10. Electric field and electric forces

香港科技大學
THE HONG KONG
UNIVERSITY OF SCIENCE
AND TECHNOLOGY

香港資優教育學苑
The Hong Kong Academy
for Gifted Education

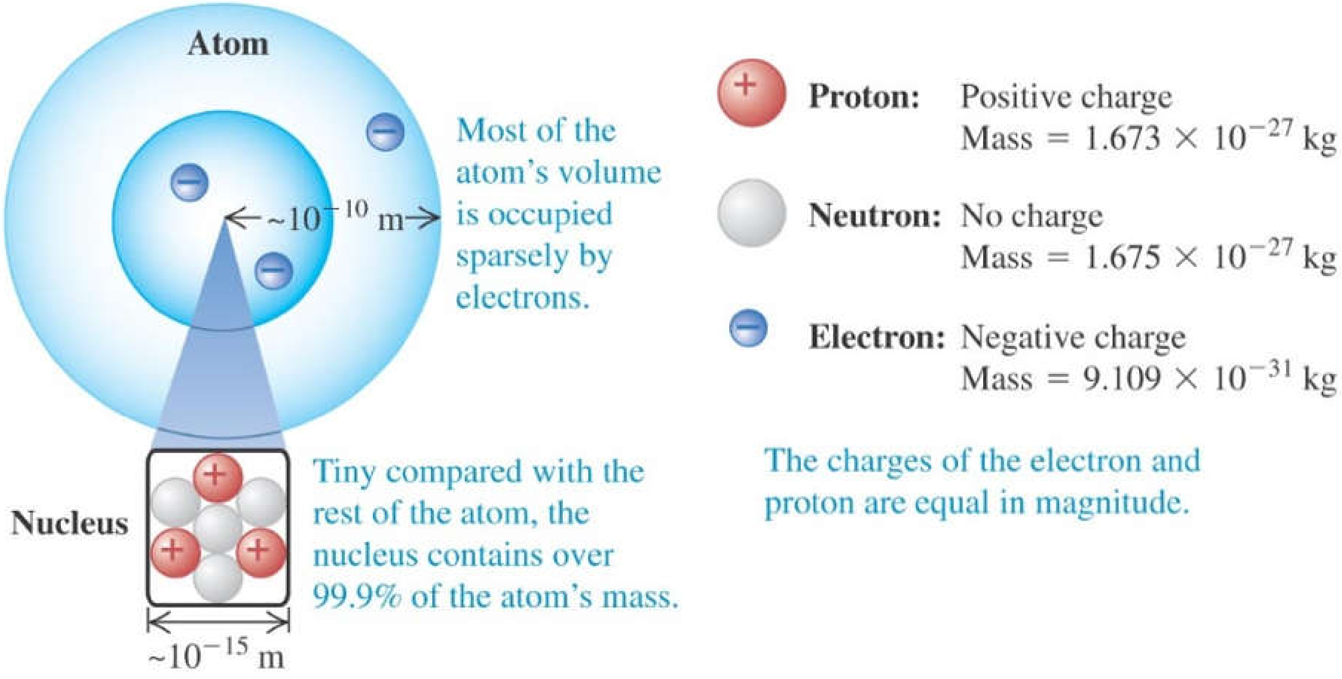
Why is E&M important?—most relevant to everyday life (from mechanical forces to biological processes) among the four fundamental interactions

The four fundamental forces:

	Relative strength	Range (meter)	Responsible for
Strong	10^{38}	10^{-15}	Binding quarks into hadrons, and neutrons and protons in nuclei
Electromagnetic	10^{36}	∞	
Weak	10^{25}	10^{-18}	Transforming neutron into proton in nuclear decay
Gravitation	1	∞	

Some simple factors from high school physics:

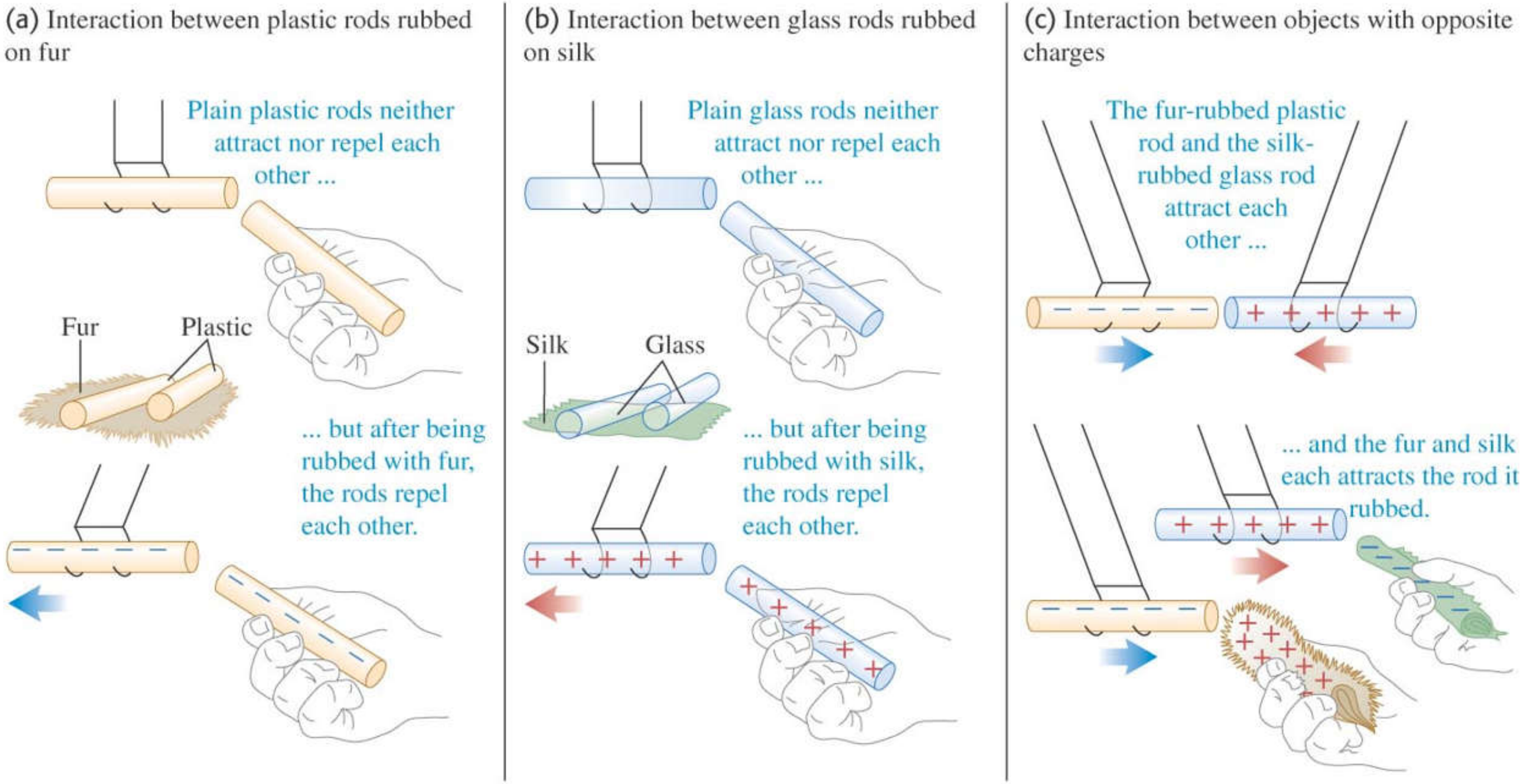
Origin of charge from atoms



A conductor permits easy movement of charge through it, while an insulator does not

Neutral object: number of positive charge = number of negative charge

Charge by rubbing (insulators)

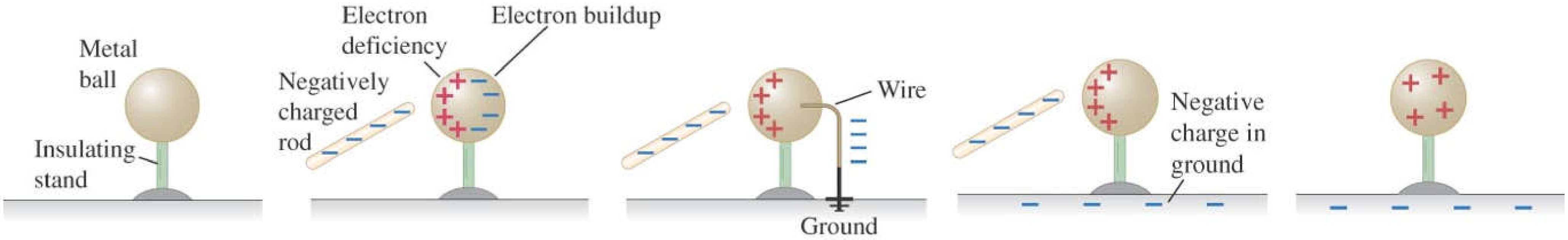


(a) Interaction between plastic rods rubbed on fur

(b) Interaction between glass rods rubbed on silk

(c) Interaction between objects with opposite charges

Charge by induction (conductors)



(a) Uncharged metal ball

(b) Negative charge on rod repels electrons, creating zones of negative and positive **induced charge**.

(c) Wire lets electron buildup (induced negative charge) flow into ground.

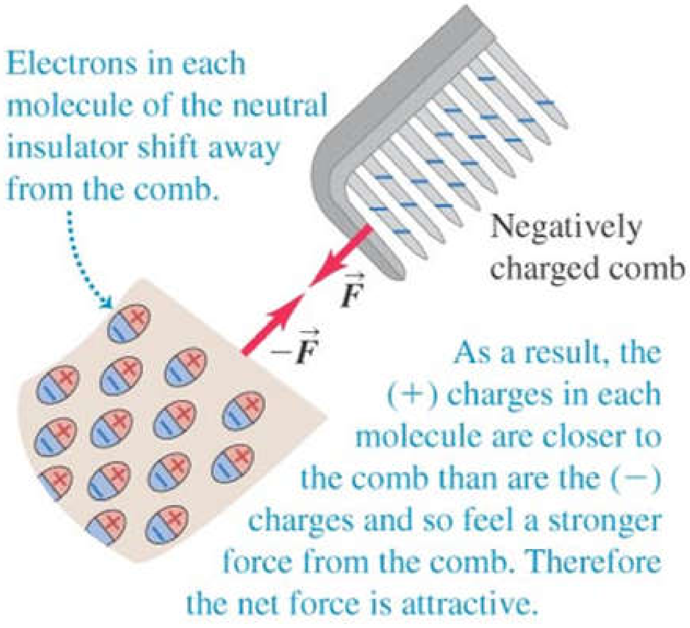
(d) Wire removed; ball now has only an electron-deficient region of positive charge.

(e) Rod removed; electrons rearrange themselves, ball has overall electron deficiency (net positive charge).

Real Life Examples

Combing hair in winter

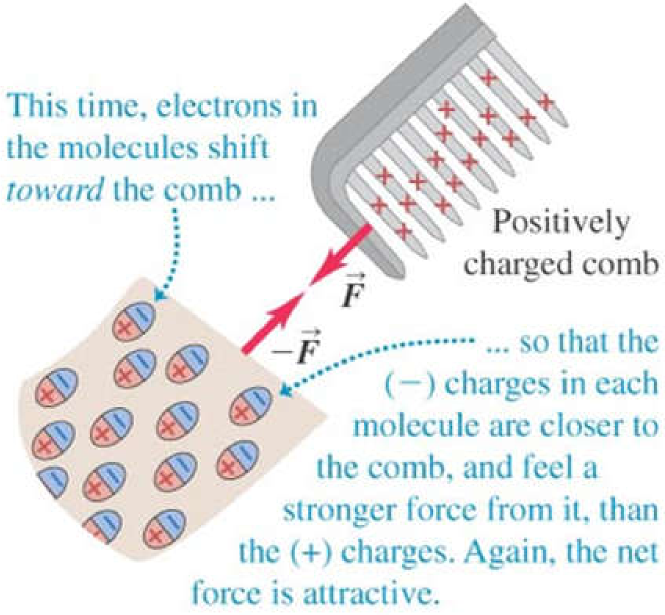
(b) How a negatively charged comb attracts an insulator



Electrons in each molecule of the neutral insulator shift away from the comb.

As a result, the (+) charges in each molecule are closer to the comb than are the (−) charges and so feel a stronger force from the comb. Therefore the net force is attractive.

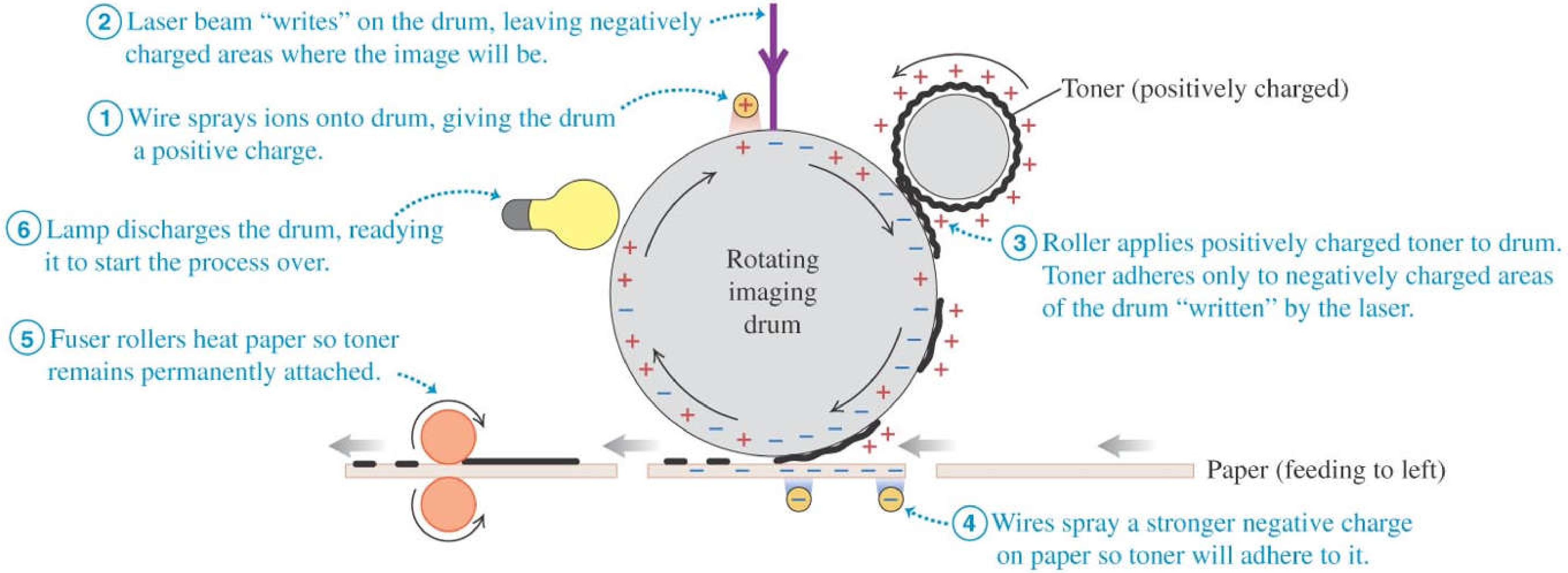
(c) How a positively charged comb attracts an insulator



This time, electrons in the molecules shift *toward* the comb ...

... so that the (−) charges in each molecule are closer to the comb, and feel a stronger force from it, than the (+) charges. Again, the net force is attractive.

Laser printing



② Laser beam "writes" on the drum, leaving negatively charged areas where the image will be.

① Wire sprays ions onto drum, giving the drum a positive charge.

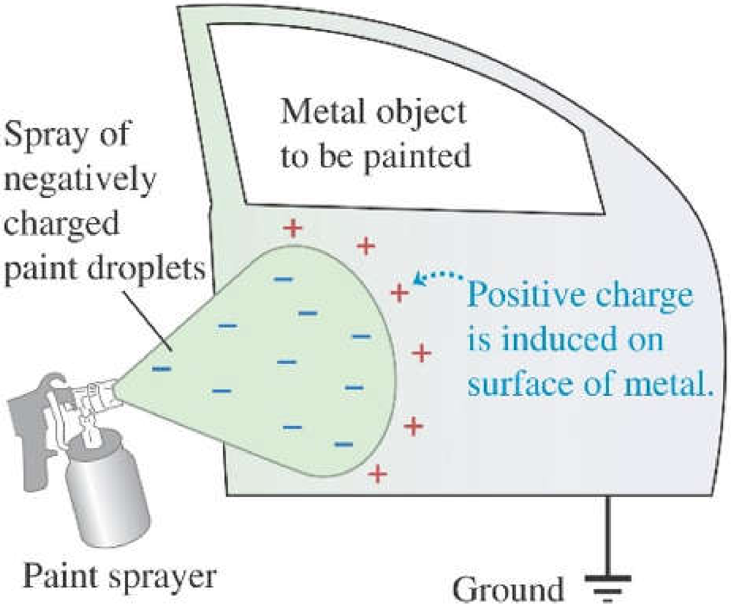
⑥ Lamp discharges the drum, readying it to start the process over.

⑤ Fuser rollers heat paper so toner remains permanently attached.

③ Roller applies positively charged toner to drum. Toner adheres only to negatively charged areas of the drum "written" by the laser.

④ Wires spray a stronger negative charge on paper so toner will adhere to it.

Electrostatic painting



Minimizes overspray and gives a smooth finish

Question: Two metal balls, one charged and one neutral, hanging on insulating threads.
If they are close to each other but not touching, they will (attract / repel / ignore) each other.
Once they touch, they will (attract / repel / ignore) each other.

Demonstration: Van de Graaf generator

Raising hair Bouncing Balloon Lightning

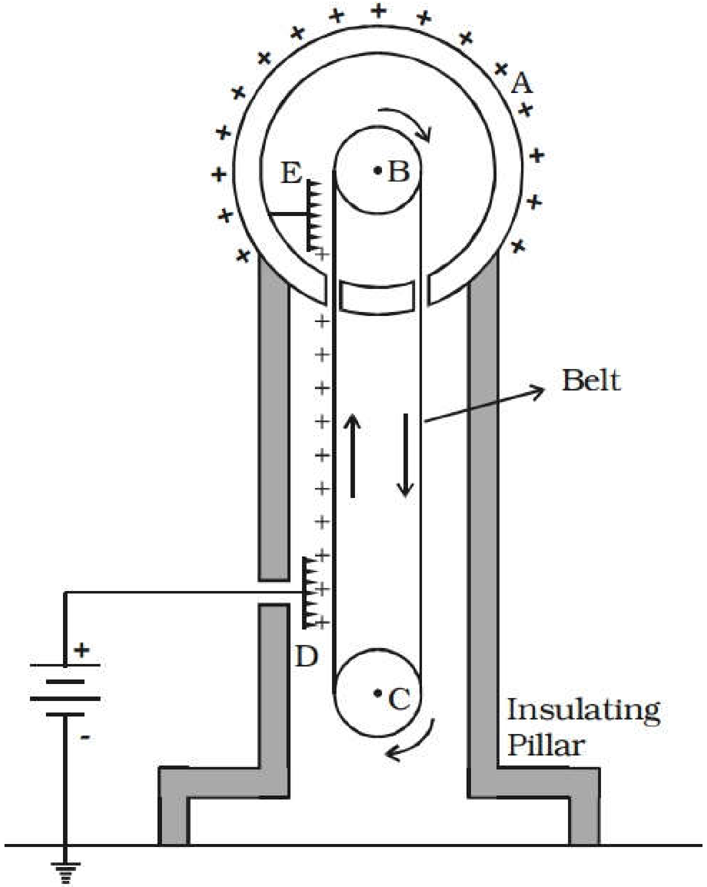
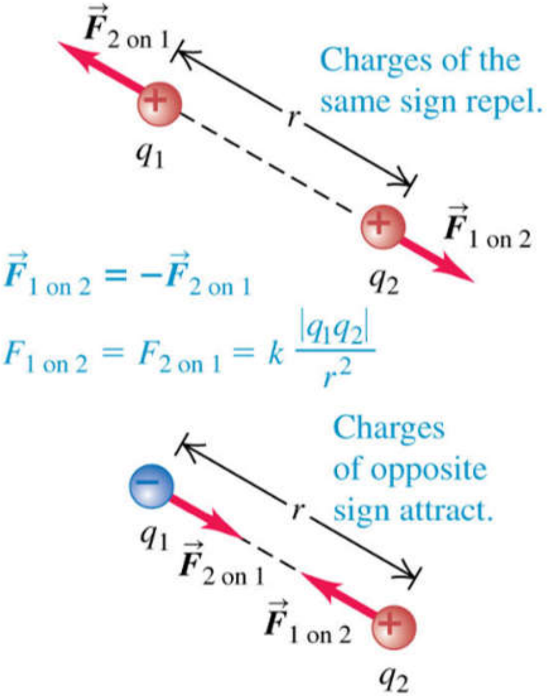


Fig 1 Van de Graaff Generator

Two very important facts about charges:

1. **Principle of conservation of charge** – the algebraic sum of all the electric charges in any closed system (no charge can escape) is constant.
2. Electron charge is the fundamental (cannot be further divided) unit of charge
   $e = 1.602176565(35) \times 10^{-19}$ C

Coulomb's Law



Magnitude of electric force between two point charges

$$F = \frac{1}{4\pi\epsilon_0}\frac{|q_1q_2|}{r^2}$$

$\epsilon_0 = 8.854 \times 10^{-12}$ C^2/Nm^2, or

$\frac{1}{4\pi\epsilon_0} = 8.988.0 \times 10^9$ Nm^2/C^2

⚠ Two 1C charge at 1m apart exert a force of 9×10^9 N on each other! 1C is a huge charge!

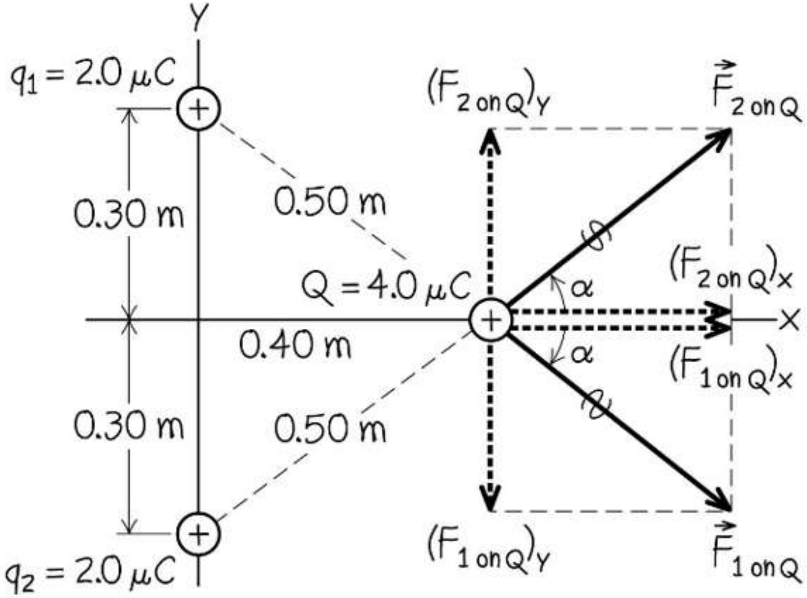
Very much like gravitation, $F = Gm_1m_2/r^2$, but much stronger

E.g. two helium nuclei (He^{2+}, $m = 6.64 \times 10^{-27}$ kg and $q = 3.2 \times 10^{-19}$C), ratio of electric to gravitational force is

$$\frac{F_e}{F_g} = \frac{\frac{1}{4\pi\epsilon_0}\frac{|q_1q_2|}{r^2}}{\frac{Gm_1m_2}{r^2}} = 3.1 \times 10^{35}$$

Example 21.4 Vector Addition of Electric Forces in a Plane

Two equal positive charges $q_1 = q_2 = 2.0\ \mu\mathrm{C}$ are located at $x = 0,\ y = 0.30$ m and $x = 0,\ y = -0.30$ m respectively. Find the total electric force experienced by a charge $Q = 4.0\ \mu\mathrm{C}$ at $x = 0.40$ m and $y = 0$?



By symmetry, $F_y = 0$

$$F_x = 2\frac{1}{4\pi\epsilon_0}\frac{|q_1q_2|}{r^2}\cos\alpha$$

$$= 2(9.0\times10^{9}\ \mathrm{Nm^2/C^2})$$

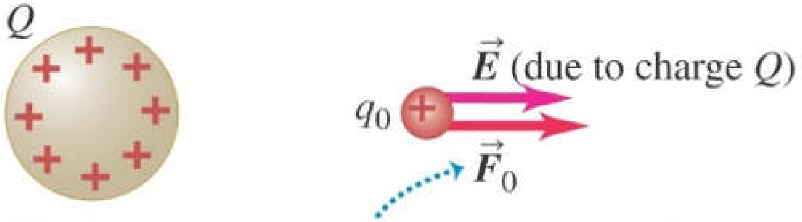
$$\times\frac{(4.0\times10^{-6}\ \mathrm{C})(2.0\times10^{-6}\ \mathrm{C})}{(0.50\ \mathrm{m})^2}$$

$$\times\frac{0.40\ \mathrm{m}}{0.50\ \mathrm{m}} = \underline{\hspace{3cm}}\mathrm{N}$$

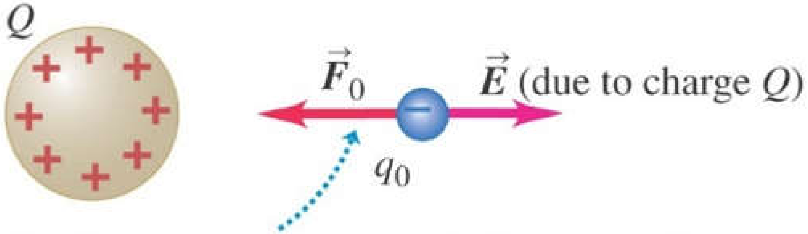
⚠ careful about the precision!

Question: If $q_2 = -2.0\ \mu\mathrm{C}$ instead, the total force on Q will be (along x direction / along – x direction / along y direction / along – y direction / zero / none of these).

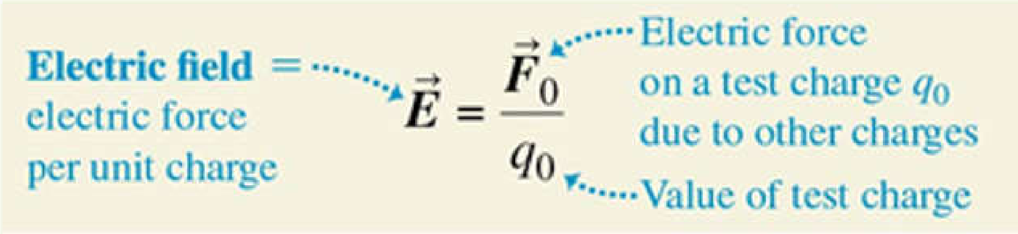
- ⚠ Q feels the charge due to *other charges* (q_1 and q_2) only, never its own charge
- ⚠ If charge Q changed, just substitute a new value into above formula. Did q_1 and q_2 set up something independent of charge Q? Yes, they set up an **electric field**.



The force on a positive test charge q_0 points in the direction of the electric field.

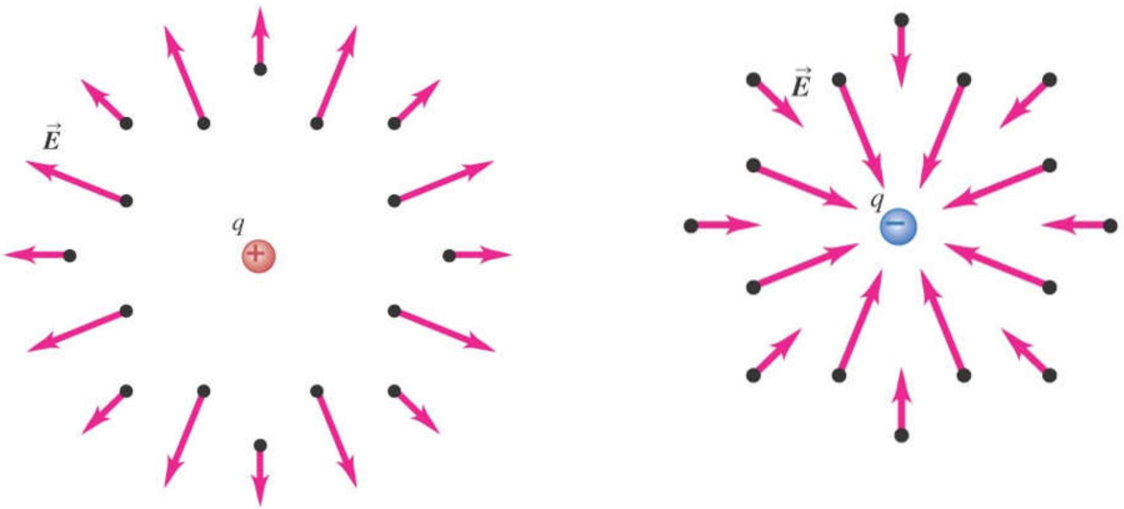


The force on a negative test charge q_0 points opposite to the electric field.



Electric Field due to a Point Charge

Always radially outward (for +ve charge) or inward (for –ve charge)



(a) The field produced by a positive point charge points *away from* the charge.

(b) The field produced by a negative point charge points *toward* the charge.

Electric field due to a point charge

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \qquad (21.7)$$

Value of point charge

Unit vector from point charge toward where field is measured

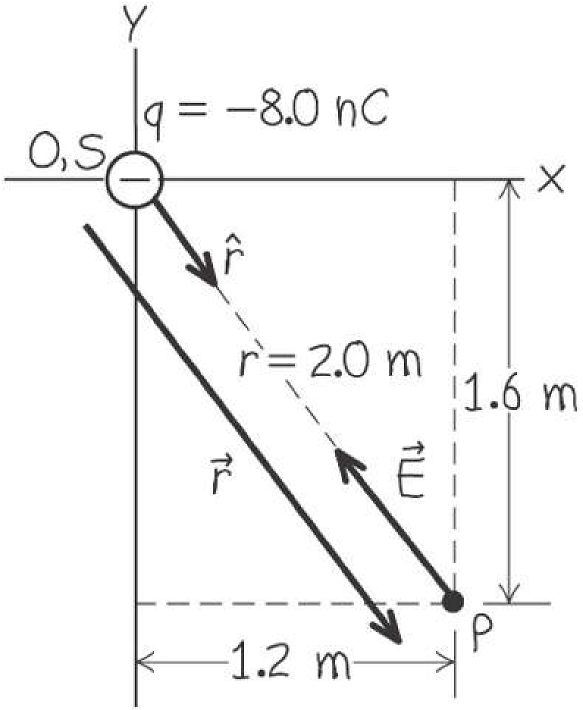
Distance from point charge to where field is measured

Electric constant

- ⚠ $\hat{r}$ is <u>always</u> pointing radially outwards
- ⚠ q can be +ve/-ve.

Example 21.6

A point charge $q = -0.8$ nC placed at the origin. The electric field at $x = 1.2$ m and $y = -1.6$ m is



EXECUTE: The distance from S to P is

$$r = \sqrt{x^2 + y^2} = \sqrt{(1.2\text{ m})^2 + (-1.6\text{ m})^2} = 2.0\text{ m}$$

The unit vector $\hat{\boldsymbol{r}}$ is then

$$\hat{\boldsymbol{r}} = \frac{\vec{\boldsymbol{r}}}{r} = \frac{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}}{r}$$

$$= \frac{(1.2\text{ m})\hat{\boldsymbol{\imath}} + (-1.6\text{ m})\hat{\boldsymbol{\jmath}}}{2.0\text{ m}} = 0.60\hat{\boldsymbol{\imath}} - 0.80\hat{\boldsymbol{\jmath}}$$

Then, from Eq. (21.7),

$$\vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}}$$

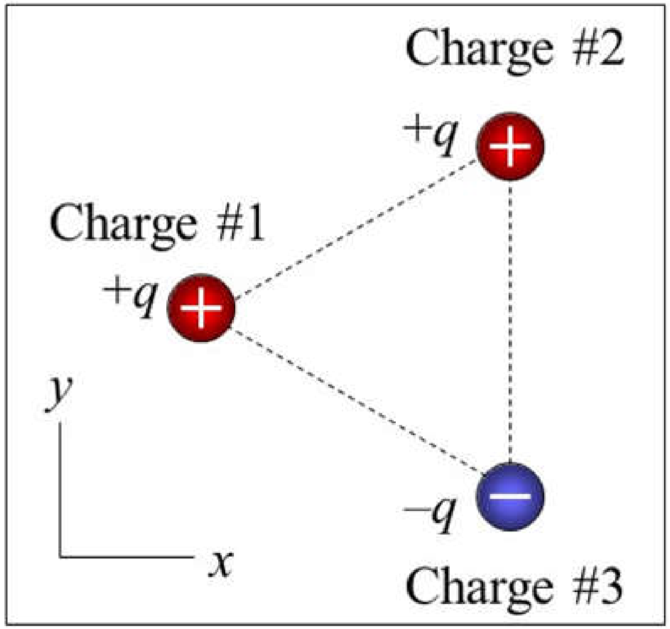
$$= (9.0 \times 10^9\,\text{N} \cdot \text{m}^2/\text{C}^2)\frac{(-8.0 \times 10^{-9}\text{ C})}{(2.0\text{ m})^2}(0.60\hat{\boldsymbol{\imath}} - 0.80\hat{\boldsymbol{\jmath}})$$

$$= (-11\text{ N/C})\hat{\boldsymbol{\imath}} + (14\text{ N/C})\hat{\boldsymbol{\jmath}}$$

Clicker Questions

Q21.3

Three point charges lie at the vertices of an equilateral triangle as shown. All three charges have the same magnitude, but charges #1 and #2 are positive ($+q$) and charge #3 is negative ($-q$). The net electric force that charges #2 and #3 exert on charge #1 is in



A. the $+x$-direction.

B. the $-x$-direction.

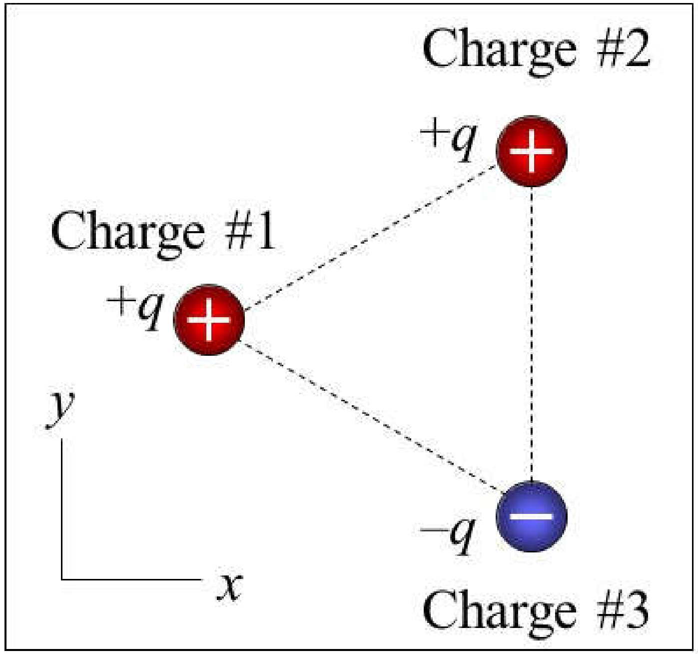
C. the $+y$-direction.

D. the $-y$-direction.

E. none of the above.

A21.3

Three point charges lie at the vertices of an equilateral triangle as shown. All three charges have the same magnitude, but charges #1 and #2 are positive ($+q$) and charge #3 is negative ($-q$). The net electric force that charges #2 and #3 exert on charge #1 is in



A. the $+x$-direction.

B. the $-x$-direction.

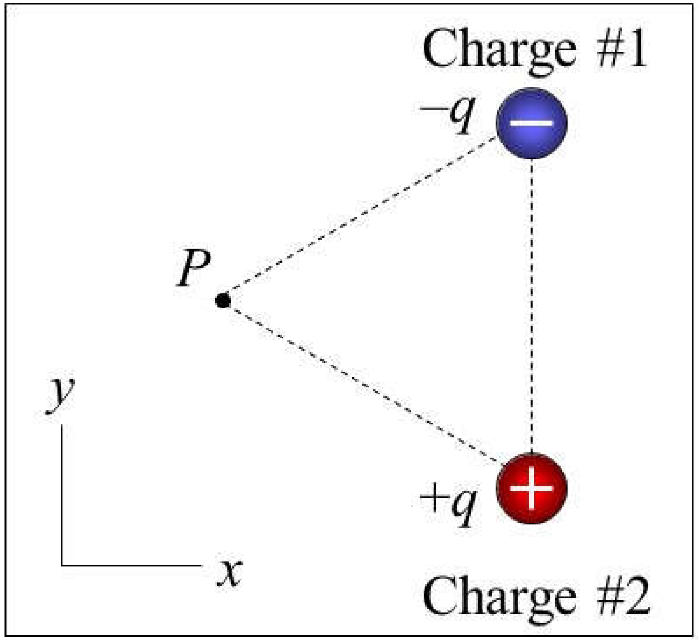
C. the $+y$-direction.

[✓ D. the $-y$-direction.]

E. none of the above.

Q21.6

Two point charges and a point P lie at the vertices of an equilateral triangle as shown. Both point charges have the same magnitude q but opposite signs. There is nothing at point P. The net electric field that charges #1 and #2 produce at point P is in



A. the $+x$-direction.

B. the $-x$-direction.

C. the $+y$-direction.

D. the $-y$-direction.

E. none of the above.

A21.6

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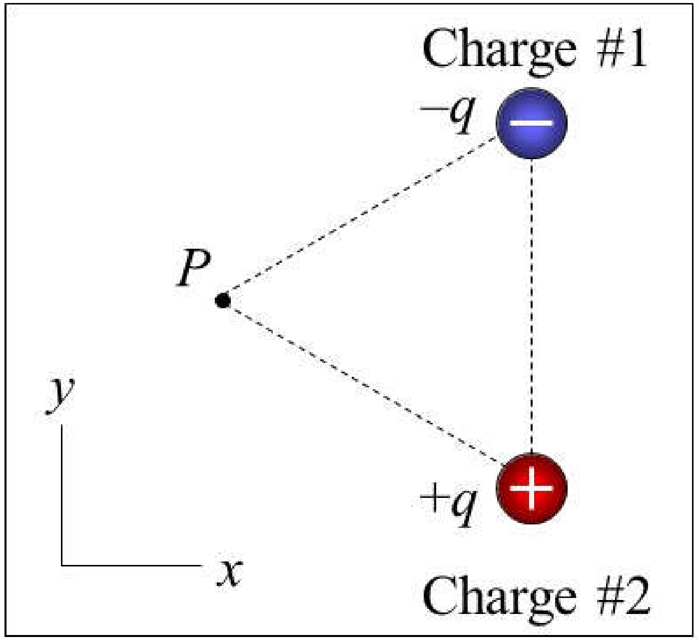


A. the $+x$-direction.

B. the $-x$-direction.

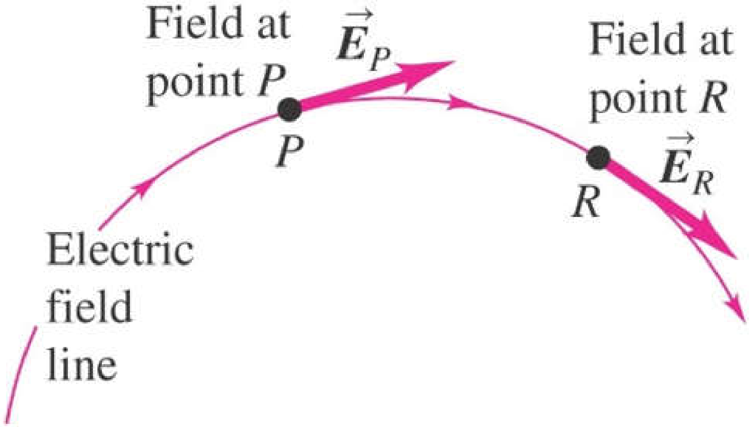
[✓ C. the $+y$-direction.]

D. the $-y$-direction.

E. none of the above.

Electric Field Lines

An elegant idea to visualize the field by Michael Faraday

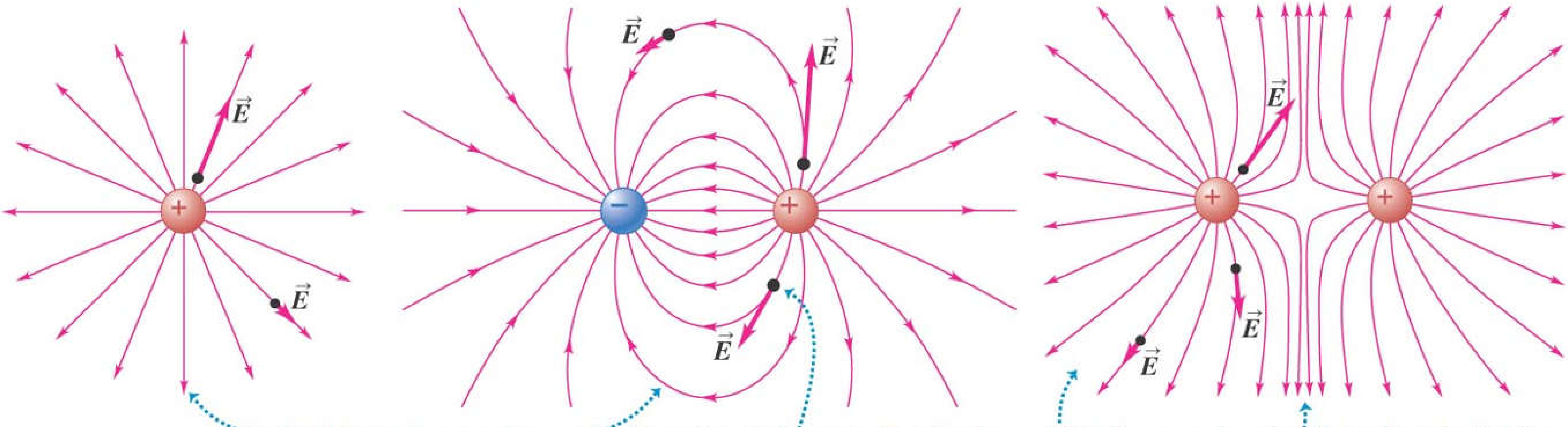


Imaginary lines whose tangent gives the direction of the electric field at that point.

They must:

1. Start from a positive charge
2. End at a negative charge
3. Repel (not cross) each other

Some examples:


(a) A single positive charge

(b) Two equal and opposite charges (a dipole)

(c) Two equal positive charges

Field lines always point *away from* (+) charges and *toward* (−) charges.

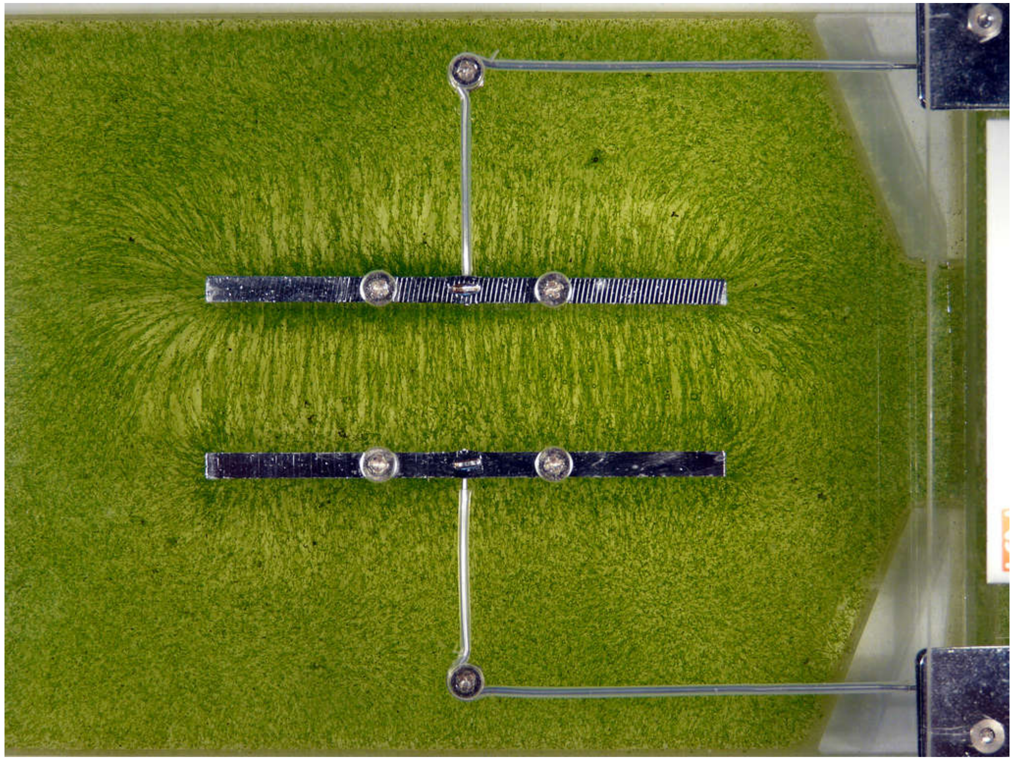
At each point in space, the electric field vector is *tangent* to the field line passing through that point.

Field lines are close together where the field is strong, farther apart where it is weaker.

- ⚠ larger density of field lines indicates strong field
- ⚠ field line is NOT trajectory of a test charge, because they indicate the direction of acceleration, not velocity

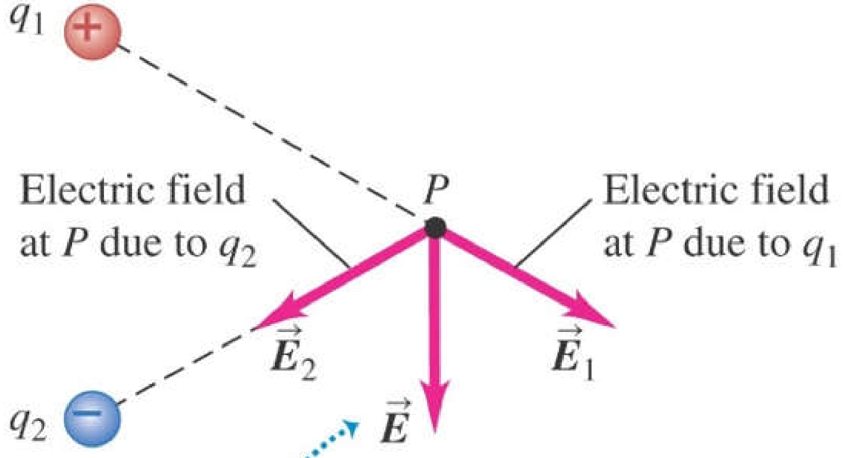
Demonstration

Electric field apparatus



Superposition of Electric Fields

Total field is the vector sum of fields due to individual point charges



The total electric field $\vec{E}$ at point P is the vector sum of $\vec{E}_1$ and $\vec{E}_2$.

$$\vec{\boldsymbol{F}}_0 = \vec{\boldsymbol{F}}_1 + \vec{\boldsymbol{F}}_2 + \vec{\boldsymbol{F}}_3 + \cdots$$

$$= q_0\vec{\boldsymbol{E}}_1 + q_0\vec{\boldsymbol{E}}_2 + q_0\vec{\boldsymbol{E}}_3 + \cdots$$

$$\vec{\boldsymbol{E}} = \frac{\vec{\boldsymbol{F}}_0}{q_0} = \vec{\boldsymbol{E}}_1 + \vec{\boldsymbol{E}}_2 + \vec{\boldsymbol{E}}_3 + \cdots$$

Example:

Example 21.8 Field of an electric dipole

Point charges $q_1 = +12$ nC and $q_2 = -12$ nC are 0.100 m apart (Fig. 21.22). (Such pairs of point charges with equal magnitude and opposite sign are called *electric dipoles*.) Compute the electric field caused by q_1, the field caused by q_2, and the total field (a) at point a; (b) at point b; and (c) at point c.

SOLUTION

IDENTIFY and SET UP: We must find the total electric field at various points due to two point charges. We use the principle of superposition: $\vec{E} = \vec{E}_1 + \vec{E}_2$. Figure 21.22 shows the coordinate system and the locations of the field points a, b, and c.

EXECUTE: At each field point, $\vec{E}$ depends on $\vec{E}_1$ and $\vec{E}_2$ there; we first calculate the magnitudes E_1 and E_2 at each field point. At a the magnitude of the field $\vec{E}_{1a}$ caused by q_1 is

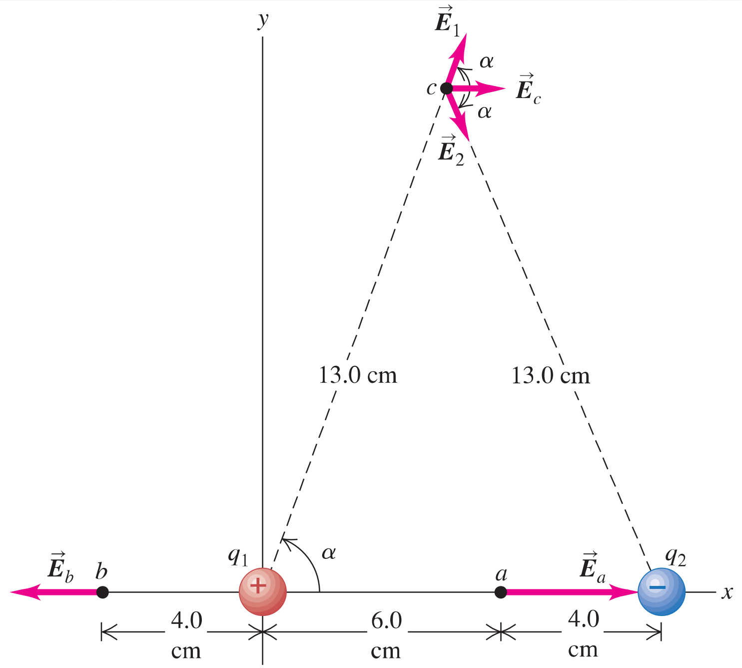
$$E_{1a} = \frac{1}{4\pi\epsilon_0}\frac{|q_1|}{r^2} = (9.0 \times 10^9\ \mathrm{N \cdot m^2/C^2})\frac{12 \times 10^{-9}\ \mathrm{C}}{(0.060\ \mathrm{m})^2}$$
$$= 3.0 \times 10^4\ \mathrm{N/C}$$

We calculate the other field magnitudes in a similar way. The results are

$$E_{1a} = 3.0 \times 10^4\ \mathrm{N/C} \qquad E_{1b} = 6.8 \times 10^4\ \mathrm{N/C}$$
$$E_{1c} = 6.39 \times 10^3\ \mathrm{N/C}$$
$$E_{2a} = 6.8 \times 10^4\ \mathrm{N/C} \qquad E_{2b} = 0.55 \times 10^4\ \mathrm{N/C}$$
$$E_{2c} = E_{1c} = 6.39 \times 10^3\ \mathrm{N/C}$$

The *directions* of the corresponding fields are in all cases *away* from the positive charge q_1 and *toward* the negative charge q_2.

21.22 Electric field at three points, a, b, and c, set up by charges q_1 and q_2, which form an electric dipole.



(a) At a, $\vec{E}_{1a}$ and $\vec{E}_{2a}$ are both directed to the right, so

$$\vec{E}_a = E_{1a}\hat{\imath} + E_{2a}\hat{\imath} = (9.8 \times 10^4\ \mathrm{N/C})\hat{\imath}$$

(b) At b, $\vec{E}_{1b}$ is directed to the left and $\vec{E}_{2b}$ is directed to the right, so

$$\vec{E}_b = -E_{1b}\hat{\imath} + E_{2b}\hat{\imath} = (-6.2 \times 10^4\ \mathrm{N/C})\hat{\imath}$$

(c) Figure 21.22 shows the directions of $\vec{E}_1$ and $\vec{E}_2$ at c. Both vectors have the same x-component:

$$E_{1cx} = E_{2cx} = E_{1c}\cos\alpha = (6.39 \times 10^3\ \text{N/C})\left(\tfrac{5}{13}\right)$$
$$= 2.46 \times 10^3\ \text{N/C}$$

From symmetry, E_{1y} and E_{2y} are equal and opposite, so their sum is zero. Hence

$$\vec{E}_c = 2(2.46 \times 10^3\ \text{N/C})\hat{\imath} = (4.9 \times 10^3\ \text{N/C})\hat{\imath}$$

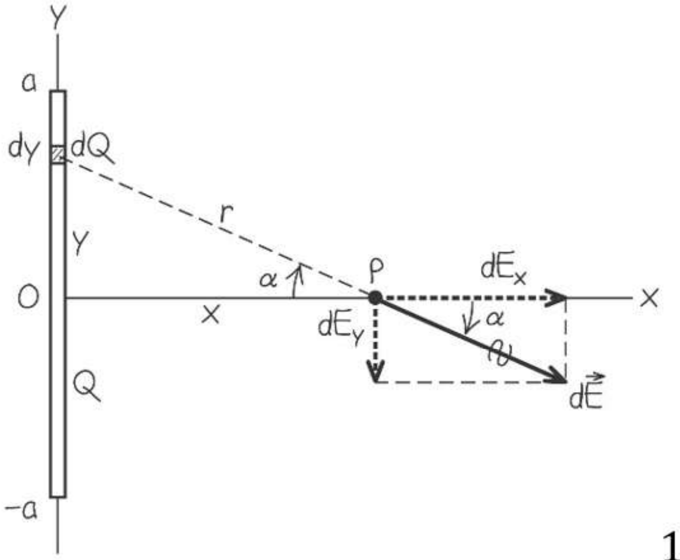
EVALUATE: We can also find $\vec{E}_c$ using Eq. (21.7) for the field of a point charge. The displacement vector $\vec{r}_1$ from q_1 to point c is $\vec{r}_1 = r\cos\alpha\hat{\imath} + r\sin\alpha\hat{\jmath}$. Hence the unit vector that points from q_1 to point c is $\hat{r}_1 = \vec{r}_1/r = \cos\alpha\hat{\imath} + \sin\alpha\hat{\jmath}$. By symmetry, the unit vector that points from q_2 to point c has the opposite x-component but the same y-component: $\hat{r}_2 = -\cos\alpha\hat{\imath} + \sin\alpha\hat{\jmath}$. We can now use Eq. (21.7) to write the fields $\vec{E}_{1c}$ and $\vec{E}_{2c}$ at c in vector form, then find their sum. Since $q_2 = -q_1$ and the distance r to c is the same for both charges,

$$\vec{E}_c = \vec{E}_{1c} + \vec{E}_{2c} = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{r}_1 + \frac{1}{4\pi\epsilon_0}\frac{q_2}{r^2}\hat{r}_2$$
$$= \frac{1}{4\pi\epsilon_0 r^2}(q_1\hat{r}_1 + q_2\hat{r}_2) = \frac{q_1}{4\pi\epsilon_0 r^2}(\hat{r}_1 - \hat{r}_2)$$
$$= \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}(2\cos\alpha\hat{\imath})$$
$$= 2(9.0 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)\frac{12 \times 10^{-9}\ \text{C}}{(0.13\ \text{m})^2}\left(\tfrac{5}{13}\right)\hat{\imath}$$
$$= (4.9 \times 10^3\ \text{N/C})\hat{\imath}$$

This is the same as we calculated in part (c).

Field of a charged line segment Example 21.10

A line segment $2a$ with charge Q distributed uniformly throughout. Divide it into an infinite number of small segments dy, each like a point charge.



Linear charge density (charge per unit length) is $\lambda = Q/2a$

dy carries a charge $dQ = \lambda dy$, creating a field at P

$$dE = \frac{1}{4\pi\epsilon_0}\frac{\lambda dy}{r^2}$$

$$dE_x = dE\cos\alpha = \frac{1}{4\pi\epsilon_0}\frac{\lambda dy}{r^2}\frac{x}{r} = \frac{1}{4\pi\epsilon_0}\frac{Q}{2a}\frac{x}{(x^2+y^2)^{3/2}}dy$$

$$E_x = \frac{1}{4\pi\epsilon_0}\frac{Qx}{2a}\underbrace{\int_{-a}^{a}\frac{dy}{(x^2+y^2)^{3/2}}}_{} = \frac{1}{4\pi\epsilon_0}\frac{Q}{x\sqrt{x^2+a^2}}$$

$= 2a/x^2\sqrt{x^2+a^2}$, check integral table of Wolfram Alpha

By symmetry, $E_y = 0$, therefore

$$\boxed{\vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0}\frac{Q}{x\sqrt{x^2+a^2}}\,\hat{\boldsymbol{\imath}}}$$

⚠ For $x \gg a$, the segment behaves like a point charge, $E = \frac{1}{4\pi\epsilon_0}\frac{Q}{x^2}$

⚠ For $x \ll a$ (infinitely long segment),

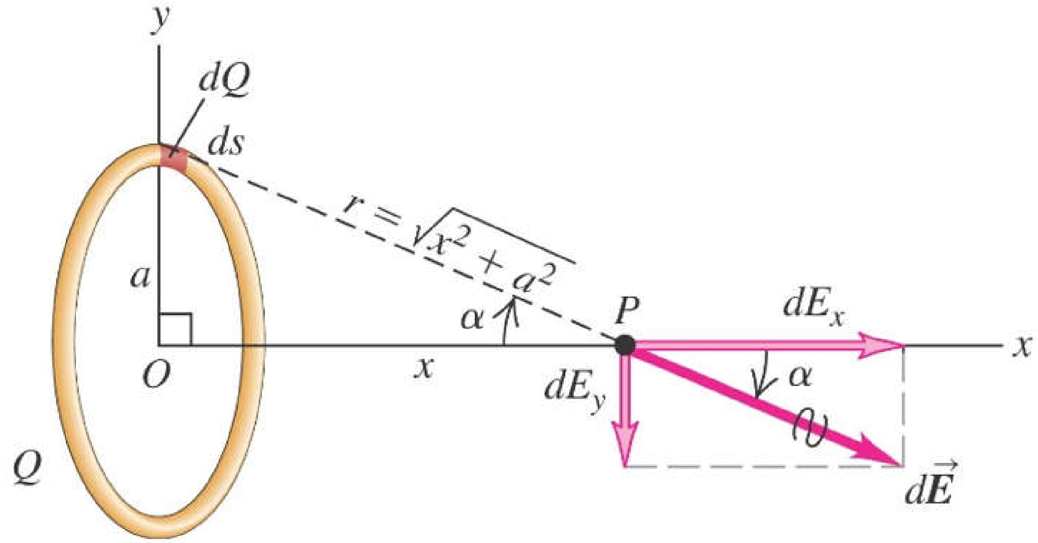
$$\boxed{E = \frac{1}{4\pi\epsilon_0}\frac{Q/a}{x} = \frac{\lambda}{2\pi\epsilon_0 x}}$$

decreases as $1/x$, as oppose to $1/x^2$ for point charge

Question: if the segment has charge $+Q$ uniformly distributed from 0 to a, and $-Q$ from 0 to $-a$, the field at *P* will be (in +x direction / in –x direction / in +y direction / in –y direction / zero).

Field due to a ring of charge along its axis Example 21.9

A ring with charge Q distributed uniformly around it. Divide into an infinite number of small arc segments ds, each like a point charge at the same distance away from P



Linear charge density (charge per unit length) is $\lambda = Q/2\pi a$

A small arc segment ds carries a charge $dQ = \lambda ds$, creating a field at P

$$dE = \frac{1}{4\pi\epsilon_0}\frac{\lambda ds}{r^2}$$

$$dE_x = dE\cos\alpha = \frac{1}{4\pi\epsilon_0}\frac{\lambda ds}{r^2}\frac{x}{r} = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi a}\frac{x}{(x^2+a^2)^{3/2}}ds$$

$$E_x = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi a}\frac{x}{(x^2+a^2)^{3/2}}\underbrace{\int ds}_{2\pi a} = \frac{1}{4\pi\epsilon_0}\frac{Qx}{(x^2+a^2)^{3/2}}$$

By symmetry, $E_y = 0$, therefore

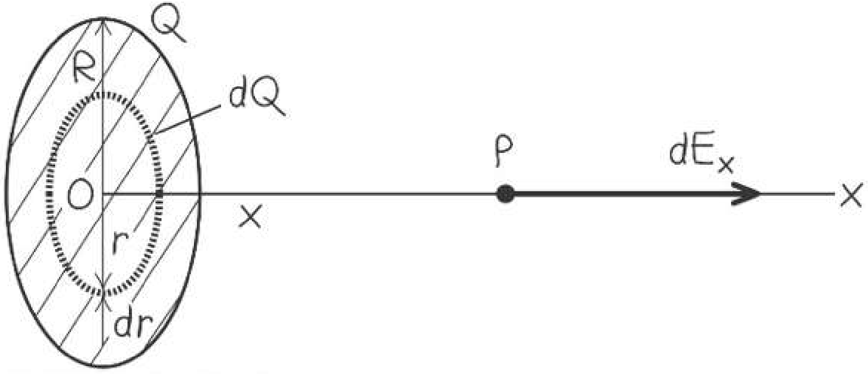
$$\boxed{\vec{\boldsymbol{E}} = \frac{1}{4\pi\epsilon_0}\frac{Qx}{(x^2+a^2)^{3/2}}\,\hat{\boldsymbol{\imath}}}$$

⚠ At the center of the ring, $x = 0$, and $\vec{\boldsymbol{E}} = 0$ by symmetry

⚠ For $x \gg a$, the ring behaves like a point charge, $E = \frac{1}{4\pi\epsilon_0}\frac{Q}{x^2}$

Field of a uniformly charged disk Example 21.11

Total charge *Q*, **surface charge density** (charge per unit area) is $\sigma = Q/\pi R^2$. Divide into an infinite number of small rings, each with radius *r* and width *dr*, charge $dQ = 2\pi r\sigma dr$



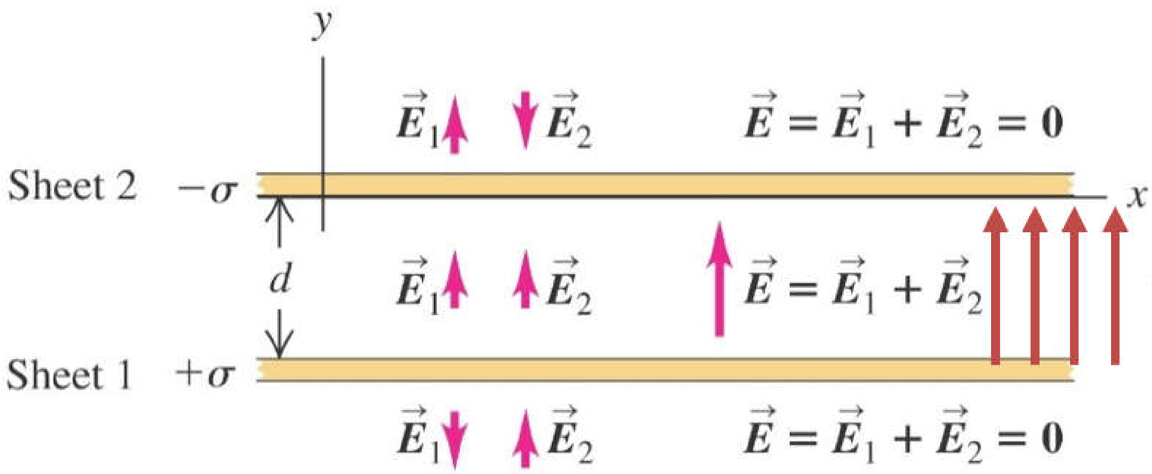
Field due to an infinitesimal ring

$$dE = \frac{1}{4\pi\epsilon_0}\frac{x(2\pi r\sigma dr)}{(x^2+r^2)^{3/2}}$$

$$E = \frac{\sigma x}{4\epsilon_0}\int_0^R \frac{2rdr}{(x^2+r^2)^{3/2}} = \frac{\sigma}{2\epsilon_0}\left[1 - \frac{1}{\sqrt{(R^2/x^2)+1}}\right]$$

If the plate is infinitely large, $R \to \infty$, $\boxed{E = \sigma/2\epsilon_0}$, ⚠ independent of distance!

Field of two oppositely charged infinite sheets Example 21.12

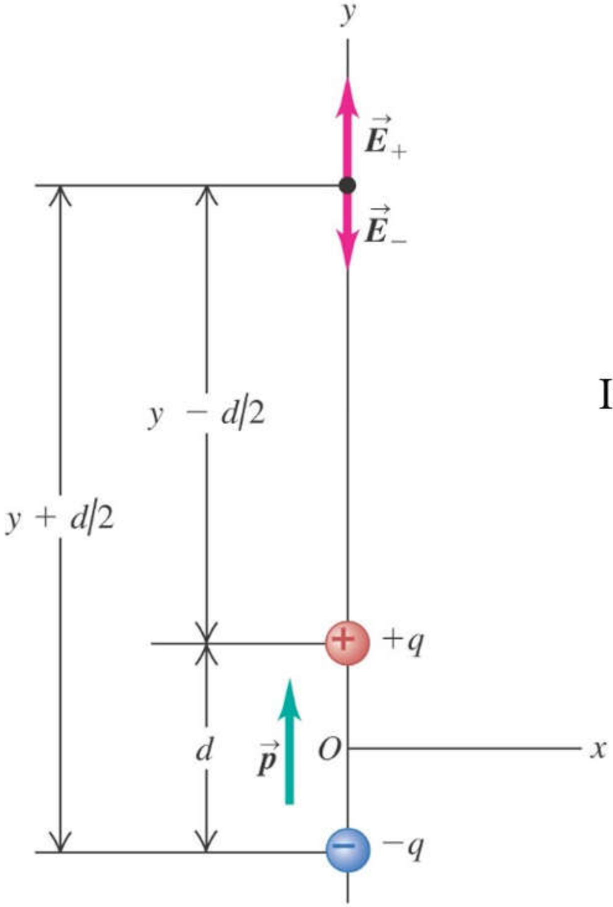


← $E = \sigma/\epsilon_0$, field lines uniform

Field of an electric dipole Example 21.14

An **electric dipole** is a pair of equal and opposite charges +q and –q at a fixed distance d apart.

Its **electric dipole moment $\vec{p}$** is defined as $p = qd$ and points from –q to +q



$$E_y = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{(y-d/2)^2} + \frac{-q}{(y+d/2)^2}\right]$$

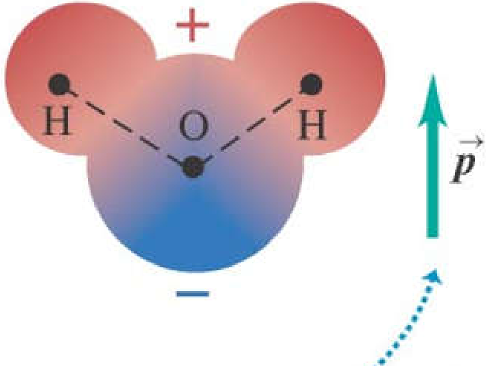
$$= \frac{1}{4\pi\epsilon_0}\frac{q}{y^2}\left[\underbrace{\left(1-\frac{d}{2y}\right)^{-2}} - \underbrace{\left(1+\frac{d}{2y}\right)^{-2}}\right]$$

$$\text{If } y \gg d \longrightarrow \underbrace{\cong 1+\frac{d}{y} \qquad \cong 1-\frac{d}{y}}_{(1+x)^n = 1+nx+\frac{n(n-1)}{2}x^2+\cdots}$$

$$\therefore \boxed{E_y \cong \frac{1}{4\pi\epsilon_0}\frac{q}{y^2}\frac{2d}{y} = \frac{1}{2\pi\epsilon_0}\frac{p}{y^3}}$$

⚠ decay as $1/y^3$ when far away

A water molecule behaves like an electric dipole

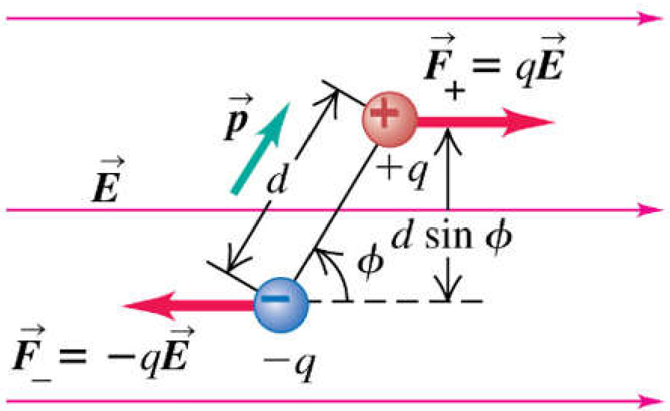


The electric dipole moment $\vec{\boldsymbol{p}}$ is directed from the negative end to the positive end of the molecule.

What if you put an electric dipole in a uniform electric field?

(How to achieve a uniform electric field?)

Define $\vec{\boldsymbol{d}}$ from $-q$ to $+q$, i.e., $\vec{\boldsymbol{p}} = q\vec{\boldsymbol{d}}$



Total force $= \vec{\boldsymbol{F}}_+ + \vec{\boldsymbol{F}}_- = 0$, center of mass does not move

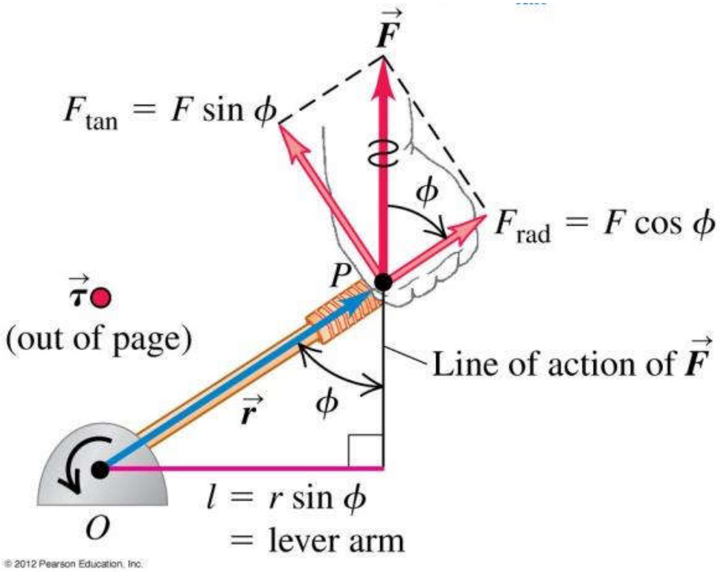
But tends to rotate, i.e., torque is non-zero

$$\vec{\boldsymbol{\tau}} = \frac{\vec{\boldsymbol{d}}}{2} \times \vec{\boldsymbol{F}}_+ + \frac{-\vec{\boldsymbol{d}}}{2} \times \vec{\boldsymbol{F}}_-$$

$$= q\vec{\boldsymbol{d}} \times \vec{\boldsymbol{E}}$$

i.e. $\boxed{\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{p}} \times \vec{\boldsymbol{E}}}$ torque on a dipole

Recall:



Define **torque** about a point O as a vector

$$\boxed{\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}}$$

⚠ $\boldsymbol{\tau}$ is ⊥to both $\boldsymbol{r}$ and $\boldsymbol{F}$

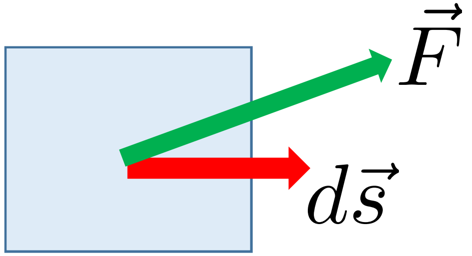
Magnitude: $$\tau = r\underbrace{F\sin\phi}_{\text{component of } \boldsymbol{F} \ \perp\text{to } \boldsymbol{r}} = \underbrace{r\sin\phi}_{\perp\text{distance from } O \text{ to line of actions of } \boldsymbol{F}} F$$

Direction gives the sense of rotation about O through the right-hand-rule.

Notation: ⊙ out of the plane into the plane

SI unit for torque: Nm (just like work done)

Work Done $\boxed{dW = \vec{F} \cdot d\vec{s}}$



Work done <u>by</u> the electric field <u>on</u> the dipole to turn it through angle $d\vec{\boldsymbol{\phi}}$ (still remember how to define the direction of $d\vec{\boldsymbol{\phi}}$?)

$$dW = \vec{\boldsymbol{\tau}} \cdot d\vec{\boldsymbol{\phi}} = -pE \sin\phi \, d\phi$$

$$W = -\int_{\phi_1}^{\phi_2} pE \sin\phi \, d\phi = pE\cos\phi_2 - pE\cos\phi_1$$

Since $W = -\Delta U$ (work done by electric field = -ve of change of potential), define the potential energy of a dipole in an electric field as $U = -pE\cos\phi$, or

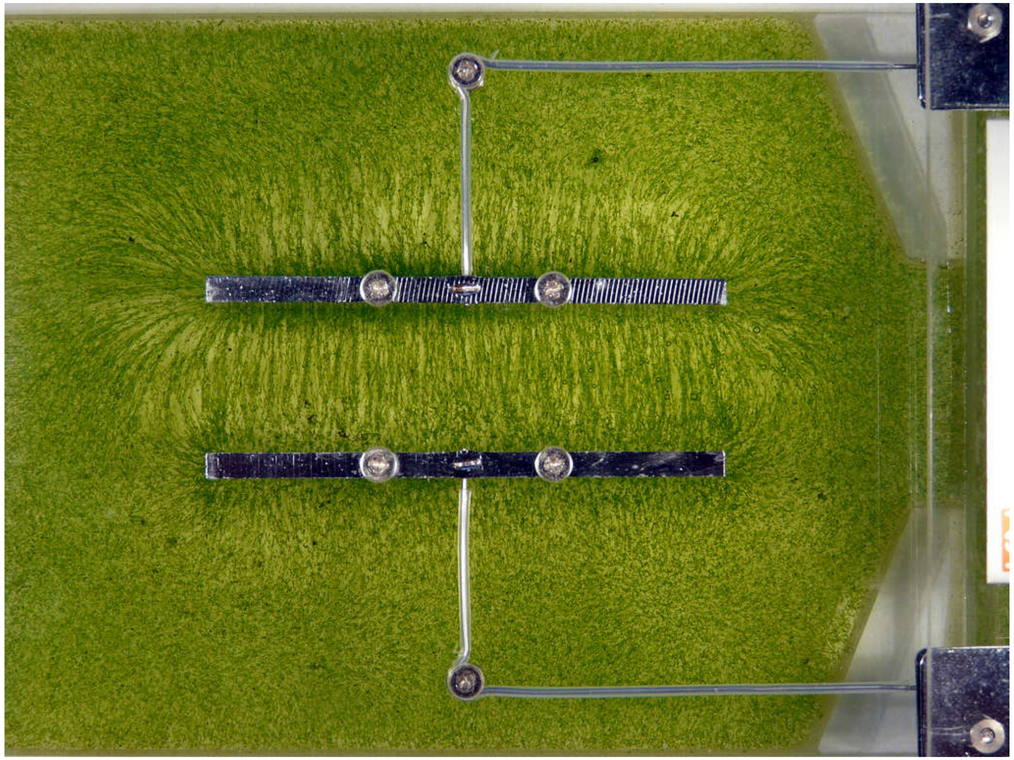
$$\boxed{U = -\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{E}}}$$

⚠ $\vec{\boldsymbol{p}}$ align with (parallel to) $\vec{\boldsymbol{E}}$ to minimize potential energy, and $\vec{\boldsymbol{\tau}} = 0$

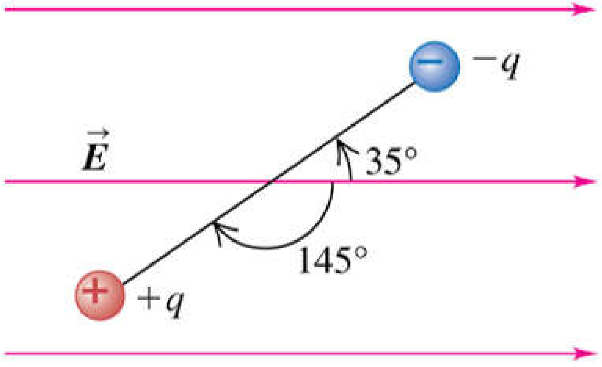
Question: if the dipole moment $\vec{\boldsymbol{p}}$ is opposite (anti-parallel) to $\vec{\boldsymbol{E}}$, then the dipole is (in stable equilibrium / in unstable equilibrium / not in equilibrium).

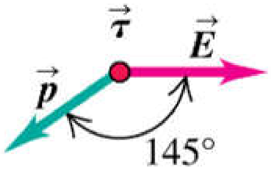
Demonstration

[Electric field apparatus]



Example 21.13: $q = 1.6 \times 10^{-19}$ C, $d = 0.125 \times 10^{-9}$ m, $E = 5.0 \times 10^{5}$ N/C





$$p = qd = (1.6 \times 10^{-19}\ \mathrm{C})(0.125 \times 10^{-9}\ \mathrm{m}) = 2.0 \times 10^{-29}\ \mathrm{Cm}$$

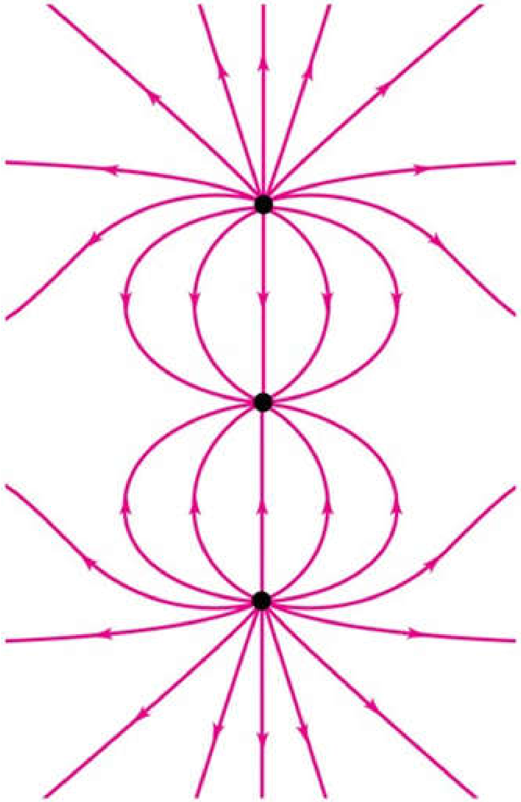
$$\tau = pE \sin\phi = (2.0 \times 10^{-29}\ \mathrm{Cm})(5.0 \times 10^{5}\ \mathrm{N/C}) \sin 145° = 5.7 \times 10^{-24}\ \mathrm{Nm}$$

$$U = -pE \cos\phi = -(2.0 \times 10^{-29}\ \mathrm{Cm})(5.0 \times 10^{5}\ \mathrm{N/C}) \cos 145° = 8.2 \times 10^{-24}\ \mathrm{J}$$

Clicker Questions

Q21.9

The illustration shows the electric field lines due to three point charges (shown by the black dots). The electric field is strongest



A. where adjacent field lines are closest together.

B. where adjacent field lines are farthest apart.

C. where adjacent field lines are parallel.

D. where the field lines are most strongly curved.

E. at none of the above locations.

A21.9

The illustration shows the electric field lines due to three point charges (shown by the black dots). The electric field is strongest

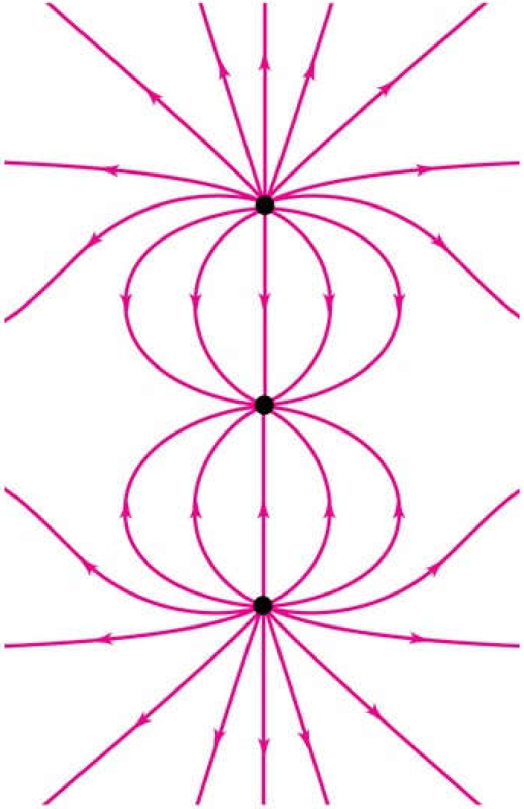
[✓ A. where adjacent field lines are closest together.]

B. where adjacent field lines are farthest apart.

C. where adjacent field lines are parallel.

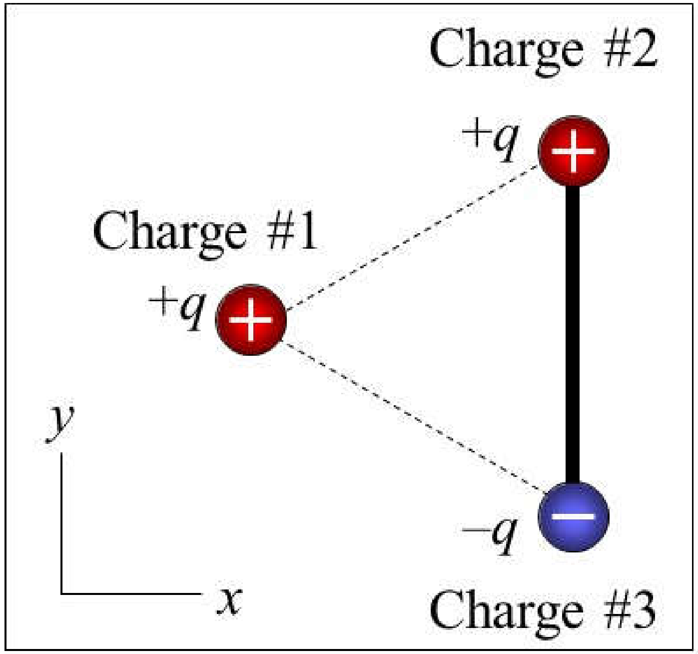
D. where the field lines are most strongly curved.

E. at none of the above locations.



Q21.11

Three point charges lie at the vertices of an equilateral triangle as shown. Charges #2 and #3 make up an electric dipole. The net electric *torque* that charge #1 exerts on the dipole is



A. clockwise.

B. counterclockwise.

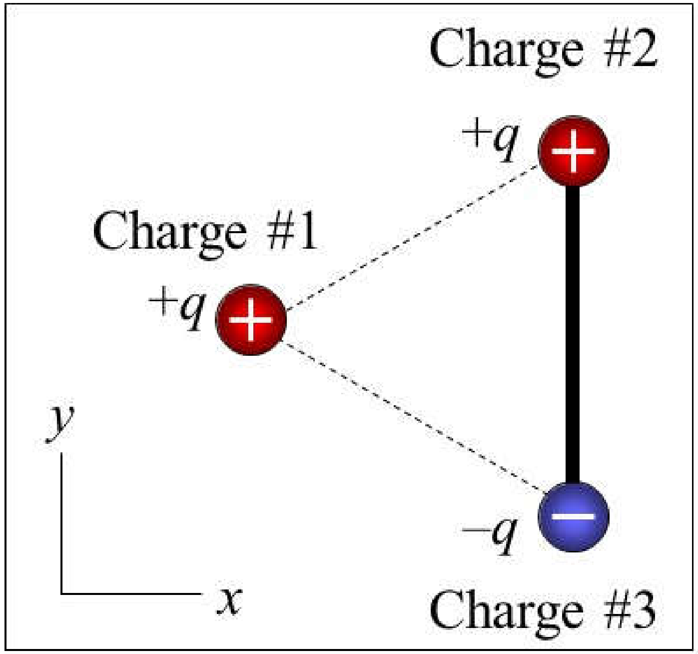
C. zero.

D. either A or B.

E. any of A, B, or C.

A21.11

Three point charges lie at the vertices of an equilateral triangle as shown. Charges #2 and #3 make up an electric dipole. The net electric *torque* that charge #1 exerts on the dipole is



[✓ A. clockwise.]

B. counterclockwise.

C. zero.

D. either A or B.

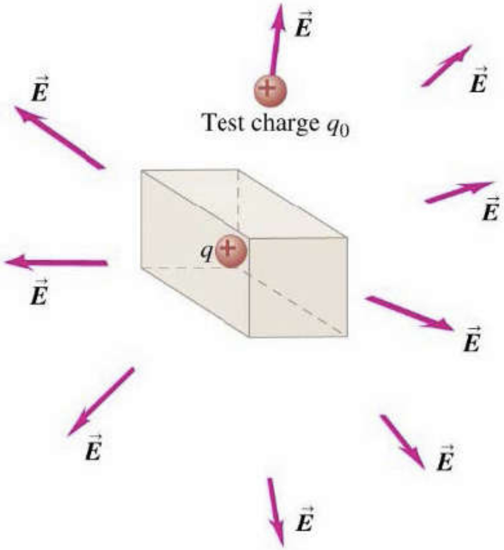
E. any of A, B, or C.

Gauss law

Know charge distribution → know electric field (by Coulomb's law)

Inverse problem: know electric field → know charge distribution? In principle yes, by mapping out the field in 3D space using a test charge

(a) A box containing an unknown amount of charge



(b) Using a test charge outside the box to probe the amount of charge inside the box

But tedious to map out the field in 3D

What if we know the field on the surfaces of the (imaginary) box only? Consider electric field lines flowing into and out of the box, called **electric flux** ("flux" meaning "flow", just like a fluid)

Where's the source/sink ? Follow the flow!

We only see water flowing out

The source of this fountain must be here!

Where’s the source/sink ? Follow the flow!

Water flowing in = water flowing out

There is NO source inside the box!

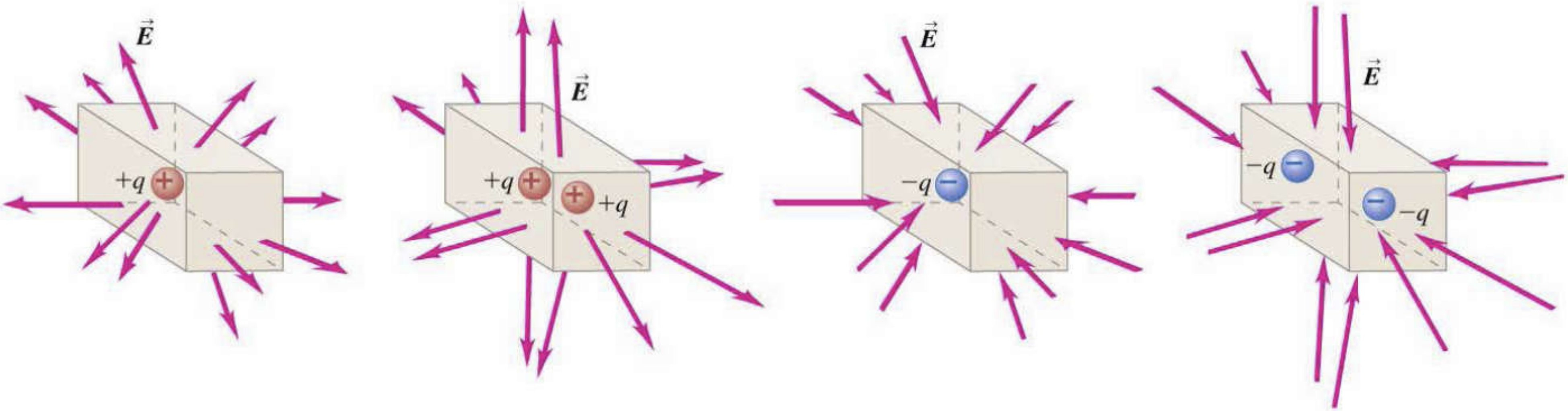
Where’s the source/sink ? Follow the flow!

We only see water flowing in

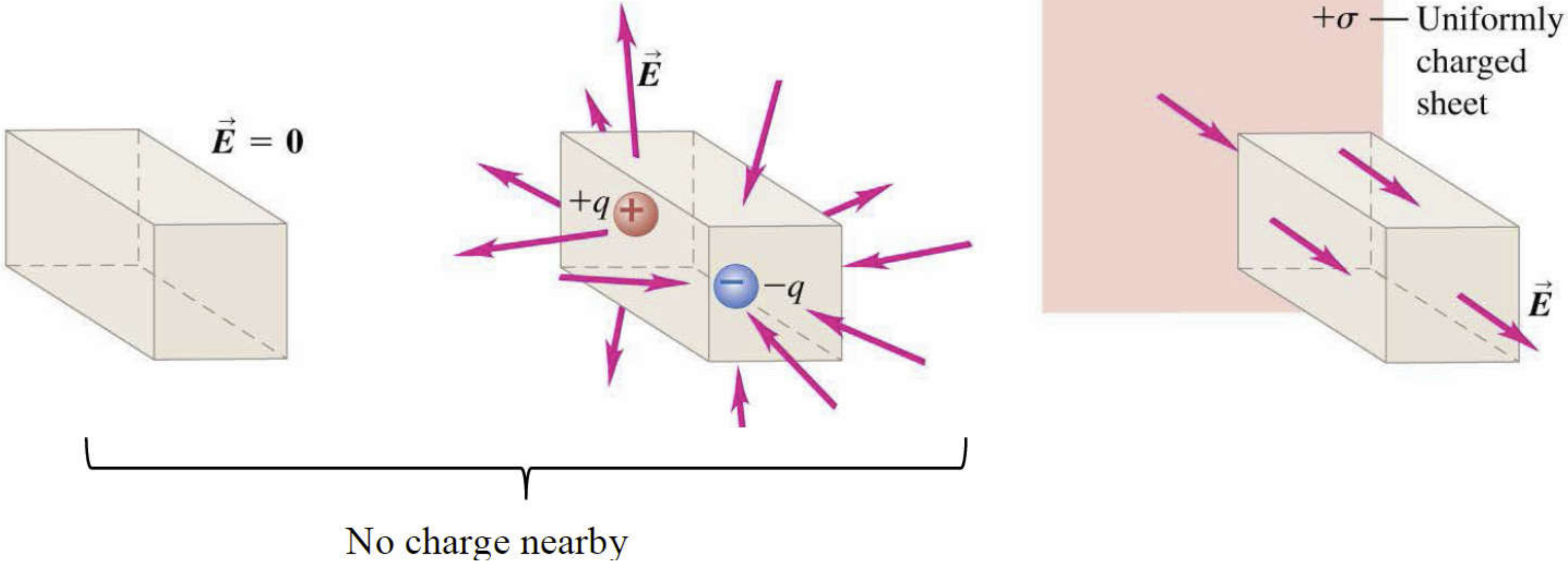
The sink of this fountain must be here!

Water flow <-> electric flux
Water source <-> positive charge
Water sink <-> negative charge

With a net charge inside box, a net electric flux flow in/out

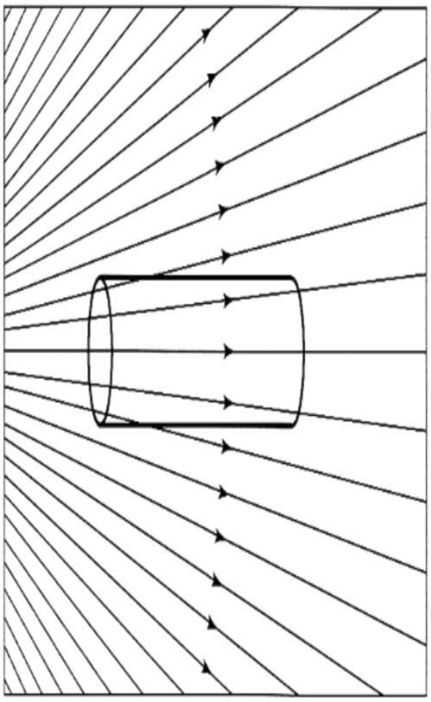


With no net charge, electric flux flowing in and out cancels, and net flux is zero

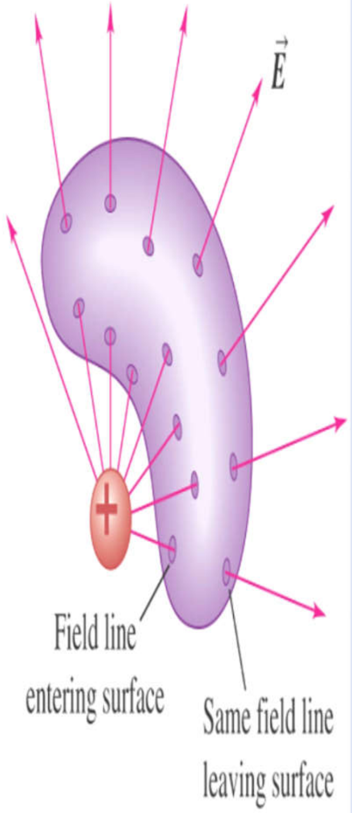


Clicker question:

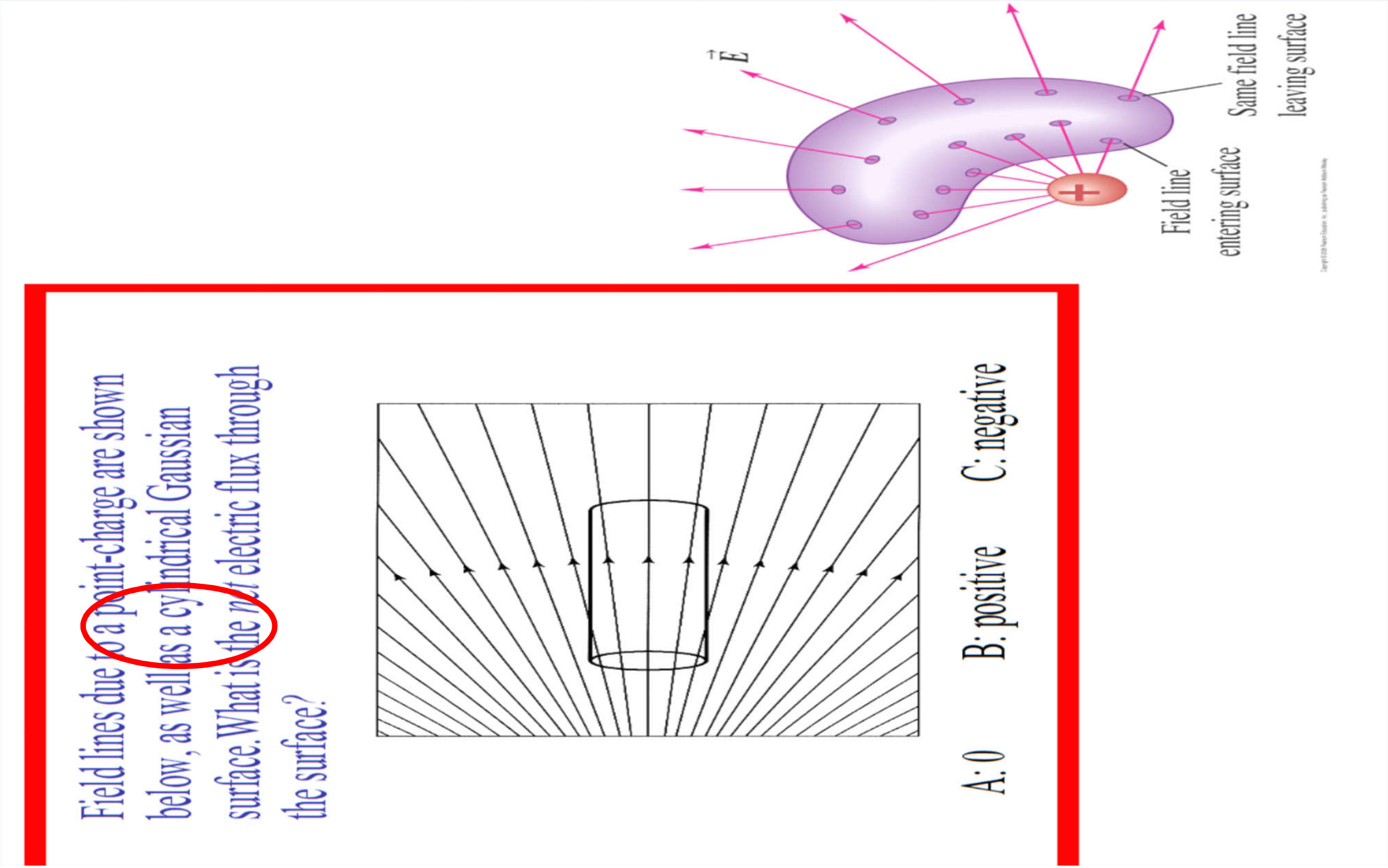
Field lines due to a point-charge are shown below, as well as a cylindrical Gaussian surface. What is the *net* electric flux through the surface?



A: 0 B: positive C: negative



Clicker question:



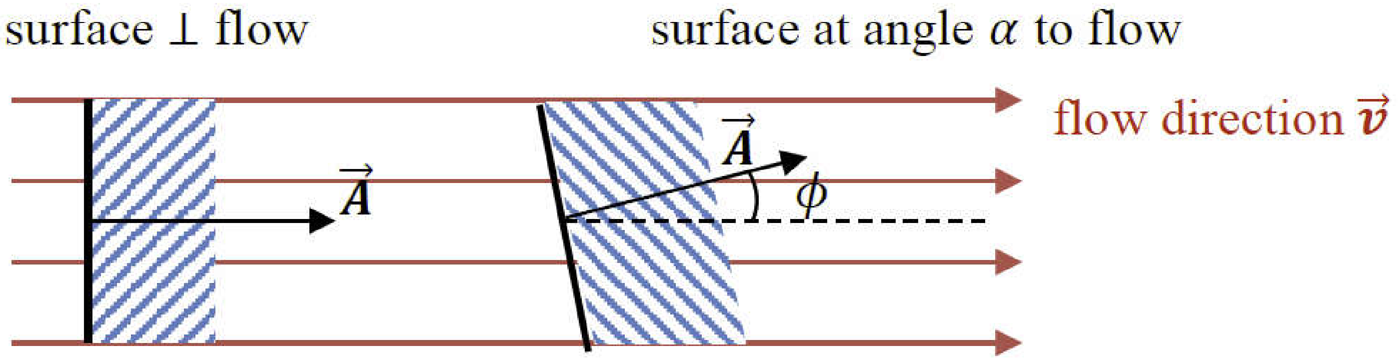
Field lines due to a point-charge are shown below, as well as a cylindrical Gaussian surface. What is the *net* electric flux through the surface?

A: 0 B: positive C: negative

To summarize:

1. A net inward/outward electric flux through a close surface means that the charge enclosed is negative/positive.
2. Charges outside the close surface give no net flux through the surface.
3. Net flux $\propto$ charge enclosed, but independent of size of closed surface (just like a fluid).

To properly define the electric flux through a surface, draw analogy with fluid flow:



volume flow through after dt is $A(vdt)$

volume flow through after dt is $A(vdt)\cos\phi = \vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{v}}\, dt$

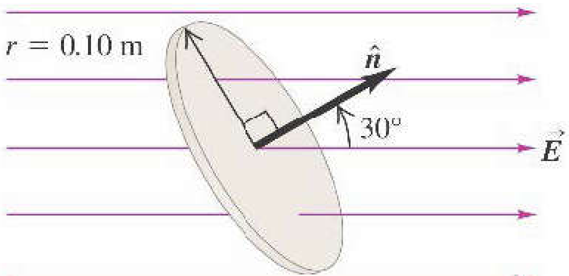
Define area vector $\vec{\boldsymbol{A}} = A\hat{\boldsymbol{n}}$, where $\hat{\boldsymbol{n}}$ is the outward (may be ambiguous) normal unit vector

Electric flux due to a uniform field through a flat surface is defines as

$$\boxed{\Phi_E = \vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{A}} = EA\cos\phi}$$

SI unit is Nm^2/C

Example 22.1 Electric flux through a disk

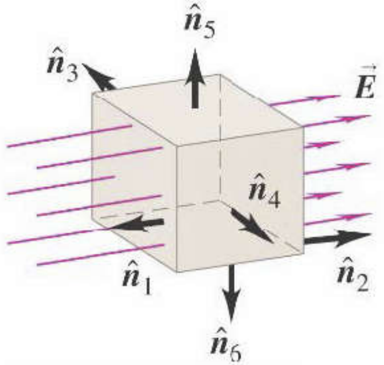


$E = 2.0 \times 10^3 \mathrm{N/C}$, electric flux through the disk

$$\Phi_E = EA\cos\phi = (2.0 \times 10^3 \mathrm{N/C})\pi(0.10\mathrm{m})^2 \cos 30° \\ = 54\ \mathrm{Nm^2/C}$$

Example 22.2 Electric flux through a cube

(a)



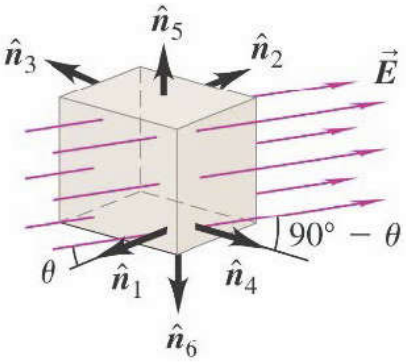
$$\Phi_{E1} = \vec{E} \cdot \hat{n}_1 A = EL^2 \cos 180° = -EL^2$$

$$\Phi_{E2} = \vec{E} \cdot \hat{n}_2 A = EL^2 \cos 0° = +EL^2$$

$$\Phi_{E3} = \Phi_{E4} = \Phi_{E5} = \Phi_{E6} = EL^2 \cos 90° = 0$$

$$\Phi_E = \Phi_{E1} + \Phi_{E2} + \Phi_{E3} + \Phi_{E4} + \Phi_{E5} + \Phi_{E6}$$
$$= -EL^2 + EL^2 + 0 + 0 + 0 + 0 = 0$$

(b)



$$\Phi_{E1} = \vec{E} \cdot \hat{n}_1 A = EL^2 \cos(180° - \theta) = -EL^2 \cos\theta$$

$$\Phi_{E2} = \vec{E} \cdot \hat{n}_2 A = +EL^2 \cos\theta$$

$$\Phi_{E3} = \vec{E} \cdot \hat{n}_3 A = EL^2 \cos(90° + \theta) = -EL^2 \sin\theta$$

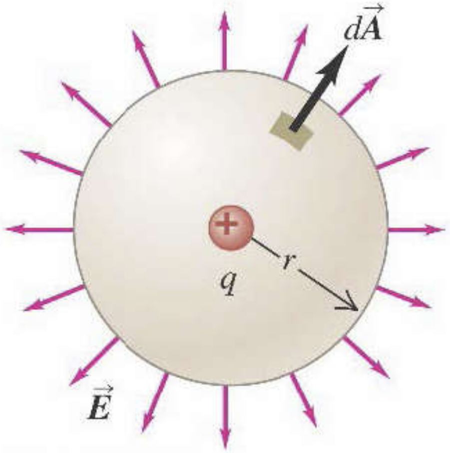
$$\Phi_{E4} = \vec{E} \cdot \hat{n}_4 A = EL^2 \cos(90° - \theta) = +EL^2 \sin\theta$$

$$\Phi_{E5} = \Phi_{E6} = EL^2 \cos 90° = 0$$

The total flux $\Phi_E = \Phi_{E1} + \Phi_{E2} + \Phi_{E3} + \Phi_{E4} + \Phi_{E5} + \Phi_{E6}$ through the surface of the cube is again zero.

⚠ no net charge enclosed leads to no net electric flux

Example 22.3 Electric flux through a sphere



Surface is not flat!!

Solution: break up surface into infinitesimal flat patches $d\vec{A}$

$$\boxed{\Phi_E = \underbrace{\sum \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}}}_{\text{sum over infinitesimal patches}} \rightarrow \underbrace{\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}}}_{\text{surface integral}}}$$

Anywhere on the surface of the sphere, $E = q/(4\pi\epsilon_0 r^2)$ and $\vec{\boldsymbol{E}} \parallel d\vec{\boldsymbol{A}}$, i.e., $\vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = EdA$

$$\Phi_E = \int \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \int EdA = E\int dA = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}4\pi r^2 = \frac{q}{\epsilon_0}$$

⚠ while E depends on r, Φ_E independent of size of close surface!

⚠ This can be generalized to any close surface and charge distribution!

Gauss's Law

The total electric flux through a closed surface (called a Gaussian surface) is equal to the net electric charge inside the surface, divided by ϵ_0

Gauss's law:

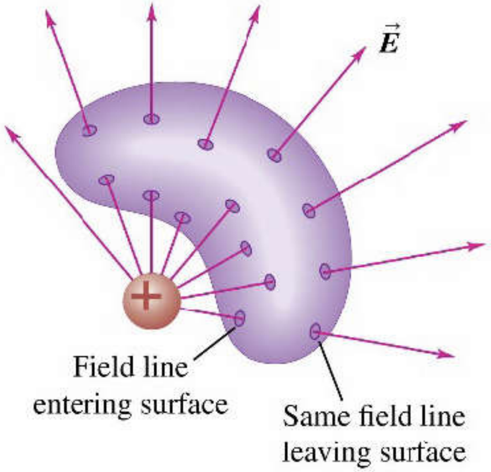
$$\Phi_E = \oint \underbrace{\vec{E} \cdot d\vec{A}} = \frac{Q_{\text{encl}}}{\epsilon_0} \qquad (22.8)$$

Total charge enclosed by surface

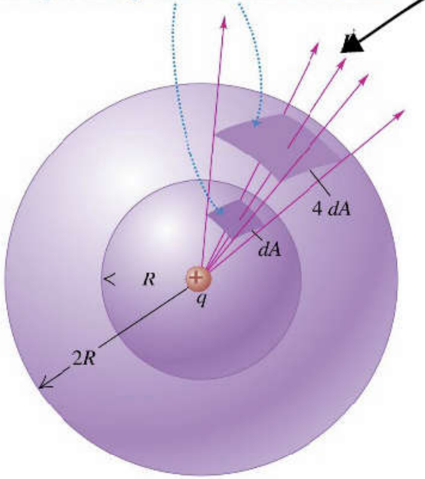
Electric flux through a closed surface of area A = surface integral of $\vec{E}$

Electric constant

How to make sense out of it?



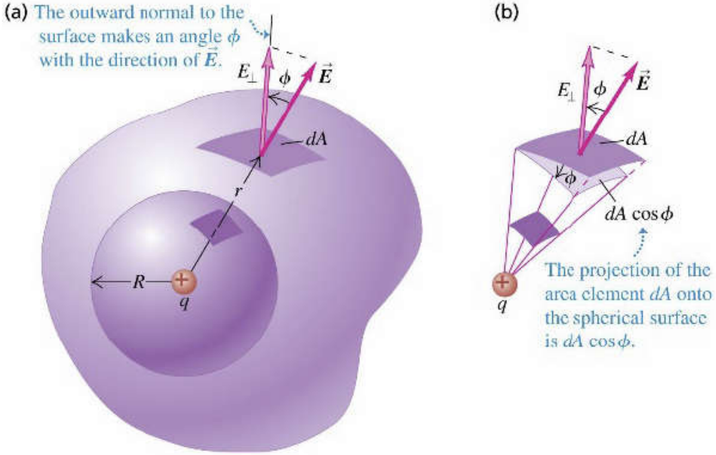
No charge enclosed, electric field lines cannot start/end inside it, flux in = flux out, no net flux



The same number of field lines and the same flux pass through both of these area elements.

$$\Phi_E = EA = \frac{1}{4\pi\epsilon_0}\overbrace{\frac{q}{R^2}(4\pi R^2)}^{R^2 \text{ cancels out, independent of size of surface}} = \frac{q}{\epsilon_0}$$

- ⚠ turns Coulomb's law (a physical law in terms of force) into a geometric law
- ⚠ $1/r^2$ in Coulomb's law is because of the dimensionality of the 3D space

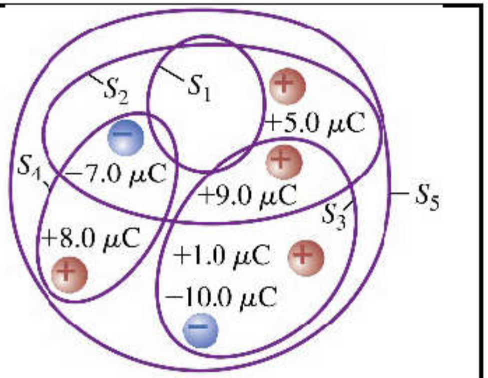


(a) The outward normal to the surface makes an angle ϕ with the direction of $\vec{E}$.

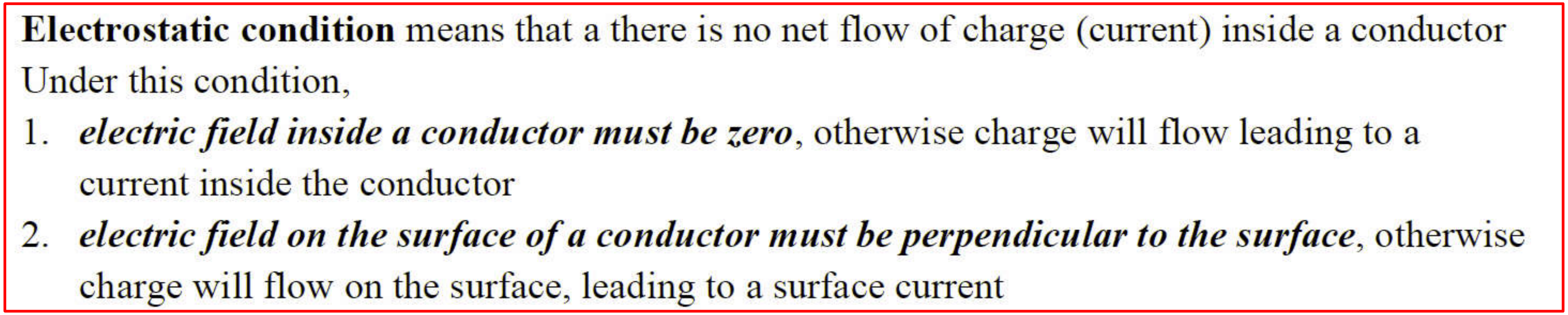
(b) The projection of the area element dA onto the spherical surface is $dA \cos\phi$.

If closed surface is not spherical, the projection $\vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}}$ makes sure that the effective tangential surface area is the same as a sphere

Question: rank the 5 Gaussian surfaces in increasing electric flux through them, from –ve to +ve



Conductor



Electrostatic condition means that a there is no net flow of charge (current) inside a conductor

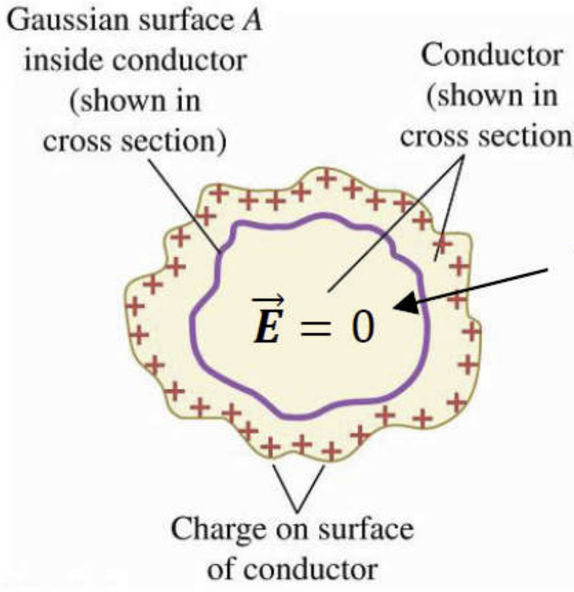
Under this condition,

1. ***electric field inside a conductor must be zero***, otherwise charge will flow leading to a current inside the conductor
2. ***electric field on the surface of a conductor must be perpendicular to the surface***, otherwise charge will flow on the surface, leading to a surface current

A consequence of Gauss's law – ***excess charge on a conductor resides entirely on its surface***

Choose a Gaussian surface inside the conductor but arbitrarily close to surface, from Gauss's law

$$\Phi_E = \oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = Q_{\mathrm{encl}}/\epsilon_0$$

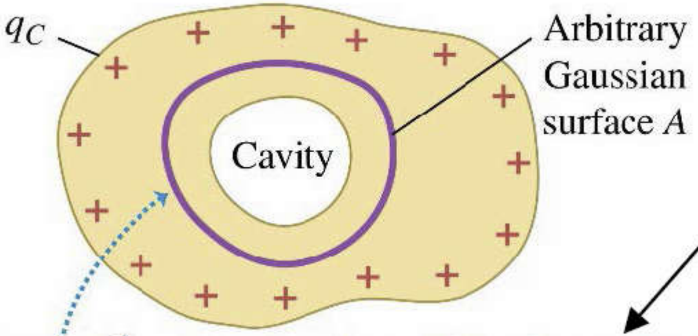


Anywhere inside a conductor, $\vec{\boldsymbol{E}} = 0$ otherwise its electrons will flow

$\vec{\boldsymbol{E}} = 0 \;\Rightarrow\; \Phi_E = 0 \;\Rightarrow\; Q_{\mathrm{encl}} = 0$, enclose no net charge!

All excess charge must reside on the surface

What if the conductor is hollow?



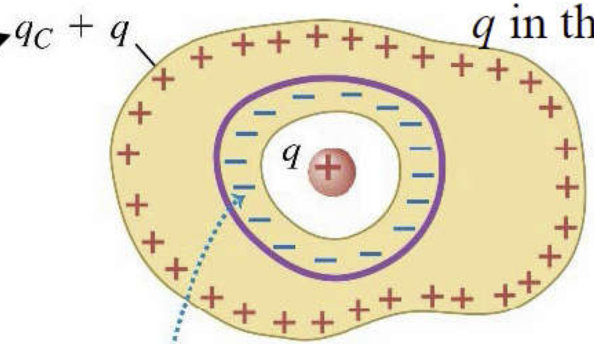
Because $\vec{E} = \mathbf{0}$ at all points within the conductor, the electric field at all points on the Gaussian surface must be zero.

Excess charge still resides on the outer surface

↓

Is it possible that equal and opposite charges reside on difference places on the inner surface? – Not possible (otherwise charge will flow) unless charge on the inner surface is induced by another charge q in the cavity

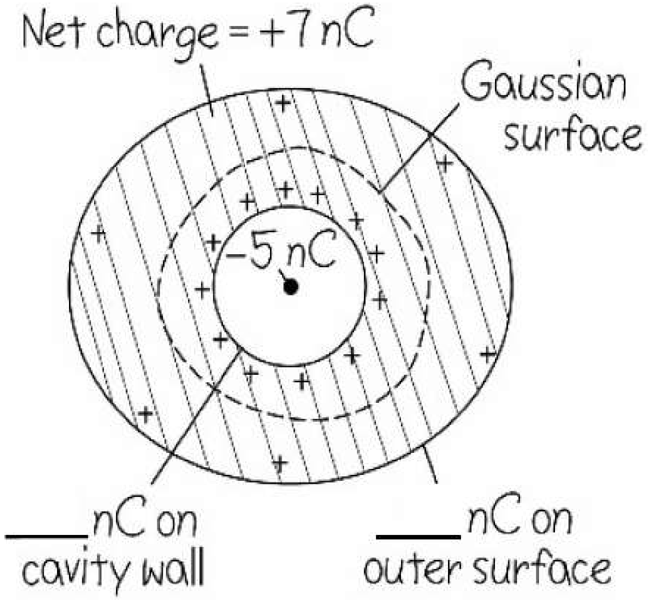
Excess charge + induced charge (can be of opposite sign) reside on the outer surface



For $\vec{E}$ to be zero at all points on the Gaussian surface, the surface of the cavity must have a total charge $-q$.

Example 22.11

A hollowed conductor has a net charge +7 nC. Inside the cavity, there is a charge -5 nC but not touching it. How much charge resides on the outer and inner surfaces?

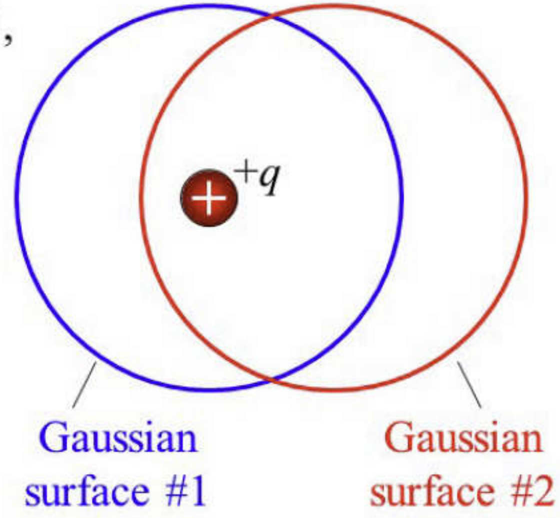


⚠ result independent of location of the -5 nC charge

Clicker Questions

Q22.2

Spherical Gaussian surface #1 has point charge $+q$ at its center. Spherical Gaussian surface #2, of the same size, also encloses the charge but is not centered on it. There are no other charges inside either Gaussian surface. Compared to the electric flux through surface #1, the flux through surface #2 is



A. greater.

B. the same.

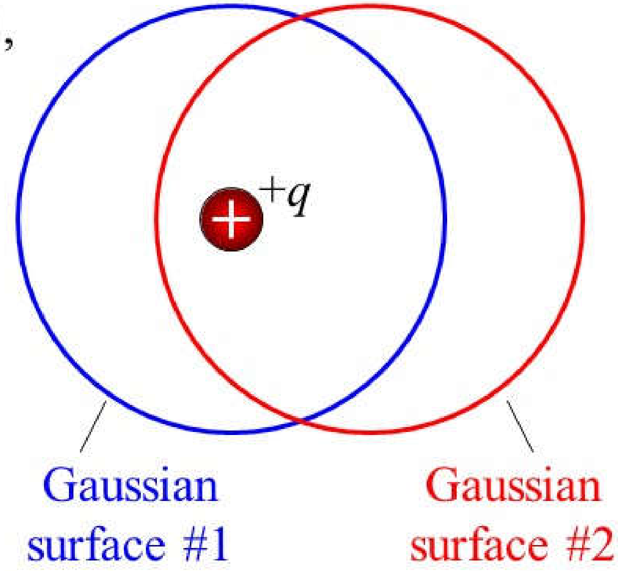
C. less, but not zero.

D. zero.

E. Not enough information is given to decide.

A22.2

Spherical Gaussian surface #1 has point charge $+q$ at its center. Spherical Gaussian surface #2, of the same size, also encloses the charge but is not centered on it. There are no other charges inside either Gaussian surface. Compared to the electric flux through surface #1, the flux through surface #2 is



A. greater.

[✓ B. the same.]

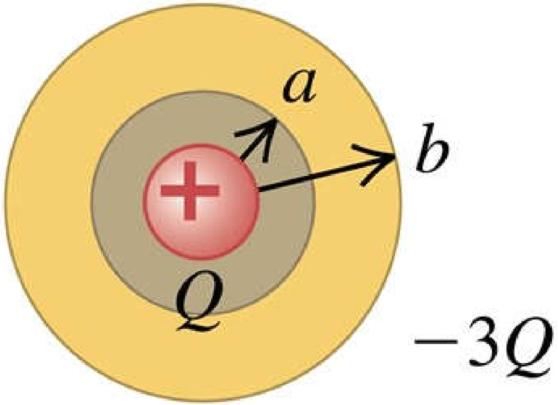
C. less, but not zero.

D. zero.

E. Not enough information is given to decide.

Q22.4

A conducting spherical shell with inner radius a and outer radius b has a positive point charge Q located at its center. The total charge on the shell is $-3Q$, and it is insulated from its surroundings. In the region $a < r < b$,



A. the electric field points radially outward.

B. the electric field points radially inward.

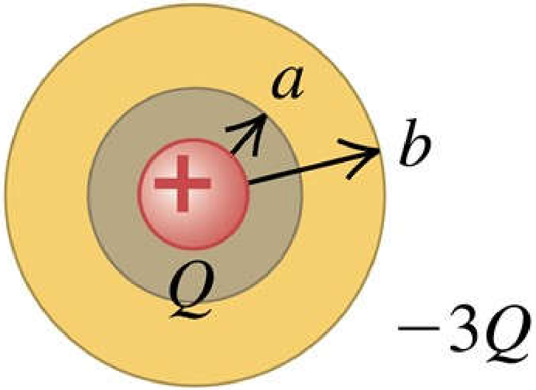
C. the electric field points radially outward in parts of the region and radially inward in other parts of the region.

D. the electric field is zero.

E. Not enough information is given to decide.

A22.4

A conducting spherical shell with inner radius a and outer radius b has a positive point charge Q located at its center. The total charge on the shell is –3Q, and it is insulated from its surroundings. In the region $a < r < b$,



A. the electric field points radially outward.

B. the electric field points radially inward.

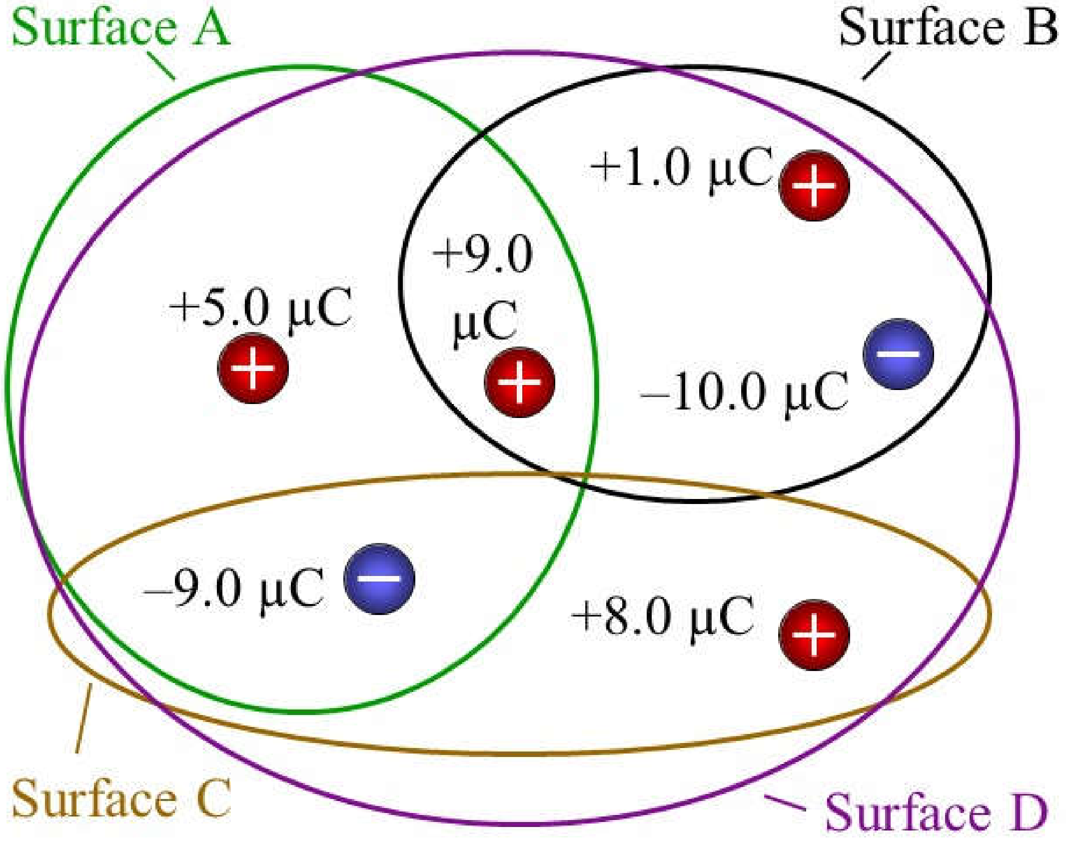
C. the electric field points radially outward in parts of the region and radially inward in other parts of the region.

[✓ D. the electric field is zero.]

E. Not enough information is given to decide.

Q-RT22.1

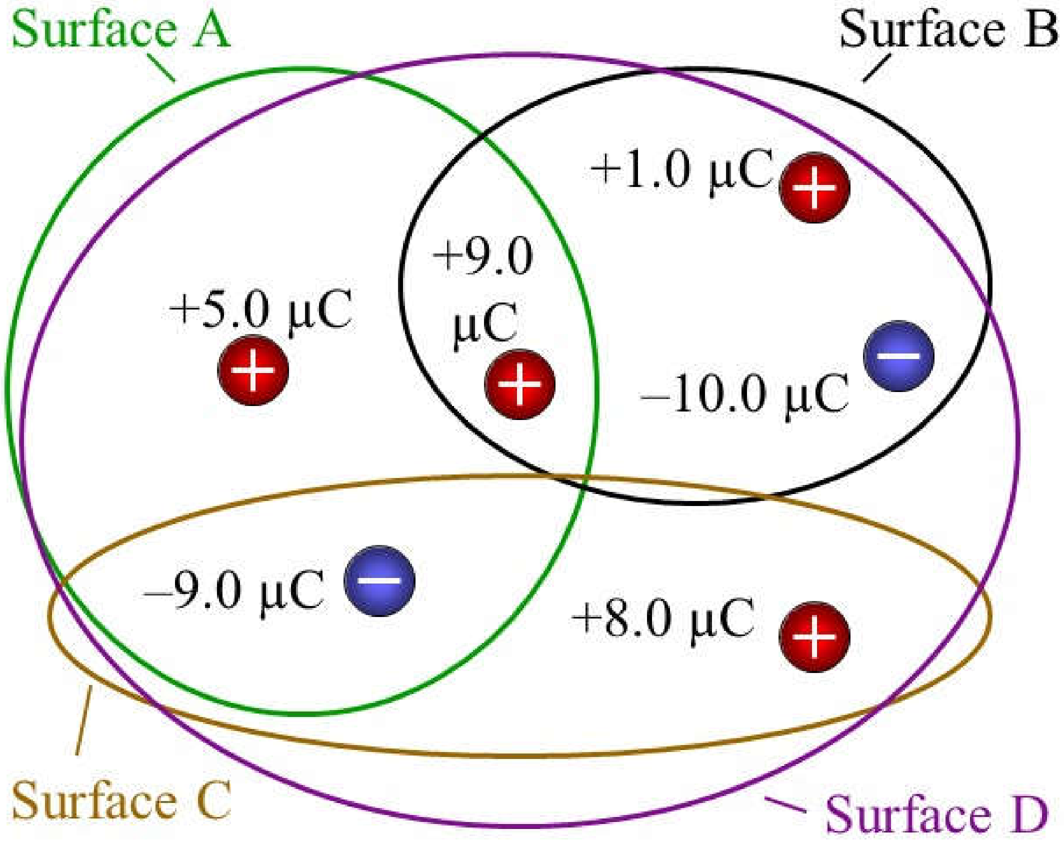
The figure shows six point charges that all lie in the same plane. Four Gaussian surfaces each enclose part of this plane, and the figure shows the intersection of each surface with the plane.



Rank surfaces A, B, C, and D in order of the electric flux through them, from most positive to most negative.

A-RT22.1

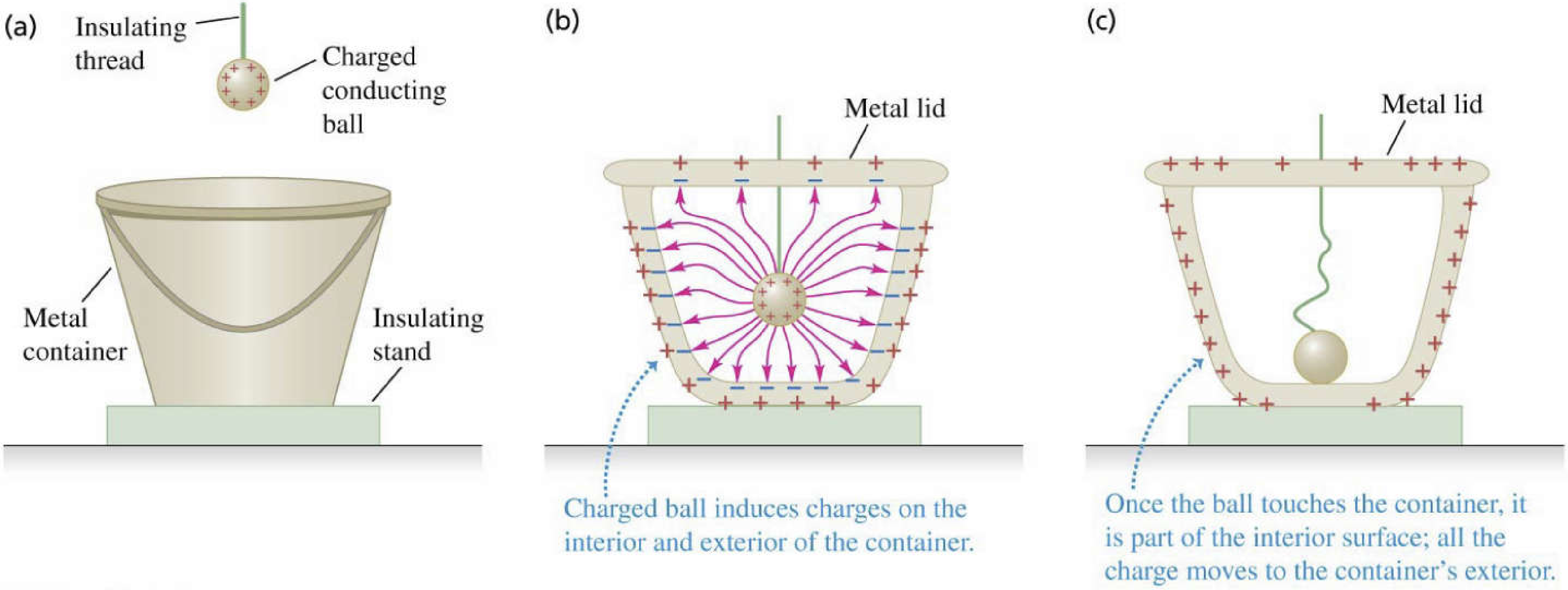
The figure shows six point charges that all lie in the same plane. Four Gaussian surfaces each enclose part of this plane, and the figure shows the intersection of each surface with the plane.



Rank surfaces A, B, C, and D in order of the electric flux through them, from most positive to most negative.

✔ Answer: ADBC

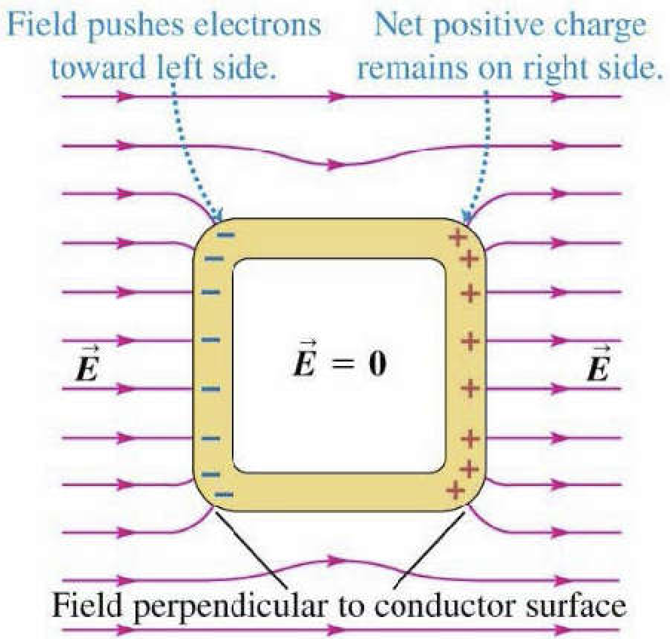
Faraday's icepail experiment



Pull the conducting ball out of the icepail → it is uncharged, thus verifying a consequence of the Gauss's law

Faraday cage

A metal cage shields its inside from external electric field



Using Gauss’s law to calculate electric field

- Gauss’s law gives the flux only, not the field.
- Only in cases where the charge distributions are symmetric we can calculate the field from the flux.
- Choose a Gaussian surface with the same symmetry as the charge distribution.

Field of a uniformly charge sphere – spherical symmetry (Example 22.9)

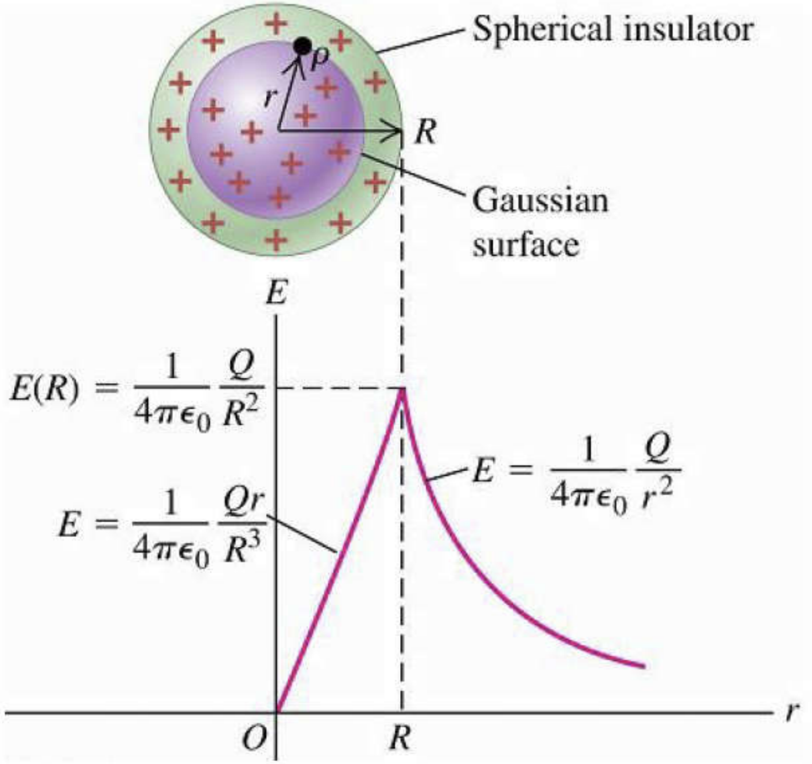
By symmetry $\vec{\boldsymbol{E}}$ is radially outwards from the center of the sphere, choose a concentric sphere as the Gaussian surface on which $\vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = EdA$, and E everywhere the same on the surface

Inside sphere, volume charge density $\rho = Q/\frac{4}{3}\pi R^3$

$$\underbrace{\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}}}_{E(4\pi r^2)} = \underbrace{Q_{\text{encl}}}_{\frac{4}{3}\pi r^3 \rho = Q\frac{r^3}{R^3}}/\epsilon_0$$

$$\Rightarrow \boxed{E = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}r}$$

- ⚠ linear in r
- ⚠ charge outside has no effect on the field on the Gaussian surface



Outside sphere, choose a larger Gaussian surface

$$E(4\pi r^2) = Q/\epsilon_0$$

$$\Rightarrow \quad \boxed{E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}}$$

⚠ as if charge concentrate at origin

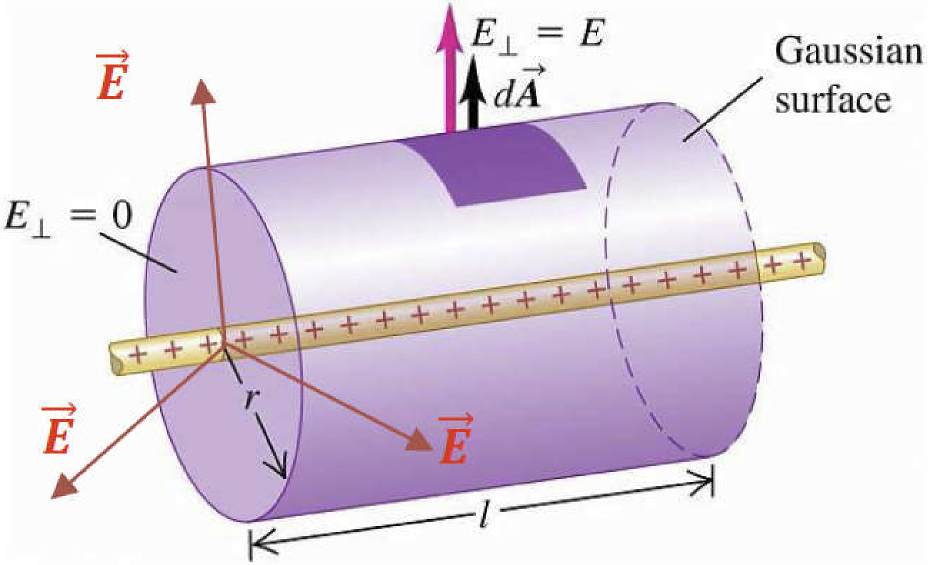
⚠ same result even if ρ is not constant, provided it is still spherically symmetry, i.e., depend on r only

If the sphere is a solid/hollow conductor, $E = 0$ inside and same as a point charge outside. (Example 22.5)

Field of a long uniform line charge – cylindrical symmetry (Example 22.6)

By symmetry $\vec{\boldsymbol{E}}$ is radially outwards ⊥ to the line, choose a cylinder as the Gaussian surface

On the curved surface, $\vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = EdA$, and E everywhere the same on the surface



Linear charge density is λ

$$\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = Q_{\text{encl}}/\epsilon_0$$

$$E(2\pi r l) = \lambda l/\epsilon_0$$

$$\boxed{E = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}}$$

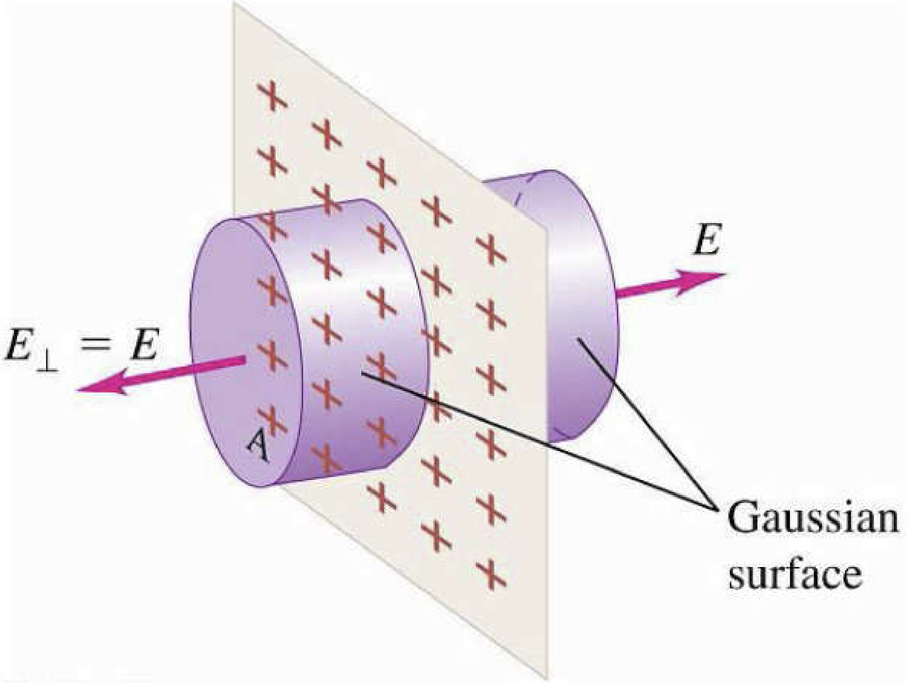
⚠ Decay as $1/r$, much slower than a point charge ($1/r^2$)

⚠ Independent of the thickness of the line charge as long as charge distribution is uniform

⚠ near the end of the line charge, $\vec{\boldsymbol{E}}$ no longer radially outward, this formula breaks down

Field of an infinite sheet of charge (Example 22.7)

By symmetry $\vec{E} \perp$ plane on both sides, choose a cylinder (or rectangular box) as the Gaussian surface



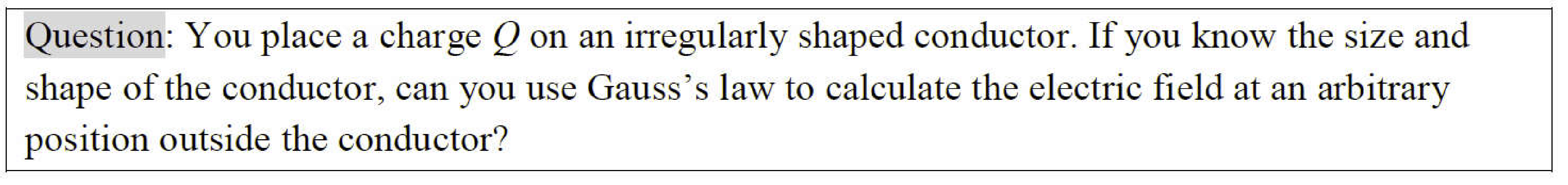
surface charge density is σ

$$\oint \vec{E} \cdot d\vec{A} = Q_{\text{encl}}/\epsilon_0$$

$$EA = \sigma A/\epsilon_0$$

$$\boxed{E = \frac{\sigma}{2\epsilon_0}}$$

⚠ same as in using the Coulomb's law, but much easier

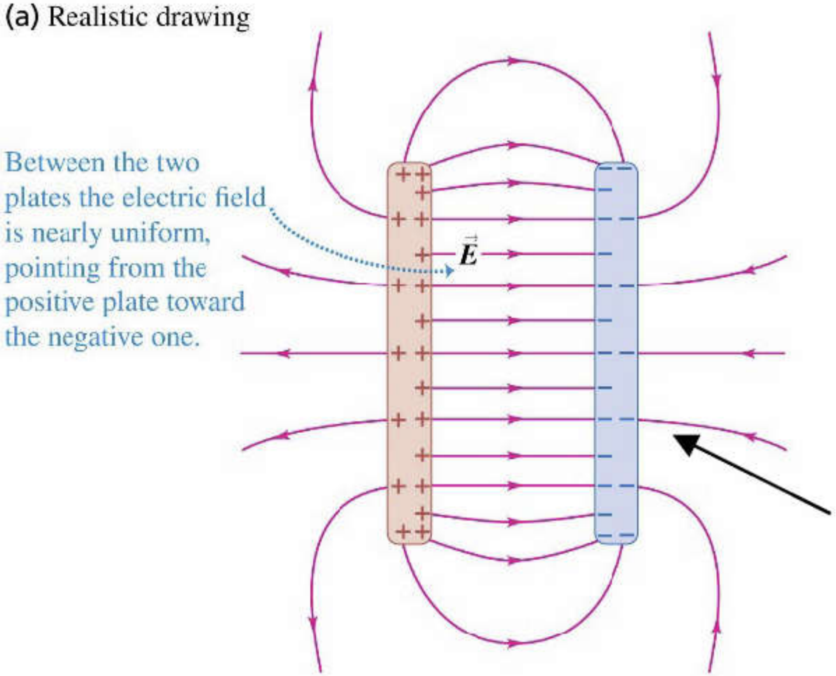


Question: You place a charge Q on an irregularly shaped conductor. If you know the size and shape of the conductor, can you use Gauss's law to calculate the electric field at an arbitrary position outside the conductor?

Answer: No, only when the charge is symmetric enough

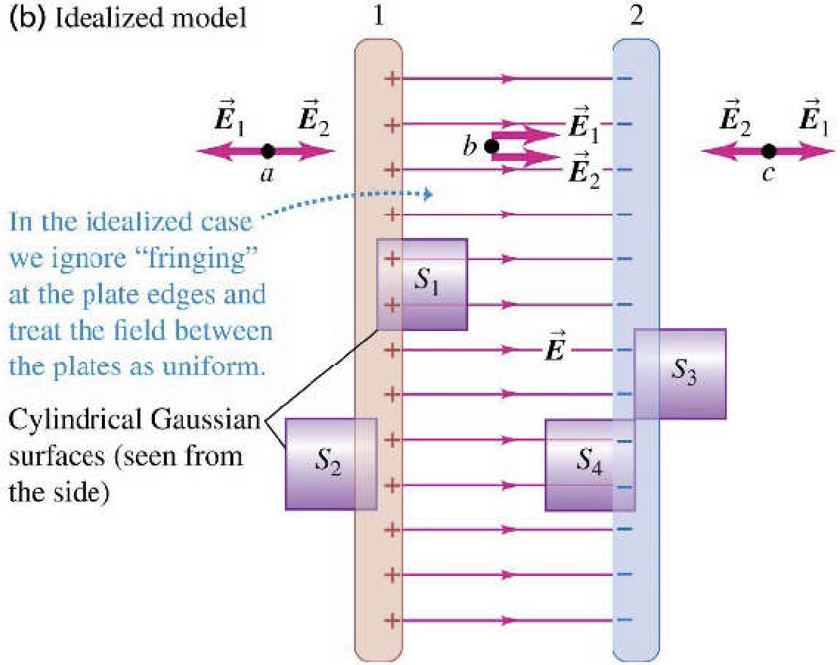
Field between oppositely charged parallel conducting plates (Example 22.8)

c.f. Example 21.12 in P. 4 of Lecture 2 notes



In real situation, charge distribution not uniform, especially near the edge. Charge reside on both the inner and outer surfaces. Electric field extend outside the overlap region between the plates, called **fringe effect**

⚠ $\vec{E}$ must be ⊥ to metal plates



(b) Idealized model

In the idealized case we ignore “fringing” at the plate edges and treat the field between the plates as uniform.

Cylindrical Gaussian surfaces (seen from the side)

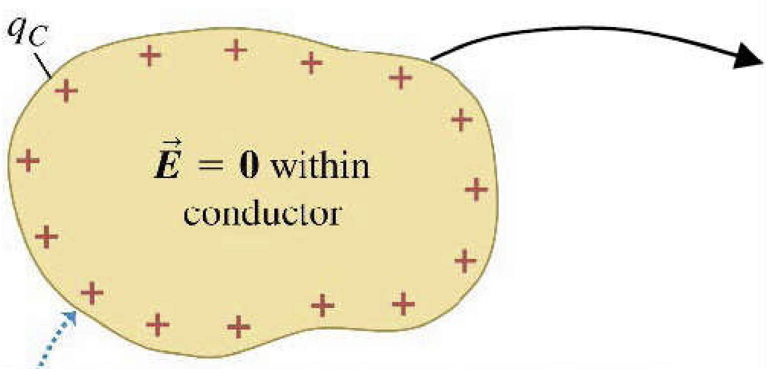
Idealized situation – away from edge, same as assuming plates are infinitely large, $\vec{\boldsymbol{E}}$ must be ⊥ to plates and everywhere the same in *y* direction, i.e., ignore fringe effect

Gaussian surfaces S_2 and S_3 shows that $\boxed{E = 0}$ *everywhere outside the overlap region*

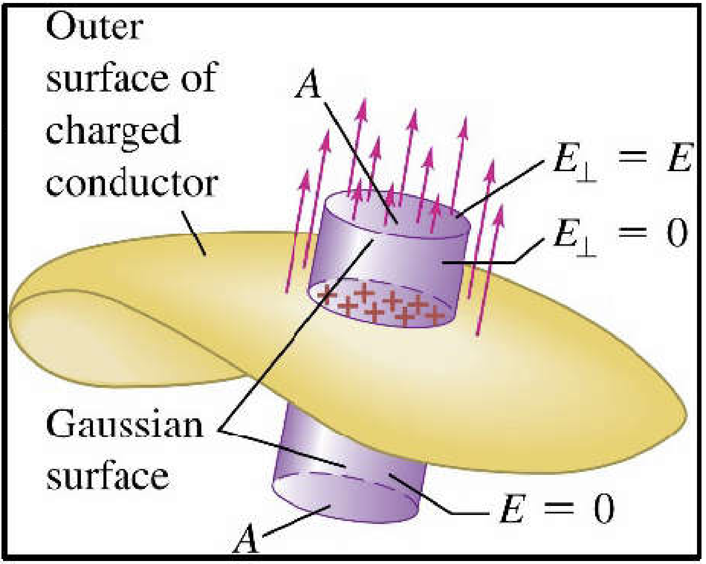
S_1 and S_4 shows that $\boxed{E = \sigma/\epsilon_0}$ *everywhere inside the overlap region*

If you don’t feel confusing enough, look at the next case 😊

Field at the surface of a conductor of arbitrary shape (not just a plate)



The charge q_C resides entirely on the surface of the conductor. The situation is electrostatic, so $\vec{E} = \mathbf{0}$ within the conductor.



$$E_\perp A + 0 = Q/\epsilon_0$$

$$\boxed{E_\perp = \frac{\sigma}{\epsilon_0}}$$

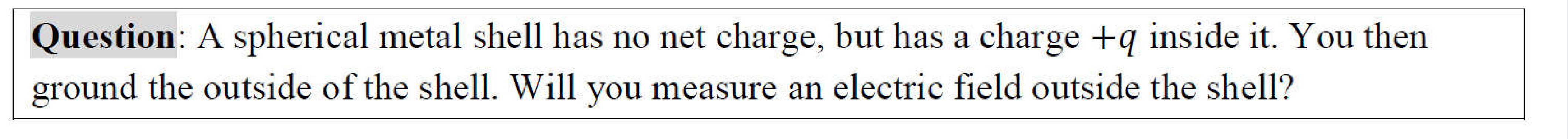
⚠ true only near surface where $\vec{E}$ ⊥ to surface

Example 22.13

The earth can be considered as a conductor. The averaged electric field measured near the earth surface is 150 N/C and is radially inwards towards the earth's center.

Surface charge density = $\epsilon_0 E = (8.85 \times 10^{-12} \mathrm{C}^2/\mathrm{N{\cdot}m}^2)(-150\ \mathrm{N/C}) = -1.33\ \mathrm{nC/m}^2$

Total charge enclosed = $\epsilon_0 E(4\pi R_E^2) = 4\pi(6.38 \times 10^6\ \mathrm{m})^2(-1.33 \times 10^{-9}\ \mathrm{C/m}^2) = -6.8 \times 10^5\ \mathrm{C}$



Question: A spherical metal shell has no net charge, but has a charge $+q$ inside it. You then ground the outside of the shell. Will you measure an electric field outside the shell?

Clicker Questions

Q22.5

There is a negative surface charge density in a certain region on the surface of a solid conductor. Just beneath the surface of this region, the electric field

A. points outward, toward the surface of the conductor.

B. points inward, away from the surface of the conductor.

C. points parallel to the surface.

D. is zero.

E. Not enough information is given to decide.

A22.5

There is a negative surface charge density in a certain region on the surface of a solid conductor. Just beneath the surface of this region, the electric field

A. points outward, toward the surface of the conductor.

B. points inward, away from the surface of the conductor.

C. points parallel to the surface.

[✓ D. is zero.]

E. Not enough information is given to decide.

Q22.6

For which of the following charge distributions would Gauss's law *not* be useful for calculating the electric field?

A. a uniformly charged sphere of radius *R*

B. a spherical shell of radius *R* with charge uniformly distributed over its surface

C. a right circular cylinder of radius *R* and height *h* with charge uniformly distributed over its surface

D. an infinitely long circular cylinder of radius *R* with charge uniformly distributed over its surface

E. Gauss's law would be useful for finding the electric field in all of these cases.

A22.6

For which of the following charge distributions would Gauss's law *not* be useful for calculating the electric field?

A. a uniformly charged sphere of radius *R*

B. a spherical shell of radius *R* with charge uniformly distributed over its surface

✓ C. a right circular cylinder of radius *R* and height *h* with charge uniformly distributed over its surface

D. an infinitely long circular cylinder of radius *R* with charge uniformly distributed over its surface

E. Gauss's law would be useful for finding the electric field in all of these cases.