Current, resistance and EMF

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Current

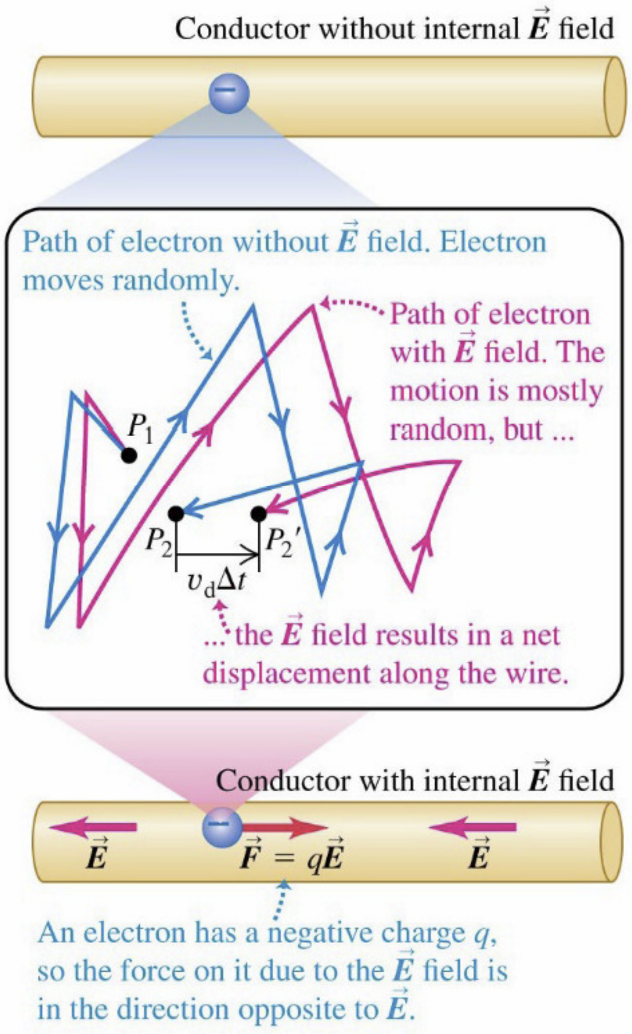
A **current** is a net drift of charge from one place to another

In a conductor:

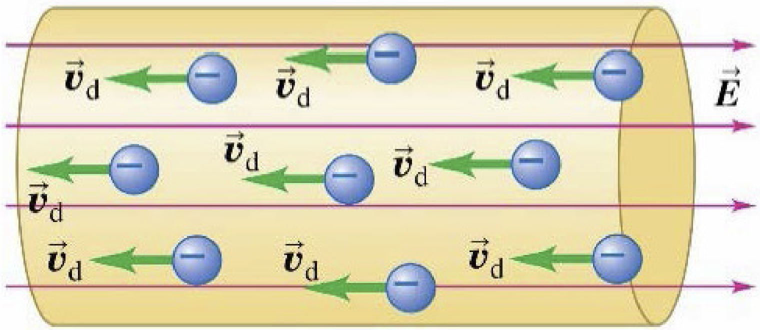
Without external $\vec{E}$ field, electrons move randomly, no net drift

With external $\vec{E}$ field, electrons have a net drift velocity $\vec{v}_d$

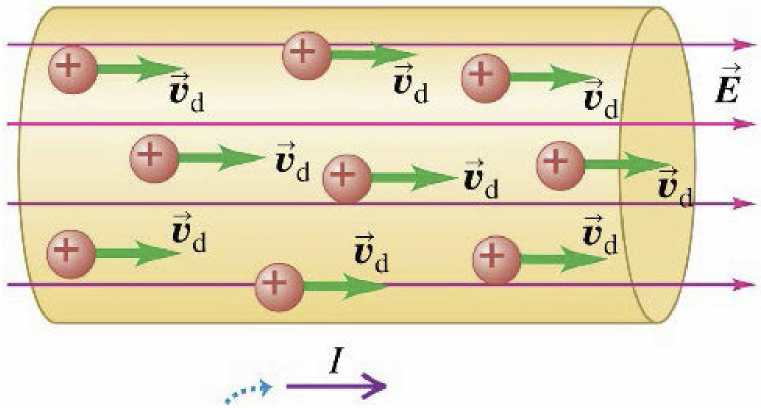
Demonstration: a current can also be produced by [oppositely charge ions drifting in opposite directions]



⚠ direction of **conventional current** is defined by assuming charge carriers to be +ve



Actual drift of charge carriers (electrons)



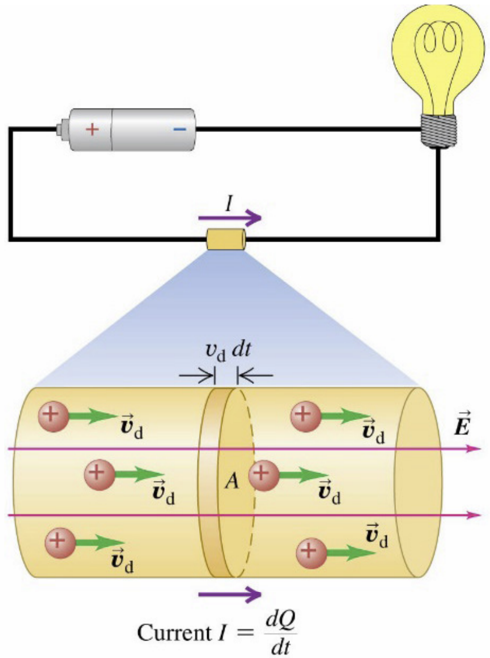
Direction of conventional current by assuming charge carriers to be +ve

Current = *charge flow through a cross section of the conductor per unit time*

$$\boxed{I = \frac{dQ}{dt}}$$

SI unit: ampere, 1 A = 1 C/s

⚠ current has *direction* (±ve along the conductor), but is a scalar



n – number of charge carriers per unit volume, or **charge carrier density**

Total charge passing through cross sectional area A in time interval dt

$$dQ = q(nAv_d dt)$$

$$\Rightarrow \quad I = \frac{dQ}{dt} = nqAv_d$$

To get rid of the geometry of the conductor, define **current density** J as the *current per unit cross sectional area*

$$\boxed{J = \frac{I}{A} = nqv_d}$$

⚠ always assume charge carrier is +ve

Example 25.1

A copper wire with diameter 1.02 mm and carries a current 1.67 A. Free-electron density of copper is $n = 8.5 \times 10^{28}\ \mathrm{m^{-3}}$.

Cross sectional area of wire is

$$A = \frac{\pi d^2}{4} = \frac{\pi(1.02 \times 10^{-3}\ \mathrm{m})^2}{4} = 8.17 \times 10^{-7}\ \mathrm{m^2}$$

Current density is

$$J = \frac{I}{A} = \frac{1.67\ \mathrm{A}}{8.17 \times 10^{-7}\ \mathrm{m^2}} = 2.04 \times 10^{6}\ \mathrm{A/m^2}$$

Drift velocity is

$$v_d = \frac{J}{ne} = \frac{2.04 \times 10^{6}\ \mathrm{A/m^2}}{(8.5 \times 10^{28}\ \mathrm{m^{-3}})(1.60 \times 10^{-19}\ \mathrm{C})} = 1.5 \times 10^{-4}\ \mathrm{m/s}$$

⚠ Compare v_d to

1. Speed of random motion of free electrons in conductor ($\sim 10^{6}\ \mathrm{m/s}$)
2. Speed of electric field spreading throughout the conductor (speed of light)

Question: If we double the cross sectional diameter of a wire, keeping the current a constant, the drift velocity of electrons will be (unchanged / doubled / four times as great / halved / one-fourth as great)

Usually define current density as a vector (*c.f.* current is a scalar)

$$\boxed{\vec{J} = nq\vec{v}_d = \frac{nq^2\tau}{m}\vec{E}}$$

Ohm's Law: $\boxed{\vec{J} \propto \vec{E}}$

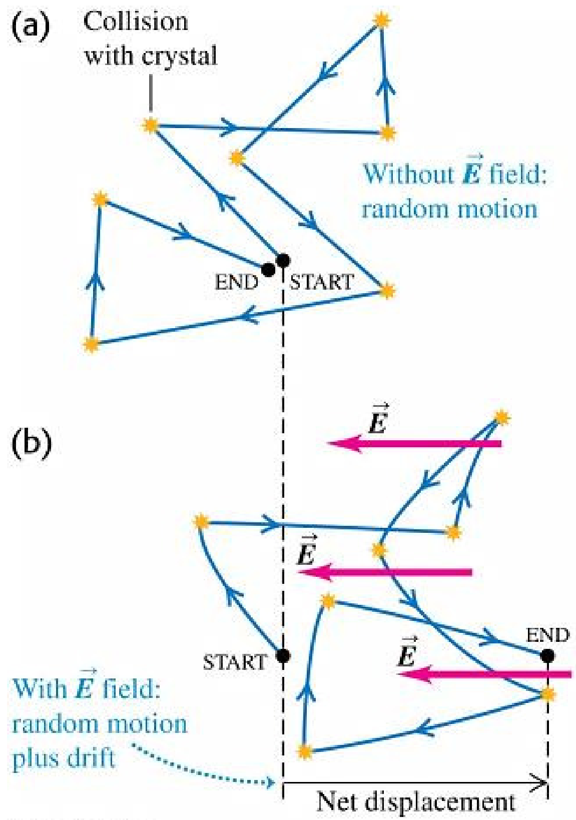
⚠ Ohm's law is *not* a universal law. Those materials (mostly good conductors) obeying Ohm's law are called **ohmic** materials, otherwise they are **non-ohmic**.

Define **resistivity** ρ of the material as the reciprocal of the proportionality constant, $J = E/\rho$
SI unit: $[E]/[J]$ = (V/m) / (A/m^2) = (V/A)m = Ωm, where 1 Ω (ohm) = 1 A / 1 V
$1/\rho$ (reciprocal of resistivity) is called the **conductivity**, SI unit: $(\Omega\text{m})^{-1}$

Theory of Metallic Conduction

Mean free time: average time between collisions

Without electric field, the average velocity of an electron Is zero



$$\underbrace{\vec{a} = \frac{\vec{F}}{m} = \frac{q\vec{E}}{m}, \quad \vec{v} = \vec{v}_0 + \vec{a}\tau}$$

$$\vec{v}_{\text{av}} = \langle \vec{v}_0 \rangle_{\text{ave}} + \vec{a}\tau = \frac{q\tau}{m}\vec{E} = \sigma E$$

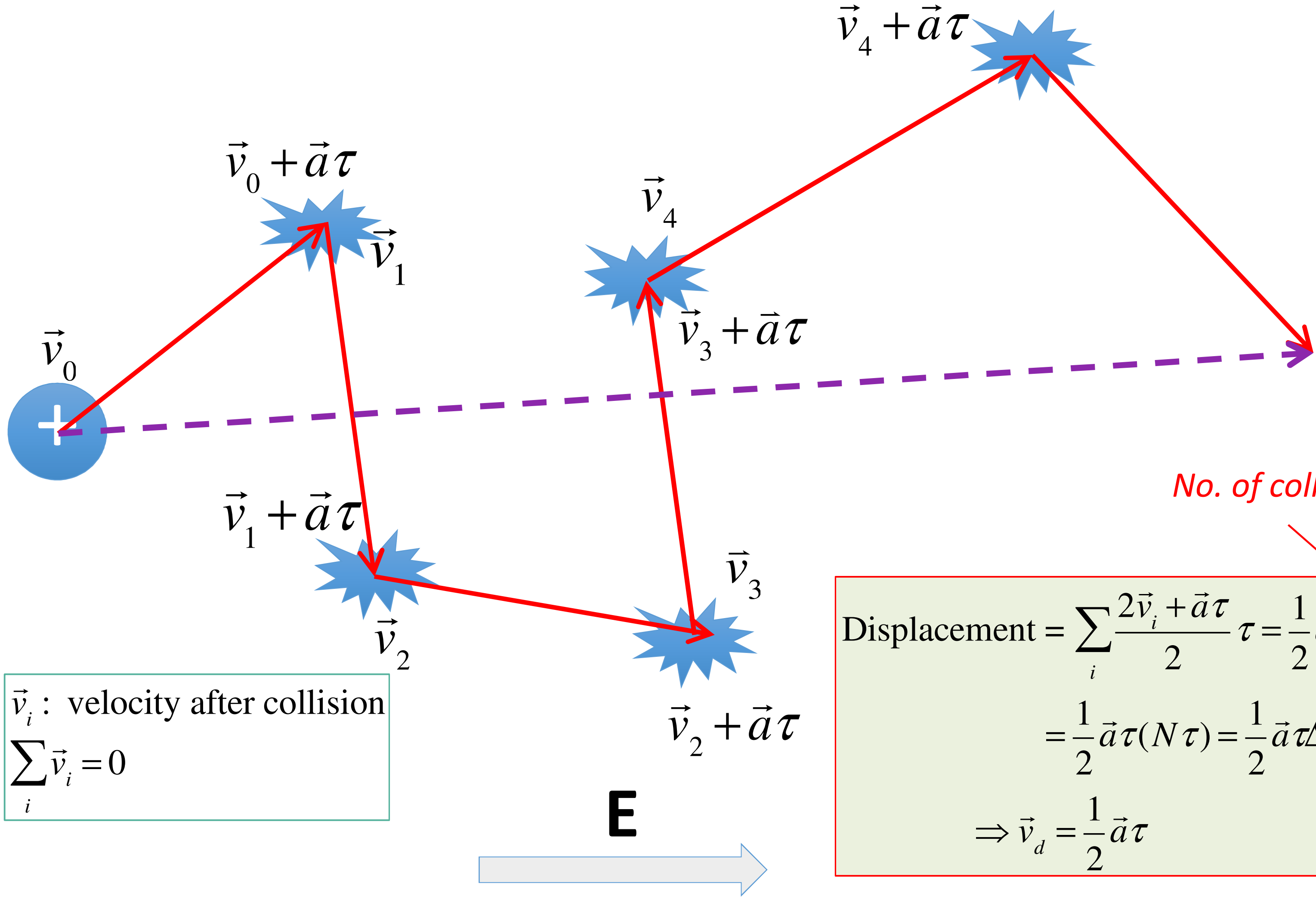
***Average* velocity of an electron**

$$\Rightarrow \vec{v}_{\text{av}} = \vec{v}_{\text{d}} = \frac{q\tau}{m}\vec{E}$$

$$\Rightarrow \vec{J} = nq\vec{v}_{\text{d}} = \frac{nq^2\tau}{m}\vec{E}$$

$$\Rightarrow \rho = \frac{E}{J} = \frac{m}{nq^2\tau} = \frac{m}{ne^2\tau}.$$

Drift velocity



$\vec{v}_i$: velocity after collision

$$\sum_i \vec{v}_i = 0$$

No. of collisions

$$\text{Displacement} = \sum_i \frac{2\vec{v}_i + \vec{a}\tau}{2}\tau = \frac{1}{2}\vec{a}\tau^2 N$$

$$= \frac{1}{2}\vec{a}\tau(N\tau) = \frac{1}{2}\vec{a}\tau\Delta t$$

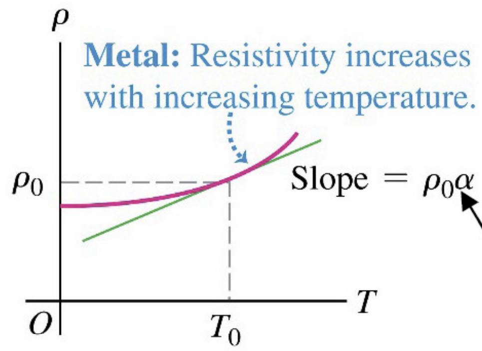
$$\Rightarrow \vec{v}_d = \frac{1}{2}\vec{a}\tau$$

Define **resistivity** ρ of the material as the reciprocal of the proportionality constant, $J = E/\rho$

SI unit: $[E]/[J]$ = (V/m) / (A/m^2) = (V/A)m = Ωm, where 1 Ω (ohm) = 1 A / 1 V

$1/\rho$ (reciprocal of resistivity) is called the **conductivity**, SI unit: $(\Omega\text{m})^{-1}$

(a)

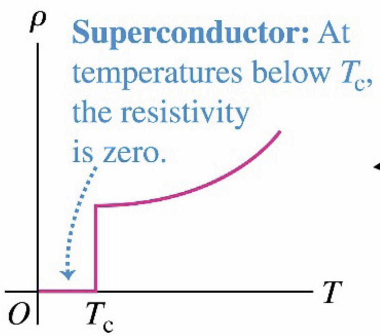


Even for good conductors like metals, ρ increases with temperature (ions vibrate more rigorously, making electrons more difficult to drift)

For a small enough temperature change, ρ can be approximated by a linear relation

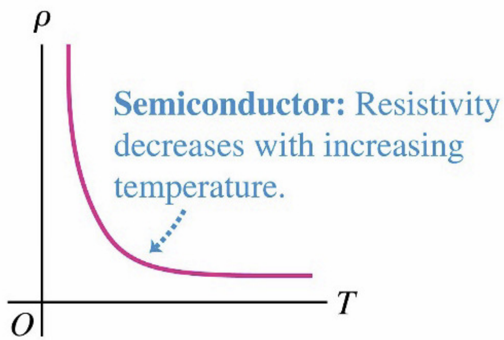
$$\frac{\rho(T) - \rho_0}{T - T_0} = \rho_0 \alpha \quad \Rightarrow \quad \boxed{\rho(T) = \rho_0[1 + \alpha(T - T_0)]}$$

(c)



These cases are more complicated

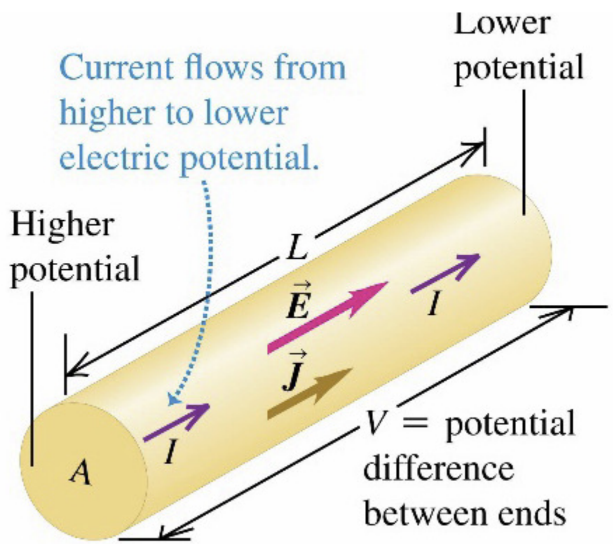
(b)



In a conductor with cross sectional area A

$$V = EL = \rho J L = \underbrace{\left(\frac{\rho L}{A}\right)}_{R} I$$

Ohm's law, $E = \rho J$



Another way to express Ohm's law: $\boxed{V = IR}$, where $\boxed{R = \rho L/A}$ is called the **resistance** of the conductor, SI unit: ohm Ω

- while ρ depends on the material of the conductor only, R also depends on its geometry
- if $\rho(T)$ linear in T, so is R with the same constant α

$$\boxed{R(T) = R_0[1 + \alpha(T - T_0)]}$$

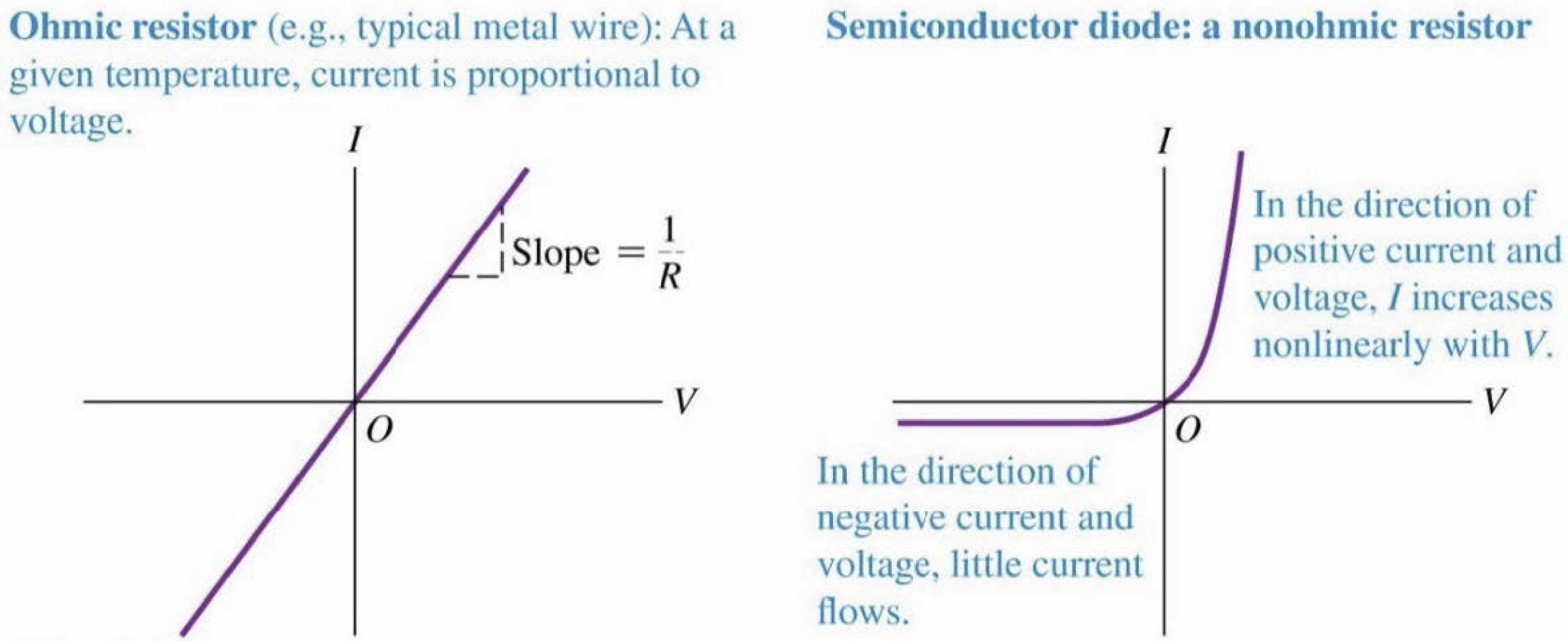
Example 25.3

A copper wire has resistance $1.05\ \Omega$ at $20\ °\mathrm{C}$. From Table 25.2, $\alpha = 0.00393\ (°\mathrm{C})^{-1}$.

Resistance at $100\ °\mathrm{C} = (1.05\ \Omega)\{1 + [0.00393\ (°\mathrm{C})^{-1}][100\ °\mathrm{C} - 20\ °\mathrm{C}]\} = 1.38\ \Omega$

⚠ A conductor dissipate more energy when it is heated up

Ohmic resistors mean R is constant (when the temperature is not changed), independent of V



Example 25.2

A copper wire of length 50.0 m and cross sectional area $8.20 \times 10^{-7}\ \mathrm{m}^2$, carrying a current 1.67 A.

Lookup from Table 25.1, resistivity of copper is $\rho = 1.72 \times 10^{-8}\ \Omega\mathrm{m}$

Electric field inside the wire

$$E = \rho J = \frac{\rho I}{A} = \frac{(1.72 \times 10^{-8}\ \Omega\mathrm{m})(1.67\ \mathrm{A})}{8.20 \times 10^{-7}\ \mathrm{m}^2} = 0.0350\ \mathrm{V/m}$$

Potential difference

$$V = EL = (0.0350\ \mathrm{V/m})(50.0\ \mathrm{m}) = 1.75\ \mathrm{V}$$

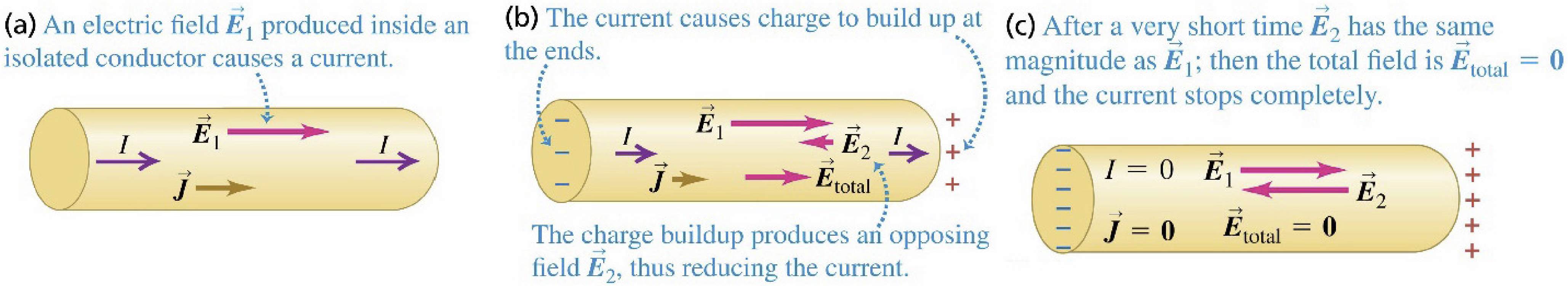
Resistance

$$R = \frac{V}{I} = \frac{1.75\ \mathrm{V}}{1.67\ \mathrm{A}} = 1.05\ \Omega$$

$$R = \frac{\rho L}{A} = 1.05\Omega$$

Electromotive force

An electric field set up in an isolated conductor (not in a complete circuit) cannot drive a steady current



To maintain a steady current, need

- ✓ a complete loop (circuit)
- ✓ a device (e.g. a battery) that provides an **electromotive force**, or emf, $\mathcal{E}$, that bumps (+ve) charge from lower to higher potential, but *it is not a force*

$\mathcal{E}$ is defined as the **energy per unit charge needed to move charge q from b to a against the electric field**.

SI unit: V (same as potential)

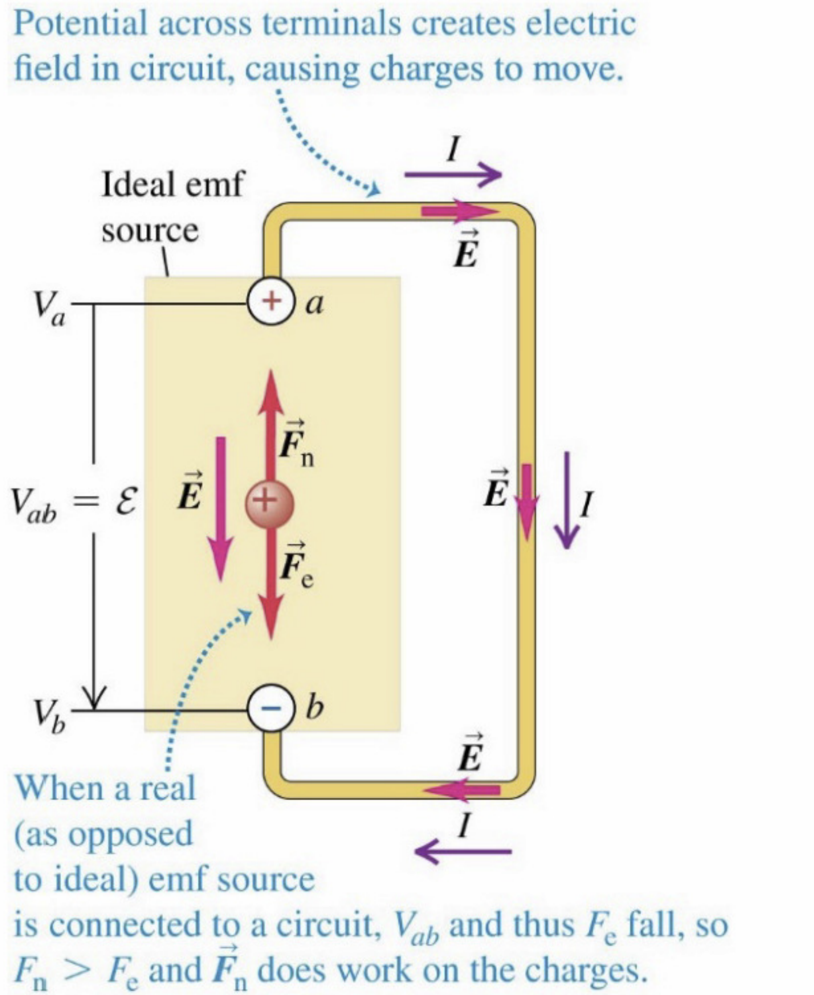
Ideally

$$q\mathcal{E} = qV_{ab} \quad \Rightarrow \quad \mathcal{E} = V_{ab}$$

In reality, the device has **internal resistance** r

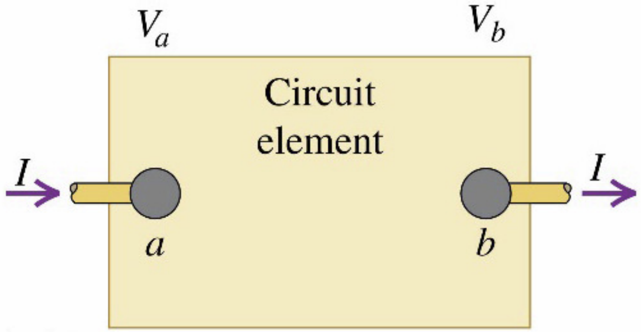
$$\mathcal{E} - Ir = V_{ab}$$

Potential drop after current passing through r



⚠ voltage across terminals of the battery V_{ab} depends on the current

Power in Electric Circuits



Workdone by electric field in moving charge dQ from a to b

$$dW = V_{ab}dQ = V_{ab}Idt$$

Power delivered to the element

$$\boxed{P = \frac{dW}{dt} = V_{ab}I}$$

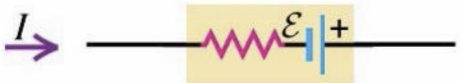
SI unit: Watt, 1 W = (1 J/C)(1 C/s) = 1 J/s

Power dissipated a resistor

$$\boxed{P = V_{ab}I = I^2R = \frac{V_{ab}^2}{R}}$$

Power delivered *by* a battery

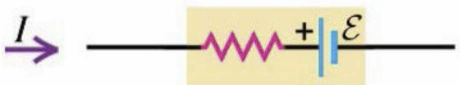
$$\boxed{P = V_{ab}I = \mathcal{E}I - I^2r}$$



Power delivered *to* a battery (charging)

power dissipated in internal resistance

$$\boxed{P = V_{ab}I = \mathcal{E}I + I^2r}$$



Example 25.5 and 25.8

To find current:

$$\mathcal{E} = Ir + IR \Rightarrow I = \frac{\mathcal{E}}{r+R} = 2\ \text{A}$$

Rate of energy conversion (e.g., chemical to electrical) inside battery

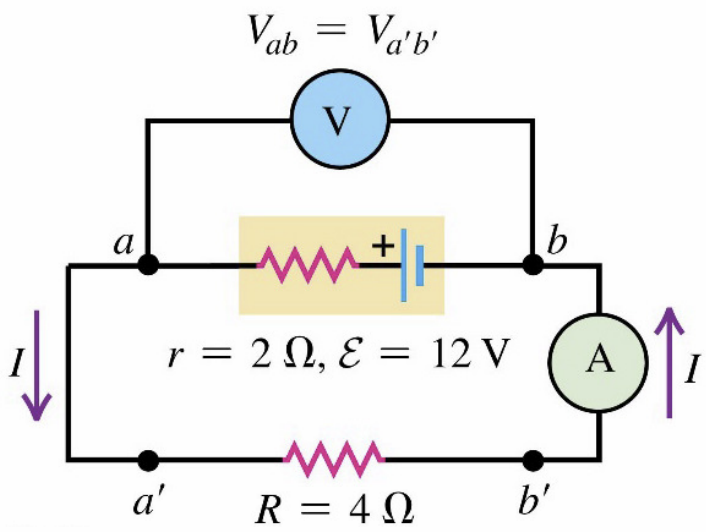
$$= \mathcal{E}I = 24\ \text{V}$$

Power delivered by battery to outside circuit

$$= \text{power dissipated in } R = I^2R = 16\ \text{W}$$

Power dissipated in internal resistance r

$$= I^2r = 8\ \text{W}$$



- ⚠ voltage across the battery terminals $V_{ab} = \mathcal{E} - Ir = 8\ \text{V}$, significantly smaller than $\mathcal{E}$ since the internal resistance r is large, *c.f.* for a car battery, $r \sim 10^{-3}\ \Omega$
- ⚠ As the battery "drains", it doesn't drain its charge (it is not a capacitor), but the stored energy
- ⚠ Rechargeable batteries are often rated in Ahr (e.g., ~3000 mAhr in typical cell phones), which is a unit of *charge*. But the nominal voltage is already implied. Total energy stored in the battery is the nominal voltage times this rating in Ahr.

Clicker Questions

Q25.1

Two copper wires of different diameter are joined end to end, and a current flows in the wire combination. When electrons move from the larger-diameter wire into the smaller-diameter wire,

A. their drift speed increases.

B. their drift speed decreases.

C. their drift speed stays the same.

D. either A or B is possible, depending on circumstances.

E. any of A, B, or C is possible, depending on circumstances.

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Q25.4

A source of emf is connected by wires to a resistor, and electrons flow in the circuit. The wire diameter is the same throughout the circuit. Compared to the *drift speed of the electrons* before entering the *source of emf*, the *drift speed of the electrons* after leaving the *source of emf* is

A. faster.

B. slower.

C. the same.

D. either A or B depending on circumstances.

E. any of A, B, or C depending on circumstances.

A25.4

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Q-RT25.1

Rank the following circuits in order from highest to lowest values of the current in the circuit.

A. a 1.4-Ω resistor connected to a 1.5-V battery that has an internal resistance of 0.10 Ω

B. an unknown resistor connected to a 12.0-V battery that has an internal resistance of 0.20 Ω and a terminal voltage of 11.0 V

C. a 1.2-Ω resistor connected to a 4.0-V battery that has a terminal voltage of 3.6 V but an unknown internal resistance

A-RT25.1

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✔ Answer: BCA

Direct-current circuits

Resistors in series

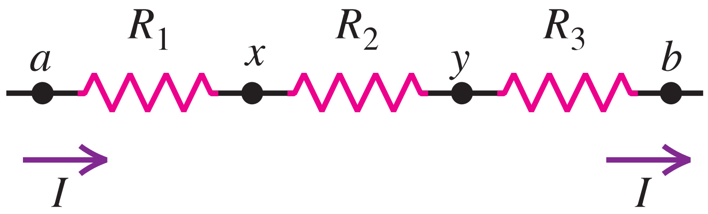
- the current must be SAME in all resistors

$$V_{ax} = IR_1 \qquad V_{xy} = IR_2 \qquad V_{yb} = IR_3$$

$$V_{ab} = V_{ax} + V_{xy} + V_{yb} = I(R_1 + R_2 + R_3)$$

$$R_{\text{eq}} = \frac{V_{ab}}{I} = R_1 + R_2 + R_3$$

(a) R_1, R_2, and R_3 in series



- The equivalent resistance:

$$R_{\text{eq}} = R_1 + R_2 + R_3 + \cdots \qquad \text{(resistors in series)}$$

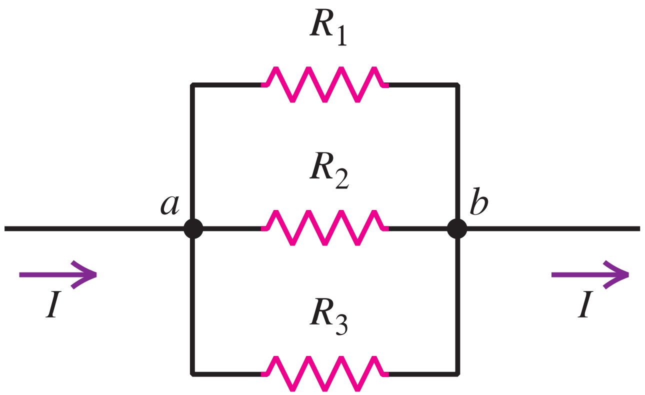
Resistors in parallel

- the potential difference (voltage) must be SAME across all resistors

$$I_1 = \frac{V_{ab}}{R_1} \qquad I_2 = \frac{V_{ab}}{R_2} \qquad I_3 = \frac{V_{ab}}{R_3}$$

$$I = I_1 + I_2 + I_3 = V_{ab}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

$$\frac{1}{R_{\mathrm{eq}}} = \frac{I}{V_{ab}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

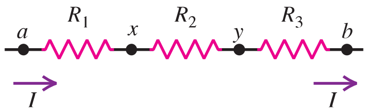


- The equivalent resistance:

$$\frac{1}{R_{\mathrm{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots \qquad \text{(resistors in parallel)}$$

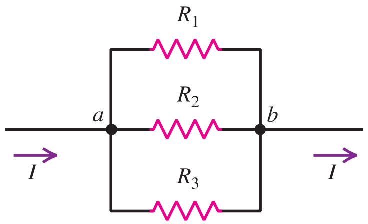
4 different ways to connect 3 resistors

(a) R_1, R_2, and R_3 in series



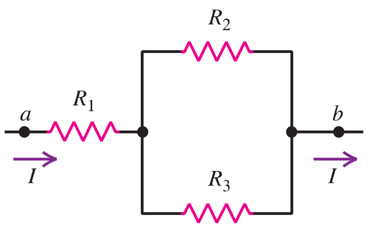
$$R_{\text{eq}} = R_1 + R_2 + R_3$$

(b) R_1, R_2, and R_3 in parallel



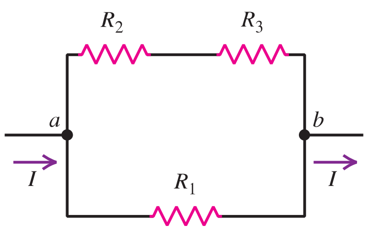
$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

(c) R_1 in series with parallel combination of R_2 and R_3



$$R_{\text{eq}} = R_1 + \frac{R_2 R_3}{R_2 + R_3}$$

(d) R_1 in parallel with series combination of R_2 and R_3



$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2 + R_3}$$

Example 26.1 Equivalent resistance

Find the equivalent resistance of the network in Fig. 26.3a and the current in each resistor. The source of emf has negligible internal resistance.

SOLUTION

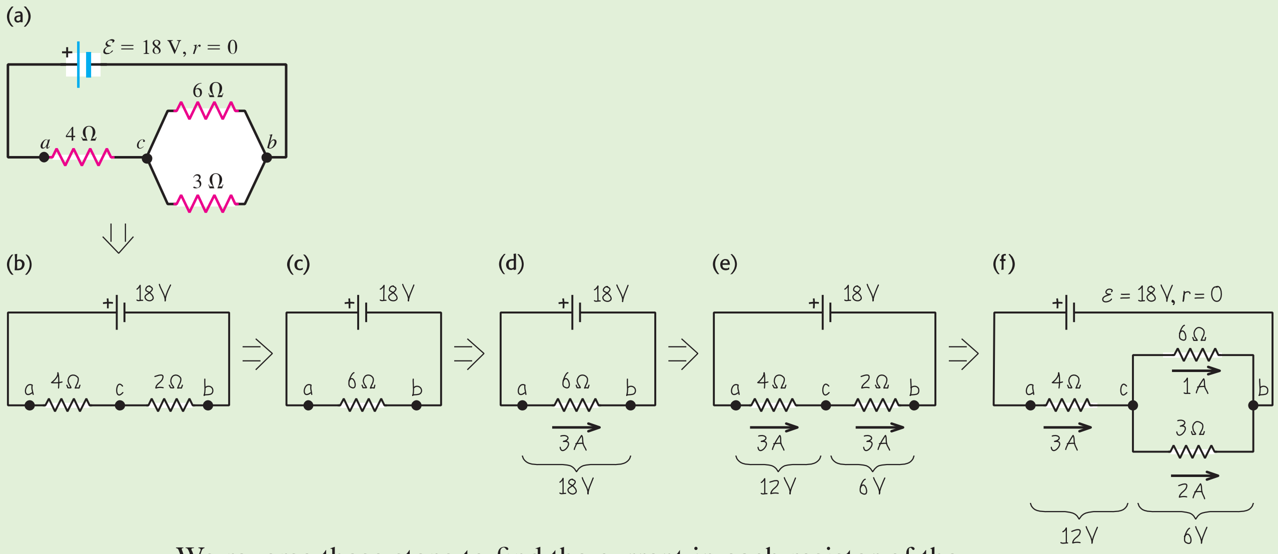
IDENTIFY and SET UP: This network of three resistors is a *combination* of series and parallel resistances, as in Fig. 26.1c. We determine the equivalent resistance of the parallel 6-Ω and 3-Ω resistors, and then that of their series combination with the 4-Ω resistor: This is the equivalent resistance R_{eq} of the network as a whole. We then find the current in the emf, which is the same as that in the 4-Ω resistor. The potential difference is the same across each of the parallel 6-Ω and 3-Ω resistors; we use this to determine how the current is divided between these.

EXECUTE: Figures 26.3b and 26.3c show successive steps in reducing the network to a single equivalent resistance R_{eq}. From Eq. (26.2), the 6-Ω and 3-Ω resistors in parallel in Fig. 26.3a are equivalent to the single 2-Ω resistor in Fig. 26.3b:

$$\frac{1}{R_{6\,\Omega+3\,\Omega}} = \frac{1}{6\ \Omega} + \frac{1}{3\ \Omega} = \frac{1}{2\ \Omega}$$

[Equation (26.3) gives the same result.] From Eq. (26.1) the series combination of this 2-Ω resistor with the 4-Ω resistor is equivalent to the single 6-Ω resistor in Fig. 26.3c.

26.3 Steps in reducing a combination of resistors to a single equivalent resistor and finding the current in each resistor.

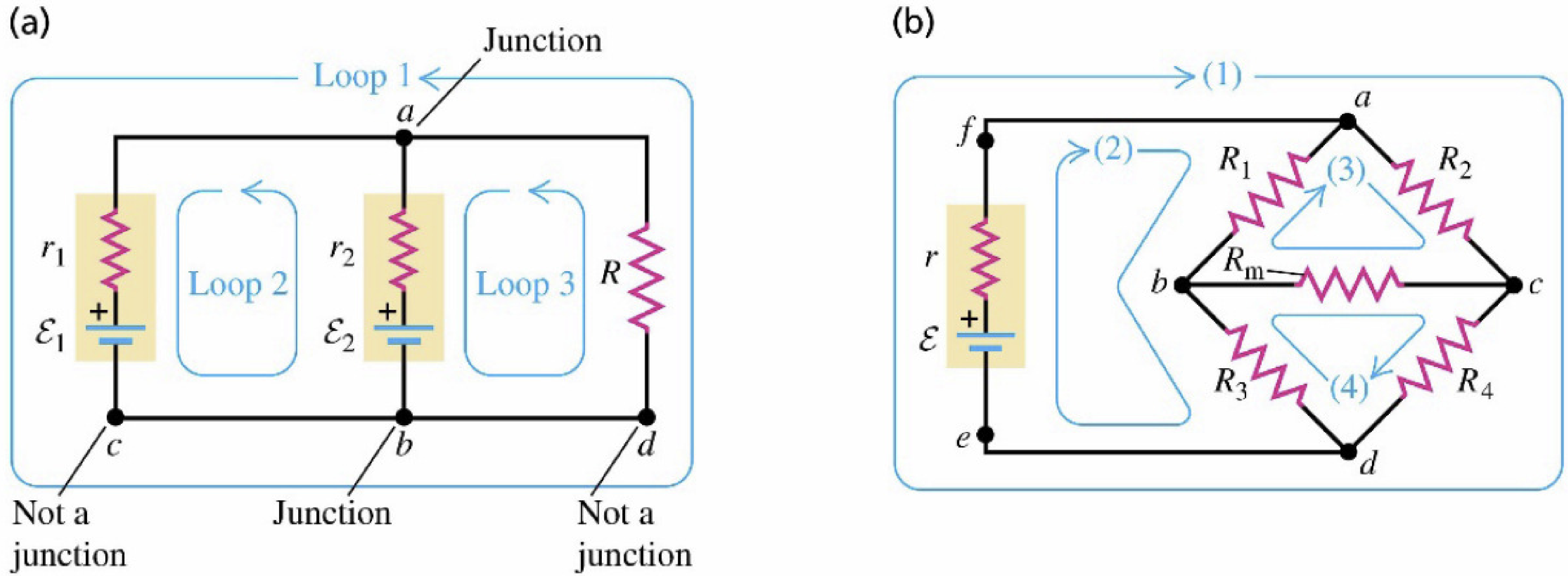


We reverse these steps to find the current in each resistor of the original network. In the circuit shown in Fig. 26.3d (identical to Fig. 26.3c), the current is $I = V_{ab}/R = (18\ \text{V})/(6\ \Omega) = 3\ \text{A}$. So the current in the 4-Ω and 2-Ω resistors in Fig. 26.3e (identical to Fig. 26.3b) is also 3 A. The potential difference V_{cb} across the 2-Ω resistor is therefore $V_{cb} = IR = (3\ \text{A})(2\ \Omega) = 6\ \text{V}$. This potential difference must also be 6 V in Fig. 26.3f (identical to Fig. 26.3a). From $I = V_{cb}/R$, the currents in the 6-Ω and 3-Ω resistors in Fig. 26.3f are respectively $(6\ \text{V})/(6\ \Omega) = 1\ \text{A}$ and $(6\ \text{V})/(3\ \Omega) = 2\ \text{A}$.

EVALUATE: Note that for the two resistors in parallel between points c and b in Fig. 26.3f, there is twice as much current through the 3-Ω resistor as through the 6-Ω resistor; more current goes through the path of least resistance, in accordance with Eq. (26.4). Note also that the total current through these two resistors is 3 A, the same as it is through the 4-Ω resistor between points a and c.

In an electric circuit

- A **junction** is a point where conductors meet
- A **loop** is any closed conducting path



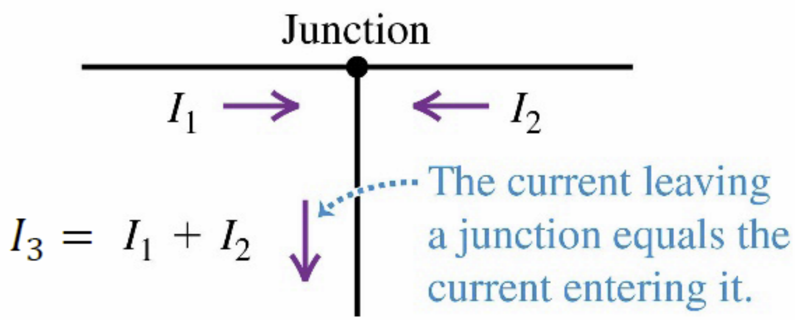
Kirchhoff's junction rule: the algebraic sum of currents at a junction is zero

$$\boxed{\sum_i I_i = 0}$$

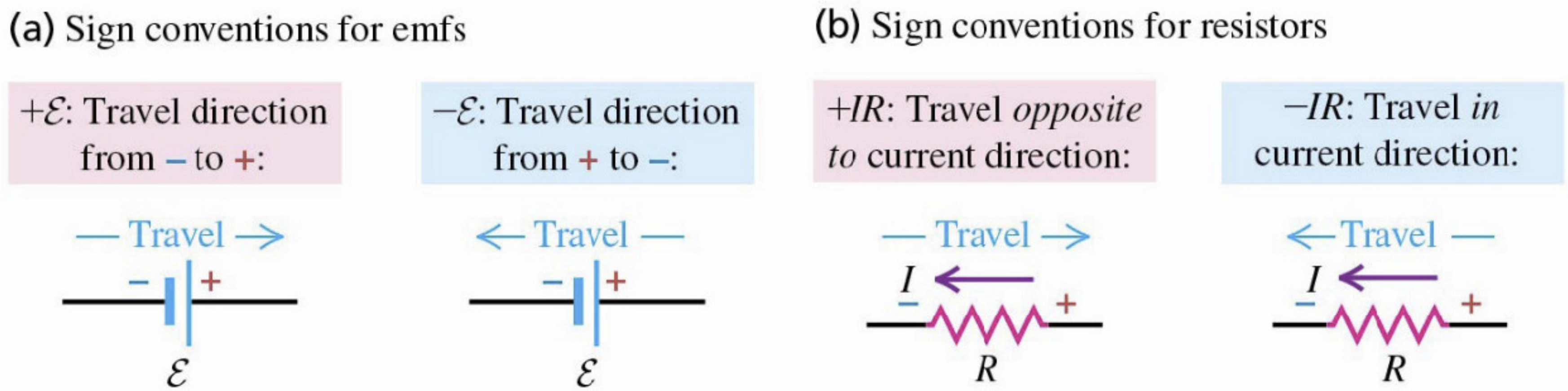
Sign convention: flow in is –ve, flow out is +ve

⚠ physical meaning: no charge accumulates at the junction, i.e., conservation of charge

(a) Kirchhoff's junction rule



While going through a loop along an arbitrary direction, the potential changes (rise or fall) after passing through each circuit element.



Kirchhoff's loop rule: the algebraic sum of potential differences in any loop must be zero

$$\boxed{\sum_i V_i = 0}$$

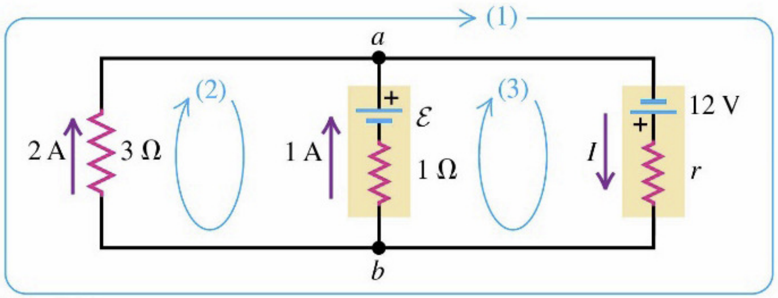
⚠ Physical meaning: potential of the same point does not change after going through a loop

Example 26.4 and 26.5 Charging a battery

3 unknowns: $\mathcal{E}$, I, and r

Need 3 independent equations

⚠ Cannot write down 3 equations using the 3 loops because (2)+(3)=(1)



Need one from junction rule, say, at a: $I - 2\ \text{A} - 1\ \text{A} = 0 \quad \Rightarrow I = 3\ \text{A}$

Loop 1: $12\ \text{V} - Ir - (2\ \text{A})(3\ \Omega) = 0 \quad \Rightarrow \quad r = 2\ \Omega$

Loop 2: $-\mathcal{E} + (1\ \text{A})(1\ \Omega) - (2\ \text{A})(3\ \Omega) = 0 \Rightarrow \quad \mathcal{E} = -5\ \text{V}$ ← -ve means its polarity is opposite to that shown, i.e., it is being charged.

For the 12 V cell:

Terminal voltage $V_{ab} = (12\ \text{V}) - Ir = 6\ \text{V}$

Net power delivered $= (12\ \text{V})I - I^2 r = 18\ \text{W}$ ← Or $= V_{ab} I = 18\ \text{W}$

For $\mathcal{E}$:

Terminal voltage $V_{ab} = -\mathcal{E} + (1\ \text{A})(1\ \Omega) = 6\ \text{V}$

Net power delivered $= \underbrace{\mathcal{E}(1\ \text{A})}_{\text{Power stored in cell as it is being charged}} - \underbrace{(1\ \text{A})^2(1\ \Omega)}_{\text{Power dissipated in internal resistor}} = -6\ \text{W}$ ← Total power consumed by cell as it is being charged

Example 26.6 Equivalent resistance of a complex network

Key: use junction rule to reduce the number of unknown currents to 3

Need three equations from loop rule

Loop 1: $(13\text{ V}) - I_1(1\ \Omega) - (I_1 - I_3)(1\ \Omega) = 0$

Loop 2: $(13\text{ V}) - I_2(1\ \Omega) - (I_2 + I_3)(2\ \Omega) = 0$

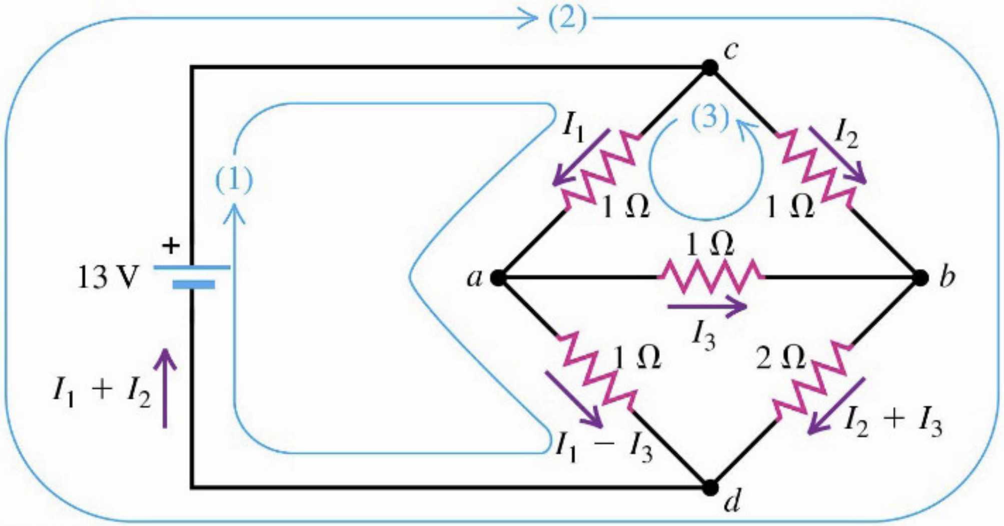
Loop 3: $-I_1(1\ \Omega) - I_3(1\ \Omega) + I_2(1\ \Omega) = 0$

Solving them gives

$$I_1 = 6\text{ A}, I_2 = 5\text{ A}, I_3 = -1\text{ A}$$

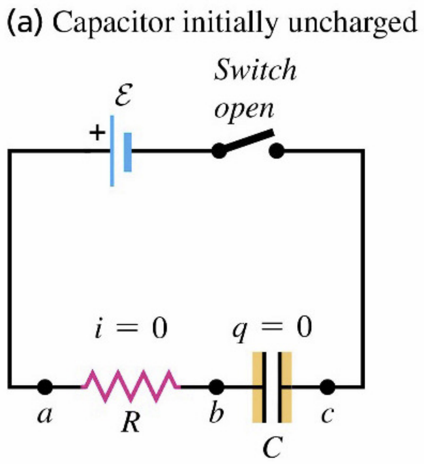
Equivalent resistance

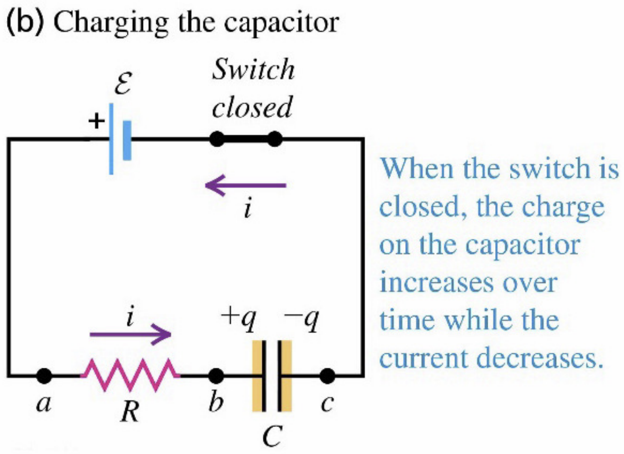
$$R = \frac{13\text{ V}}{I_1 + I_2} = 1.2\ \Omega$$



R-C Circuits

Charging a capacitor:





Qualitatively:

1. On closing switch, capacitor initially uncharged and cost no energy to move charge from one plate to another, i.e., acts like a conductor. Initial current is $i(0) = \mathcal{E}/R$
2. As charge q builds up in capacitor, need more energy to move charge from one plate to another, builds up a potential difference q/C that oppose the current, $i(t)$ decreases.
   ⚠ while C tend to "oppose" a current as it charges up, it is different from a resistor. Energy in a resistor is **dissipative**, once lost you cannot get it back. But energy deposited to a capacitor is **non dissipative** and can be recovered when it discharges.
3. After a long time, when charge builds up to $Q_f = C\mathcal{E}$, $\mathcal{E}$ cannot drive any more charge and $i = 0$

Quantitatively, from Kirchhoff's rule

$$\mathcal{E} - iR - \frac{q}{C} = 0$$

Since $i = dq/dt$, get a *differential equation*

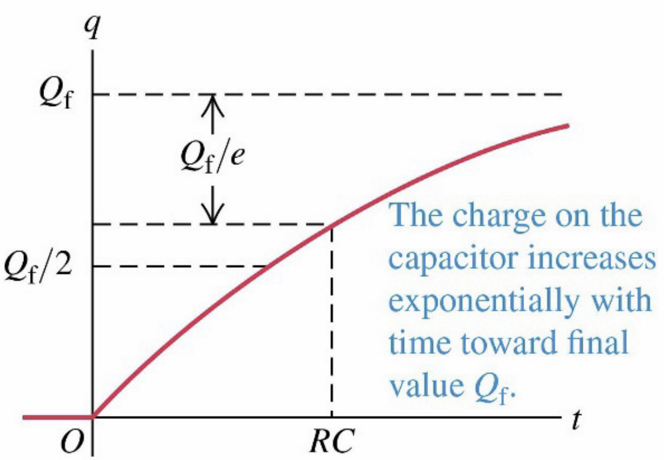
$$R\frac{dq}{dt} = -\frac{1}{C}(q - C\mathcal{E}) \quad \Rightarrow \quad \frac{dq}{q - C\mathcal{E}} = -\frac{dt}{RC}$$

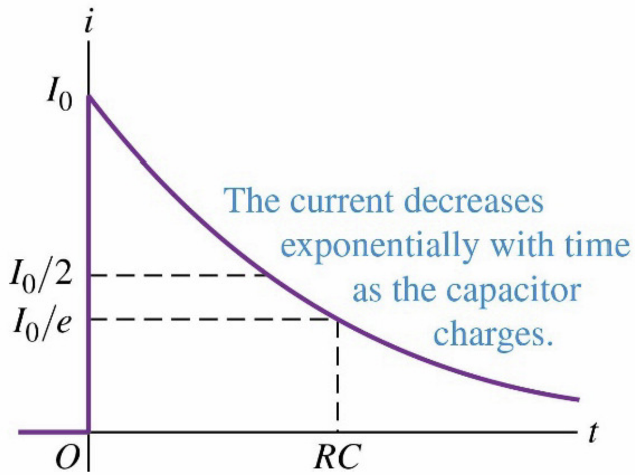
$$\ln(q(t) - C\mathcal{E}) = -\frac{t}{RC} + const$$

With initial condition $q(0) = 0$, $const = \ln(-C\mathcal{E})$

$$\ln\frac{C\mathcal{E} - q(t)}{C\mathcal{E}} = -\frac{t}{RC}$$

$$\Longrightarrow \quad \boxed{q(t) = C\mathcal{E}\left(1 - e^{-\frac{t}{RC}}\right)} \quad \text{and} \quad \boxed{i(t) = \frac{dq}{dt} = \frac{\mathcal{E}}{R}e^{-t/RC} = i(0)e^{-t/RC}}$$





Check: $q(\infty) = C\mathcal{E}$, $i(\infty) = 0$ as expected.

For exponential increase/decrease, **time constant** τ is the time taken to either increase to a factor $(1 - 1/e)$ of the final value, or decrease to a factor $1/e$ of the initial value

For an R-C circuit, $\boxed{\tau = RC}$

⚠ τ measures how fast or slow the exponential change is, e.g., if R large, τ large and the change is slow, like a damping effect

⚠ No need to memorize these formulae, just use qualitative reason to decide whether they should be exponentially increasing or decreasing

Power delivery and dissipation while charging a capacity

Instantaneous power
(multiply i to Kirchhoff's rule)

$$i(t)\mathcal{E} - i^2(t)R - i(t)\left(\frac{q(t)}{C}\right) = 0$$

Delivered by $\mathcal{E}$

$$\int_0^\infty i\mathcal{E}dt = \frac{\mathcal{E}^2}{R}\int_0^\infty e^{-\frac{t}{RC}}dt$$
$$= C\mathcal{E}^2 = \mathcal{E}q(\infty)$$

Dissipated in R

$$\int_0^\infty i^2 R dt$$
$$= R\left(\frac{\mathcal{E}}{R}\right)^2\int_0^\infty e^{-\frac{2t}{RC}}dt = \frac{1}{2}C\mathcal{E}^2$$

⚠ *Independent of* R!

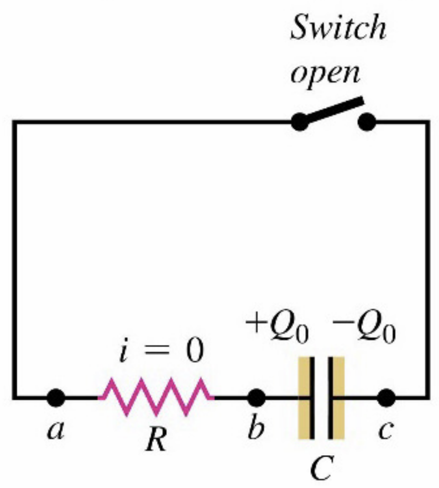
Stored in C

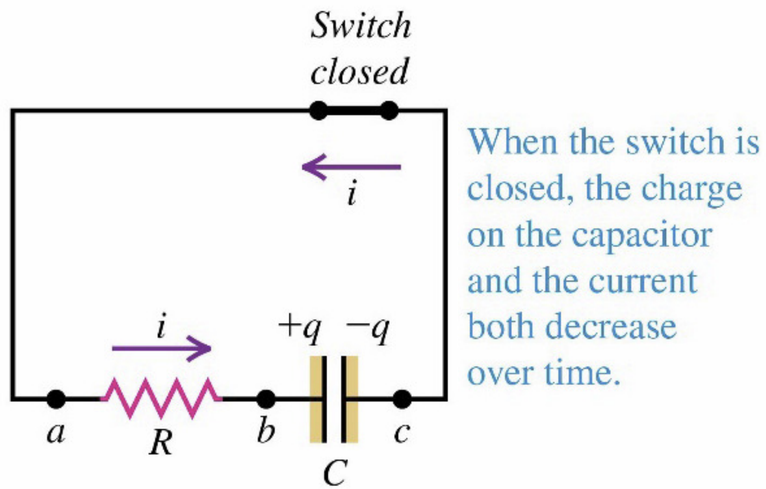
$$\int_0^\infty i\left(\frac{q}{C}\right)dt = \frac{1}{C}\int_0^\infty q\frac{dq}{dt}dt$$
$$= \frac{1}{C}\int_0^{C\mathcal{E}} q dq = \frac{1}{2}C\mathcal{E}^2$$

⚠ Half the total energy supplied by emf is lost in charging a capacitor, no matter how small R is

Discharging a capacitor:

(a) Capacitor initially charged (b) Discharging the capacitor





From similar qualitative argument, both q and i decrease in *magnitude*

$$-iR - \frac{q}{C} = 0 \quad \Longrightarrow \quad i = \frac{dq}{dt} = -\frac{q}{RC}$$

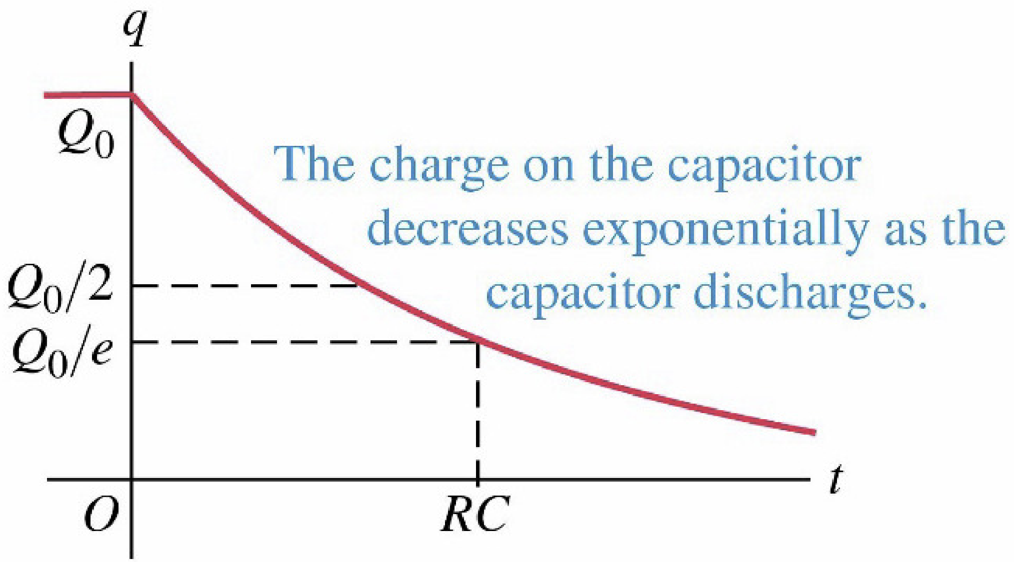
$$\ln q = -\frac{t}{RC} + const$$

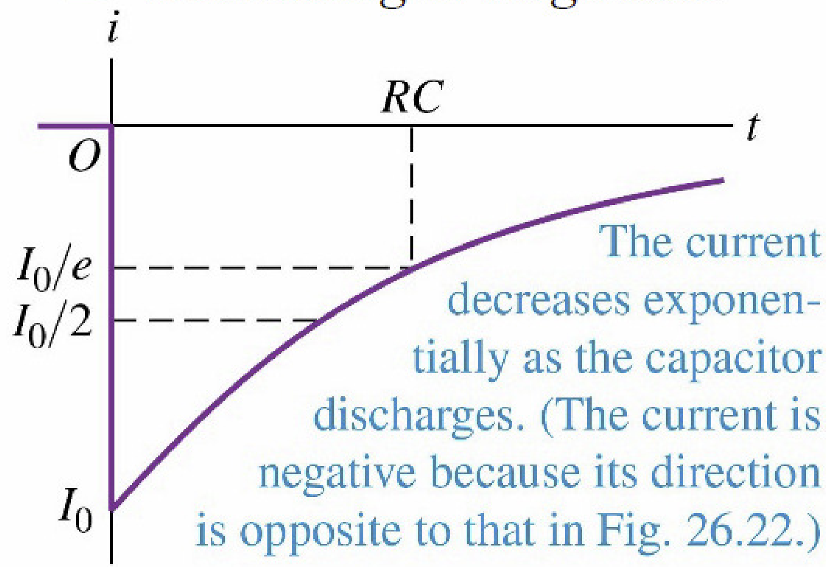
With initial condition $q(0) = Q_0$, $const = \ln Q_0$

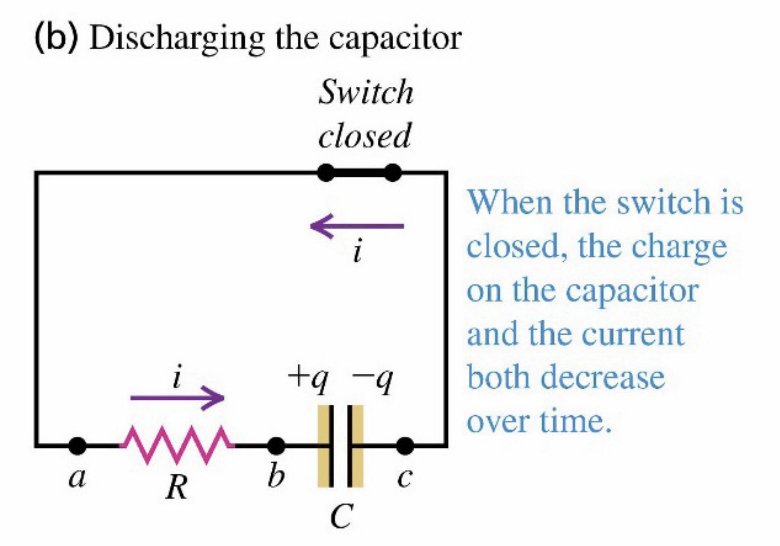
$$\ln \frac{q(t)}{Q_0} = -\frac{t}{RC}$$

$$\Rightarrow \quad \boxed{q(t) = Q_0 e^{-t/RC}} \quad \text{and} \quad \boxed{i(t) = -\frac{Q_0}{RC} e^{-t/RC} = i(0) e^{-t/RC}}$$

⚠ *Decreasing in magnitude*







Question: if we choose i in the opposite direction in the discharge circuit, we get $i = q/RC$ and then $q(t) = Q_0 e^{t/RC}$ which is nonsense. What is wrong?
Hint: anything wrong with $i = dq/dt$ in this case?

Example 26.12

A 10 MΩ resistor is connected in series with a 1.0 μF capacitor and a battery with emf 12.0 V. The capacitor is initially uncharged.

Time constant is $\tau = RC = (10 \times 10^{7}\ \Omega)(1.0 \times 10^{-6}\ \text{F}) = 10\ \text{s}$

Initial current is $i(0) = I_0 = \mathcal{E}/R = 1.2\ \mu\text{A}$

After one time constant 10 s, fraction of initial current I_0 still flowing

$$= \frac{i(10\ \text{s})}{I_0} = e^{-\frac{10\ \text{s}}{10\ \text{s}}} = \frac{1}{e} = 0.37$$

After 4.6 time constants, $t = 46$ s, fraction of initial current I_0 still flowing

$$= \frac{i(46\ \text{s})}{I_0} = e^{-\frac{46\ \text{s}}{10\ \text{s}}} = 0.010$$

⚠ Decreasing R, i.e., τ, can make the change (both charging and discharging) faster

Clicker Questions

Q26.10

A battery, a capacitor, and a resistor are connected in series. Which of the following affect(s) the rate at which the capacitor charges?

A. the emf ε of the battery

B. the capacitance C of the capacitor

C. the resistance R of the resistor

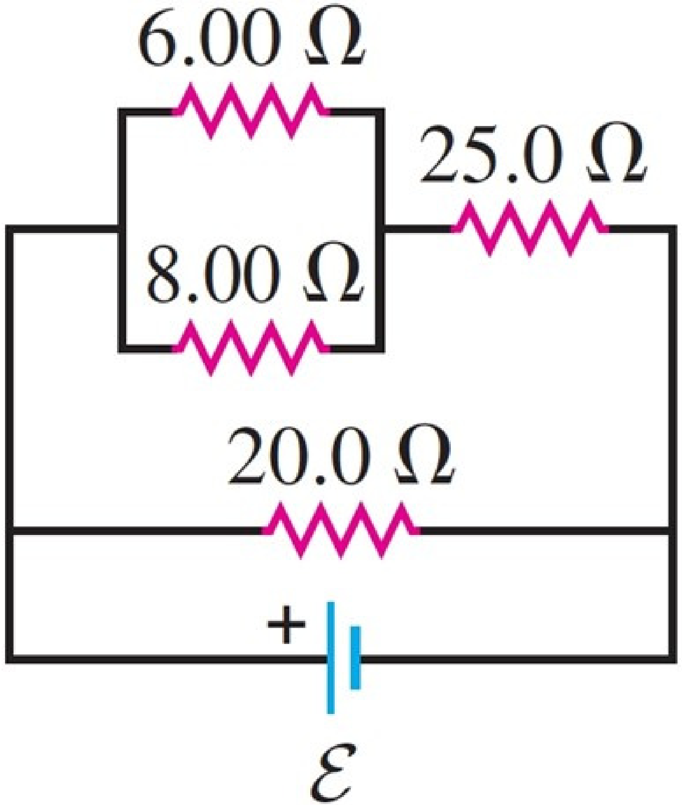
D. both A and B

E. all of A, B, and C

Ans. E

Q-RT26.1

Four resistors are connected to a source of emf as shown. ***Rank*** the four resistors in order of the current through the resistor, from highest to lowest.



A. the 6.00-Ω resistor

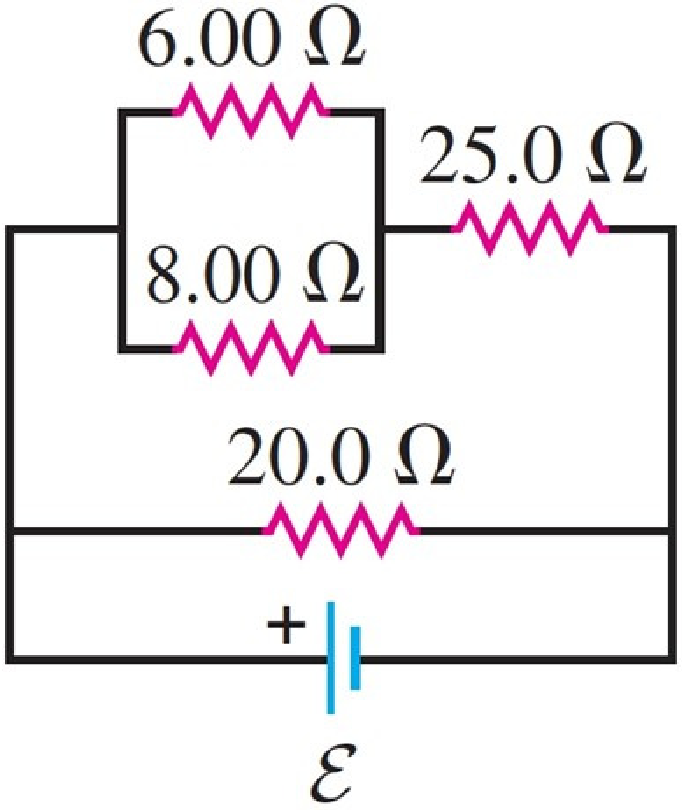
B. the 8.00-Ω resistor

C. the 20.0-Ω resistor

D. the 25.0-Ω resistor

A-RT26.1

Four resistors are connected to a source of emf as shown. ***Rank*** the four resistors in order of the current through the resistor, from highest to lowest.



A. the 6.00-Ω resistor

B. the 8.00-Ω resistor

C. the 20.0-Ω resistor

D. the 25.0-Ω resistor

✔ Answer: CDAB