Electric Potential

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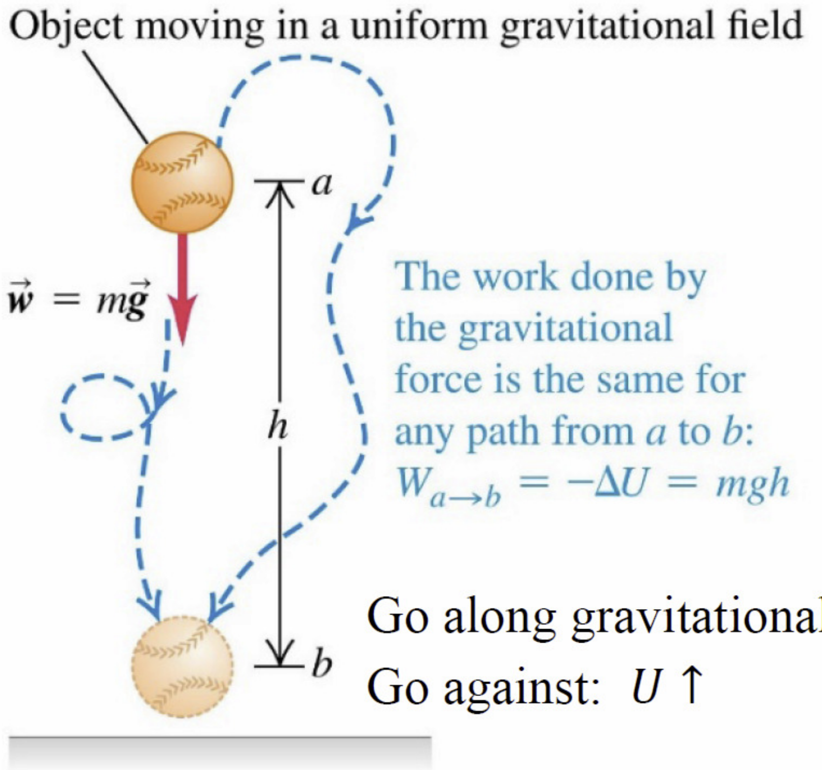
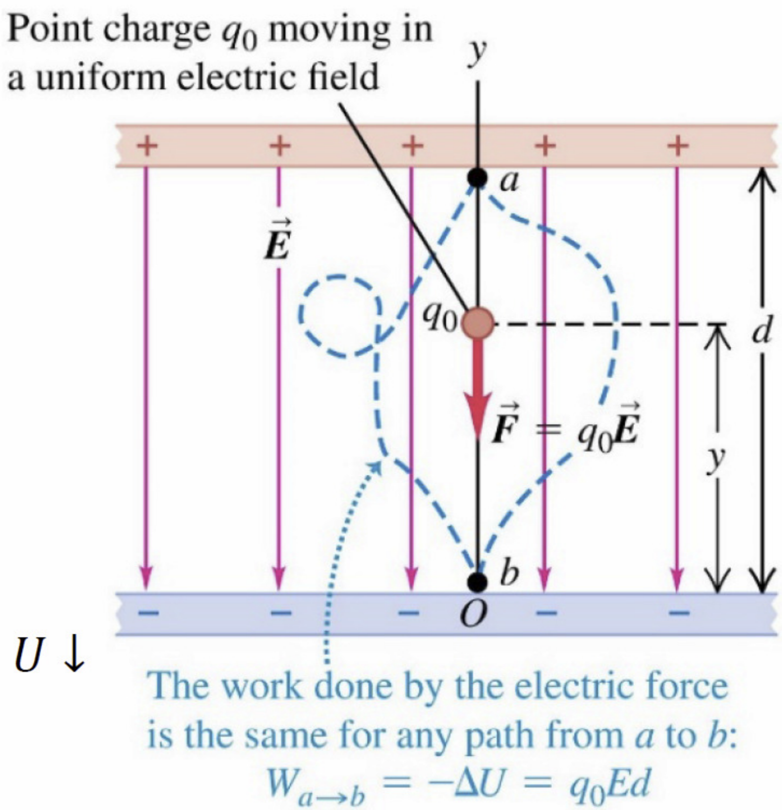
Analogy between a uniform gravitational field and electric field

Gravitation – a *conservative* field (from PHYS 1112)



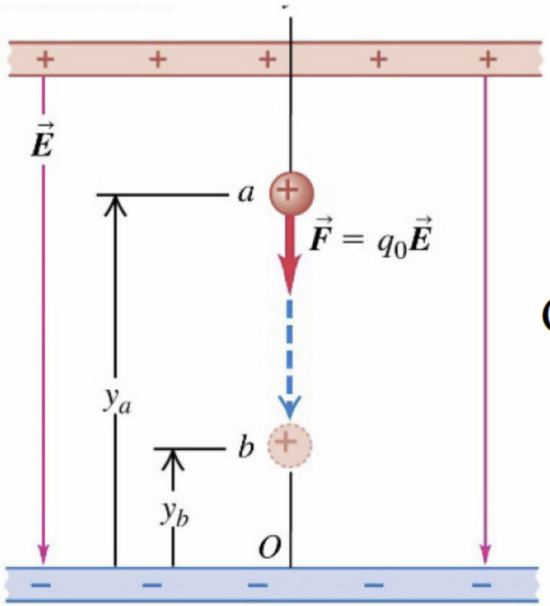
Go along gravitational force: $U \downarrow$

Go against: $U \uparrow$

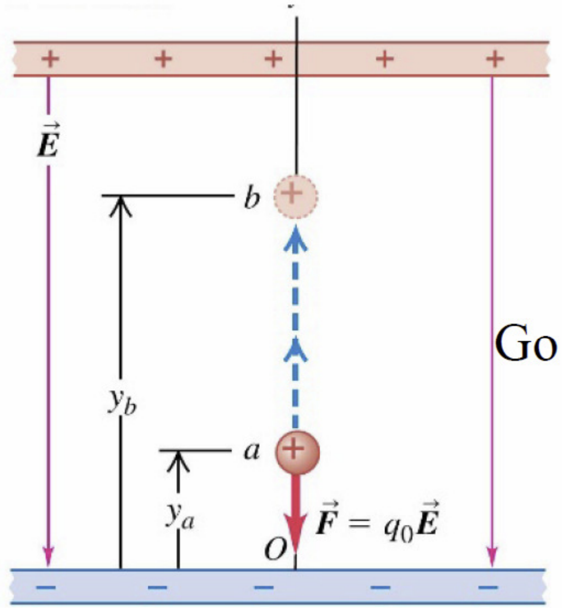


By analogy, **electric field is also conservative** (path independent)

⚠ Complication: q_0 can be +ve/-ve



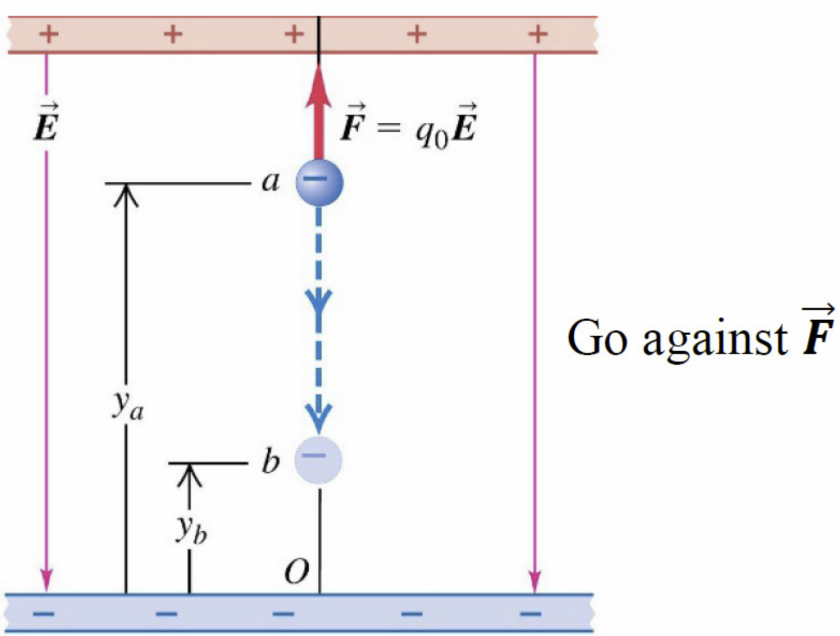
Go along $\vec{F}$

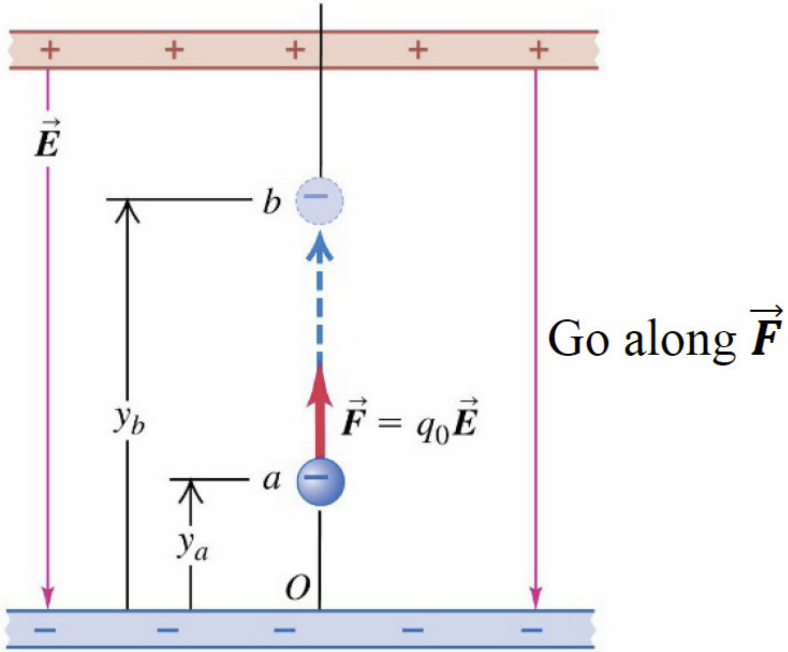


Go against $\vec{F}$

$W = \vec{F} \cdot \vec{s} > 0 \implies \Delta U = -W < 0, \therefore U \downarrow$

$W = \vec{F} \cdot \vec{s} < 0 \implies \Delta U = -W > 0, \therefore U \uparrow$





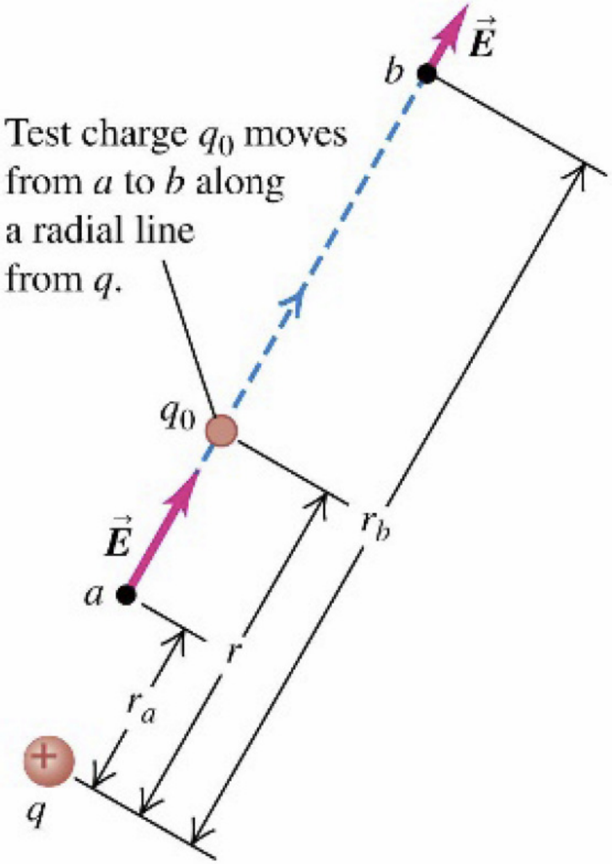
$W = \vec{F} \cdot \vec{s} < 0 \implies \Delta U = -W > 0, \therefore U \uparrow$

$W = \vec{F} \cdot \vec{s} > 0 \implies \Delta U = -W < 0, \therefore U \downarrow$

Conclusion:

U increases as q_0 goes *against* electric force, decreases as it goes *along* electric force

In a radial field set up by a fixed charge q, test charge q_0 moves from a to b



if a and b along the same radial line, $d\vec{\boldsymbol{l}} = d\vec{\boldsymbol{r}}$

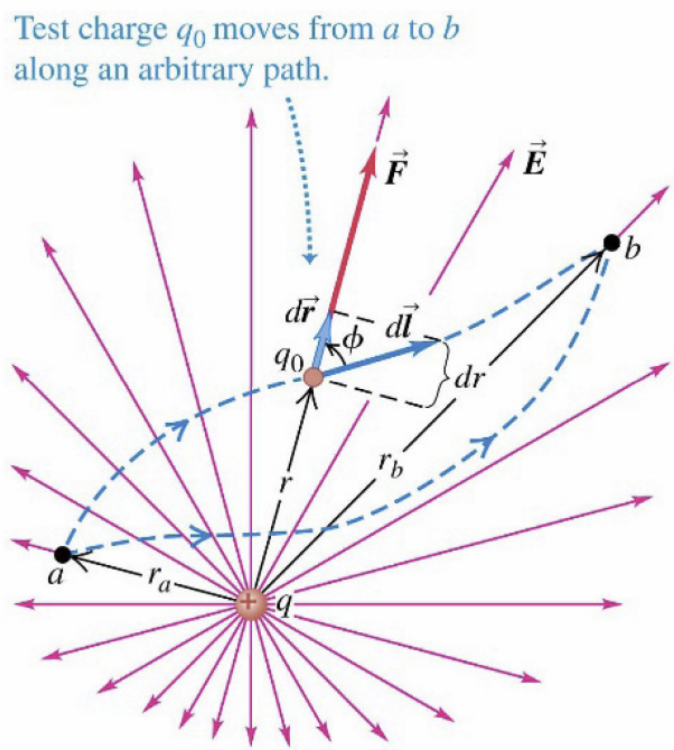
⚠ $d\vec{\boldsymbol{l}}$ is along the path whereas $d\vec{\boldsymbol{r}}$ is an outward vector

$$W_{a\to b} = \int_a^b (q_0\vec{\boldsymbol{E}}) \cdot d\vec{\boldsymbol{l}} = \int_a^b (q_0\vec{\boldsymbol{E}}) \cdot d\vec{\boldsymbol{r}}$$

$$= \int_a^b q_0 \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right) dr = \frac{qq_0}{4\pi\epsilon_0}\left(-\frac{1}{r}\right)\Big|_{r_a}^{r_b}$$

$$= \frac{qq_0}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

$$= -(U(r_b) - U(r_a))$$



If a and b not along the same radial line

$$dW = \vec{F} \cdot d\vec{l} = F dl \cos\phi = F dr$$

$$W_{a\to b} = \int_a^b q_0 \vec{E} \cdot d\vec{l} = \int_a^b \frac{qq_0}{4\pi\epsilon_0} \frac{1}{r^2} dr = \frac{qq_0}{4\pi\epsilon_0} \left(\frac{1}{r_a} - \frac{1}{r_b} \right)$$

Exactly the same as before!

$$= -(U(r_b) - U(r_a))$$

⚠ path independent, confirms that electric field is **conservative**

⚠ $\Delta U = -W_{a\to b} < 0$, $U \downarrow$ since q_0 goes *along* electric force

From $W_{a\to b} = -\Delta U = (U_a - U_b)$, define potential energy in this case to be

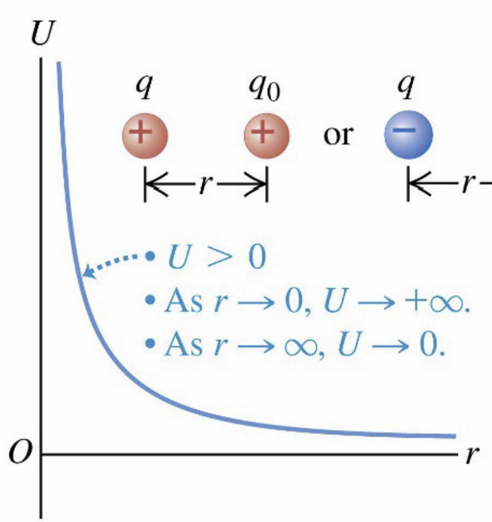
$$\boxed{U = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r}}$$

⚠ $U = 0$ at $r = \infty$ (zero level of potential energy is arbitrary)

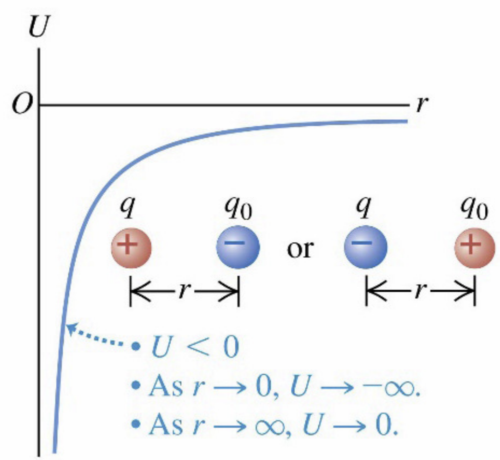
⚠ $U(r) = U(r) - U(\infty) = W_{r\to\infty}$, i.e., **$U(r)$ is the workdone by the field set up by q to bring q_0 from r to ∞**, or equivalently, **the work needed to bring q_0 from ∞ to r against the electric force** (preferred meaning)

⚠ U does not belong to test charge q_0 only. It belongs to q_0 and the field (set up by q) as a single system. Therefore can regard **U as the potential energy of the two charges q_0 and q at distance r apart**

(a) q and q_0 have the same sign.



(b) q and q_0 have opposite signs.



Example 23.1

A positron e^+ has mass 9.11×10^{-31} kg and charge $+1.6 \times 10^{-19}$ C. An alpha particle α has charge 3.20×10^{-19} C and mass 6.64×10^{-27}kg. α can be considered to be at rest because it is a lot heavier than e^+. Initially e^+ is at 1.00×10^{-10} m from α when it is moving directly away at 3.00×10^6 m/s.

When e^+ is at 2.00×10^{-10} m from α

$$\tfrac{1}{2}mv_a^2 + \frac{1}{4\pi\epsilon_0}\frac{q_\alpha q_{e^+}}{r_a} = \tfrac{1}{2}mv_b^2 + \frac{1}{4\pi\epsilon_0}\frac{q_\alpha q_{e^+}}{r_b}$$

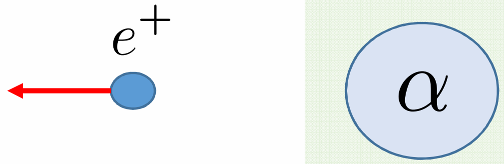
$$v_b = 3.8 \times 10^6 \text{ m/s}$$

⚠ e^+ accelerates, because they are repulsive

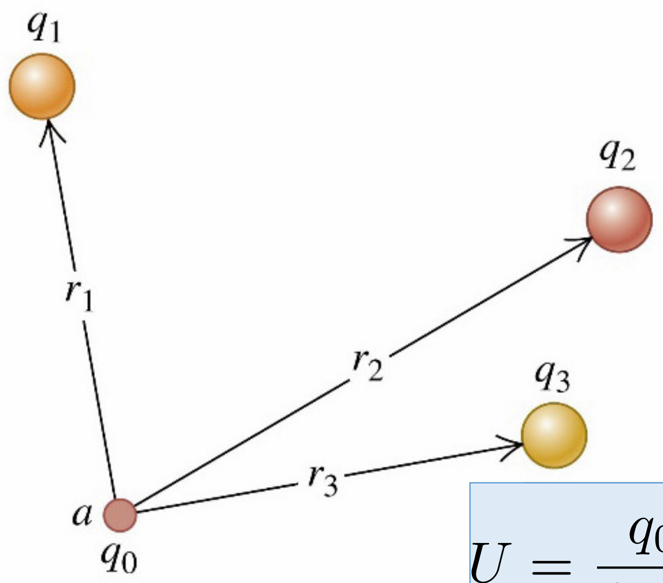
Replace e^+ by an electron e^-, with the same mass but opposite charge. It decelerates until at a point r_d where it stops and turn back towards α, $v_d = 0$,

$$\tfrac{1}{2}mv_a^2 + \frac{1}{4\pi\epsilon_0}\frac{q_\alpha q_{e^-}}{r_a} = \frac{1}{4\pi\epsilon_0}\frac{q_\alpha q_{e^-}}{r_d}$$

$$r_d = 9.0 \times 10^{-10} \text{ m}$$



If the field on q_0 is due to several charges, each produces a field $\vec{\boldsymbol{E}}_i$. U is the workdone by the total electric field to move q_0 from that position to ∞



$$U = \int q_0 \left(\sum \vec{\boldsymbol{E}}_i\right) \cdot d\vec{\boldsymbol{r}}$$

$$= \sum \underbrace{\int (q_0 \vec{\boldsymbol{E}}_i) \cdot d\vec{\boldsymbol{r}}}$$

as if in the absence of other charges, each one is $q_0 q / 4\pi\epsilon_0 r_i$

$$\boxed{U = \frac{q_0}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i}{r_i}}$$ *electric potential energy is additive*

A very different situation – no test charge (forget about q_0): Suppose charges $q_1, q_2, ..$ initially at infinity, bring them in place one by one by external force

Bring in q_1: No energy needed

Bring in q_2: $\dfrac{q_2}{4\pi\epsilon_0}\dfrac{q_1}{r_{12}}$

Bring in q_3: $\dfrac{q_3}{4\pi\epsilon_0}\left(\dfrac{q_1}{r_{13}}+\dfrac{q_2}{r_{13}}\right)$

Bring in q_4: $\dfrac{q_4}{4\pi\epsilon_0}\left(\dfrac{q_1}{r_{14}}+\dfrac{q_2}{r_{14}}+\dfrac{q_3}{r_{14}}\right)$

$\vdots$ $\vdots$

Total energy needed to assemble the charge configuration

$$\boxed{U=\frac{1}{4\pi\epsilon_0}\sum_{j=2}\sum_{i=1}^{j-1}\frac{q_iq_j}{r_{ij}}=\frac{1}{4\pi\epsilon_0}\sum_{i<j}\frac{q_iq_j}{r_{ij}}}$$

⚠ $i \neq j$, a charge doesn't interact with itself

⚠ $i > j$ guarantee each pair counted once only

Define the **potential energy of a system of charge** as the *total energy needed to assemble it from infinite distance apart*

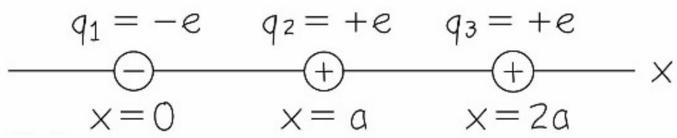
$$\boxed{U=\frac{1}{4\pi\epsilon_0}\sum_{i<j}\frac{q_iq_j}{r_{ij}}}$$

⚠ note the difference (both the form and interpretation) between this and the previous expression

Example 23.2

Two point charges, $q_1 = -e$ at $x = 0$, and $q_2 = +e$ at $x = a$.
Work needed to bring a third point charge $q_3 = +e$ from infinity to $x = 2a$

$$= \frac{q_3}{4\pi\epsilon_0}\left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right) = +\frac{e^2}{8\pi\epsilon_0 a}$$



⚠ +ve work means have to push against effective repulsion on q_3

Total potential energy of the three-charge system

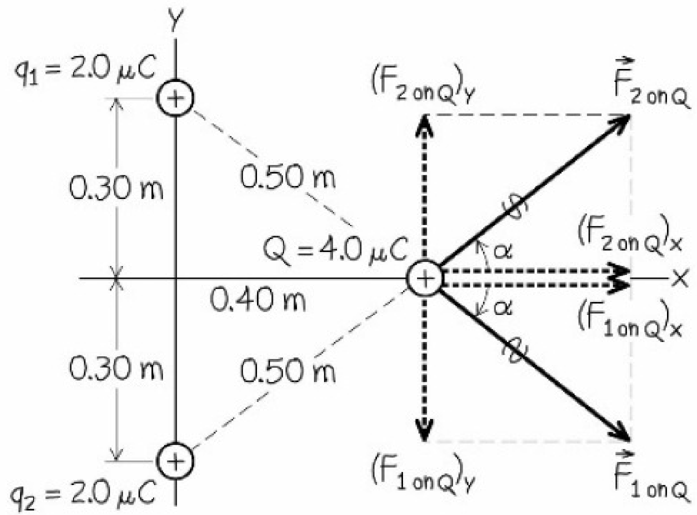
$$= \frac{1}{4\pi\epsilon_0}\sum_{i>j}\frac{q_i q_j}{r_{ij}} = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right) = -\frac{e^2}{8\pi\epsilon_0 a}$$

⚠ -ve total energy means the system is stable (compare to when they are at infinity apart)

Question:

The total potential energy of this system is (+ve / -ve / zero)

The total amount of work needed to move these charges infinitely far from each other is (+ve / -ve / zero)



Electric Potential

Idea: just like we factor out the test charge from electric force, $\vec{\boldsymbol{F}} = q_0\vec{\boldsymbol{E}}$, factor out q_0 from U

Define **electric potential** V by $\boxed{U = q_0 V}$

e.g. For a field set up by a point charge

$$\boxed{V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}}$$

For a field set up by a system of charges

$$\boxed{V = \frac{1}{4\pi\epsilon_0}\sum\frac{q_i}{r_i}}$$

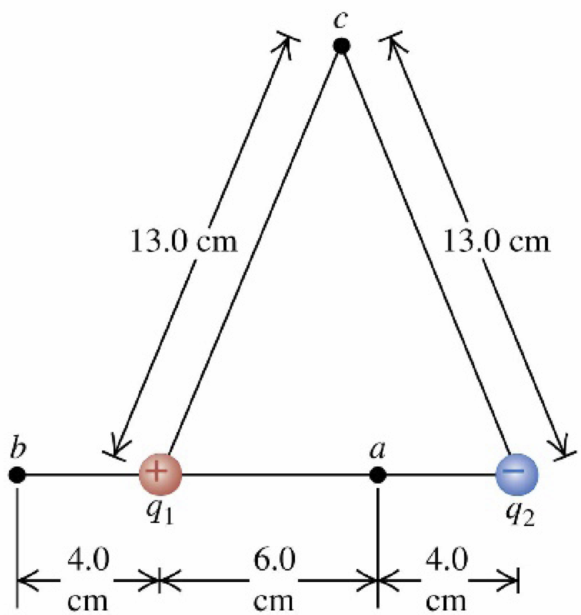
Unit: J/C = volt V, named after Volta

Example 23.4

An electric dipole with $q_1 = +12$ nC and $q_2 = -12$ nC. Compare the potential at the points a, b, and c.

At c

$$V_c = \frac{1}{4\pi\epsilon_0}\left(\frac{+12\text{ nC}}{13.0\text{ cm}} + \frac{-12\text{ nC}}{13.0\text{ cm}}\right) = 0.00\text{ V}$$



Question: Does $V = 0$ at a point implies $\vec{\boldsymbol{E}} = 0$ at that point?

$$\frac{W_{a\to b}}{q_0} = -\frac{\Delta U}{q_0} = -\left(\frac{U_b}{q_0} - \frac{U_a}{q_0}\right) = -(V_b - V_a) = V_a - V_b \equiv V_{ab}$$

V_{ab} is the *potential difference between a and b*, or the *potential of a relative to b*

V_{ab} (in V) = **workdone by electric force to move 1 C of charge from *a* to *b***, or equivalently

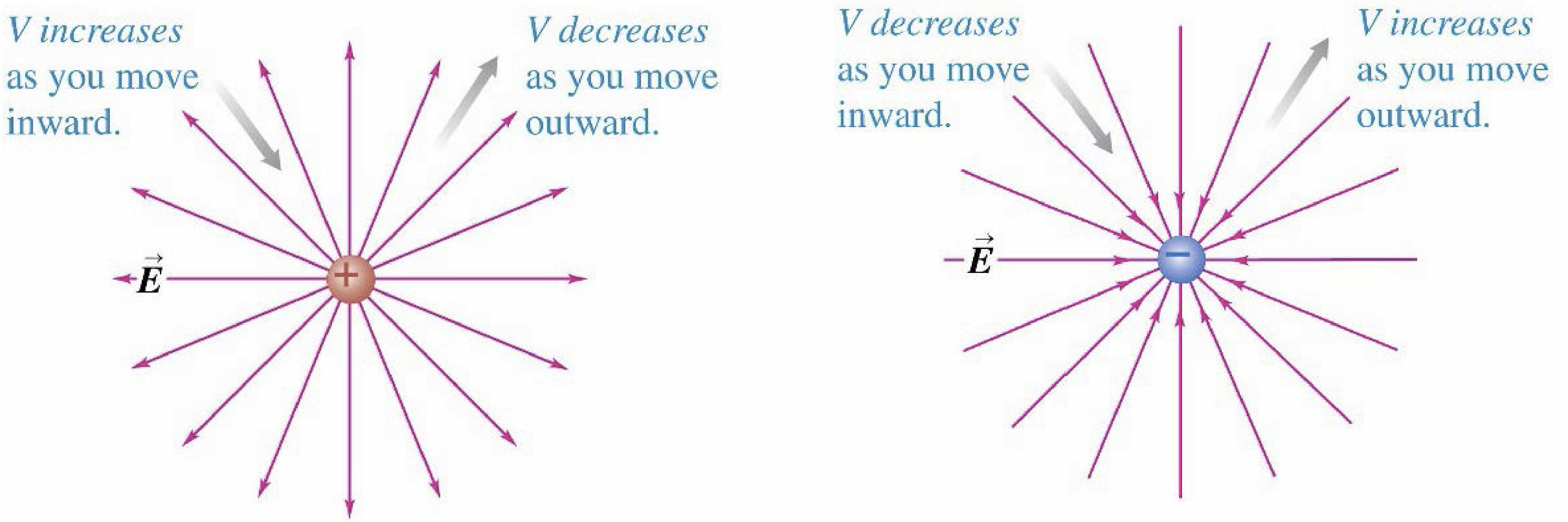
V_{ab} (in V) = **work needed to move 1 C of charge from *b* to *a* against the electric force**

Finding Electric Potential from Electric Field

$$W_{a\to b} = \int_a^b \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{l}} = q_0 \int_a^b \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}} \;\Rightarrow\; \boxed{V_{ab} = V_a - V_b = \int_a^b \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}} = -\int_b^a \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}}}$$

⚠ since $W_{a\to b}$ is path independent, so is V_{ab}

⚠ go against $\vec{\boldsymbol{E}}$ *increase* potential (*c.f.* go against $\vec{\boldsymbol{F}}$ *increases* U)



⚠ another unit for electric field, V/m = N/C

Electron Volt (eV) – a unit of *energy*

1 eV is the **energy needed to move one electron charge *e* through a potential difference 1 V**

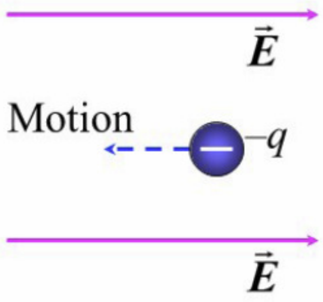
$W_{a\to b} = q_0 V_{ab} \quad \Rightarrow \quad 1\ \text{eV} = (1.602 \times 10^{-19}\ \text{C})(1\ \text{V}) = 1.602 \times 10^{-19}\ \text{J}$, another unit of energy

e.g. an α particle (charge $2e$) move through a potential difference 1000 V, change in potential energy is $2e(1\ \text{V}) = 2000\ \text{eV} = 2(1.602 \times 10^{-19}\ \text{C})(1000\ \text{V}) = 3.204 \times 10^{-16}\ \text{J}$

Clicker Questions

Q23.4

When a negative charge moves opposite to the direction of the electric field,



A. the field does positive work on it and the potential energy increases.

B. the field does positive work on it and the potential energy decreases.

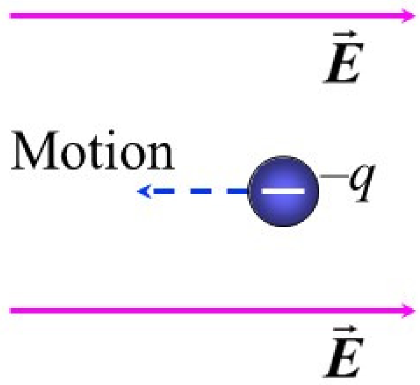
C. the field does negative work on it and the potential energy increases.

D. the field does negative work on it and the potential energy decreases.

E. the field does zero work on it and the potential energy remains constant.

A23.4

When a negative charge moves opposite to the direction of the electric field,



A. the field does positive work on it and the potential energy increases.

B. the field does positive work on it and the potential energy decreases.

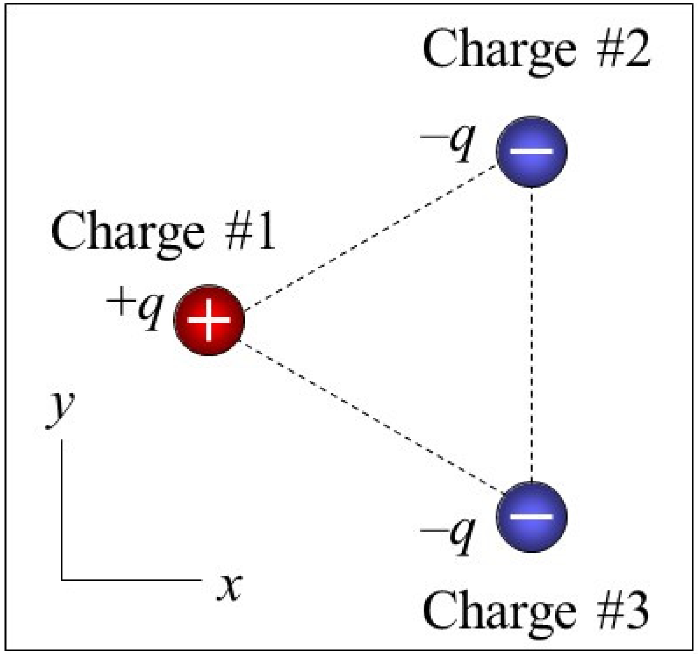
C. the field does negative work on it and the potential energy increases.

D. the field does negative work on it and the potential energy decreases.

E. the field does zero work on it and the potential energy remains constant.

Q23.8

The electric potential due to a point charge approaches zero as you move farther away from the charge. If the three point charges shown here lie at the vertices of an equilateral triangle, the electric potential at the center of the triangle is



A. positive.

B. negative.

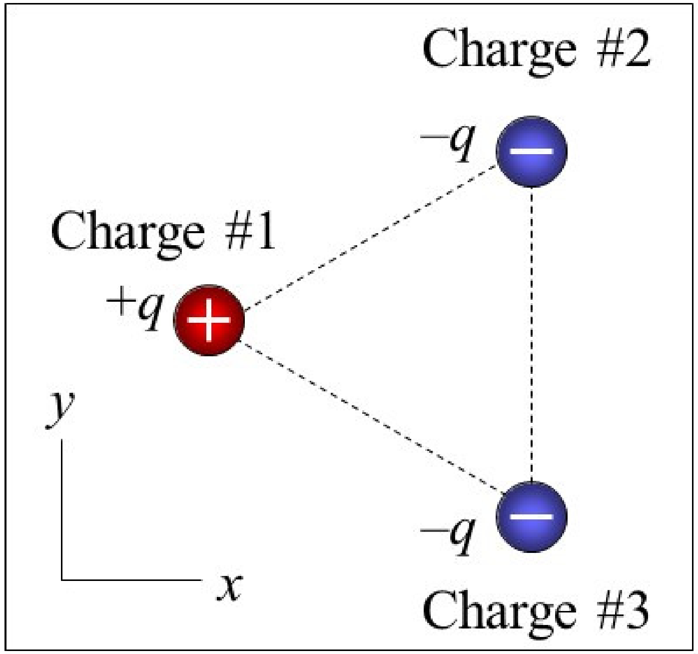
C. zero.

D. either positive or negative.

E. either positive, negative, or zero.

A23.8

The electric potential due to a point charge approaches zero as you move farther away from the charge. If the three point charges shown here lie at the vertices of an equilateral triangle, the electric potential at the center of the triangle is



A. positive.

✔ B. negative.

C. zero.

D. either positive or negative.

E. either positive, negative, or zero.

Calculate Electric Potential of a System (if $\vec{E}$ can be found readily, e.g. using Gauss's law)

$$V_{ab} \equiv V_a - V_b = \int_a^b \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}} = -\int_b^a \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}}$$

Potential of a relative to b

Workdone by electric force to move one unit charge from a to b

Workdone by external force to move one unit charge from b to a

If take V to be zero at infinity, $\boxed{V_a = \int_a^\infty \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}} = -\int_\infty^a \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}}}$

Potential of a Charged Conducting Sphere Example 23.6 and 23.8

Outside sphere, $r > R$:

Again $\vec{\boldsymbol{E}}$ radially outward

$$V(r) = V(r) - V(\infty) = \int_r^\infty \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}} = \int_r^\infty \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr = -\frac{1}{4\pi\epsilon_0}\frac{q}{r}\bigg|_r^\infty = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

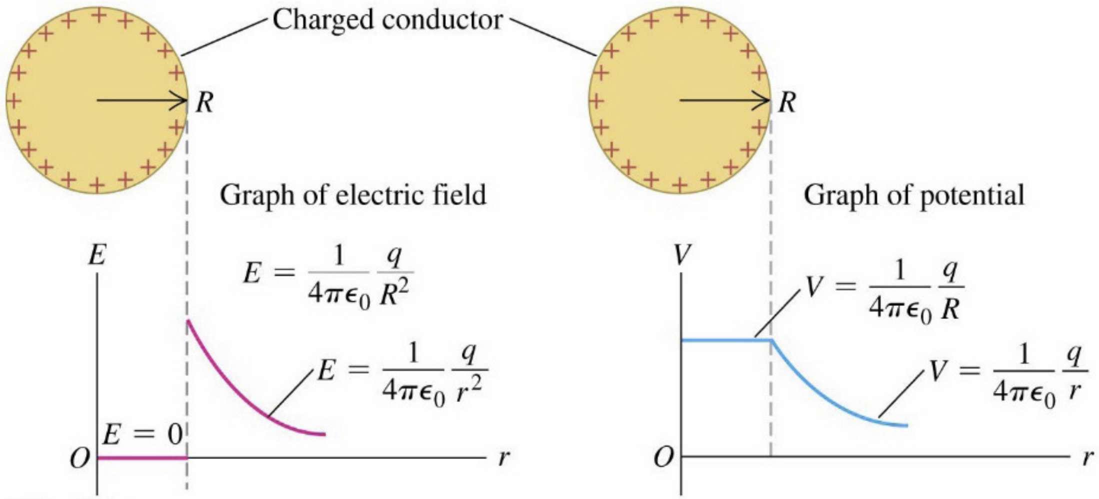
⚠ same as a point charge q located at the center of the sphere

Inside sphere, $r < R$:

$\vec{\boldsymbol{E}} = 0$ (*why?*), $V(r) = V(R) = \frac{1}{4\pi\epsilon_0}\frac{q}{R}$, the constant value at the surface

⚠ **inside and on the surface of the sphere, the potential is the same** and $\propto 1/R$

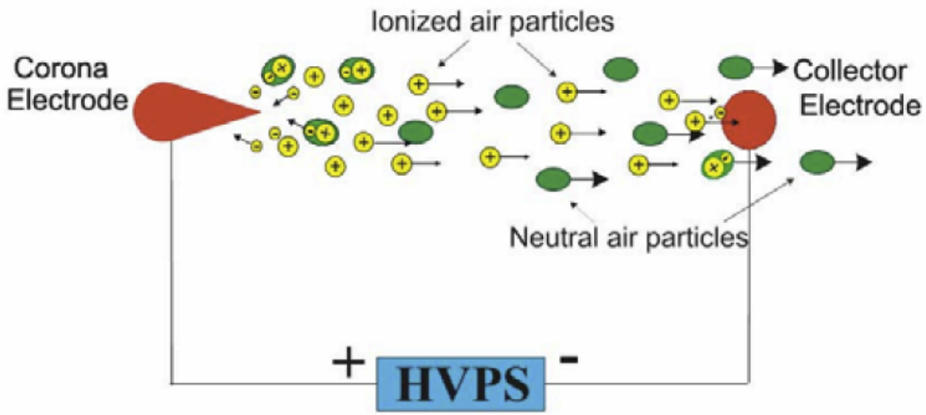
⚠ on the sphere's surface, $E = V/R$



Corona Discharge

If E too large, air molecules will be ionized, leading to an **electric breakdown**

Sharp conductor (R small) – large E at the same potential, easier to cause breakdown



http://thefutureofthings.com/upload/image/articles/2007/ionic-wind/corona-discharge.jpg

Consequences:

1. Lightning rods have round shape at the tip
2. The larger the metal sphere in a van de Graaff generator, the higher V it can build up

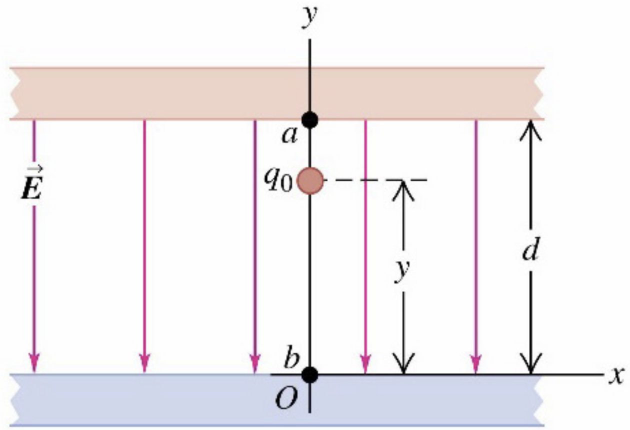
Oppositely Charged Parallel Plates Example 23.9

Already know: E uniform between plates if they are infinitely large

$$V_y - V_b = \int_y^b \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}} = Ey$$

In particular, $V_a - V_b = Ed$

⚠ if (arbitrarily) set $V_b = 0$, then $V_y = Ey$ and $V_a = Ed$



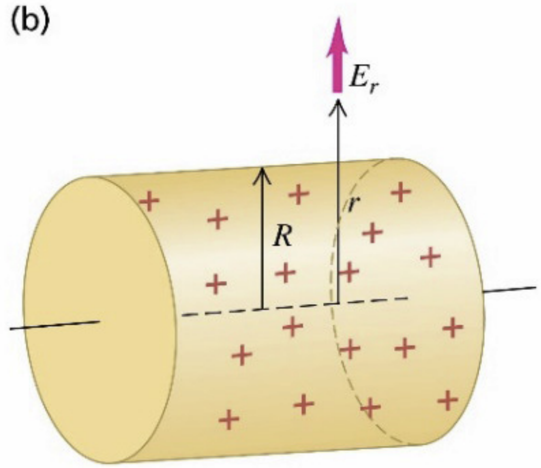
Infinite Charged Conducting Cylinder Example 23.10

Already know: $E = 0$ inside cylinder, and radially outward outside cylinder

Inside cylinder: V constant and equals to the value on the surface (just like the case of conducting sphere)

Outside:

$$V(r) = V(r) - V(\infty) = \int_r^{\infty} \vec{E} \cdot d\vec{l} = \int_r^{\infty} \frac{\lambda}{2\pi\epsilon_0 r} dr = \frac{\lambda}{2\pi\epsilon_0} \ln\frac{\infty}{r} \ ??$$



⚠ cannot choose $V(\infty) = 0$, true in cases where charge distribution extends to infinity

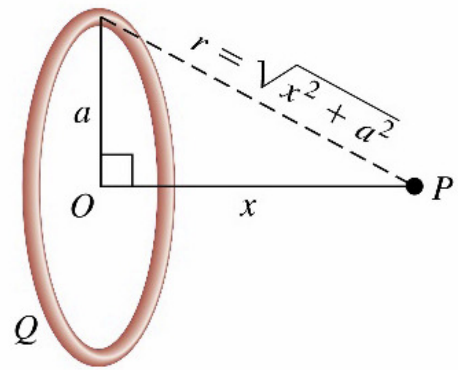
If choose $V(R) = 0$ (on the cylinder surface) instead

$$V(r) = -\int_R^{r} \vec{E} \cdot d\vec{l} = -\int_R^{r} \frac{\lambda}{2\pi\epsilon_0 r} dr = \frac{\lambda}{2\pi\epsilon_0} \ln\frac{R}{r} < 0$$

In the above examples, $V_{ab} = \int_a^b \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}}$ to calculate V because E can be found easily using Gauss's law. Otherwise we go back to $= \frac{1}{4\pi\epsilon_0}\sum \frac{q_i}{r_i}$, assuming $V(\infty) = 0$

A Ring of Charge Example 23.11

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r} = \frac{1}{4\pi\epsilon_0 r}\int dq = \frac{Q}{4\pi\epsilon_0 r} = \frac{1}{4\pi\epsilon_0}\frac{Q}{\sqrt{x^2+a^2}}$$



- ⚠ as $x \to \infty$, ring is just like a point charge, $V = \frac{1}{4\pi\epsilon_0}\frac{Q}{x}$
- ⚠ compare to calculating E, no need to find vector sum

A Finite Line of Charge Example 23.12

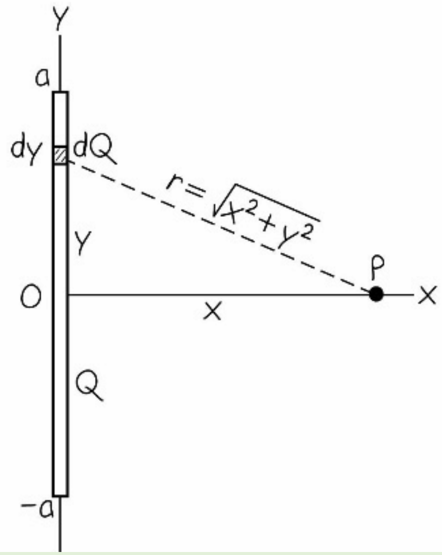
Charge of segment dy is $dQ = \left(\frac{Q}{2a}\right) dy$

$$dV = \frac{1}{4\pi\epsilon_0}\frac{Q}{2a}\frac{dy}{\sqrt{x^2+y^2}}$$

$$\Rightarrow \qquad V = \frac{1}{4\pi\epsilon_0}\frac{Q}{2a}\int_{-a}^{a}\frac{dy}{\sqrt{x^2+y^2}}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{Q}{2a}\ln\left(\frac{\sqrt{a^2+x^2}+a}{\sqrt{a^2+x^2}-a}\right)$$

may try
Wolfram Alpha

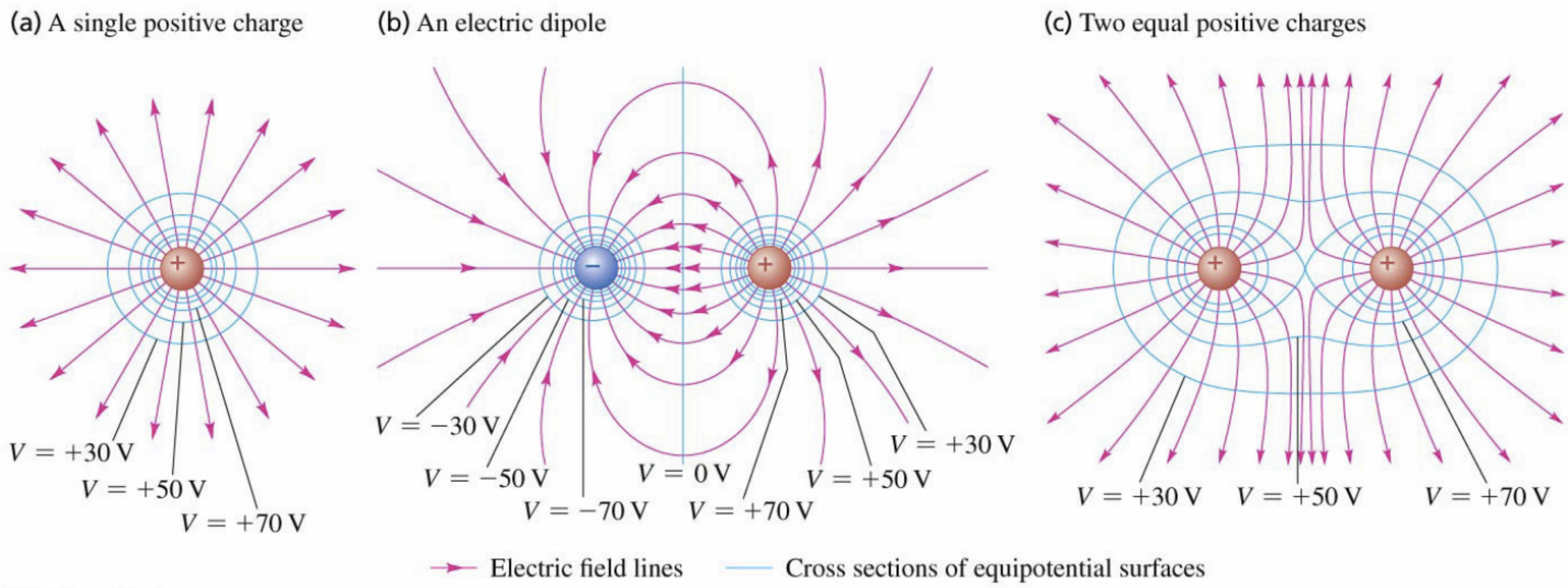


Question: does this result make sense for an infinitely long line, $a \to \infty$?

Equipotential Surface

Electric field is a vector field, visualized as vectors (electric field lines)

Potential is a scalar field, visualized as **equipotential surfaces** (on which every point has the potential), *c.f.* a contour plot



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峰景
Hong Villa
花園
Gdn
寶雲山
Bowen Hill
PEAK ROAD
山頂道
Interocean Court
26
金馬倫
Cameron
嘉樂苑
Cameron Hse
MAGAZINE GAP RD
馬己仙峽道
MAGAZINE GAP ROAD
Katoomba
摘星閣
Sky Court
COOMBE RD
甘道
香港仔郊野公園
Aberdeen Country Park
馬己仙峽
MAGAZINE GAP
太古大廈
Tai Koo Hse
Oasis
警隊博物館
Police Museum
Coombe Apts
嘉樂園
Carolina Gard
COOMBE ROAD
灣仔峽道
灣仔峽
WAN CHAI GAP
司徒拔道
STUBBS ROAD
WAN CHAI GAP RD
鄧鏡波書院
Tang King Po Coll
胡忠大廈
Wu Chung Hse
鳳凰閣
Phoenix Crt
琥苑
Amber Gdn
竹林苑
Bamboo Grove
豐樂新邨
Merry Garden
Guildford Court
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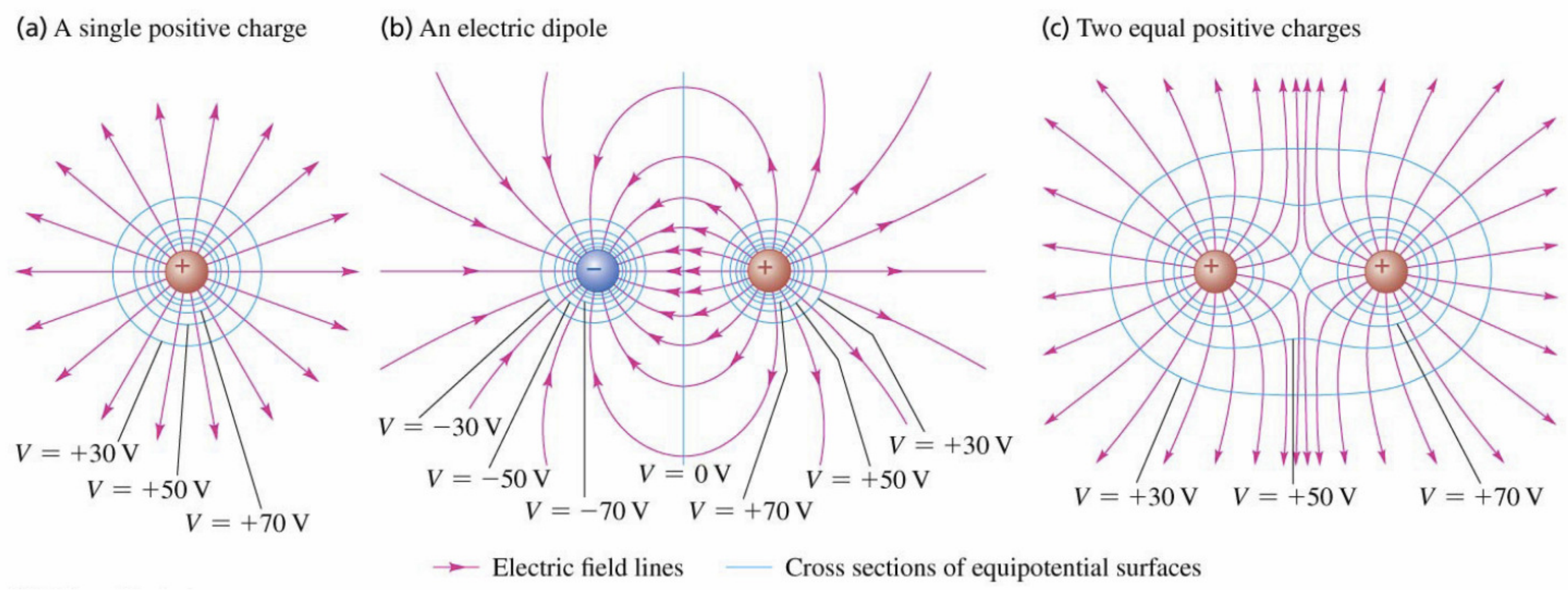


Properties of equipotential surfaces

1. No work is done when a test charge moves along an equipotential surface
2. Consequently **field lines must be perpendicular to equipotential surfaces**
3. Field lines go from high potential to low potential surfaces
4. A positive test charge "falls" from high potential to low potential, *c.f.* a mass fall down the hill
5. Equipotential surfaces closer together if electric field is stronger, *c.f.* a steeper hill

⚠ while field lines repel each other and cannot cross, equipotential surfaces can cross each other

⚠ E need not be constant on the same equipotential surface

Conductor Revisited

Consider a conductor of arbitrary shape, which may have a cavity inside. It may carry a surplus charge, but there is no free charge inside the cavity. It is assumed to be in electrostatic condition (no net current).

We already know from studying $\vec{\boldsymbol{E}}$ that

1. Inside the solid conductor (not including the cavity), $\vec{\boldsymbol{E}} = 0$
2. On the conductor surfaces (inner and outer), $\vec{\boldsymbol{E}}$ perpendicular to surface

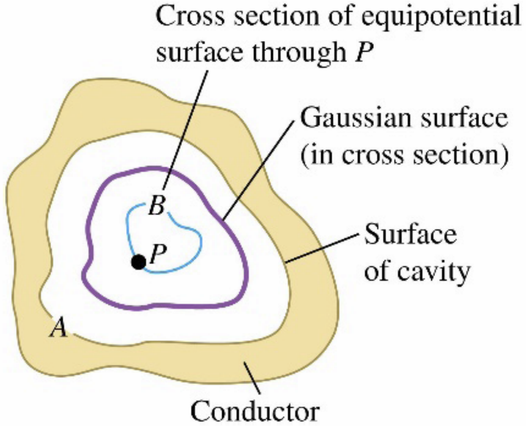
The potential anywhere on the surfaces, inside the conductor, and inside the cavity are all the same

To prove it:

1. On the surfaces (both inside and outside), $E_\perp = 0$, potential difference between any two points $V_a - V_b = \int E_\perp dl = 0$
2. Any two points inside the conductor (not including the cavity), $V_a - V_b = \int \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}} = 0$

⚠ already seen points 1 and 2 in a metal sphere

3. Inside the cavity, prove by contradiction:
   - Inner surface A is equipotential, V_A
   - **Assume a point $\boldsymbol{P}$ inside the cavity has different potential $\boldsymbol{V_P \neq V_A}$.** Construct an equipotential surface B through P
   - B cannot touch surface A (*why*?)
   - Anywhere between A and B, $\vec{\boldsymbol{E}}$ must point either from A to B if $V_A > V_B$, or B to A if $V_A < V_B$
   - Construct a Gaussian surface sitting between surfaces A and B. It has non-zero flux but encloses no charge, **a contradiction**!!



Potential Gradient – electric field from potential

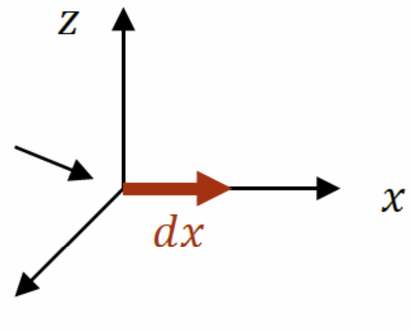
$$V_a - V_b = \int_b^a dV = -\int_b^a \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}}$$

True for any path and any endpoints, ∴ $dV = -\vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}} = -(E_x dx + E_y dy + E_z dz)$

If vary x but hold y and z constant, $dy = dz = 0$

$$dV = \frac{\partial V}{\partial x} dx = -E_x dx \quad \Rightarrow \quad E_x = -\frac{\partial V}{\partial x}$$

Differentiate $V(x, y, z)$ as if y and z are constant



Likewise

$$E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}$$

Therefore

$$\boxed{\vec{\boldsymbol{E}} = -\left(\frac{\partial V}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial V}{\partial y}\vec{\boldsymbol{j}} + \frac{\partial V}{\partial z}\vec{\boldsymbol{k}}\right) \equiv -\nabla V}$$

$\nabla \equiv \frac{\partial}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y}\vec{\boldsymbol{j}} + \frac{\partial}{\partial z}\vec{\boldsymbol{k}}$ is called the *gradient operator*, or *grad*

∇V (a *vector* !!)is called the **potential gradient**, i.e., *the rate of change* of V

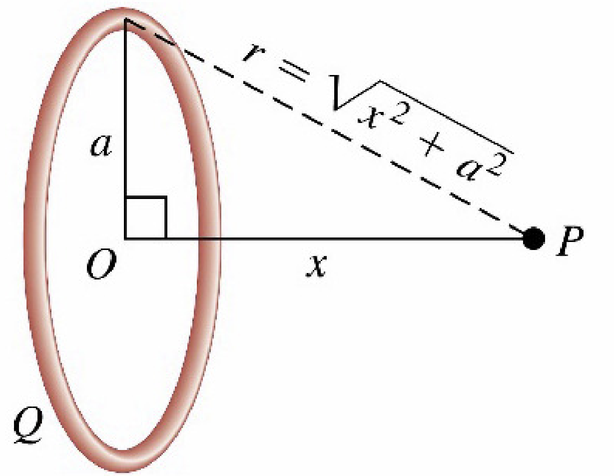
⚠ already know that E stronger in places where equipotential surfaces are closer

Example 23.14

We previous found that for P along the axis,

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{\sqrt{x^2+a^2}}$$

$$E_x = -\frac{\partial V}{\partial x} = \frac{1}{4\pi\epsilon_0}\frac{Qx}{(x^2+a^2)^{3/2}}, \qquad E_y = E_z = 0$$



⚠ if V depend on x only, then $\vec{\boldsymbol{E}}$ is along the x direction, $\vec{\boldsymbol{E}} = E_x\hat{\boldsymbol{\imath}}$. Likewise for y and z.

Likewise if V depends on radial distance r only, then $\vec{\boldsymbol{E}}$ is radially outward/inward,

$\vec{\boldsymbol{E}} = E_r \hat{\boldsymbol{r}} = -(\partial V / \partial r)\hat{\boldsymbol{r}}$

e.g. for a point charge

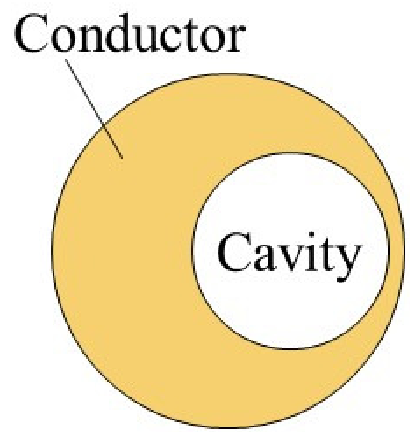
$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \quad \Rightarrow \; E_r = -\frac{\partial V}{\partial r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$$

Question: Suppose $V(x, y, z) = A + Bx + Cy^3 + Dxy$, where A, B, C and D are positive constants. Which of the followings is/are correct?

1) Increase A will increase the magnitude of $\vec{\boldsymbol{E}}$ at all points;
2) Increase A will decrease the magnitude of $\vec{\boldsymbol{E}}$ at all points;
3) $\vec{\boldsymbol{E}}$ has no z component;
4) $\vec{\boldsymbol{E}}$ at the origin is zero.

Q23.11

A solid spherical conductor has a spherical cavity in its interior. The cavity is *not* centered on the center of the conductor. If there is a net positive charge on the conductor, the electric field in the cavity



A. points generally from the center of the conductor toward the outermost surface of the conductor.

B. points generally from the outermost surface of the conductor toward the center of the conductor.

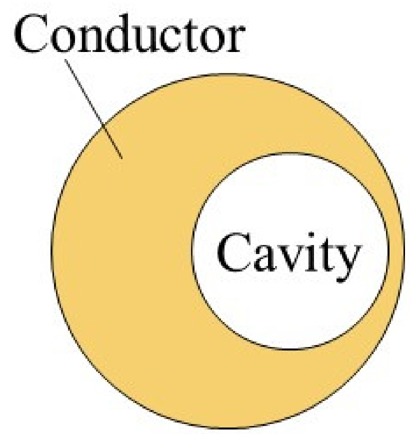
C. is uniform and nonzero.

D. is zero.

E. cannot be determined from information given.

A23.11

A solid spherical conductor has a spherical cavity in its interior. The cavity is *not* centered on the center of the conductor. If there is a net positive charge on the conductor, the electric field in the cavity



A. points generally from the center of the conductor toward the outermost surface of the conductor.

B. points generally from the outermost surface of the conductor toward the center of the conductor.

C. is uniform and nonzero.

D. is zero.

E. cannot be determined from information given.

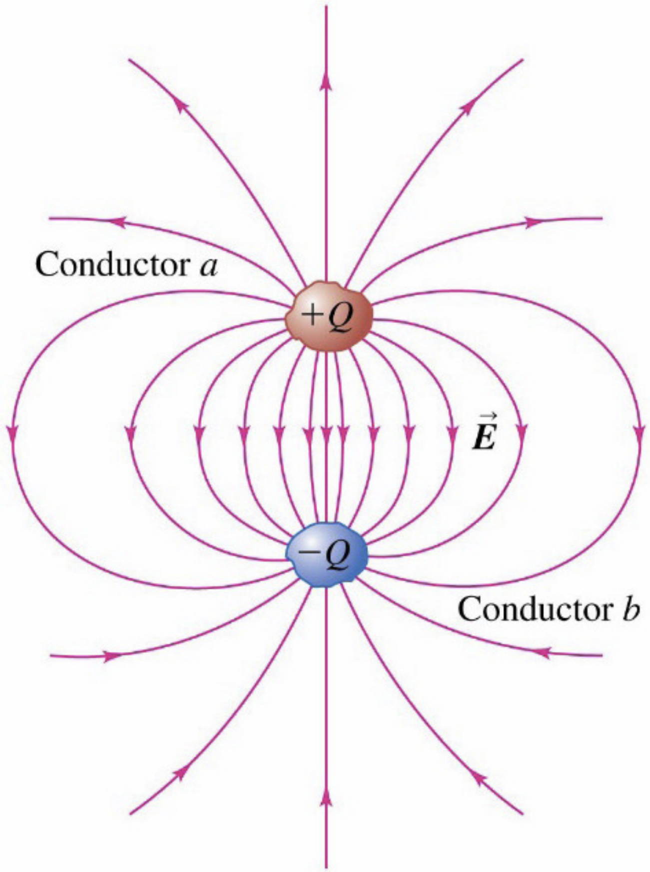
Capacitance and Dielectrics

Any two conductors insulated from each other form a **capacitor**

In practice:

1. Conductors contain equal and opposite charges $\pm Q$
2. Although overall neutral, we refer to Q as the **charge stored in the capacitor**
3. **Potential difference**, or **voltage**, of the capacitor is the potential of conductor carrying $+Q$ relative to that carrying $-Q$

$$V_{ab} = V_a - V_b$$

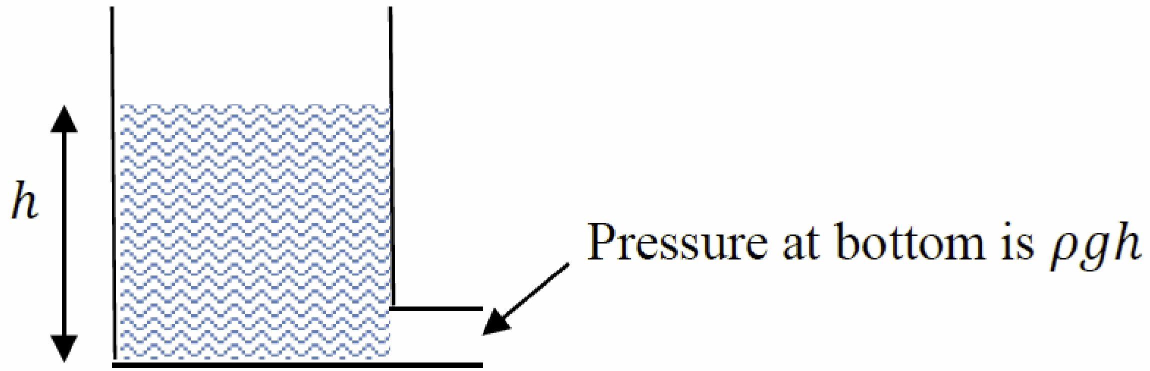


⚠ electric field at any point $\propto Q \Rightarrow V_{ab} \propto Q$

Define the proportionality constant as the **capacitance** C

$$\boxed{C \equiv \frac{Q}{V_{ab}}}$$

c.f. a tank to hold water

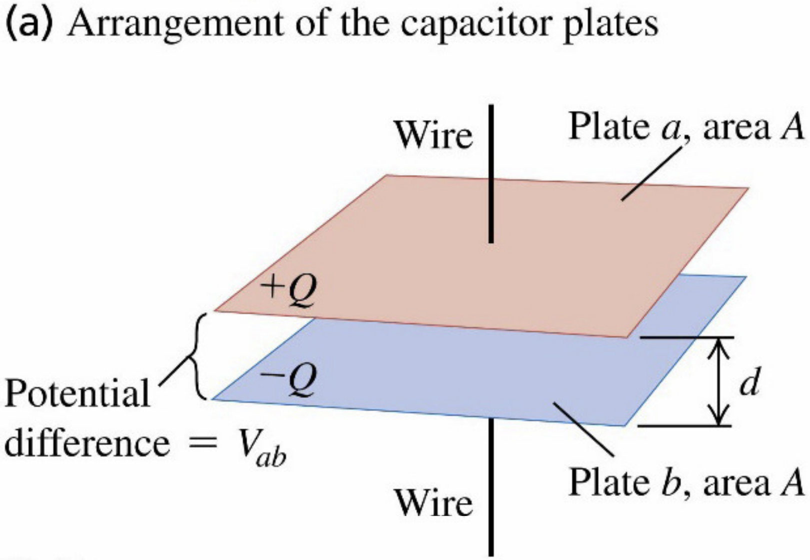


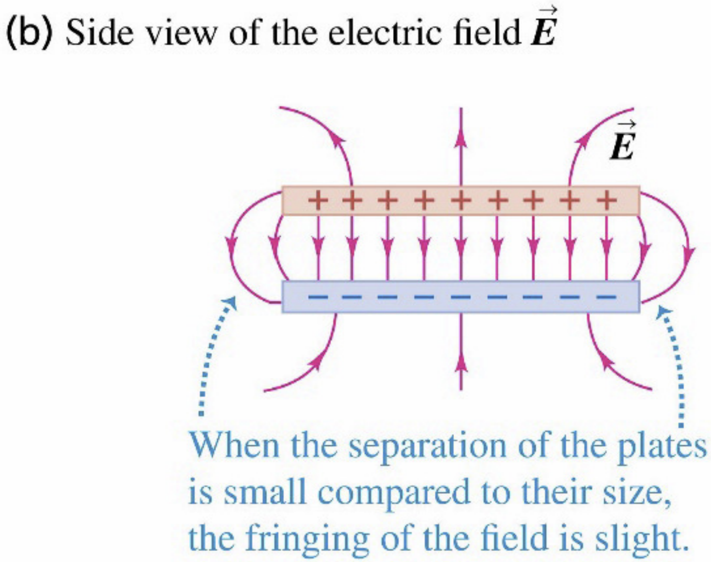
SI Unit: farad F, 1F = 1 C/V

1F is a huge capacitor (because 1 C is a large amount of charge). More practical units of capacitance are μF (micro, 10^{-6} F) and pF (pico, 10^{-12} F)

- ⚠ C measures the ability of the conductors to hold charge at a certain voltage
- ⚠ C depend on the geometry of the two conductors only

Parallel-Plate Capacitor





Assume plates infinitely large, i.e., $d \ll \sqrt{A}$, ignore edge effect, $E = \sigma/\epsilon_0 = Q/\epsilon_0 A$

$$V_{ab} = Ed = \frac{1}{\epsilon_0}\frac{Qd}{A}$$

$$\boxed{C = \frac{Q}{V_{ab}} = \epsilon_0 \frac{A}{d}}$$

⚠ for a 1 F parallel-plate capacity whose plates are 1.0 mm apart,

$$A = \frac{Cd}{\epsilon_0} = 1.1 \times 10^8\ \mathrm{m}^2 \quad !!!!$$

Spherical Capacitor Example 24.3

Two concentric conductor shells, inside shell has $+Q$ and outer has $-Q$

Use Gauss's law to find field between the shells

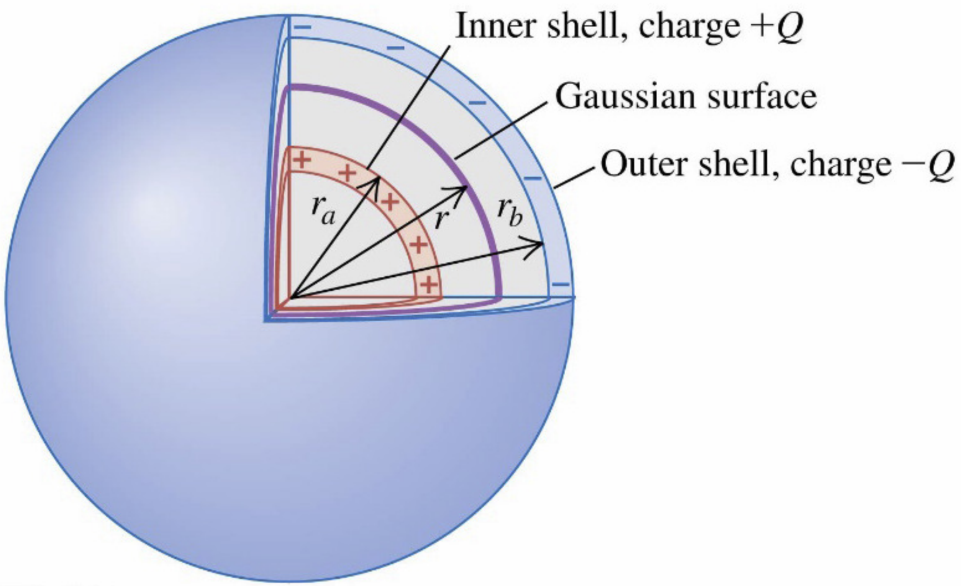
$$\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \frac{Q}{\epsilon_0}$$

$$E(4\pi r^2) = \frac{Q}{\epsilon_0} \Rightarrow E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$$

$$V_{ab} = \int_a^b \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{r}} = \int_{r_a}^{r_b} E dr$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

$$= \frac{Q}{4\pi\epsilon_0}\frac{r_b - r_a}{r_a r_b}$$



$$\boxed{C = 4\pi\epsilon_0 \frac{r_a r_b}{r_b - r_a}}$$

⚠ outer shell with charge $-Q$ has no effect on E, V_{ab}, nor C. It shields the field from outside.

⚠ same as parallel-plate capacity if $d \rightarrow r_b - r_a$ and $A \rightarrow 4\pi r_a r_b$ (**geometric mean** of inner and outer shell surfaces, $4\pi r_a^2$ and $4\pi r_b^2$).

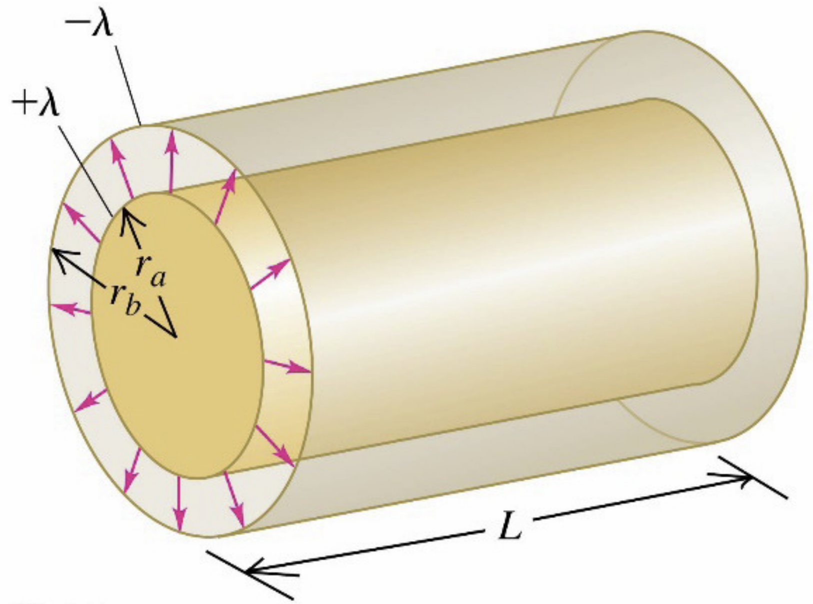
Cylindrical Capacitor Example 24.4

Coaxial conductors with charge per unit length $\pm\lambda$

$$E(2\pi r L) = \frac{\lambda L}{\epsilon_0} \quad \Rightarrow \quad E = \frac{\lambda}{2\pi\epsilon_0 r}$$

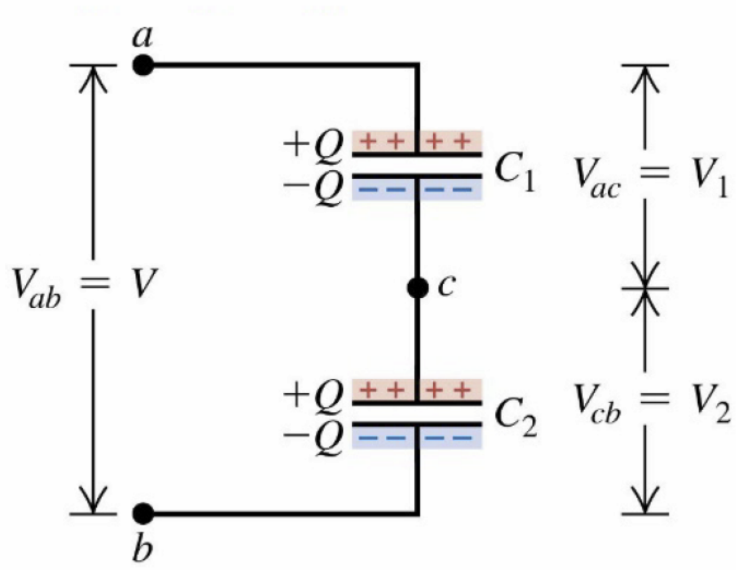
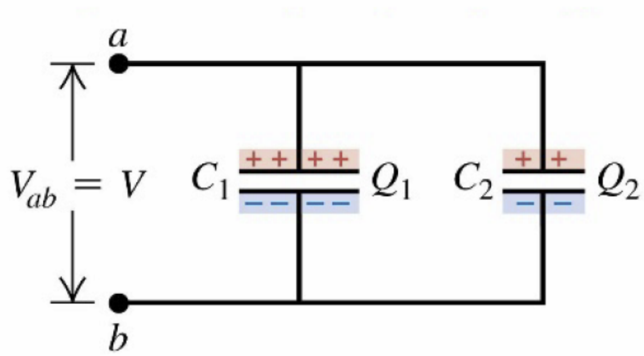
$$V_{ab} = \int_{r_a}^{r_b} E dr = \frac{\lambda}{2\pi\epsilon_0} \ln\frac{r_b}{r_a}$$

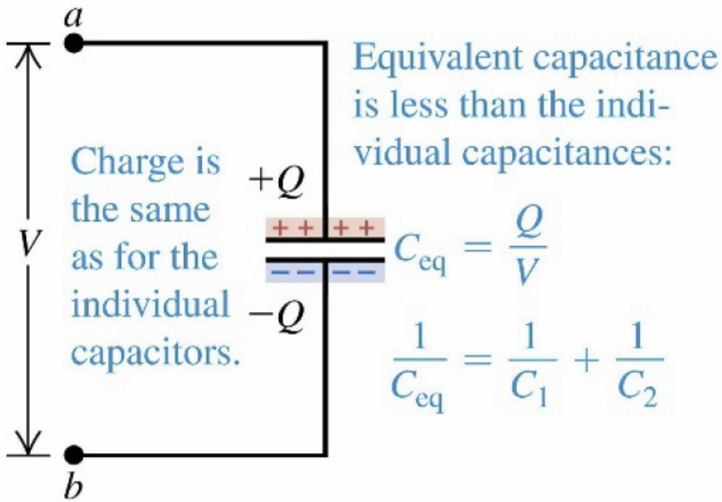
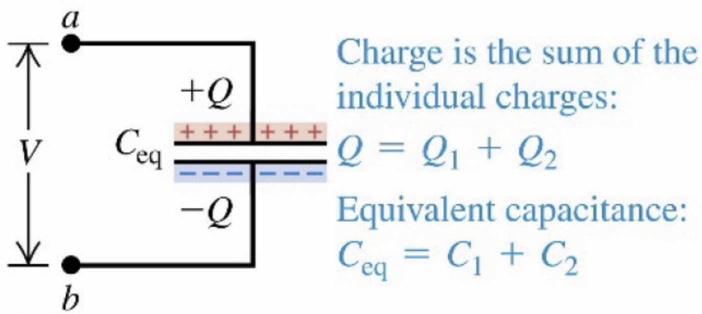
$$\boxed{C = 2\pi\epsilon_0 \frac{L}{\ln(r_b/r_a)}}$$



e.g. a typical TV coaxial cable has capacitance 69 pF/m

Capacitors in series and parallel

	Two capacitors in series	Two capacitors in parallel
Network		
V across each capacitor	Different unless $C_1 = C_2$	same
Q in each capacitor	same	Different unless $C_1 = C_2$

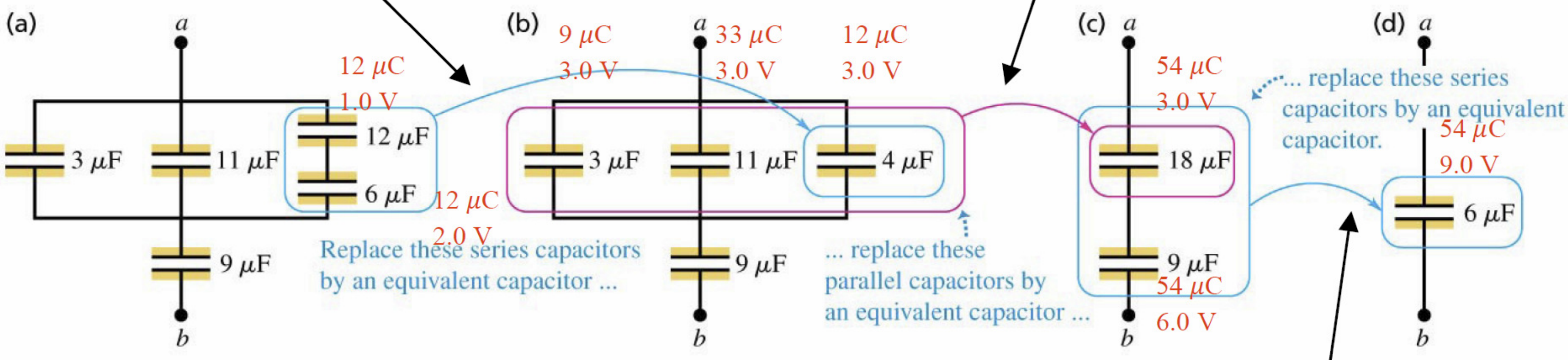
Equivalent capacitor		
Total voltage	$V_1 + V_2 = \frac{Q}{C_1} + \frac{Q}{C_2}$	V
Total charge	Q	$Q_1 + Q_2 = C_1 V + C_2 V$
Equivalent capacitance	$\boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}}$	$\boxed{C_{eq} = C_1 + C_2}$

Example 24.6 A capacitor network

Work out the equivalent capacity from (a) → (b) → (c) → (d)

$$\frac{1}{C_{\text{eq}}} = \frac{1}{12\ \mu\text{F}} + \frac{1}{6\ \mu\text{F}} \Rightarrow C_{\text{eq}} = 4\ \mu\text{F}$$

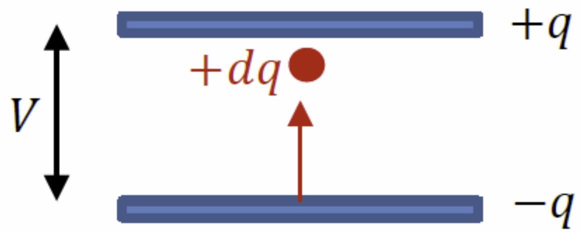
$$3\ \mu\text{F} + 11\ \mu\text{F} + 4\ \mu\text{F} = 18\ \mu\text{F}$$



$$\frac{1}{C_{\text{eq}}} = \frac{1}{18\ \mu\text{F}} + \frac{1}{9\ \mu\text{F}} \Rightarrow C_{\text{eq}} = 6\ \mu\text{F}$$

If given $V_{ab} = 9.0$ V, work out the charge and voltage of each capacitor in (a) in the reversed manner (d) → (c) → (b) → (a) as shown in red.

Energy Stored in Capacitors



Start from uncharged conductors, add charge slowly

At some stage when charge is q and voltage V, workdone by electric field to move dq across a potential difference V

$$= -V\,dq = -\frac{q}{C}\,dq$$

Total workdone by electric field to build up charge from 0 to Q

$$W = -\int_0^Q \frac{q}{C}\,dq = -\frac{Q^2}{2C} = -(U(Q) - U(0))$$

If define $U(0) = 0$ (potential energy of an uncharged capacitor is zero)

$$\boxed{U = \frac{Q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}QV}$$

⚠ $U \neq QV$ because the voltage is not constant during the charging process

⚠ U is also the work needed (provided by external agent) to charge the capacitor, i.e., $-W$

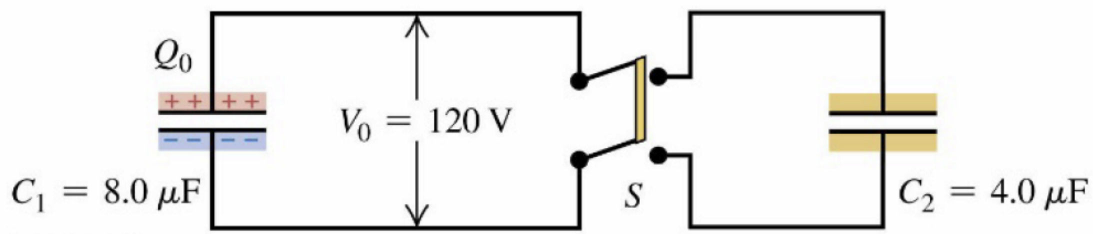
Example 24.7

With switch S open:

C_1 is at $V_0 = 120$ V, while C_2 uncharged

charge of C_1 is $Q_0 = C_1 V_0 = 960\ \mu\text{C}$

energy stored in C_1 is $U_0 = \frac{1}{2} C_1 V_0^2 = 0.058$ J



With switch S close and charge stop flowing:

Conservation of charge $Q_0 = Q_1 + Q_2$

$$V_1 = V_2 \quad \Longrightarrow \quad \frac{Q_1}{C_1} = \frac{Q_2}{C_2}$$

We get $Q_1 = 640\ \mu\text{C}$, $Q_2 = 320\ \mu\text{C}$, $V_1 = V_2 = 80$ V

Energy stored in C_1 and $C_2 = \frac{1}{2} V_1 Q_1 + \frac{1}{2} V_2 Q_2 = 0.038$ J

Question: where goes the energy difference $0.058\ \text{J} - 0.038\ \text{J} = 0.020$ J?

Electric-Field Energy

Charging a capacitor builds up electric field. Can consider U as energy stored in the electric field

For parallel-plate capacitor, energy density (per unit volume) of electric field in between conductors

$$u = \frac{\frac{1}{2}CV^2}{Ad} = \frac{\frac{1}{2}(\epsilon_0 A/d)(Ed)^2}{Ad} \Rightarrow \boxed{u = \frac{1}{2}\epsilon_0 E^2}$$

⚠ this formula is *not* limited to a parallel-plate capacitor. It holds for any electric field, whether uniform or not

⚠ provide another way to **interpret electric potential energy as stored locally in the field**

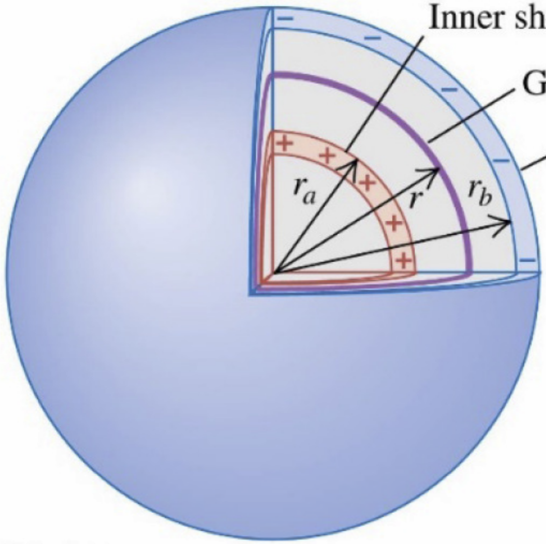
Energy stored in spherical conductor Example 24.9

Electric field radially outward between the shells. From Gauss's law

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$$

And capacitance is

$$C = 4\pi\epsilon_0 \frac{r_a r_b}{r_b - r_a}$$



Two ways to interpret the potential energy of this capacitor

as work needed to assemble charge	as energy stored in electric field
$U = \frac{Q^2}{2C} = \frac{Q^2}{8\pi\epsilon_0}\frac{r_b - r_a}{r_a r_b}$	$U = \int_{r_a}^{r_b}\left(\frac{1}{2}\epsilon_0 E^2\right)(4\pi r^2 dr) = \frac{Q^2}{8\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) = \frac{Q^2}{8\pi\epsilon_0}\frac{r_b - r_a}{r_a r_b}$

Question: You have a 4 μF and a 8 μF capacitors.

a) The voltage of the 4 μF capacitor is greater than that of the 8 μF one if they are (in series / in parallel / either series or parallel / neither series nor parallel).

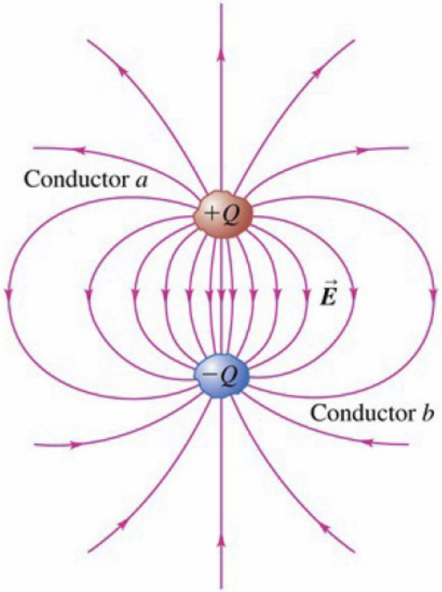
b) The charge of the 4 μF capacitor is greater than that of the 8 μF one if they are (in series / in parallel / either series or parallel / neither series nor parallel).

c) The energy stored in the 4 μF capacitor is greater than that of the 8 μF one if they are (in series / in parallel / either series or parallel / neither series nor parallel).

Clicker Questions

Q24.1

The two conductors a and b are insulated from each other, forming a capacitor. You increase the charge on a to $+2Q$ and increase the charge on b to $-2Q$, while keeping the conductors in the same positions. As a result of this change, the capacitance C of the two conductors



A. becomes four times as great.

B. becomes twice as great.

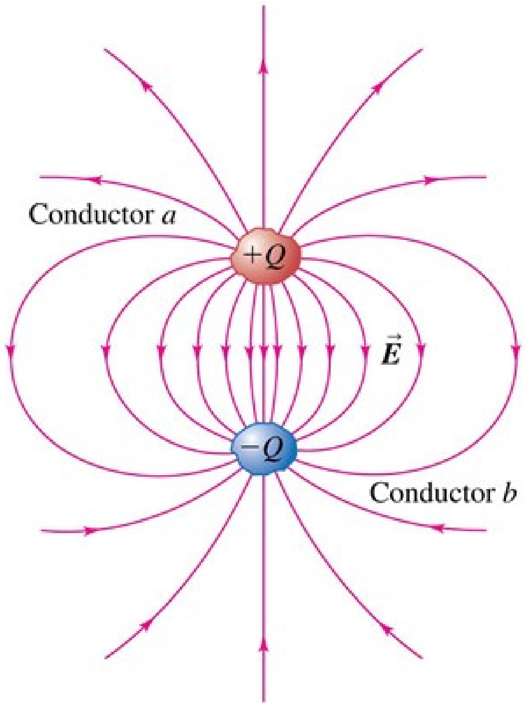
C. remains the same.

D. becomes half as great.

E. becomes one-quarter as great.

A24.1

The two conductors *a* and *b* are insulated from each other, forming a capacitor. You increase the charge on *a* to $+2Q$ and increase the charge on *b* to $-2Q$, while keeping the conductors in the same positions. As a result of this change, the capacitance C of the two conductors



A. becomes four times as great.

B. becomes twice as great.

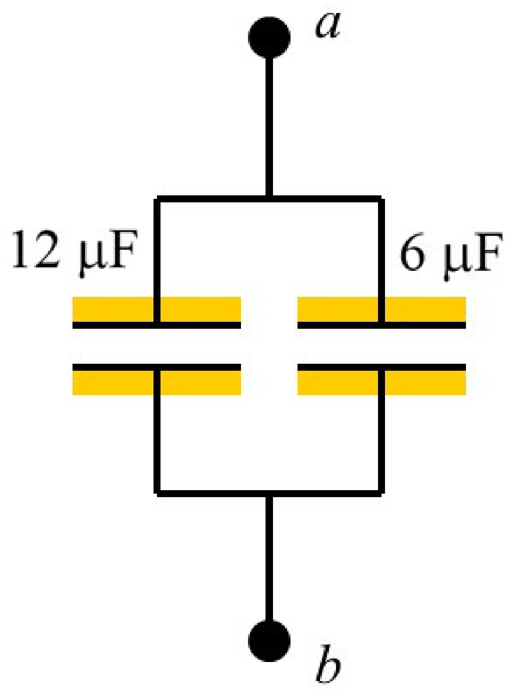
✔ C. remains the same.

D. becomes half as great.

E. becomes one-quarter as great.

Q24.5

A 12-μF capacitor and a 6-μF capacitor are connected together as shown. What is the equivalent capacitance of the two capacitors as a unit?



A. $C_{eq} = 18\ \mu F$

B. $C_{eq} = 9\ \mu F$

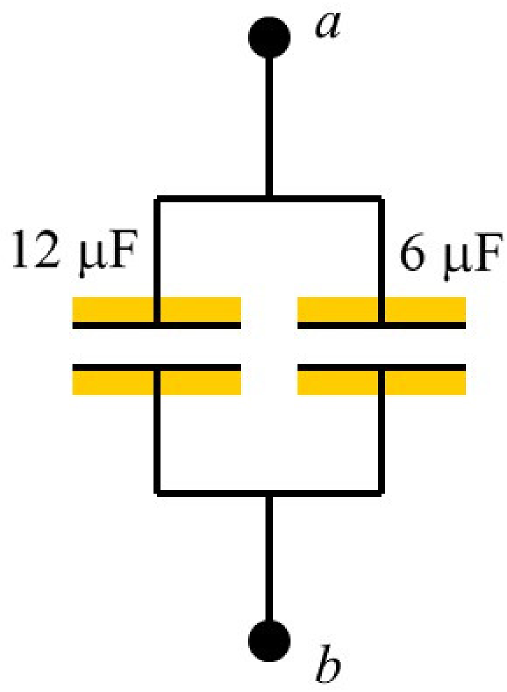
C. $C_{eq} = 6\ \mu F$

D. $C_{eq} = 4\ \mu F$

E. $C_{eq} = 2\ \mu F$

A24.5

A 12-μF capacitor and a 6-μF capacitor are connected together as shown. What is the equivalent capacitance of the two capacitors as a unit?



A. $C_{eq} = 18\ \mu F$

B. $C_{eq} = 9\ \mu F$

C. $C_{eq} = 6\ \mu F$

D. $C_{eq} = 4\ \mu F$

E. $C_{eq} = 2\ \mu F$

Q24.7

You reposition the two plates of a capacitor so that the capacitance doubles. There is vacuum between the plates. If the charges $+Q$ and $-Q$ on the two plates are kept constant in this process, the energy stored in the capacitor

A. becomes four times as great.

B. becomes twice as great.

C. remains the same.

D. becomes half as great.

E. becomes one-quarter as great.

A24.7

You reposition the two plates of a capacitor so that the capacitance doubles. There is vacuum between the plates. If the charges $+Q$ and $-Q$ on the two plates are kept constant in this process, the energy stored in the capacitor

A. becomes four times as great.

B. becomes twice as great.

C. remains the same.

✔ D. becomes half as great.

E. becomes one-quarter as great.

Dielectric

A **dielectric** is a non-conducting material, e.g., plastic, air, …

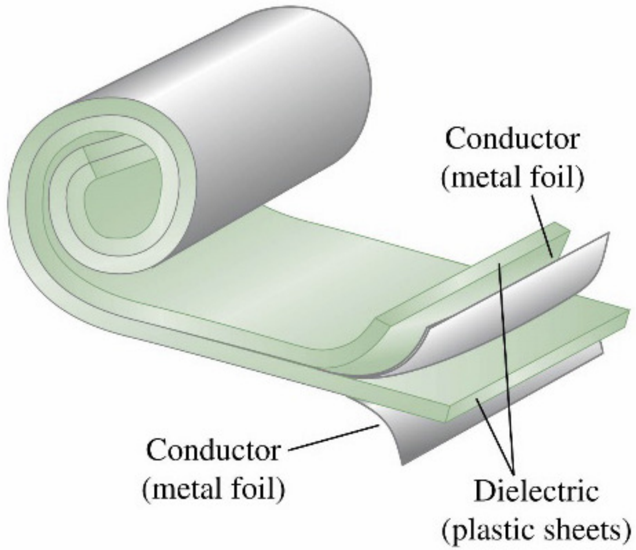
Practical capacitors have an insulating material (or dielectric) sandwiched between the conductors

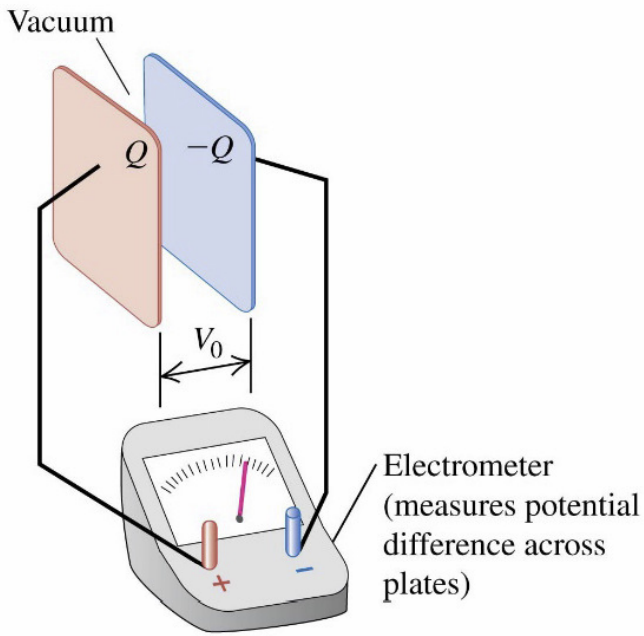
Purpose:

1. Separate the two conductors
2. More difficult to breakdown than air, sustain a higher voltage ,i.e, stronger electric field
3. Increase the capacitance

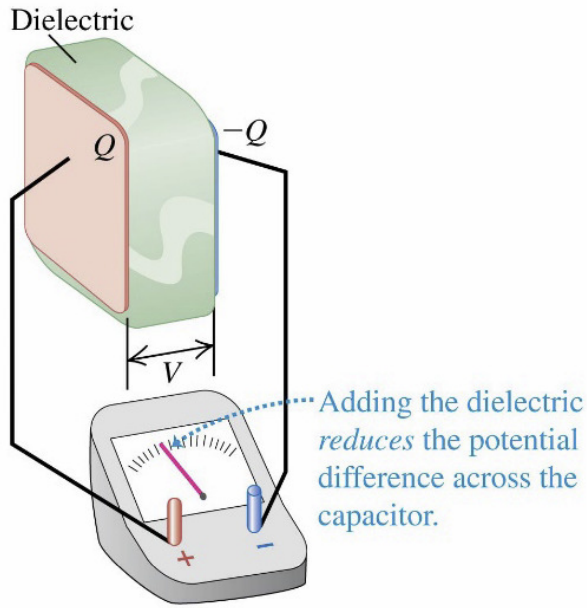
(a)

(b)





$$C_0 = Q/V_0$$



$$C = Q/V$$

With Q remain constant (capacitor isolated), voltage V *decreases* upon insertion of dielectric, i.e., capacitance $C = Q/V$ *increases*

Define **dielectric constant**

$$\boxed{K = \frac{C}{C_0} > 1}$$

or $K = V_0/V$ provided Q remains the same

TABLE 24.1 Values of Dielectric Constant K at 20°C

Material	K	Material	K
Vacuum	1	Polyvinyl chloride	3.18
Air (1 atm)	1.00059	Plexiglas®	3.40
Air (100 atm)	1.0548	Glass	5–10
Teflon	2.1	Neoprene	6.70
Polyethylene	2.25	Germanium	16
Benzene	2.28	Glycerin	42.5
Mica	3–6	Water	80.4
Mylar	3.1	Strontium titanate	310

- ⚠ Dielectric constant of vacuum is 1
- ⚠ Dielectric constant of air is almost the same as vacuum
- ⚠ Pure water has large dielectric constant, but dissolved impurities conduct electricity easily making it unsuitable for capacitors

Under a strong electric field, a dielectric may be partially ionized and becomes conducting, called **dielectric breakdown**, e.g., corona discharge, or even lightning, is the dielectric breakdown of air

The maximum field a dielectric can sustain before breakdown is called the **dielectric strength**

TABLE 24.2 **Dielectric Constant and Dielectric Strength of Some Insulating Materials**

Material	Dielectric Constant, K	Dielectric Strength, E_m (V/m)
Polycarbonate	2.8	3×10^7
Polyester	3.3	6×10^7
Polypropylene	2.2	7×10^7
Polystyrene	2.6	2×10^7
Pyrex glass	4.7	1×10^7

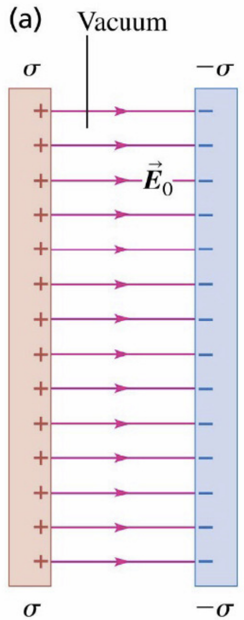
c.f. Air $3 \times 10^6 V/m$

Induced Charge on Dielectric

Again assume capacitor is isolated, the charge density σ *on the conductors* remains unchanged, and dielectric is assumed to be uncharged

$$V = \frac{V_0}{K} \quad \Rightarrow \quad E = \frac{V}{d} = \frac{E_0}{K}$$

Smaller E, charge must be induced on dielectric

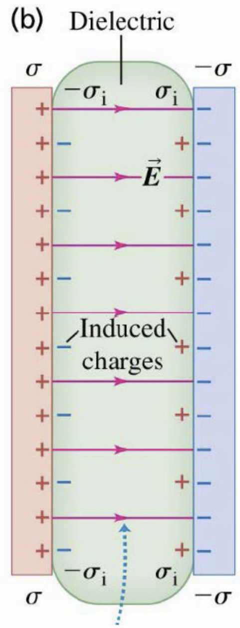


Gauss's law

$$E_0 = \frac{\sigma}{\epsilon_0}$$

$$C_0 = \frac{\sigma A}{V_0} = \frac{\epsilon_0 A}{d}$$

Energy density $u = \frac{1}{2}\epsilon_0 E^2$



Gauss's law

$$E = \frac{\sigma - \sigma_i}{\epsilon_0} = \frac{E_0}{K}$$

$$\Rightarrow \boxed{\sigma_i = \sigma\left(1 - \frac{1}{K}\right)}$$

$$E = \frac{\sigma}{\epsilon}$$

where $\epsilon = K\epsilon_0$, the **permittivity** of the dielectric

c.f., ϵ_0 is the **permittivity in vacuum**

$$C = \frac{\sigma A}{V} = \frac{\epsilon A}{d}$$

$$u = \frac{1}{2}K\epsilon_0 E^2 = \frac{1}{2}\epsilon E^2$$

⚠ just replace $\epsilon_0 \rightarrow \epsilon$

Example 24.10 and 24.11

A parallel-plate capacitor with $A = 2.00 \times 10^{-1}\ \mathrm{m}^2$ and $d = 1.00 \times 10^{-2}$ m, originally charged to a voltage $V_0 = 3.00 \times 10^3$ V and then disconnected. After inserting a plastic that completely fill the space between the plates, the voltage drops to 1.00×10^3 V.

$$K = \frac{V_0}{V} = 3.00$$

Energy stored before inserting plastic

$$U_0 = \frac{1}{2} C_0 V_0^2 = \frac{\epsilon_0 A}{2d} V_0^2 = \frac{(8.85 \times 10^{-12}\ \mathrm{C}^2/\mathrm{Nm}^2)(2.00 \times 10^{-1}\ \mathrm{m}^2)}{2(1.00 \times 10^{-2}\ \mathrm{m})} (3.00 \times 10^3\ \mathrm{V})^2$$

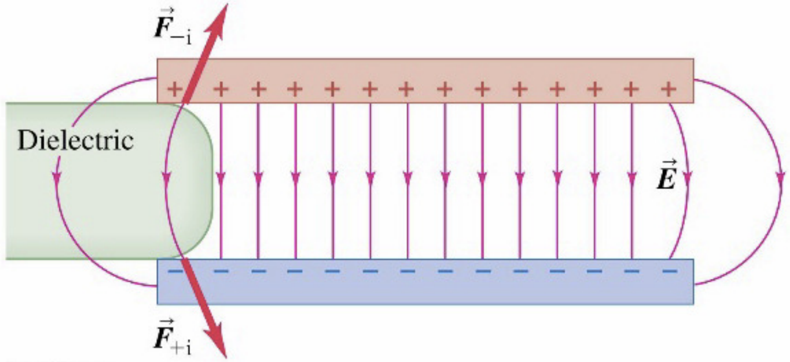
$$= 7.97 \times 10^{-4}\ \mathrm{J}$$

Energy stored after inserting plastic

$$U = \frac{1}{2} C V^2 = \frac{1}{2} (K C_0) \left(\frac{V_0}{K}\right)^2 = \frac{U_0}{K} = 2.66 \times 10^{-4}\ \mathrm{J}$$

Where goes the energy? How can the field do work when $\vec{\boldsymbol{E}} \perp$ displacement?

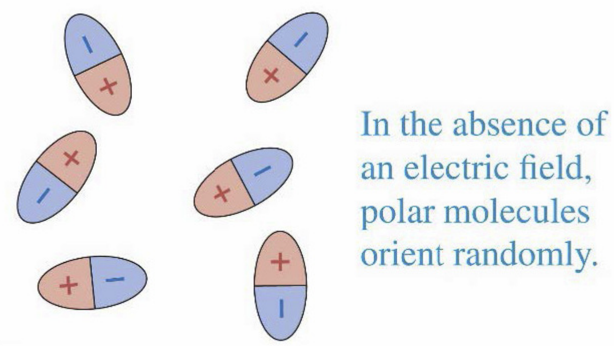
Answer: edge (fringe) effect. While inserting, field at boundary polarize the dielectric and pull it in, doing +ve work, so potential energy decreases, $W = -\Delta U$



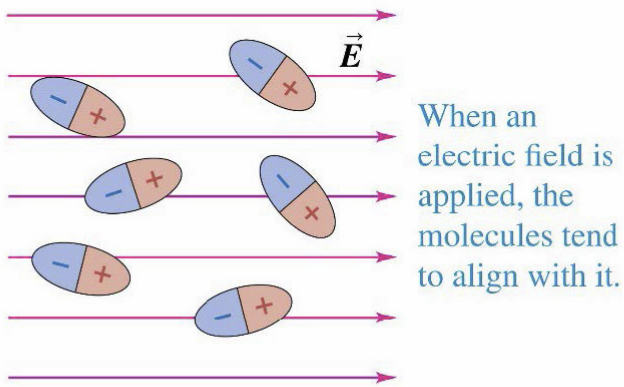
Question: a parallel-plate capacitor with a slab of dielectric in between carries a charge Q. It is isolated so that Q remains unchanged. If you pull the dielectric slab out, the energy stored in the capacitor will (increase / decrease / remain the same).

Molecular Model of Induced Charge

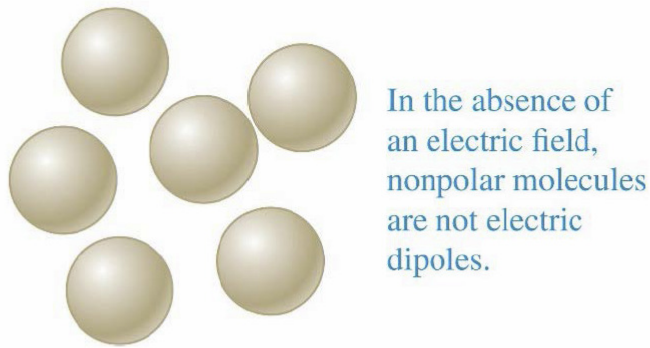
Polar – molecules are permanent dipoles even without external field, but are randomly oriented



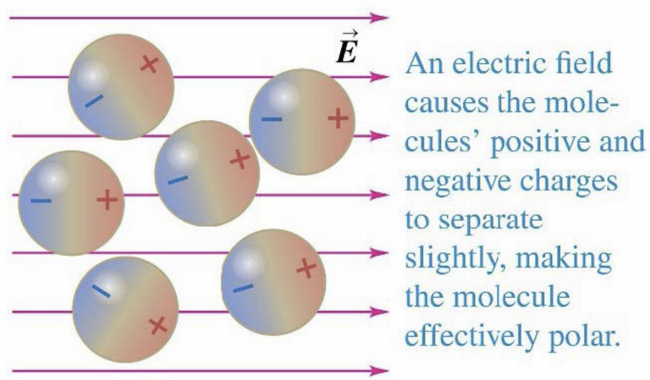
External field partially align dipoles, causing induced surface charge, i.e., the dielectric is **polarized**

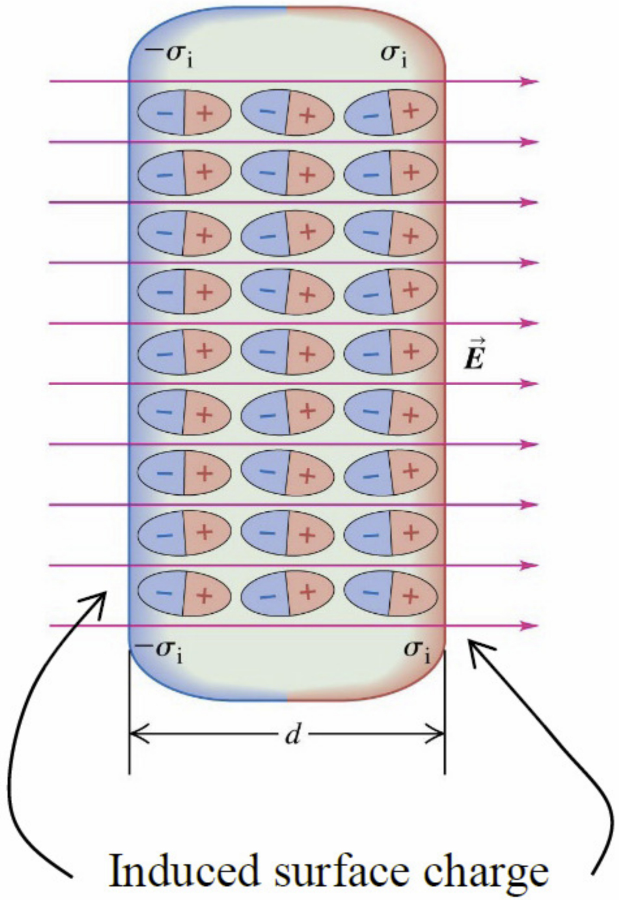


Nonpolar – molecules are not dipoles without external field



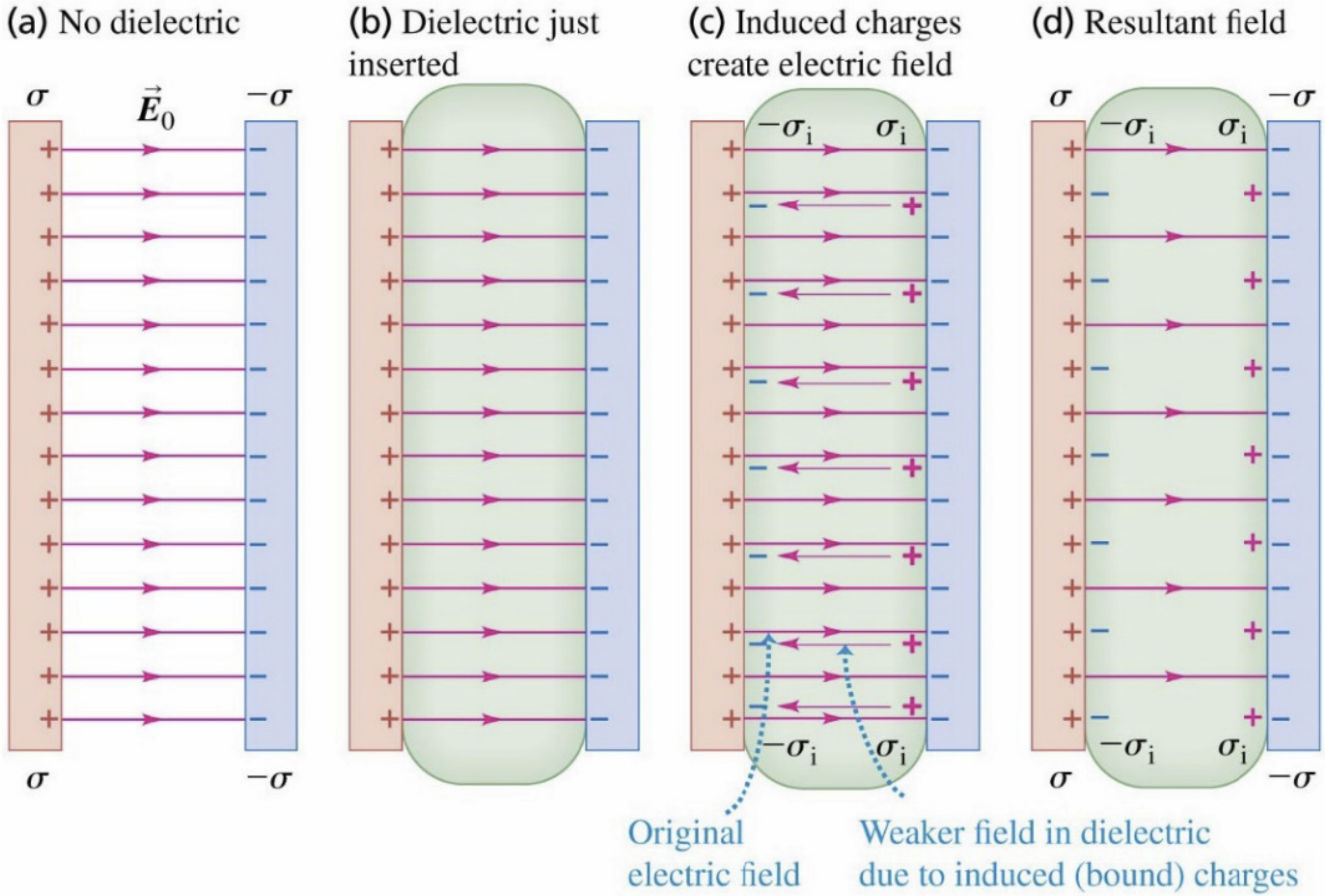
External field creates **induced dipoles**, polarize the dielectric and leads to induced surface charge





$-\sigma_i$
σ_i
$\vec{E}$
$-\sigma_i$
σ_i
d
Induced surface charge

Induced charge creates a field in the dielectric *opposing* the external field, making the total field weaker, *c.f.* $E = E_0/K$



Distinguish between two types of charge:

Free charge – reside on the conductor, can be added or removed

Bound charge – induced on dielectric, cannot move

Gauss's Law in Dielectric

Consider boundary between conductor and dielectric
Free charge σ – those residing on conductor
Bound charge σ_i – those residing on dielectric
By Gauss's law:

$$EA = \frac{(\sigma - \sigma_i)A}{\epsilon_0}$$

Using $\sigma_i = \sigma\left(1 - \frac{1}{K}\right)$, i.e., $\sigma - \sigma_i = \sigma/K$

$$EA = \frac{\sigma A}{K\epsilon_0}$$

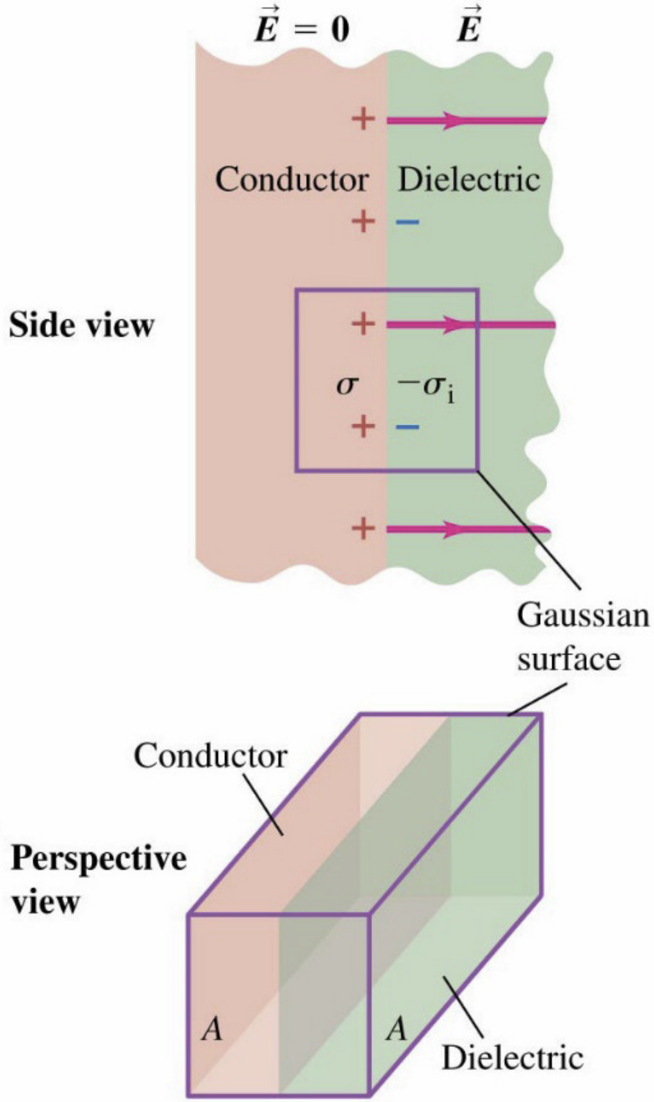
General form of the Gauss's law with a dielectric medium

$$\boxed{\oint K\vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \frac{Q_{\text{encl-free}}}{\epsilon_0}}$$

⚠ $Q_{\text{encl-free}}$ is the free charge only, excluding bound charge
⚠ If K is a constant in the dielectric, then

$$\boxed{\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \frac{Q_{\text{encl-free}}}{\epsilon}}$$

Again replace $\epsilon_0 \to \epsilon$



Example 24.12

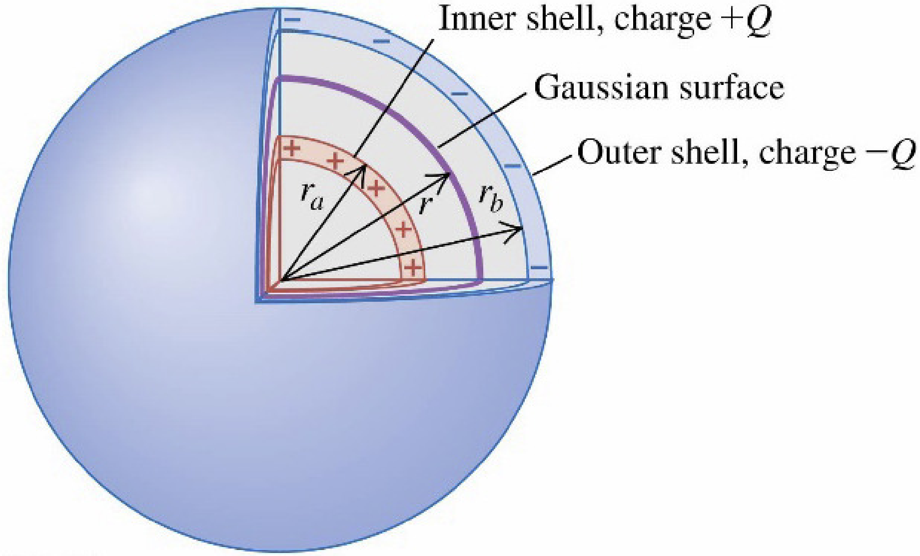
If space between concentric shells is filled with oil of dielectric constant K

From Gauss's law

$$E(4\pi r^2) = \frac{Q}{K\epsilon_0} = \frac{Q}{\epsilon}$$

And

$$C = \frac{Q}{\int_{r_b}^{r_a} E dr} = \frac{4\pi\epsilon r_a r_b}{r_b - r_a}$$



Question: A single point charge q is embedded in a dielectric medium with dielectric constant K. At a point inside the dielectric at a distance r from the point charge, the magnitude of the electric field is ($\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$ / $\frac{K}{4\pi\epsilon_0}\frac{q}{r^2}$ / $\frac{1}{4\pi K\epsilon_0}\frac{q}{r^2}$ / none of these).

Clicker Questions

Q24.10

You slide a slab of dielectric between the plates of a parallel-plate capacitor. As you do this, the *charges* on the plates remain constant. What effect does adding the dielectric have on the *energy stored* in the capacitor?

A. The stored energy increases.

B. The stored energy decreases.

C. The stored energy remains the same.

D. Two of A, B, and C are possible.

E. All three of A, B, or C are possible.

A24.10

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E. All three of A, B, or C are possible.

Q24.12

You slide a slab of dielectric between the plates of a parallel-plate capacitor. As you do this, the *potential difference* between the plates remains constant. What effect does adding the dielectric have on the *energy stored* in the capacitor?

A. The stored energy increases.

B. The stored energy decreases.

C. The stored energy remains the same.

D. Two of A, B, and C are possible.

E. All three of A, B, or C are possible.

A24.12

You slide a slab of dielectric between the plates of a parallel-plate capacitor. As you do this, the *potential difference* between the plates remains constant. What effect does adding the dielectric have on the *energy stored* in the capacitor?

✔ A. The stored energy increases.

B. The stored energy decreases.

C. The stored energy remains the same.

D. Two of A, B, and C are possible.

E. All three of A, B, or C are possible.

Q-RT24.1

An isolated parallel-plate capacitor has charges Q and $-Q$ on its two plates. A dielectric slab with $K = 4.00$ is then inserted into the space between the plates, and completely fills the space. ***Rank*** the following electric-field magnitudes in order from largest to smallest.

A. the field between the plates before the slab is inserted

B. the resultant field between the plates after the slab is inserted

C. the field between the plates due to the bound charges

A-RT24.1

An isolated parallel-plate capacitor has charges Q and $-Q$ on its two plates. A dielectric slab with $K = 4.00$ is then inserted into the space between the plates, and completely fills the space. ***Rank*** the following electric-field magnitudes in order from largest to smallest.

A. the field between the plates before the slab is inserted

B. the resultant field between the plates after the slab is inserted

C. the field between the plates due to the bound charges

✔ Answer: ACB