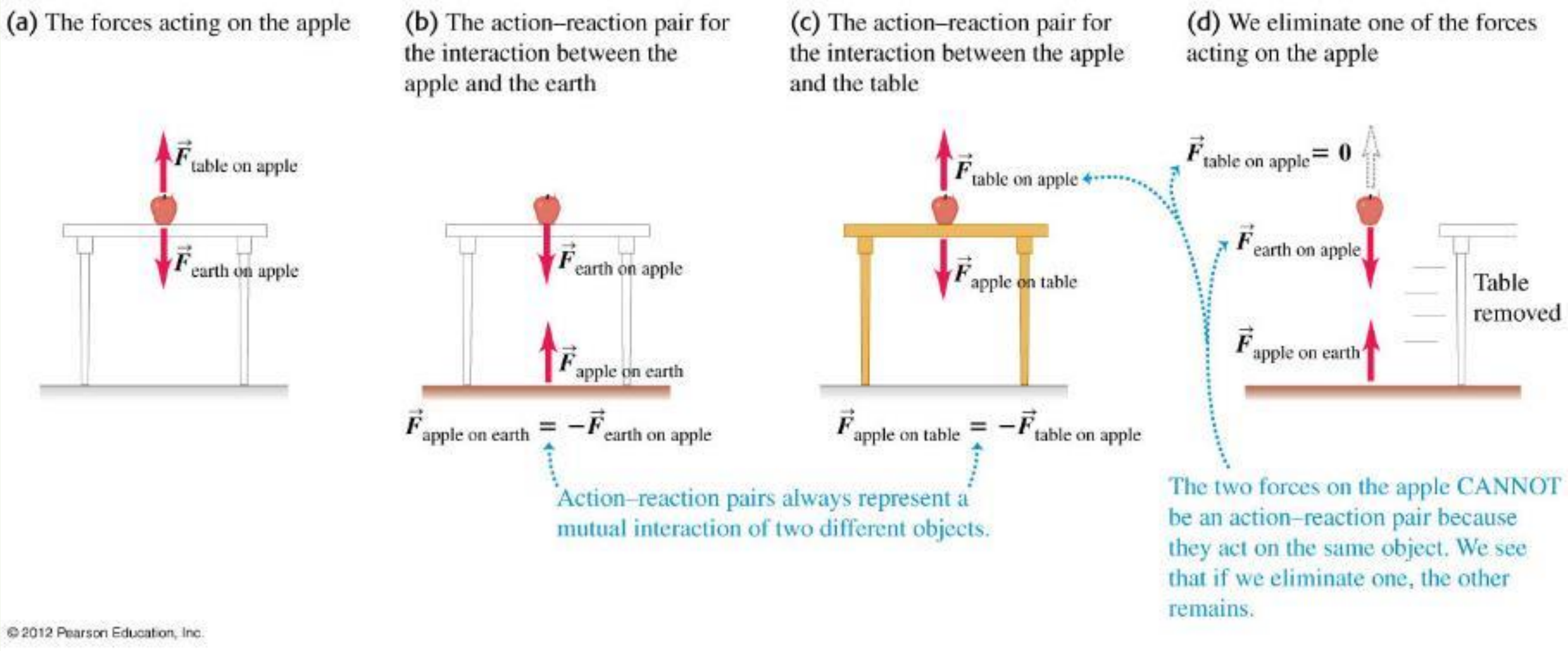
The Newton law looks the same in two initial reference frames!!

Newton’s third law of motion

- If body A exerts a force on body B (an “action”), then body B exerts a force on body A (a “reaction”).
- These two forces have the same magnitude but are opposite in direction.
- These two forces act on different bodies.

Example 4.9

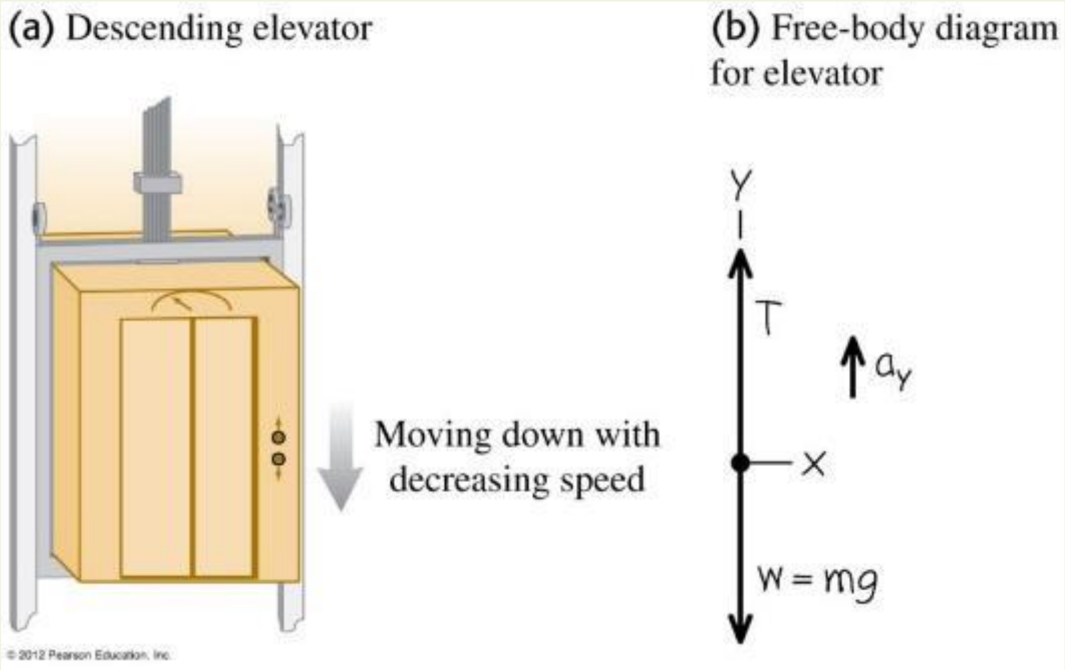
Action and reaction forces acting on an apple sitting on a table



Example: Tension in an elevator cable

An elevator, mass 800 kg, moving downwards at 10.0 m/s

If it comes to a stop in a distance of 25.0 m



To find deceleration a_y

$$v_y^2 = v_{0y}^2 + 2a_y(y - y_0)$$

0 m/s   -10.0 m/s   -25.0 m

$$\Longrightarrow a_y = 2.00\ \text{m/s}^2$$

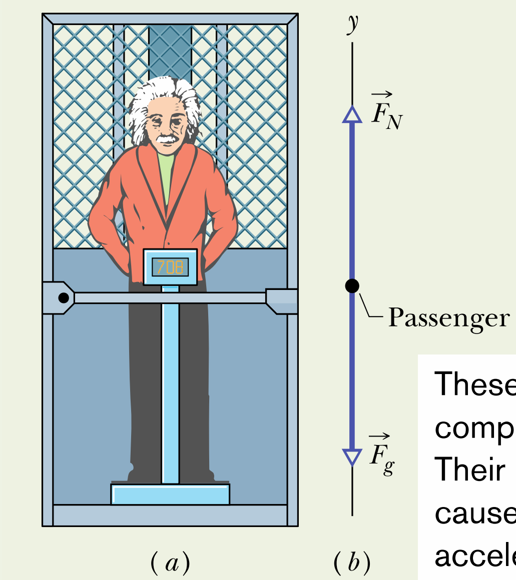
Tension in the cable

$$\sum F_y = T - w = ma_y$$

$$\Rightarrow\ T = m(g + a_y) = 9440\ \text{N}$$

Example: Forces within an elevator cab

A passenger of mass m = 72.2kg stands on a platform scale in elevator cab. We are concerned with the scale readings when the cab is stationary and when it is moving up or down.



These forces compete. Their net force causes a vertical acceleration.

The reading is equal to the magnitude of the normal force $\mathbf{F_N}$ on the passenger from the scale. The only other force acting on the passenger is the gravitational force $\mathbf{F_g}$. Using Newton's 2nd law, we have (take upward as positive)

$$F_N - mg = ma \quad \text{or} \quad F_N = m(g+a)$$

Therefore,

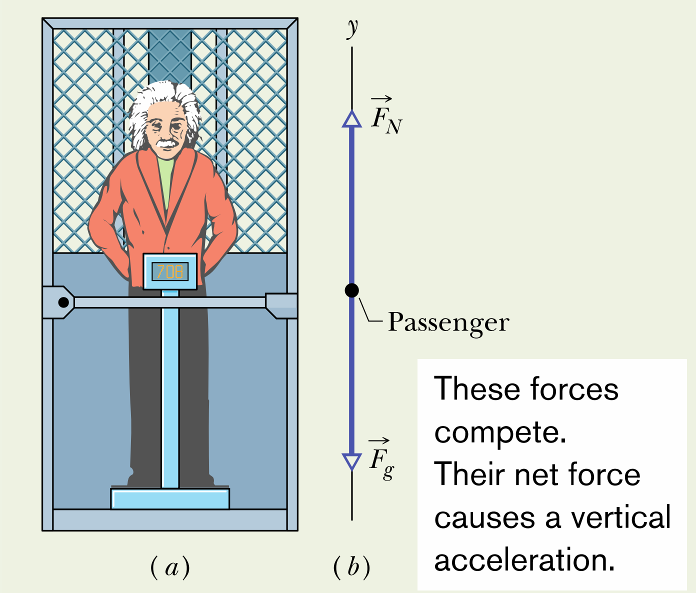
if a>0 (accelerating upward), $F_N > mg$

if a<0 (accelerating downward), $F_N < mg$

In particular, if a=-g (free fall), $F_N = 0$

(You appear weightless in a non-inertia reference frame)

More importantly, the passenger is stationary relative to the elevator cab. (i.e. his acceleration relative to the frame of the cab is zero.)



But

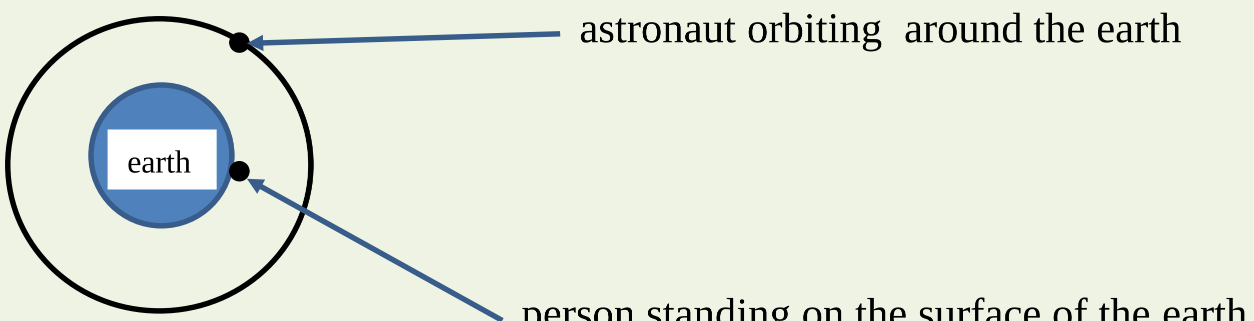
$F'_{net} = F_N - F_g \neq 0$

Hence, we conclude that

$F'_{net} \neq ma'$

The Newton's 2nd law doesn't hold in the frame of the accelerating elevator cab (the non-inertia reference frame)

Question



Both are under the gravitational attraction of the earth. Why does the person has weight but the astronaut is weightless?

Summary

Newton's First Law:

Objects in motion tend to stay in motion and objects at rest tend to stay at rest unless acted upon by an unbalanced force.

Newton's Second Law:

Force equals mass times acceleration ($F = ma$).

Newton's Third Law:

For every action there is an equal and opposite reaction.

Some particular Forces – Gravitational Force

The gravitational force $\mathbf{F_g}$ on a body is a certain type of pull that is directed toward a second body (Earth).

$$\vec{F}_g = -mg\hat{y} \text{ (take upward as positive)}$$

The **weight W** of a body is the magnitude of the net force required to prevent the body from falling freely, as measured by someone on the ground.

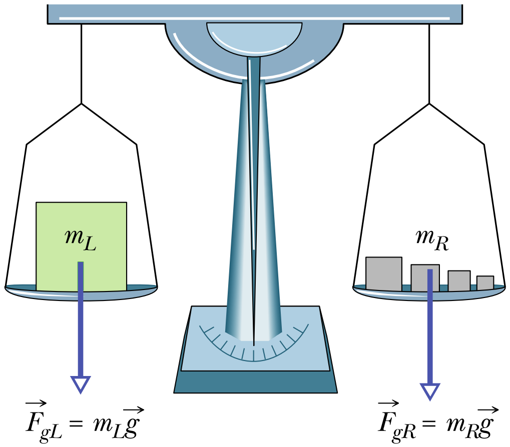
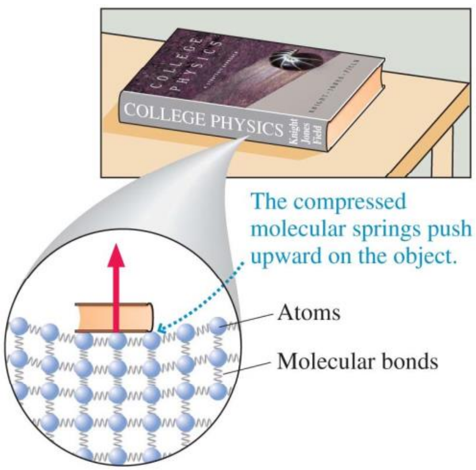


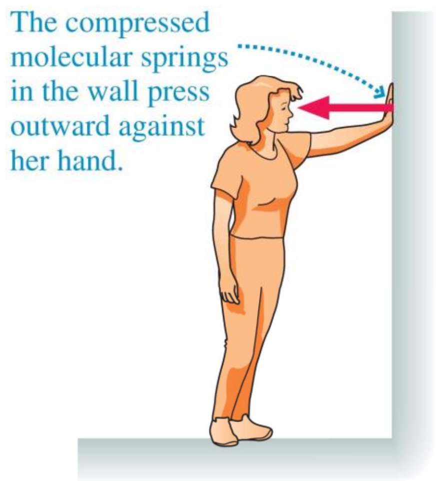
Fig. 5-5 An equal-arm balance. When the device is in balance, the gravitational force $\vec{F}_{gL}$ on the body being weighed (on the left pan) and the total gravitational force $\vec{F}_{gR}$ on the reference bodies (on the right pan) are equal. Thus, the mass m_L of the body being weighed is equal to the total mass m_R of the reference bodies.

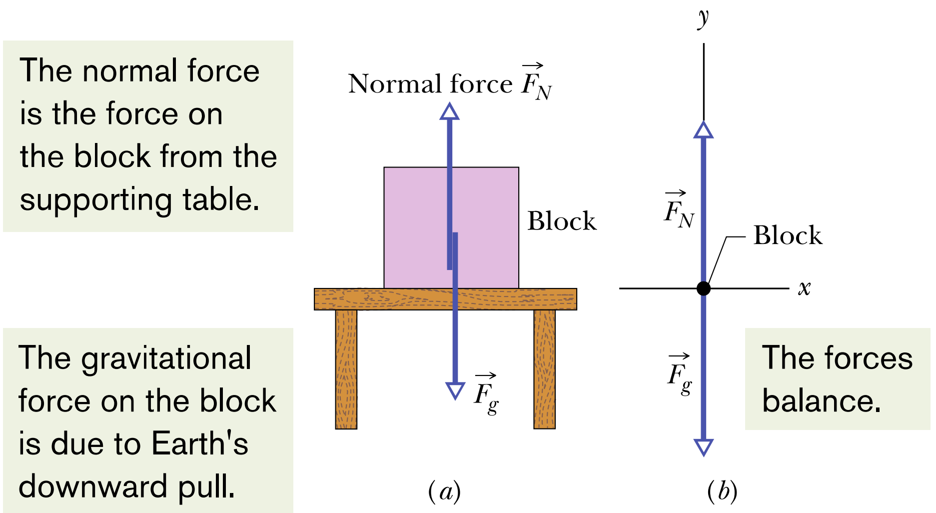
Caution: A body's weight is not its mass. If you move a body to a point where g is different (eg, on the moon), the body's mass (intrinsic property) is the same but the weight is different.

Some particular Forces – Normal force

When a body presses against a surface, the surface (even a seemingly rigid one) deforms and pushes on the body with a normal force F_N that is perpendicular to the surface.







$$\vec{F}_N + \vec{F}_g = m\vec{a}$$

$F_N - F_g = ma$ (upward as positive)

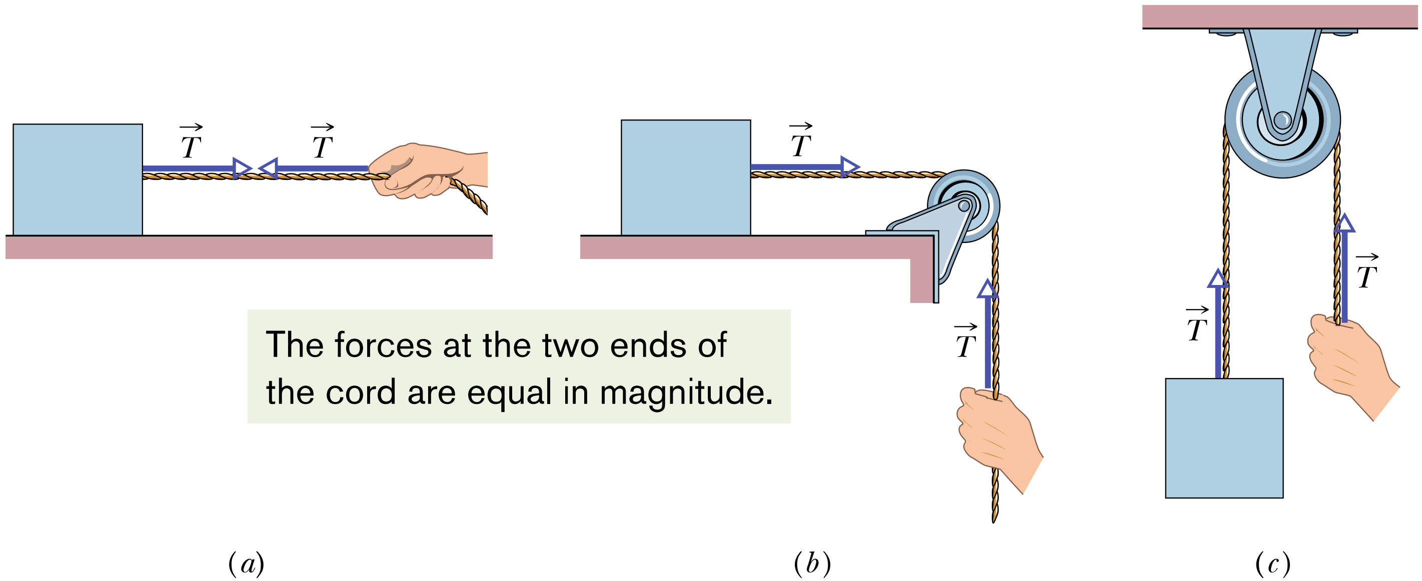
$F_N = m(a+g)$ (For example, if the table is in an accelerating elevator)

In particular, if a=0, we have

$$F_N = mg$$

Some particular Forces – Tension

The tension force is the force that is transmitted through a string, rope, cable or wire when it is pulled tight by forces acting from opposite ends. The tension force is directed along the length of the wire and pulls equally on the objects on the opposite ends of the wire.

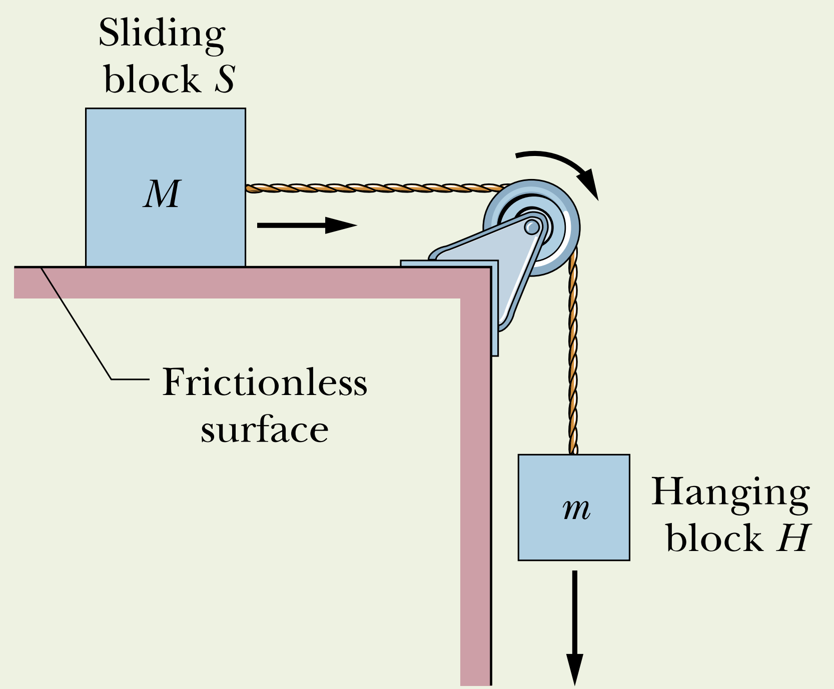


Action-reaction forces??

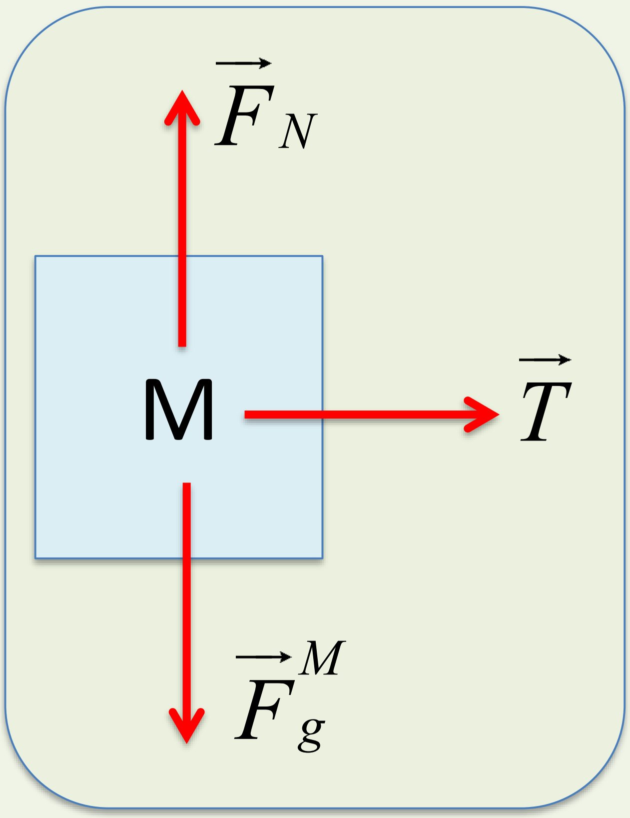
Example 1

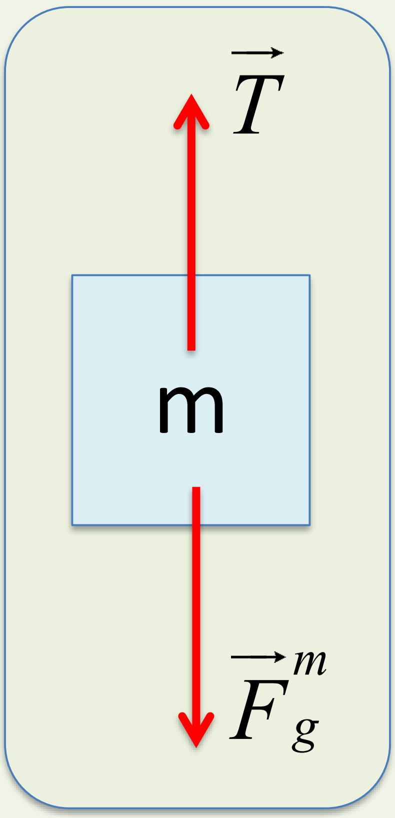
A block S (the sliding block) with mass M. The block is free to move along a horizontal frictionless surface and connected, by a cord that wraps over a frictionless pulley, to a second block H (the hanging block), with mass m. The cord and pulley have negligible masses compared to the blocks (they are "massless"). The hanging block H falls as the sliding block S accelerates to the right. Find

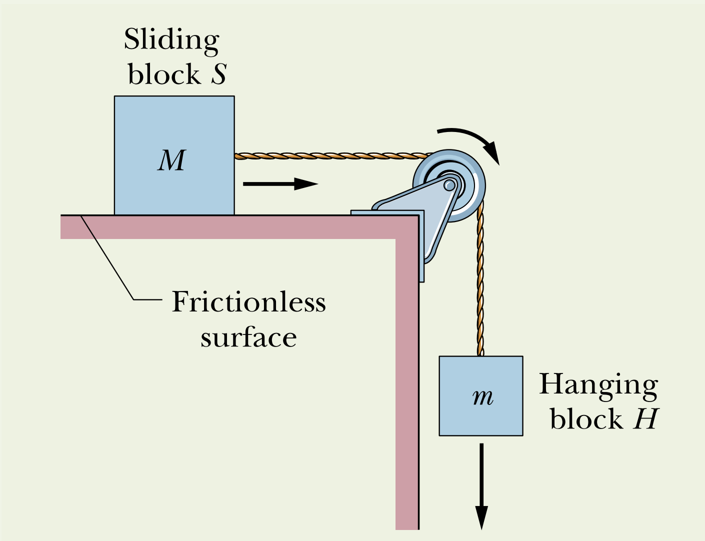
(a) the acceleration of block S,
(b) the acceleration of block H,
(c) the tension in the cord.



Free Body Diagram - In every problem where the Newton's 2nd Law applies. it is fundamental to draw what is called the Free Body Diagram. This diagram must show all the external forces acting on a body. We isolate the body and the forces due to that strings and surfaces are replaced by arrows; of course, the friction forces and the force of gravity must be included. If there are several bodies, a separate diagram should be drawn for each one.







We assume that the cord does not stretch, so that if block H falls 1 mm in a certain time, block S moves 1 mm to the right in that same time. This means that the blocks move together and their accelerations have the same magnitude a.

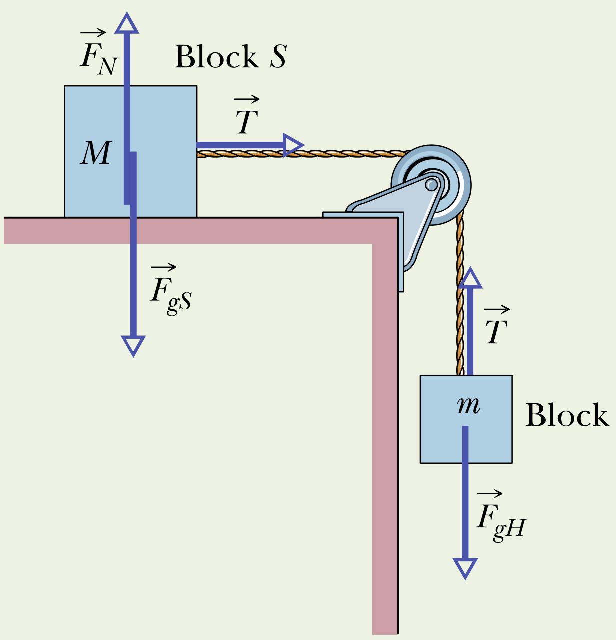
We first apply the Newton’s 2nd law on block S,

Vertical: $F_N - F_g^M = Ma_y = 0 \Rightarrow F_N = Mg$

Horizontal: $T = Ma$

Next, we apply the Newton’s 2nd law on block H, (*downward as +ve*)

$$F_g^m - T = ma \Rightarrow mg - T = ma$$



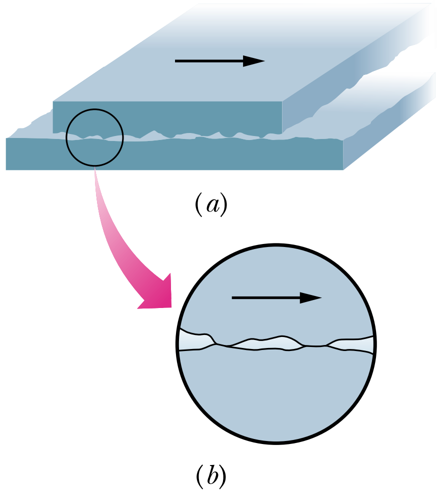
Hence, we have

$$\text{mg-Ma=ma} \Rightarrow a = \frac{m}{m+M}g$$

And the tension $T = \dfrac{mM}{m+M}g$

Some particular Forces – Friction

If we either slide or attempt to slide a body over a surface, the motion is resisted by a bonding between the body and the surface. The resistance is considered to be a single force $\mathbf{F_f}$, called either the frictional force or simply friction. This force is directed along the surface, opposite the direction of the intended motion



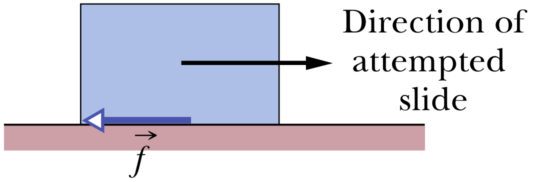
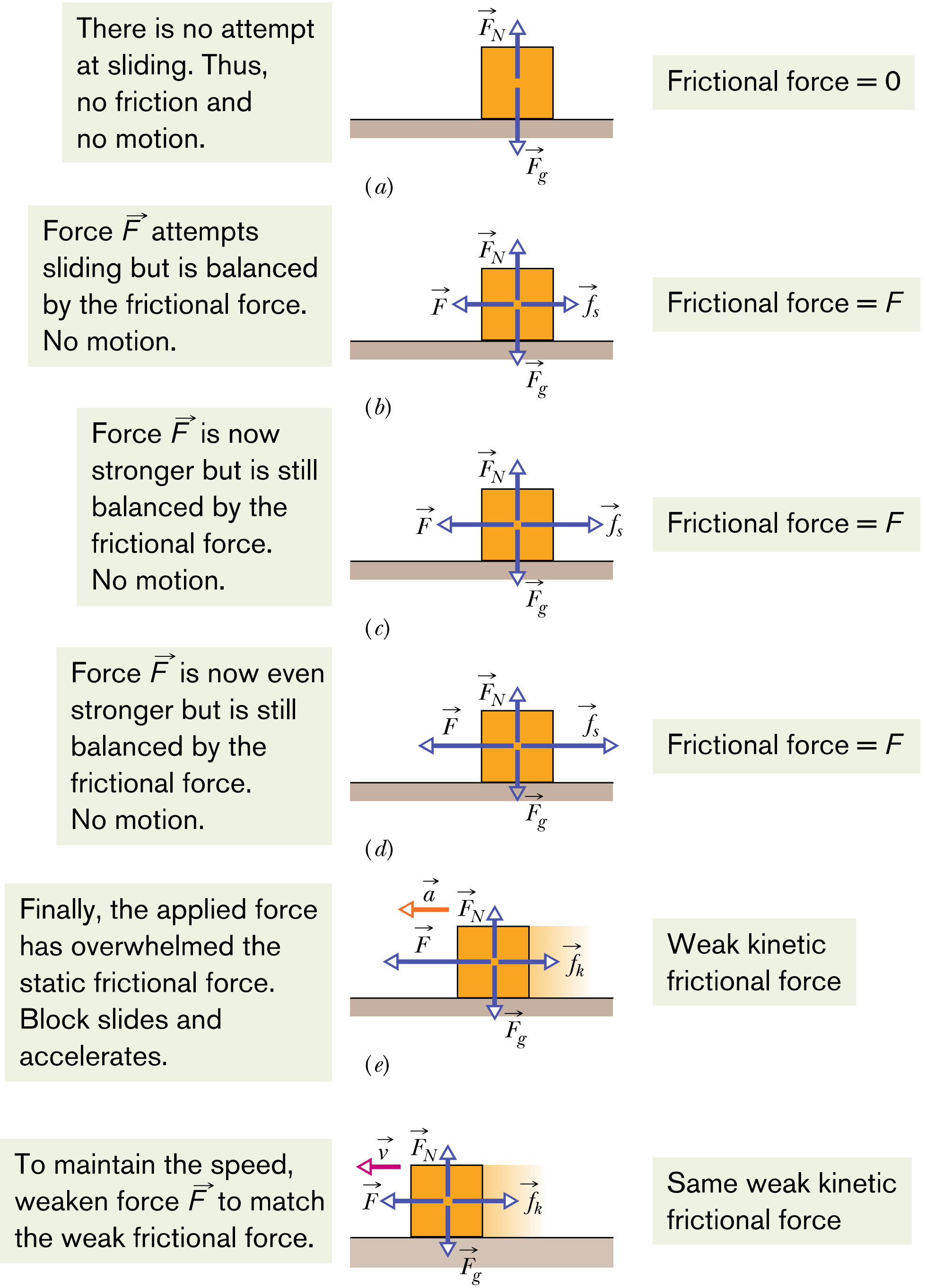
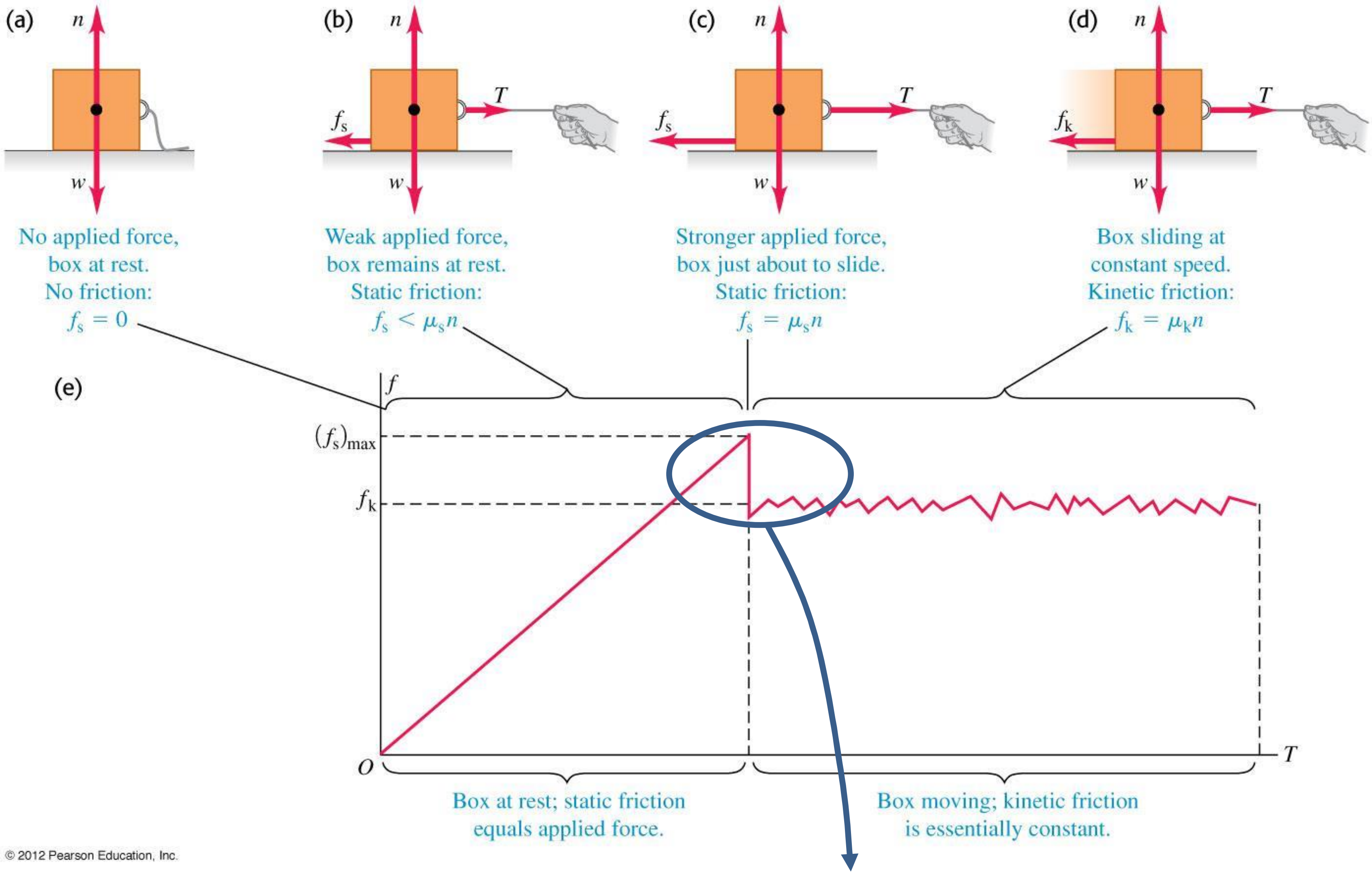


Fig. 5-8 A frictional force $\vec{f}$ opposes the attempted slide of a body over a surface.

Properties of Friction

1. If the body does not move, then the static frictional force $\mathbf{F_s}$ and the component of $\mathbf{F}$ that is parallel to the surface balance each other. They are equal in magnitude, and $\mathbf{F_s}$ is directed opposite that component of $\mathbf{F}$.
2. The maximum value of the static friction is given by, $\mathbf{F_{s,max}=\mu_s F_N}$ where $\boldsymbol{\mu}_s$ is the coefficient of static friction.
3. If the body begins to slide along the surface, the magnitude of the frictional force rapidly decreases to a value $\mathbf{F_k}$ given by, $\mathbf{F_k=\mu_k F_N}$ where $\boldsymbol{\mu}_k$ is the coefficient of kinetic friction.





Interpretation: easier to keep the block moving than to start it moving

Static & Kinetic Friction Coefficients

Material	Coefficient of Static Friction μ_S	Coefficient of Kinetic Friction μ_S
Rubber on Glass	2.0+	2.0
Rubber on Concrete	1.0	0.8
Steel on Steel	0.74	0.57
Wood on Wood	0.25 – 0.5	0.2
Metal on Metal	0.15	0.06
Paper on paper	0.28	
Synovial Joints in Humans	0.01	0.003

Example 2

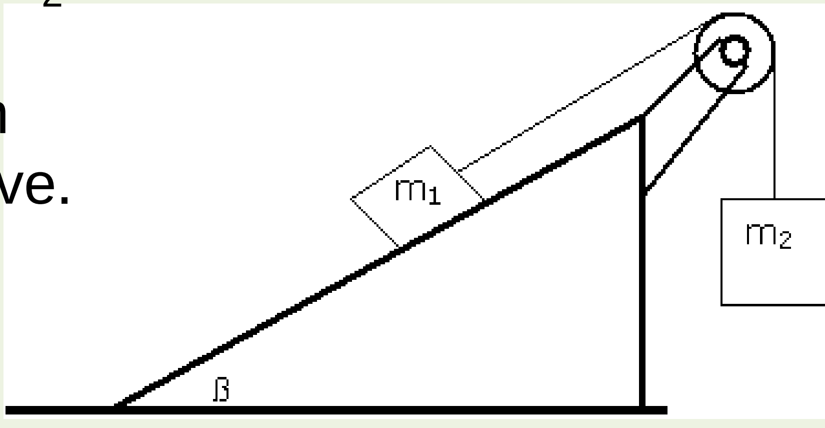
The pulley is frictionless and weightless. The block of mass m_1 is on the plane, inclined at an angle β with the horizontal. The block of mass m_2 is connected to m_1 by a string.

1. Assuming there is no friction, show a formula for the acceleration of the system in terms of m_1, m_2, β and g.

2. What condition is required for m_1 to go up the incline?

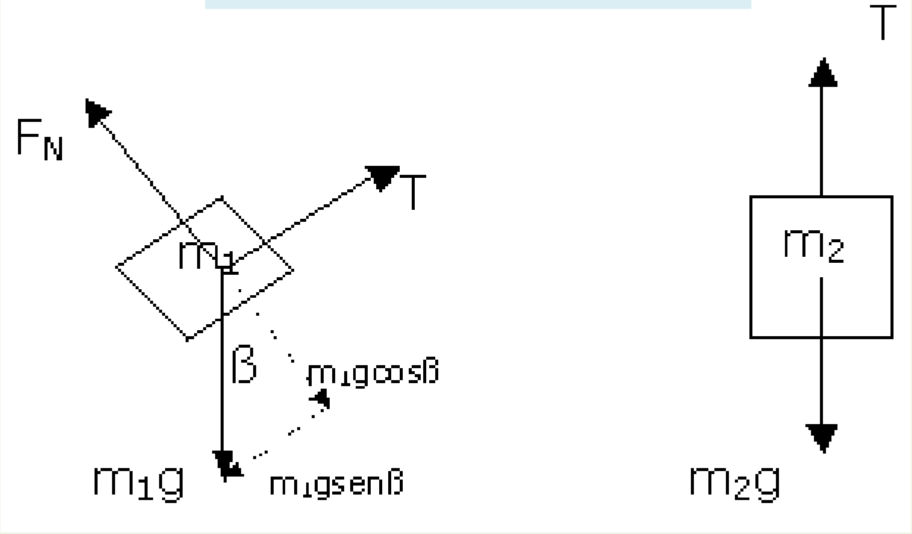
3. Assume that the coefficient of kinetic friction between m_1 and the plane is 0.2, m_1=2kg, m_2=2.5kg and the angle β=30°. Calculate the acceleration of m_1 and m_2.

4. What is the maximum value of friction coefficient so the system can still move.



Free Body Diagram - In every problem where the Second Newton's Law applies it is fundamental to draw what is called the Free Body Diagram. This diagram must show all the external forces acting on a body. We isolate the body and the forces due to that strings and surfaces are replaced by arrows; of course, the friction forces and the force of gravity must be included. If there are several bodies, a separate diagram should be drawn for each one.

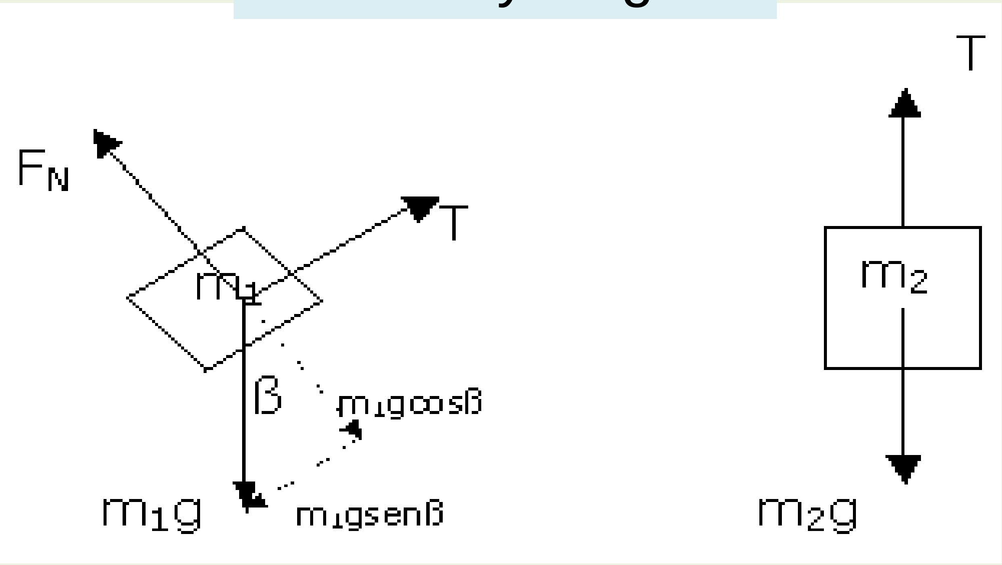
Free Body diagram



Key Observations:

- The (tension) force that m_1 exerts on m_2 through the rope has the same magnitude T. This is so because a rope only changes the direction of a force, not its magnitude assuming a weightless rope.
- The magnitude of the acceleration is the same at both ends of the rope assuming an inextensible rope.

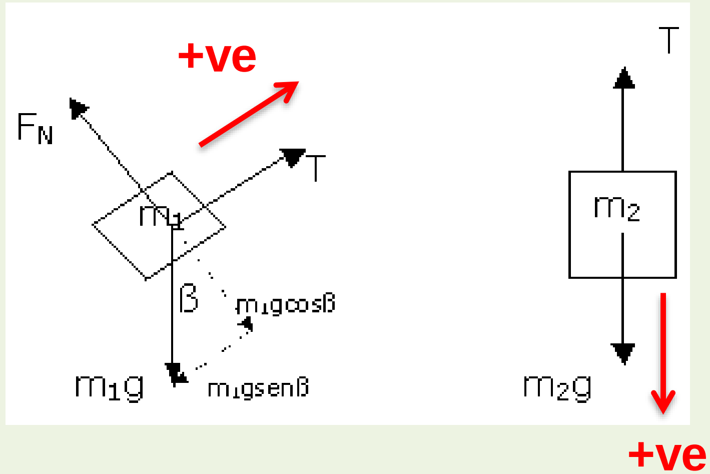
Free Body diagram



Components of forces

Free Body diagram

Notice from the diagram the weight of m_1 has been split into the components $m_1g\sin\beta$ parallel to the incline, and $m_1g\cos\beta$ perpendicular to it.



Without friction

1) Let's assume the direction of the acceleration makes m_1 to go upward.

Sum of forces on m_1 in the dirction of the incline plane: $T - m_1 g \sin b = m_1 a$

Sum of verticeal forces on m_2 : $m_2 g - T = m_2 a$

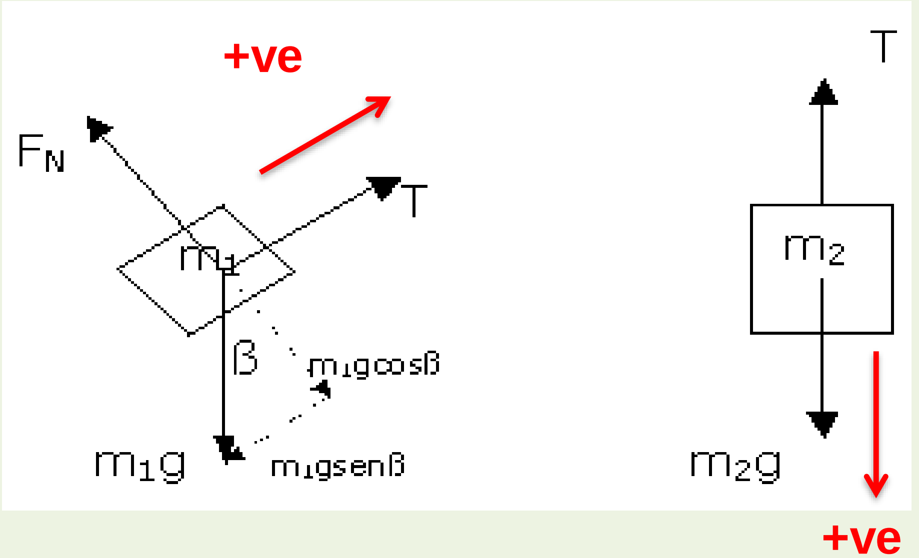
Adding both equations we get $m_2 g - m_1 g \sin b = a(m_1 + m_2)$

$$a = g\frac{m_2 - m_1 \sin b}{m_1 + m_2}$$

The acceleration of the masses is:

Free Body diagram

$$a = g\frac{m_2 - m_1 \sin b}{m_1 + m_2}$$

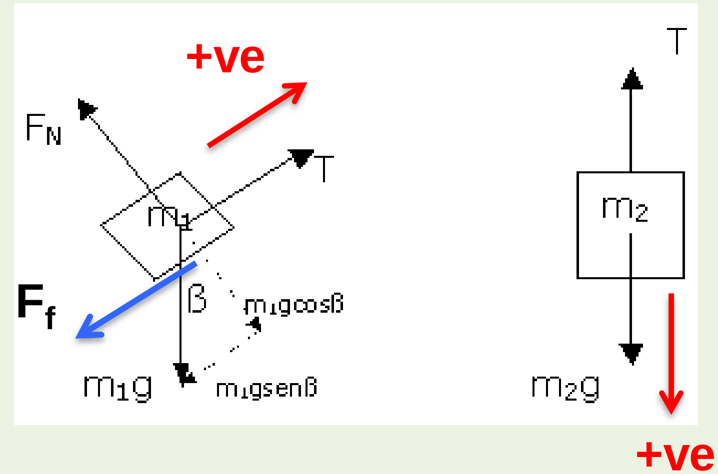


2) For a to be positive (i.e. m_1 going up): $m_2 > m_1 \sin b$

For a to be negative (i.e. m_1 going down): $m_2 < m_1 \sin b$

3) Now appears a friction force, always in an opposite direction to the movement. The magnitude of this friction force is $F_f = \mu F_N$. Where μ is the coefficient of kinetic friction.

Free Body diagram



$$F_N - m_1 g\cos\beta = 0 \quad \text{OR} \quad F_N = m_1 g\cos\beta$$

The friction force is then $F_f = \mu m_1 g\cos\beta$.

Hence the sum of forces on m_1 on the incline plane is now:

$$T - m_1 g\sin\beta - \mu m_1 g\cos\beta = m_1 a$$

The sume of vertical forces on m_2 is:

$$m_2 g - T = m_2 a$$

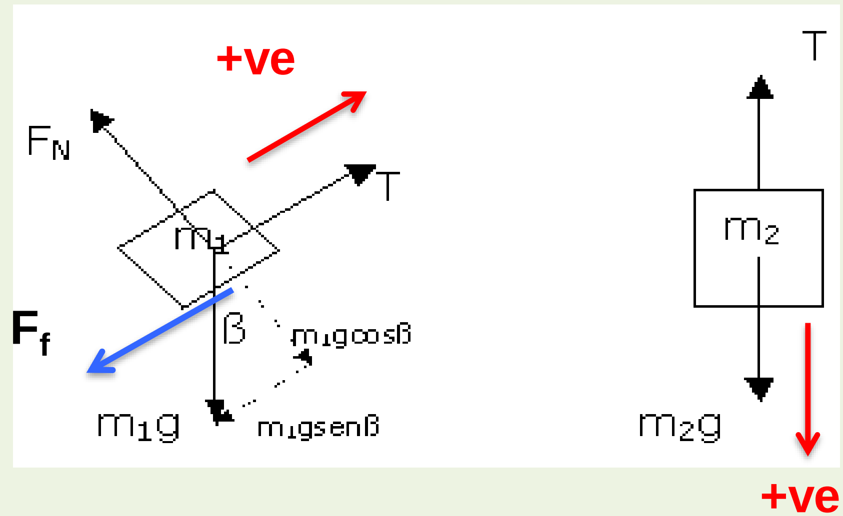
$$\therefore \quad a = \frac{m_2 g - m_1 g(\sin\beta + \mu\cos\beta)}{m_1 + m_2}$$

Replacing values, we have $a = 2.51\,m/s^2$

The acceleration of the masses is:

$$a = \frac{m_2 g - m_1 g(\sin\beta + \mu\cos\beta)}{m_1 + m_2}$$

Free Body diagram



As the coefficient of friction μ increases, the acceleration decreases until the acceleration becomes zero. The condition is obtained when:

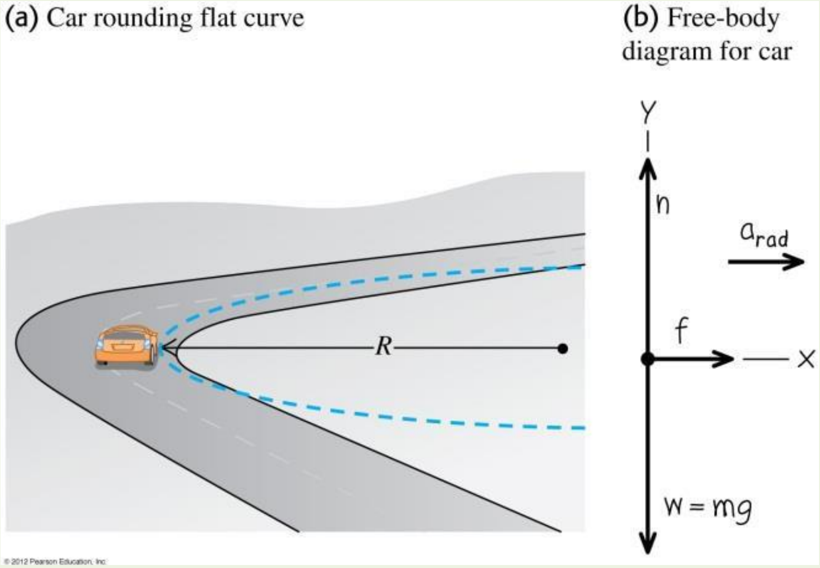
$$m_2 g - m_1 g(\sin\beta + \mu\cos\beta) = 0$$

$$\Rightarrow \mu = \frac{m_2 g - m_1 g \sin\beta}{m_1 \cos\beta}$$

Replacing values we get $\mu = 0.87$.

Example 3: Why banked curves in a racing track help?

On a flat curve



What supplies the centripetal force?

(Static / Kinetic) friction!

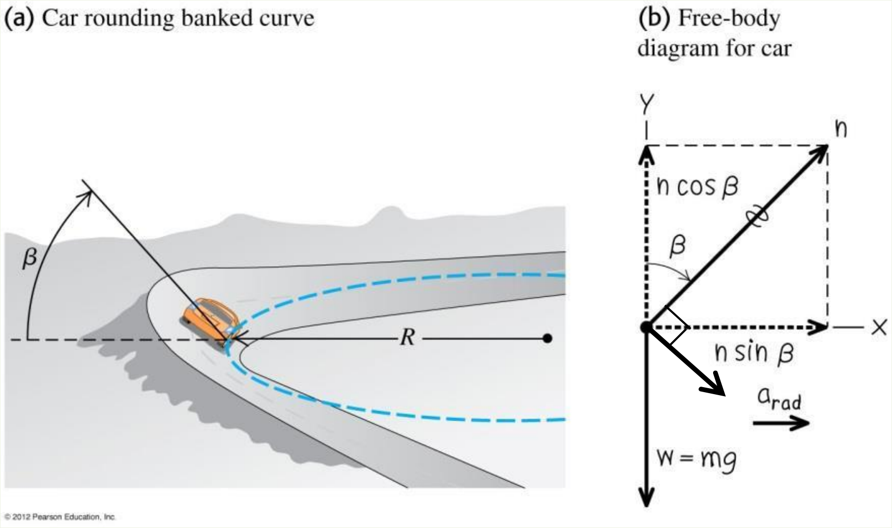
Max. speed without skidding:

$$f = f_{max} = m\frac{v_{max}^2}{R} \Rightarrow v_{max} = \sqrt{\mu_s g R}$$

$$f_{max} = \mu_s n = \mu_s m g$$

Example 3: Why banked curves in a racing track help?

If banked at angle β



What supplies the centripetal force? n and f!

$$\Sigma F_x = n \sin\beta + f\cos\beta = mv^2/R$$

$$\Sigma F_y = n\cos\beta - f\sin\beta - mg = 0$$

$$\Rightarrow f = m\left(\frac{v^2}{R}\cos\beta - g\sin\beta\right), n = m\left(\frac{v^2}{R}\sin\beta + g\cos\beta\right)$$

$$f \le \mu_s n \Rightarrow v \le v_{max} = \sqrt{\frac{\tan\beta + \mu_s}{1 - \mu_s \tan\beta} gR} \ge \sqrt{\mu_s gR}$$

Challenging Question:

What happen to the friction f if $v < \sqrt{gR\tan\beta}$? How would you interpret this situation?