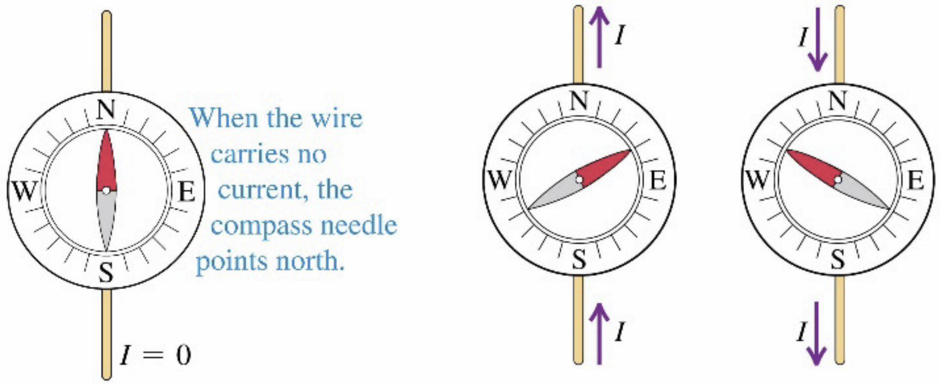
Magnetic field and magnetic forces

Magnetic Field

Ancient Chinese wisdom – a bar magnet (compass) points along a magnetic field

Demonstration:

Oersted's experiment



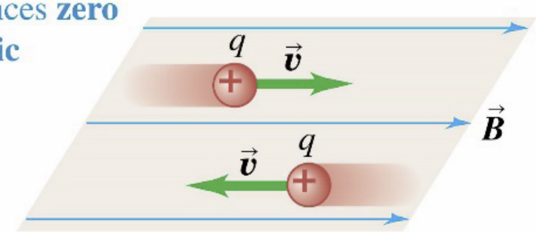
In order to experience a magnetic force, a particle must be

1. *Charged*, with charge q
2. *Moving*, with velocity $\vec{\boldsymbol{v}}$

The **magnetic force** is given by $\boxed{\vec{\boldsymbol{F}} = q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}}$

(a)

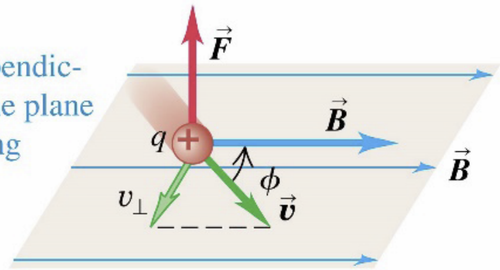
A charge moving **parallel** to a magnetic field experiences **zero magnetic force.**



(b)

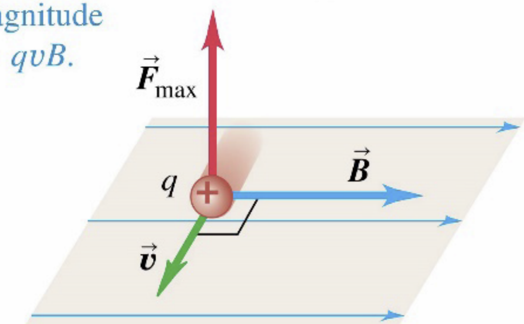
A charge moving at an angle ϕ to a magnetic field experiences a magnetic force with magnitude $F = |q|v_{\perp}B = |q|vB \sin\phi$.

$\vec{\boldsymbol{F}}$ is perpendicular to the plane containing $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{B}}$.



(c)

A charge moving **perpendicular** to a magnetic field experiences a maximal magnetic force with magnitude $F_{max} = qvB$.



Magnitude of magnetic force is

$$F = |q|v_{\perp}B$$
$$= |q|vB\sin\phi$$

$$\boxed{\vec{\boldsymbol{F}} = q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}}$$

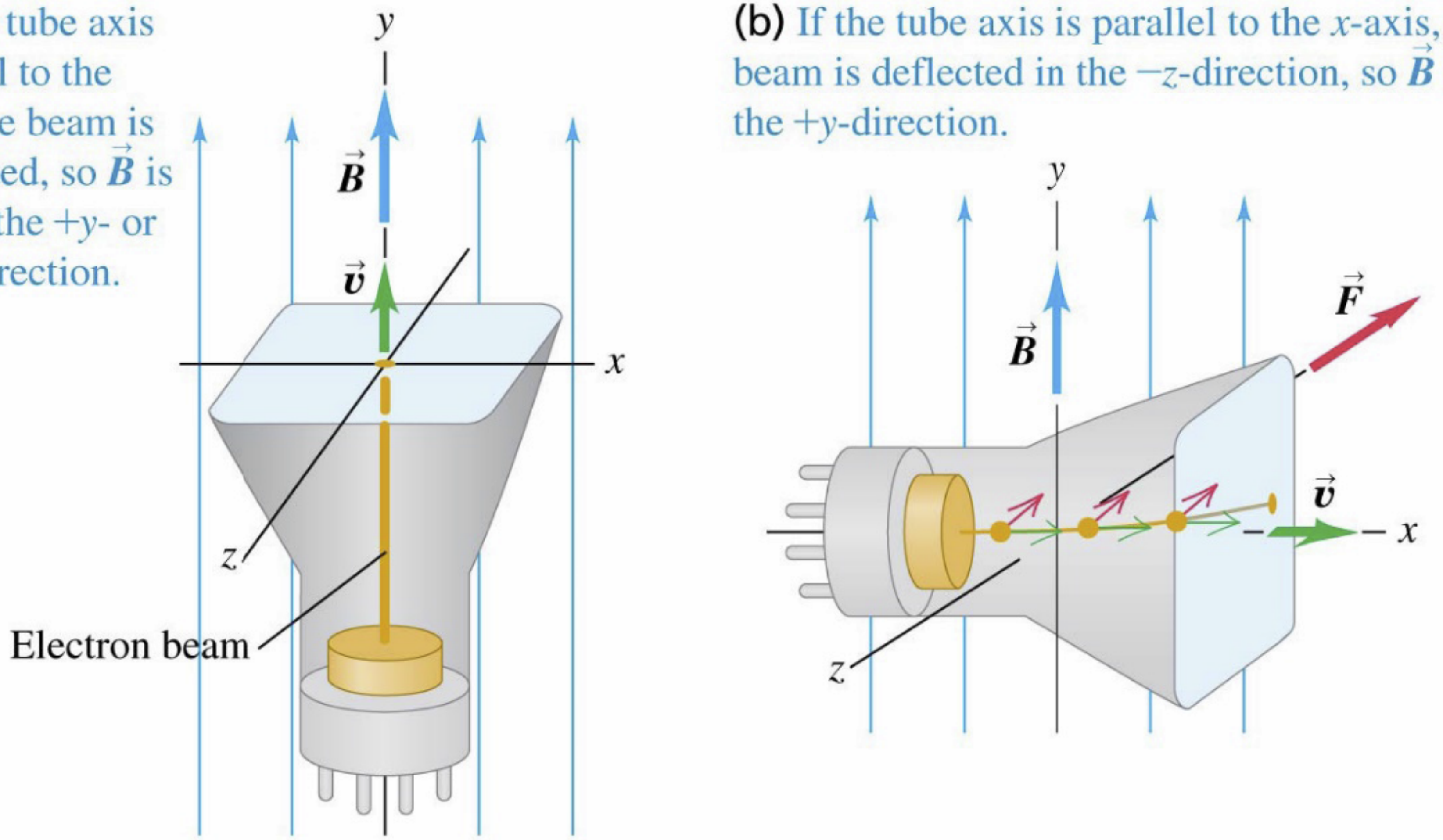
This can be regarded as the **definition of magnetic field $\vec{\boldsymbol{B}}$**

SI unit: $\mathrm{N/(C\ ms^{-1}) = N/(A\ m) = T}$ tesla

⚠ 1 T is a large magnetic field, a smaller unit is gauss G, $1\ \mathrm{G} = 10^{-4}\ \mathrm{T}$

⚠ Earth's magnetic field is ~ 1 G

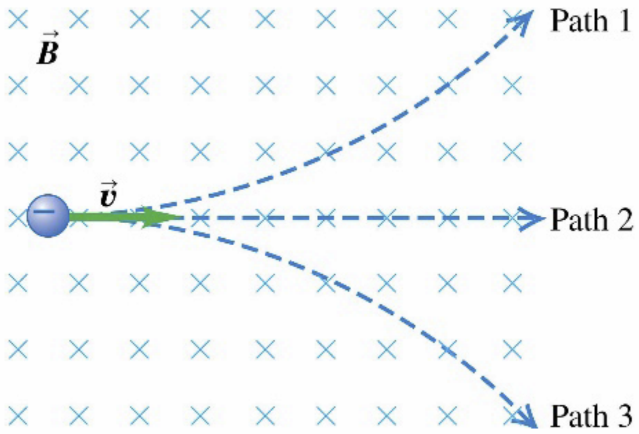
One can use a cathode ray tube to determine the direction of a uniform magnetic field:



(a) If the tube axis is parallel to the y-axis, the beam is undeflected, so $\vec{B}$ is in either the $+y$- or the $-y$-direction.

(b) If the tube axis is parallel to the x-axis, the beam is deflected in the $-z$-direction, so $\vec{B}$ is in the $+y$-direction.

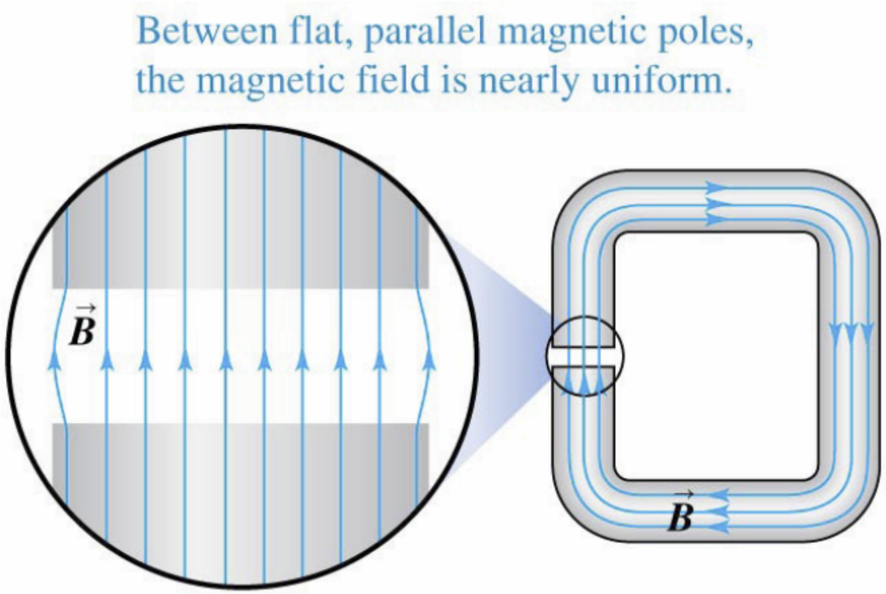
Question: the correct path of the particle in the magnetic field is Path (1 / 2 / 3)



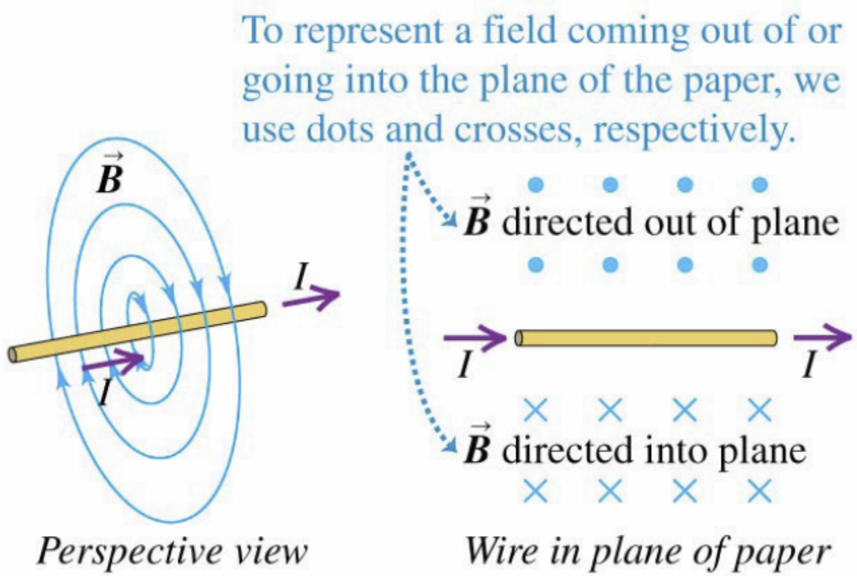
Magnetic Field Lines

Visualize a vector field $\vec{B}$ by lines, *c.f.* electric field $\vec{E}$

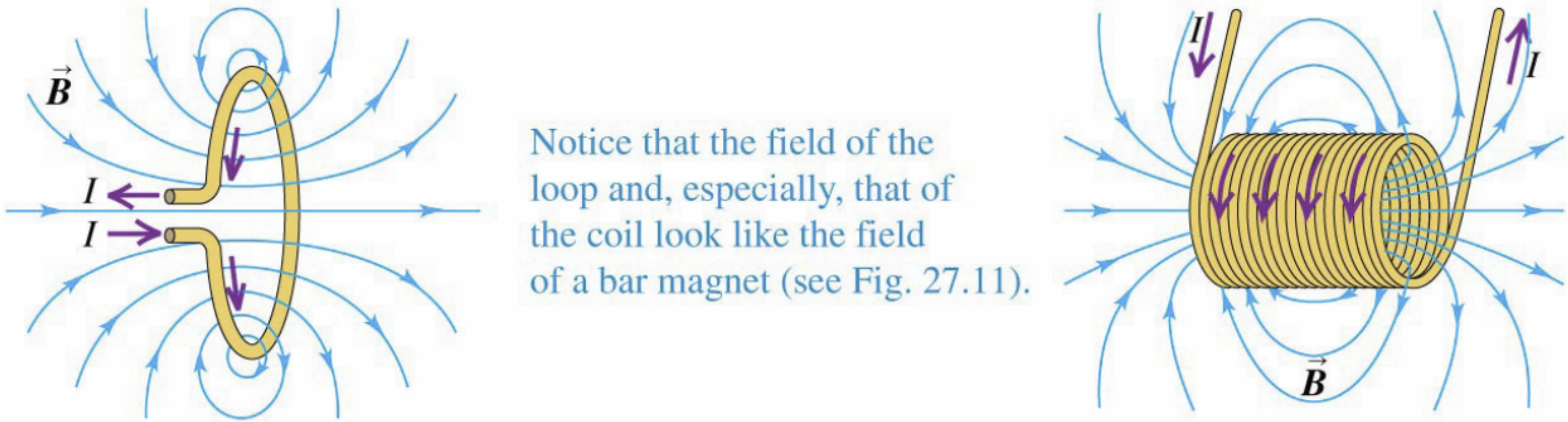
(a) Magnetic field of a C-shaped magnet



(b) Magnetic field of a straight current-carrying wire



(c) Magnetic fields of a current-carrying loop and a current-carrying coil (solenoid)



Define **magnetic flux** (field lines flowing through a surface) in exactly the same way as electric flux

For a uniform magnetic field, $\Phi_B = \vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{A}}$

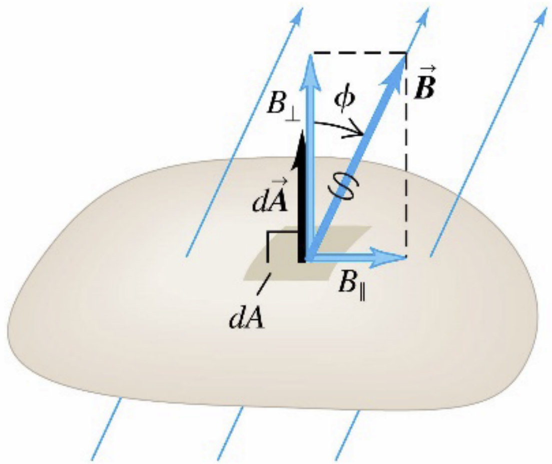
For non-uniform magnetic field

$$d\Phi_B = \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{A}} = B \cos\phi \, dA = B_\perp dA = B dA_\perp$$

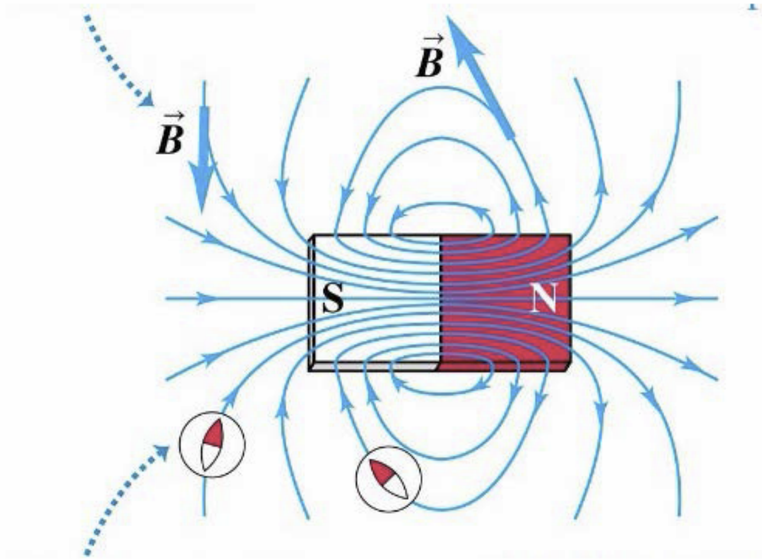
$$\boxed{\Phi_B = \int \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{A}}}$$

$$\boxed{B = \frac{d\Phi_B}{dA_\perp}}$$

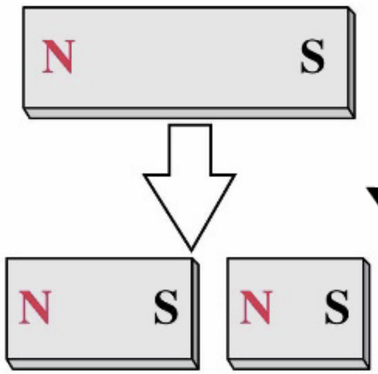
SI unit: weber Wb, 1 Wb = 1 T m^2



⚠ Unlike electric field lines which start at +ve charge and end at –ve charge, magnetic field lines have no beginning nor end, because N and S poles cannot be separated. *Magnetic field lines form close loops*.



Breaking a magnet in two ...



... yields two magnets, not two isolated poles.

Therefore, **Gauss's law for magnetism** is

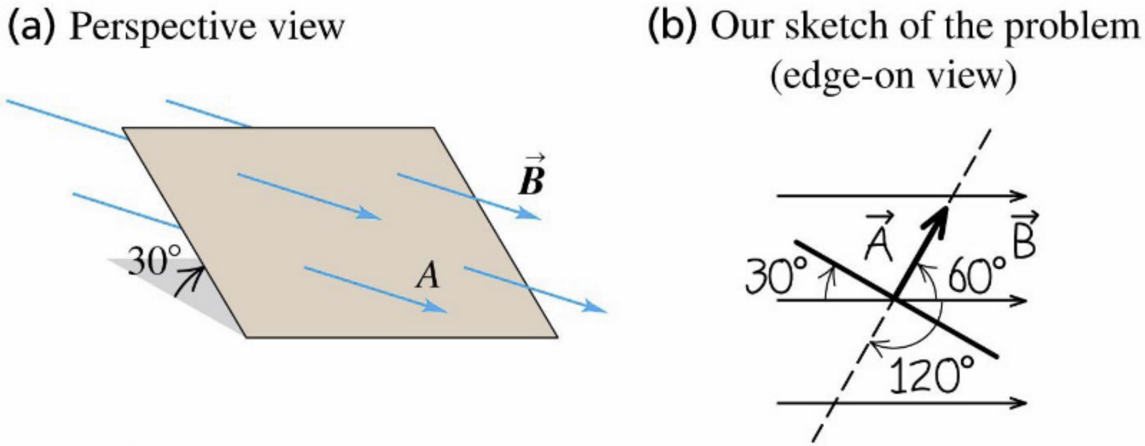
$$\oint \vec{B} \cdot d\vec{A} = 0$$

No magnetic monopole

Example 27.2

Surface area $A = 3.0\ \text{cm}^2$, magnetic flux through the surface is $+0.90\ \text{mWb}$. Magnitude of the magnetic field is

$$B = \frac{\Phi_B}{A\cos\phi} = \frac{+0.90\times10^{-3}\ \text{Wb}}{(3.0\times10^{-4}\ \text{m}^2)(\cos 60°)} = 6.0\ \text{T}$$



Motion of Charged Particles in a Magnetic Field

$$\vec{F} = m\vec{a} = q\vec{v} \times \vec{B}$$

$\vec{a} \perp \vec{v}$, therefore $\vec{a}$ is the centripetal acceleration that changes direction of $\vec{v}$, but never change its magnitude. **Motion of a charged particle under the action of a magnetic field *alone* is always motion with constant speed with no change in energy.**

⚠ Magnetic force never do work on a charged particle ($\because \vec{F} \perp \vec{v}$), therefore cannot define a scalar potential function through $W = -\Delta U$ like in the case of electric field.

For uniform B field

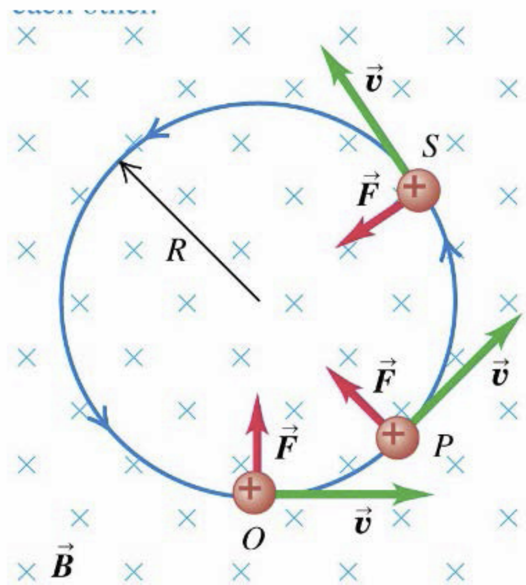
1. If $\vec{\boldsymbol{v}} \perp \vec{\boldsymbol{B}}$

$\vec{\boldsymbol{F}}$ always in the same plane as $\vec{\boldsymbol{v}}$, therefore *2D motion*

$\vec{\boldsymbol{a}}$ always normal to path, therefore *uniform circular motion*

$$F = |q|vB = \frac{mv^2}{R}$$

← Centripetal acceleration v^2/R



Radius of circle

$$R = \frac{mv}{|q|B}$$

Angular frequency of the uniform circular motion

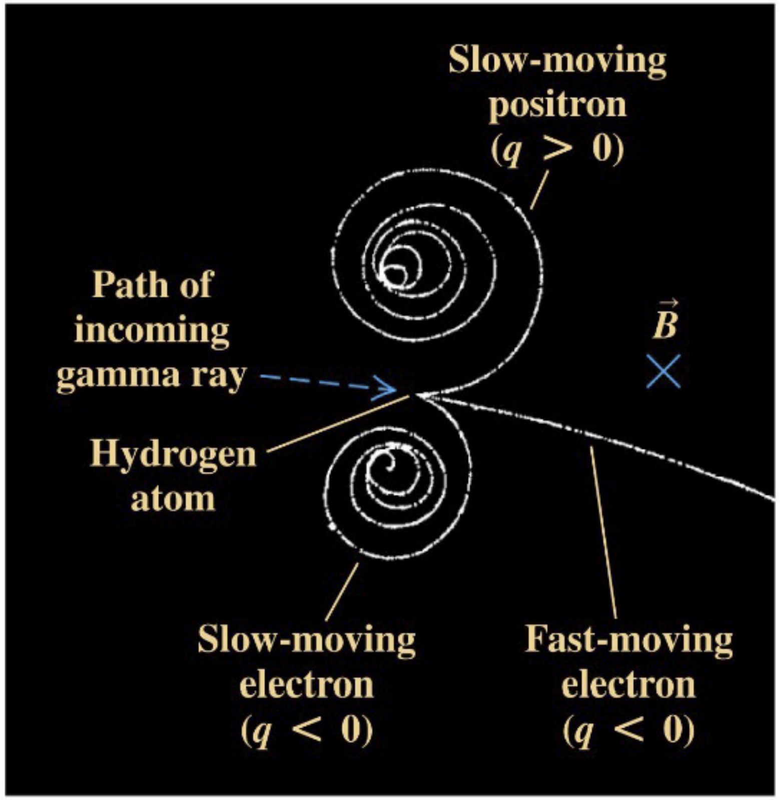
$$\omega = \frac{v}{R} = v\frac{|q|B}{mv} = \frac{|q|B}{m}$$

The frequency $f = \omega/2\pi = |q|B/2\pi m$ is called the **cyclotron frequency**

⚠ A cyclotron accelerates charge particles using an E field and at the same time keep them in circular orbits using a B field

Example – a bubble chamber

Used in studying particle physics. Filled with superheated liquid hydrogen. Charged particles fly through causing liquid to vaporize, forming tiny bubbles behind that mark its path.



A gamma ray photon γ knock out a fast electron from H atom, and itself changed into an electron e^- plus a positron e^+ (antiparticle of electron, with same mass but opposite charge).

This process is called **pair production**

$$\gamma \longrightarrow e^- + e^+$$

Question:

1. Can you identify which one is the path of e^+?
2. Which one is the path of the fast electron knocked out from an H atom?
3. Why the paths spiral instead of remaining as circles?

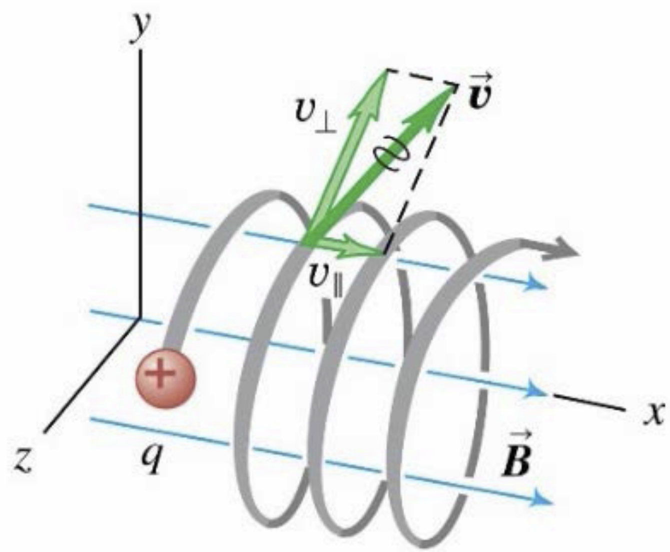
2. If $\vec{\boldsymbol{v}} \nparallel \vec{\boldsymbol{B}}$

$v_{\perp} \Rightarrow$ uniform circular motion $\perp \vec{\boldsymbol{B}}$

$v_{\parallel} \Rightarrow$ constant drift along $\vec{\boldsymbol{B}}$

Combine to trace out a **helical path** of radius

$$R = \frac{mv_{\perp}}{|q|B}$$



Example 27.4

In the above helical path case, suppose the particle is a proton with $m = 1.67 \times 10^{-27}$ kg, $q = 1.6 \times 10^{-19}$ C, and $B = 0.500$ T.

If at $t = 0$, $v_x = 1.50 \times 10^5$ m/s, $v_y = 0$, $v_z = 2.00 \times 10^5$ m/s, magnetic force on proton

$$\vec{\boldsymbol{F}} = q\left(v_x\hat{\boldsymbol{\imath}} + v_z\widehat{\boldsymbol{k}}\right) \times B\hat{\boldsymbol{\imath}} = qv_zB\left(\widehat{\boldsymbol{k}} \times \hat{\boldsymbol{\imath}}\right) = qv_zB\hat{\boldsymbol{\jmath}} = (1.60 \times 10^{-14}\ \text{N})\hat{\boldsymbol{\jmath}}$$

Period of the circular motion

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = \frac{2\pi m}{qB} = 1.31 \times 10^{-7}\text{s}$$

Radius of the helix

$$R = \frac{mv_z}{|q|B} = 4.18 \times 10^{-3}\ \text{m}$$

Pitch of the helix (distance travelled along the helix axis per revolution)

$$= Tv_x = 0.0197\ \text{m}$$

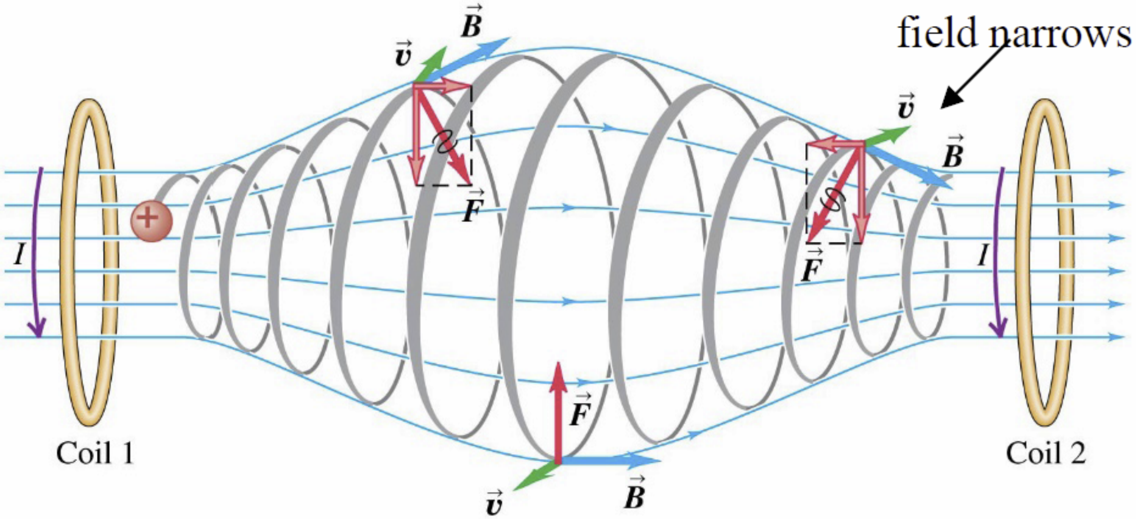
For non-uniform B field

Consider two examples:

Magnetic confinement in a Thermonuclear Reactor

A **magnetic bottle** in a thermonuclear fusion reactor (H + H ⟶ He) traps charged particles at very high temperature ~10^6 K (called **plasma**) using a non-uniform magnetic field

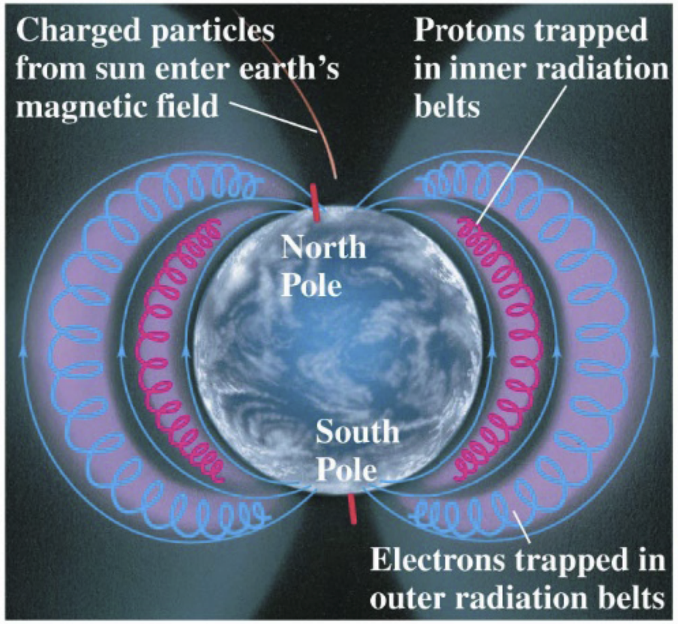
⚠ Particles are turned back by the magnetic field near the two ends as B field narrows



Aurora

Earth's magnetic field protects the earth from harmful charged particles from the sun. It acts as a magnetic bottle to trap charged particles in the **van Allen radiation belt** outside the atmosphere. During strong solar flare, too many charged particles, not all are turned back at the poles. Some enter the atmosphere and collide with air molecules, resulting in **aurora**

(a)



(b)

Clicker Questions

Q27.2

When a charged particle moves through a magnetic field, the direction of the magnetic force on the particle at a certain point

A. is in the direction of the magnetic field at that point.

B. is opposite to the direction of the magnetic field at that point.

C. is perpendicular to the magnetic field at that point.

D. is none of the above.

E. depends on the sign of the particle’s electric charge.

A27.2

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✔ C. is perpendicular to the magnetic field at that point.

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E. depends on the sign of the particle's electric charge.

Q27.7

A positively charged particle moves in the positive z-direction. The magnetic force on the particle is in the positive y-direction. What can you conclude about the z-component of the magnetic field at the particle's position?

A. $B_z > 0$

B. $B_z < 0$

C. $B_z = 0$

D. Either A or B is possible, but not C.

E. Any of A, B, or C is possible.

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✔ E. Any of A, B, or C is possible.

Q27.8

Under what circumstances is the total magnetic flux through a closed surface *positive?*

A. if the surface encloses the north pole of a magnet, but not the south pole

B. if the surface encloses the south pole of a magnet, but not the north pole

C. if the surface encloses both the north and south poles of a magnet

D. more than one of the above

E. none of the above

A27.8

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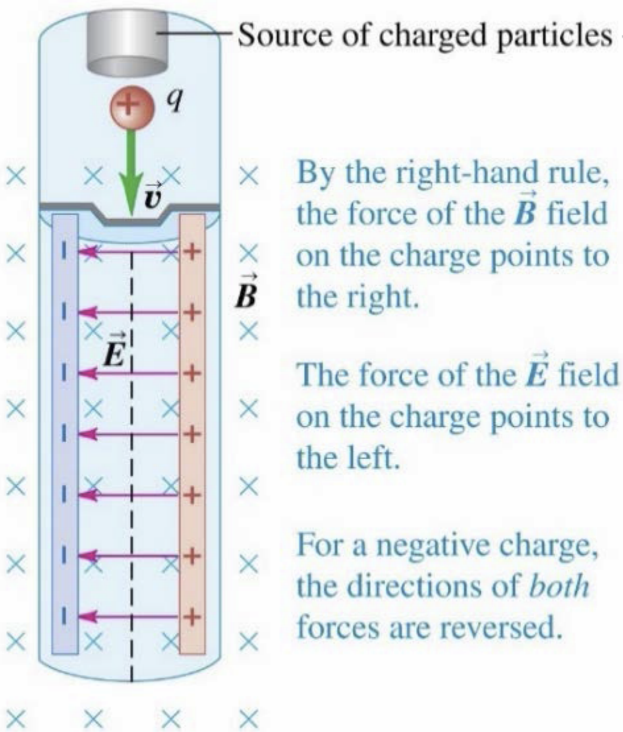
✔ E. none of the above

Lorentz force

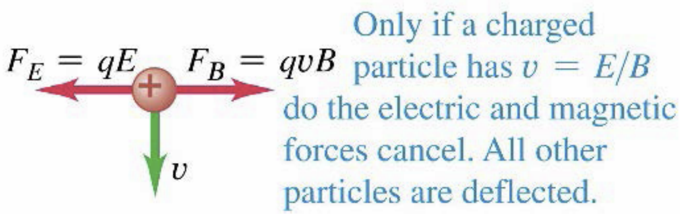
When under an electric *and* magnetic field, a charged particle experiences a combined electric and magnetic force called **Lorentz force**

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

Velocity Selector



From a heated cathode, or radioactive source

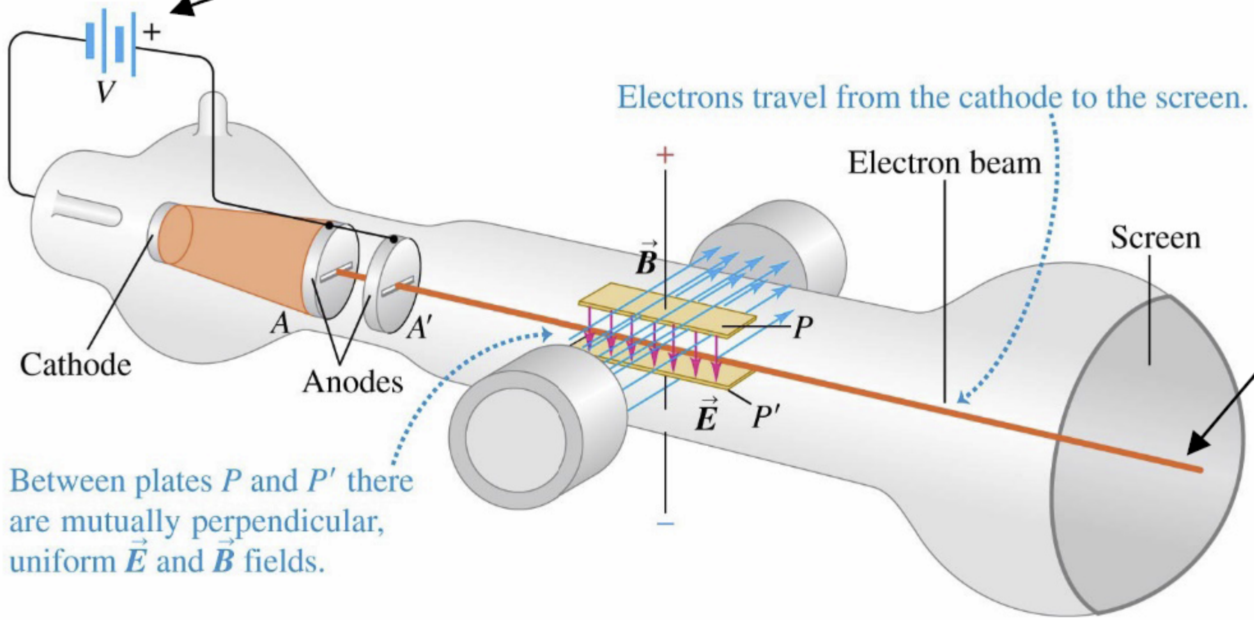


Undeviated charge particles have speed

$$v = \frac{E}{B}$$

Thomson's *e/m* Experiment (1897 at Cavendish Lab, Cambridge; Nobel Prize 1906)
First experiment to identity the charge particle that we call *electron* today

Accelerate charged particle to kinetic energy eV



Beam undeviated,

$$v = E/B$$

$$eV = \frac{1}{2}mv^2 = \frac{mE^2}{2B^2}$$

$$\boxed{\frac{e}{m} = \frac{E^2}{2VB^2}}$$

⚠ Millikan measured electron charge e in his Oil Drop Experiment (1909, Nobel Prize 1923). Combined with e/m get the mass of electron

Mass Spectrometer

After passing through velocity selector,

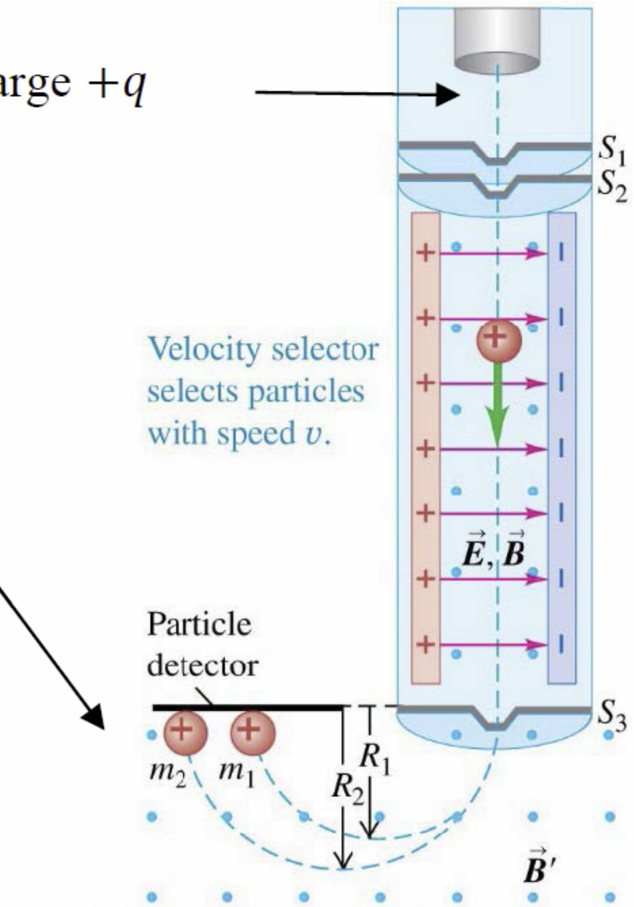
$$v = \frac{E}{B}$$

Ions with charge $+q$

Radius of ions' path inside uniform magnetic field $\vec{\boldsymbol{B}}'$

$$\frac{mv^2}{R} = qvB' \;\Rightarrow\; R = \frac{mv}{qB'}$$

$$\Rightarrow \boxed{m = \frac{qB'R}{v}}$$



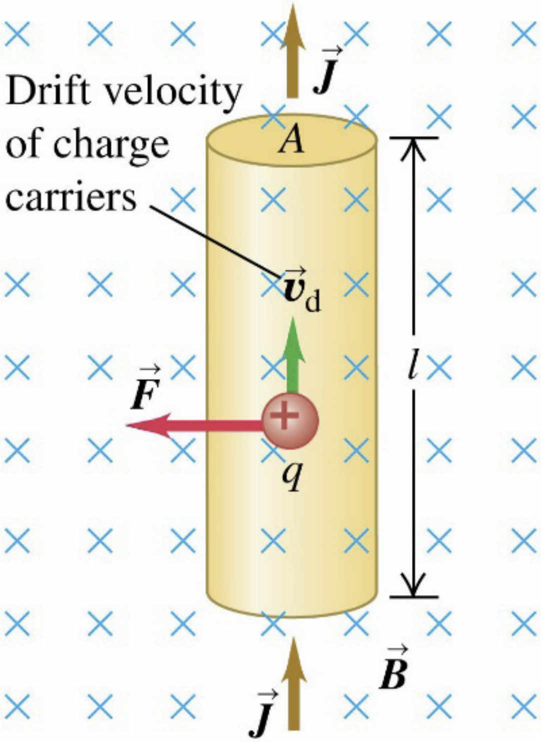
⚠ This set up can be used to identify ions, e.g., in a leak detector, Example 27.6. To detect He^+ ion (mass 6.65×10^{-27} kg and charge $+1.60 \times 10^{-19}$ C), suppose $B' = 0.0818$ T, and the speed of ions after passing through the velocity selector is $v = 1.00 \times 10^5$ m/s.

$$R = \frac{mv}{qB'} = \frac{(6.65 \times 10^{-27}\ \text{kg})(1.00 \times 10^5\ \text{m/s})}{(+1.60 \times 10^{-19}\ \text{C})(0.0818\ \text{T})} = 5.08\ \text{cm}$$

should look for signal from a particle detector located at a point $2 \times 5.08 = 10.16$ cm from slits S_3

Magnetic Force on a Current-Carrying Conductor

You already know from high school that $F = BIl\sin\theta$. Let's derive its vector form.



Conductor segment carrying a current density $\vec{J}$

Total charge in the conductor segment is $n(Al)q$

Total magnetic force on the segment

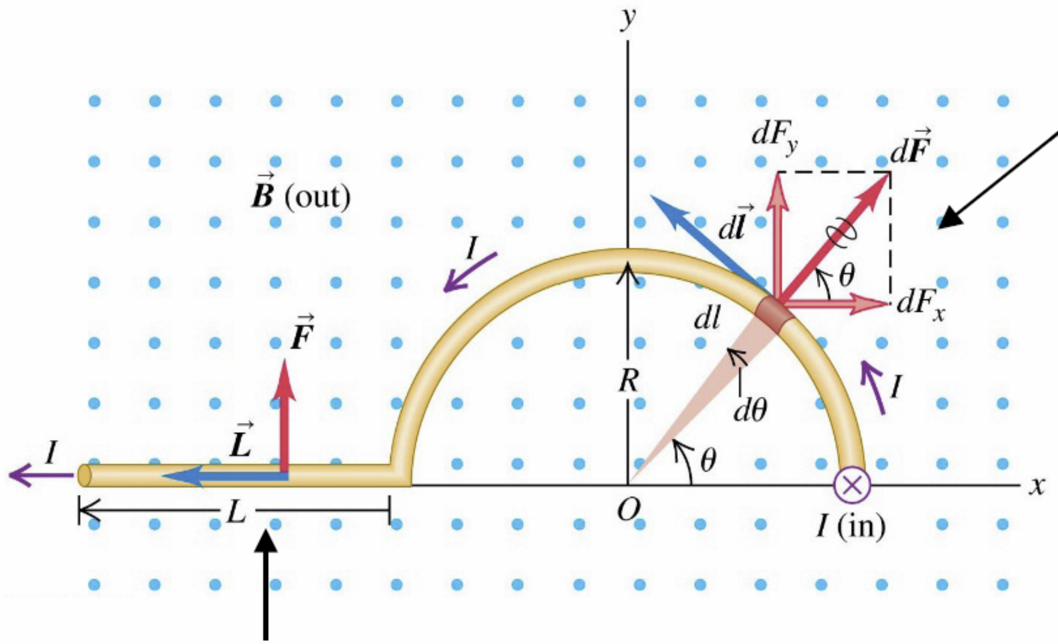
$$\vec{F} = (nAlq)\vec{v}_d \times \vec{B} = \underbrace{lA(nq\vec{v}_d)}_{l(A\vec{J}) = I\vec{l}} \times \vec{B}$$

Define $\vec{l}$ as the length vector of the segment along the direction of I,

$$\boxed{\vec{F} = I\,\vec{l} \times \vec{B}}$$

⚠ If conductor is not straight, break it up into infinitesimal segments $d\vec{\boldsymbol{l}}$, force on infinitesimal segment is $\boxed{d\vec{\boldsymbol{F}} = I\, d\vec{\boldsymbol{l}} \times \vec{\boldsymbol{B}}}$

Example 27.8



Force on semi-circular segment:

By symmetry, dF_x sum up to zero

$$dF_y = I(dl)B \sin\ \theta$$

With $dl = Rd\theta$

$$dF_y = IRB \sin\theta\, d\theta$$

Total force on this segment

$$\vec{\boldsymbol{F}} = \left(\int_0^{\pi} IRB \sin\ \theta\, d\theta\right)\hat{\boldsymbol{j}}$$

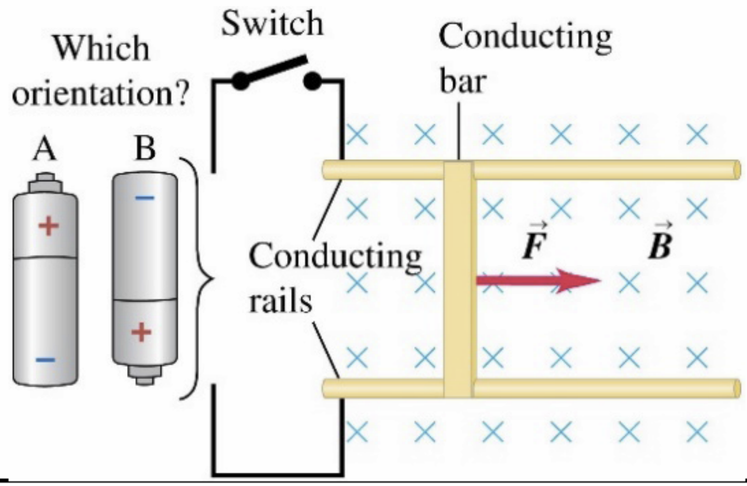
$$= 2IRB\hat{\boldsymbol{j}}$$

⚠ Same as for a straight segment of length $2R$

Force on straight segment

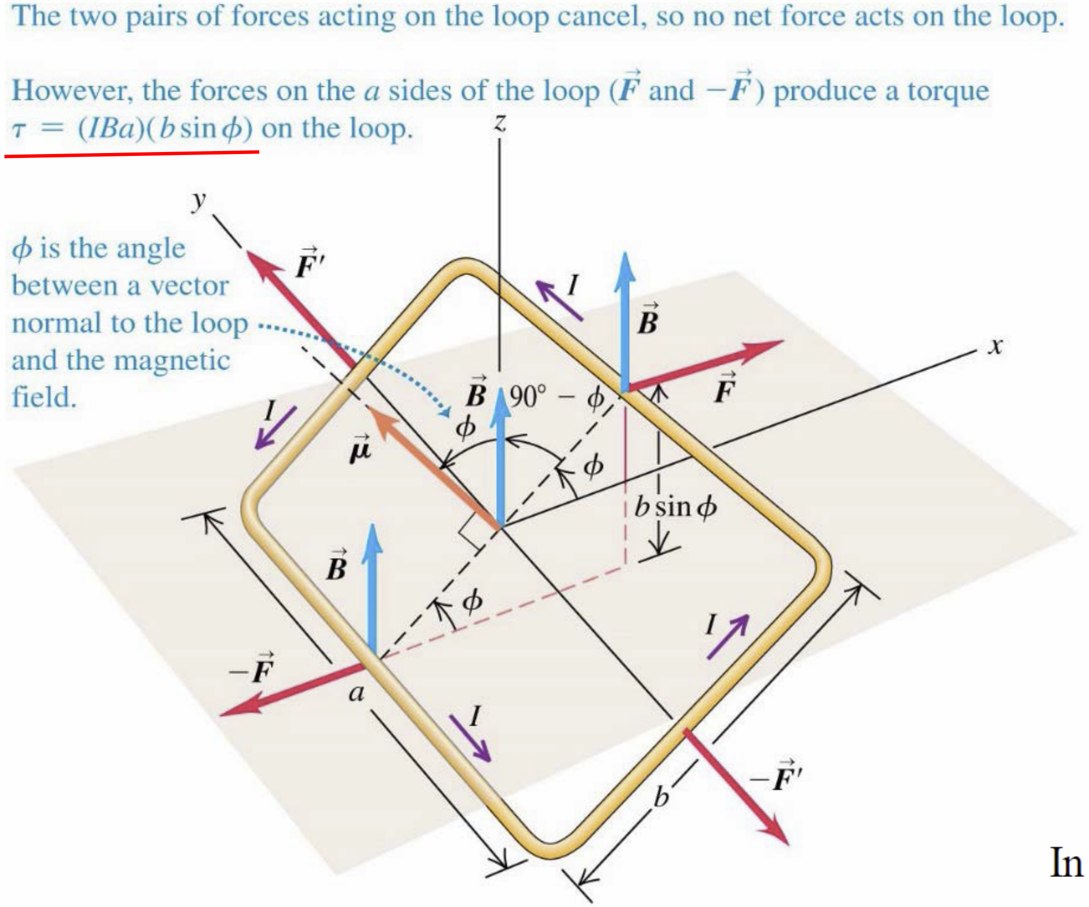
$$= I\vec{\boldsymbol{L}} \times \vec{\boldsymbol{B}} = ILB\hat{\boldsymbol{j}}$$

Question: in order to produce a force indicated in the diagram, the battery should be inserted with polarity as in (A / B).

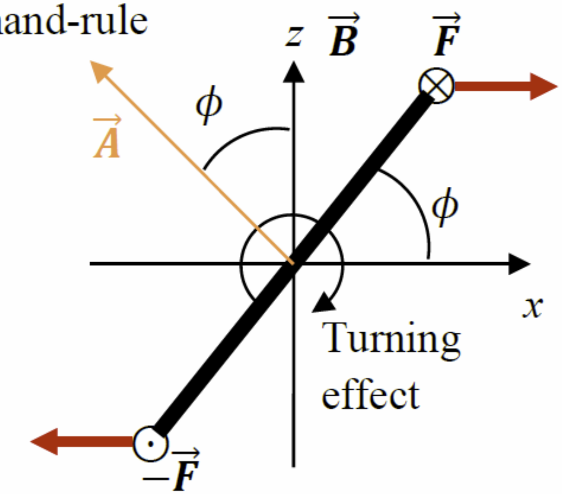


Torque and Turning Effect on a Current Loop

You already learned the turning effect in high school. Let's focus on the vector form



Area vector $\vec{A}$ defined by right-hand-rule



$$|\vec{\tau}| = 2|\vec{r} \times \vec{F}| = 2\left(\frac{b}{2}\right) F \sin\phi$$

$$= IabB \sin\phi$$

In vector form $\boxed{\vec{\tau} = I\vec{A} \times \vec{B}}$

Usually define the **magnetic dipole moment** to be $\vec{\mu} = I\vec{A}$, then the torque is $\boxed{\vec{\tau} = \vec{\mu} \times \vec{B}}$

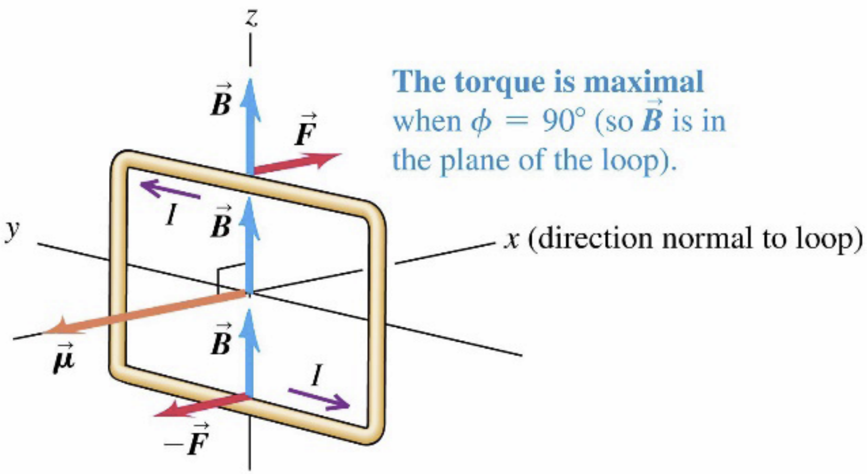
Whatever possess a magnetic dipole moment (such as a current carrying loop) is called a **magnetic dipole**

⚠ If the loop has N turns, the total magnetic moment sum up to be $\vec{\boldsymbol{\mu}} = NI\vec{\boldsymbol{A}}$

⚠ *c.f.* torque on an electric dipole with dipole moment $\vec{\boldsymbol{p}}$ is $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{p}} \times \vec{\boldsymbol{E}}$

⚠ Since potential energy of an electric dipole in an electric field is $-\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{E}}$, the potential energy of a magnetic dipole in a magnetic field is $\boxed{U = -\vec{\boldsymbol{\mu}} \cdot \vec{\boldsymbol{B}}}$

(b)

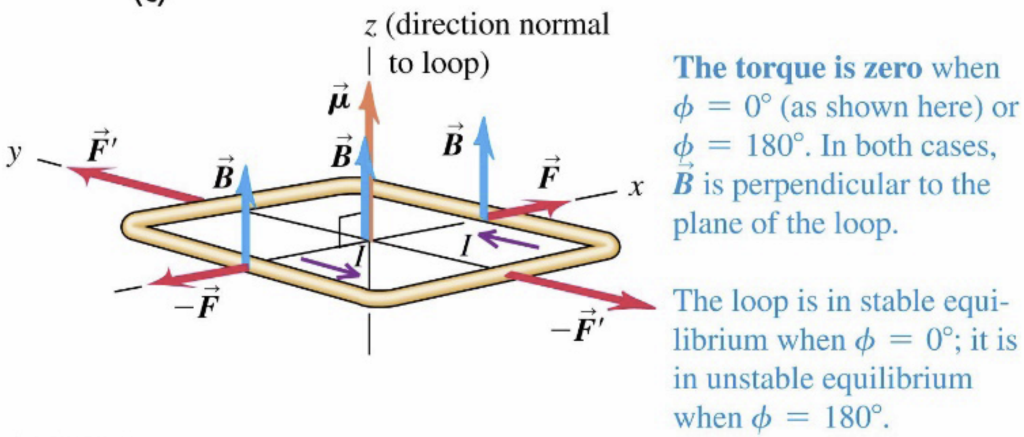


$\phi = 90°, \vec{\mu} \perp \vec{B}$

$\tau = IAB$, maximum

$U = 0$

(c)



$\phi = 0°, \vec{\mu} \parallel \vec{B}$

$\tau = 0$

$U = -IAB$, minimum

Stable equilibrium

$\phi = 180°, \vec{\mu}$ anti- $\parallel \vec{B}$

$\tau = 0$

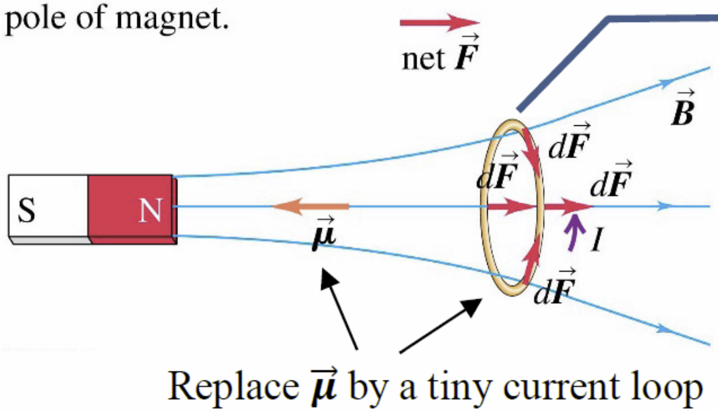
$U = IAB$, maximum

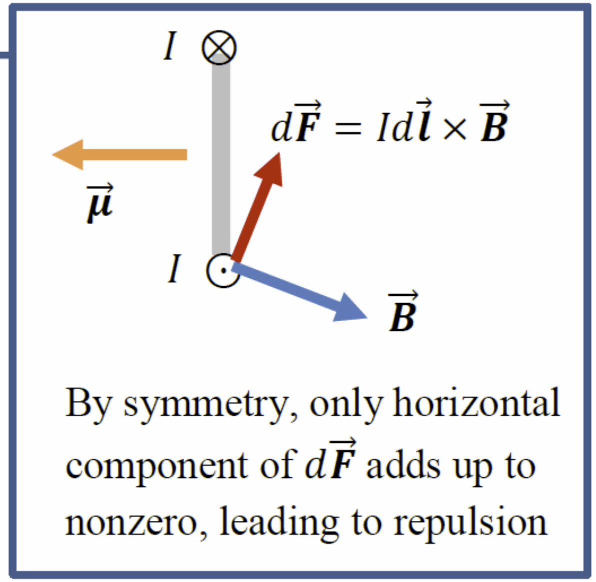
Unstable equilibrium

Magnetic Dipole Moment in a Nonuniform Magnetic Field

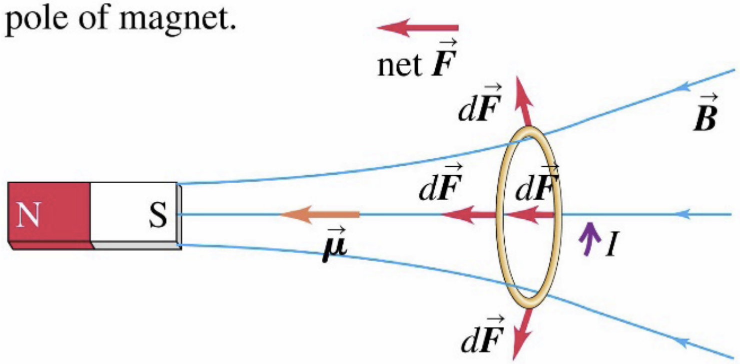
Consider a magnetic dipole moment as a tiny current carrying loop

(a) Net force on this coil is away from north pole of magnet.





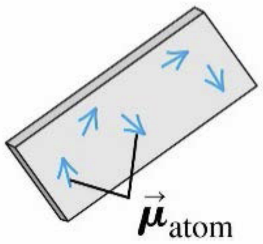
(b) Net force on same coil is toward south pole of magnet.



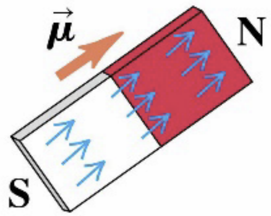
Flipping the magnet leads to attraction

⚠ Can consider $\vec{\mu}$ as a bar magnet, vector points from S to N pole

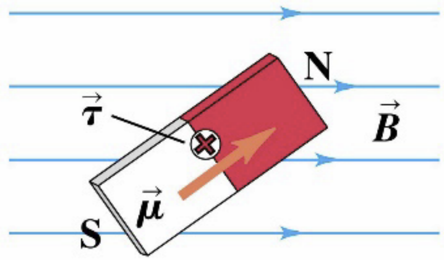
Magnetic materials like iron, each atom is a tiny magnetic moment



Unmagnetized iron, $\vec{\boldsymbol{\mu}}$ of individual atoms are in random orientation, adding up to zero

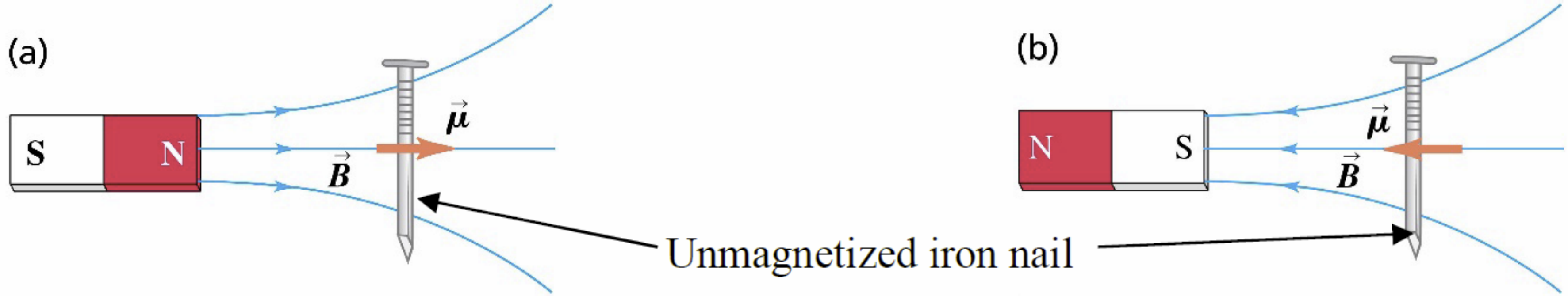


Once magnetized (a bar magnet), $\vec{\boldsymbol{\mu}}$ of individual atoms are locked in certain orientation, adding up to a non-zero *permanent* magnetic moment $\vec{\boldsymbol{\mu}}$



$\vec{\boldsymbol{\mu}}$ align with the external field ($U = -\vec{\boldsymbol{\mu}} \cdot \vec{\boldsymbol{B}}$), like a compass needle

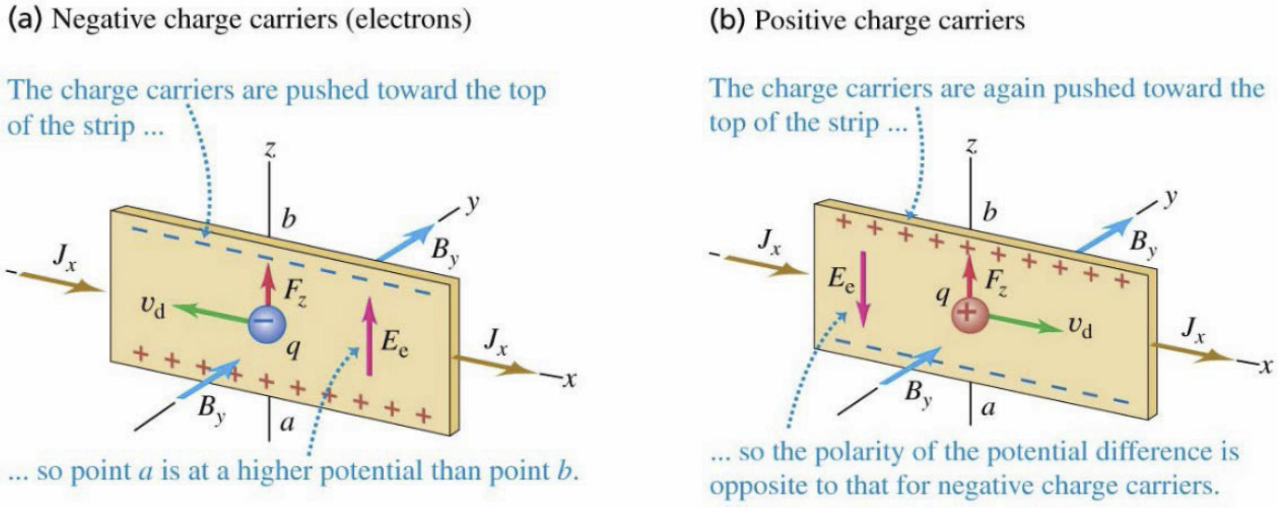
⚠ For unmagnetized iron, external magnetic field can partly align $\vec{\mu}$ of individual atoms, giving it a nonzero magnetic moment (*c.f.* electric dipole induced by electric field)



The Hall Effect

(You learned it in high school if you have taken 1x Physics)

Current density J_x through a flat sheet of conductor under a B field



Initially B field cause q to drift along z direction, creating a E field and a Hall emf $V_{ba} = E_z d$ along the z direction which depends on the sign of q

After reaching steady state, i.e., no more drift along z

$$qE_z + qv_d B_y = 0$$

Current density is $J_x = nqv_d$, therefore

$$nq = -\frac{J_x B_y}{E_z}$$

$$nq = -\frac{J_x B_y}{E_z}$$

⚠ B_x and B_z have no effect here, why?

⚠ The Hall effect can be used to measure:

1. The sign of q
2. The B field if you know n, and conversely n if you know B
3. The drift velocity v_d

Example: A copper ($n = 11.6 \times 10^{28}$ m^{-3}) strip $l_z = 1.5$ cm wide and $l_y = 2.0$ mm thick. A 75 A current runs in the *x* direction, and measures a Hall emf of $V_{ba} = 0.81$ μV. The magnetic field is

$$B = \frac{ne(V_{ba}/l_z)}{I/(l_y l_z)} = 0.40 \text{ T}$$

Question: A copper wire of square cross section is oriented vertically. The four sides of the wire face north, south, east, west respectively. There is a uniform magnetic field from east to west, and the current in the wire runs downwards. Then the (N / S/ E / W) side of the wire is at the highest electric potential.

Clicker Questions

Q27.11

A circular loop of wire carries a constant current. If the loop is placed in a region of uniform magnetic field, the *net magnetic force* on the loop

A. is perpendicular to the plane of the loop, in a direction given by a right-hand rule.

B. is perpendicular to the plane of the loop, in a direction given by a left-hand rule.

C. is in the same plane as the loop.

D. is zero.

E. depends on the magnitude and direction of the current and on the magnitude and direction of the magnetic field.

A27.11

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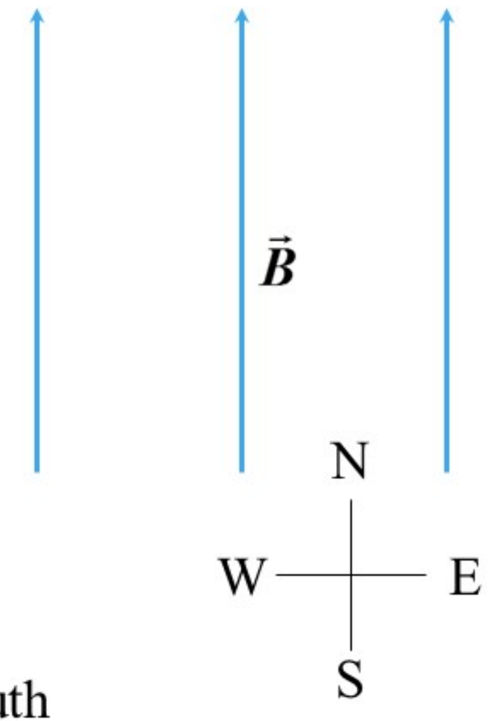
C. is in the same plane as the loop.

✔ D. is zero.

E. depends on the magnitude and direction of the current and on the magnitude and direction of the magnetic field.

Q-RT27.1

A uniform, horizontal 1.00-mT magnetic field points north. Four charged particles, A, B, C, and D, each move in a horizontal plane in the presence of this field. Each particle moves at 3.00×10^3 m/s. ***Rank*** the four particles in order of the magnitude of the force exerted on them by this field, from largest to smallest.



A. particle with $q = +3.00$ nC moving due south

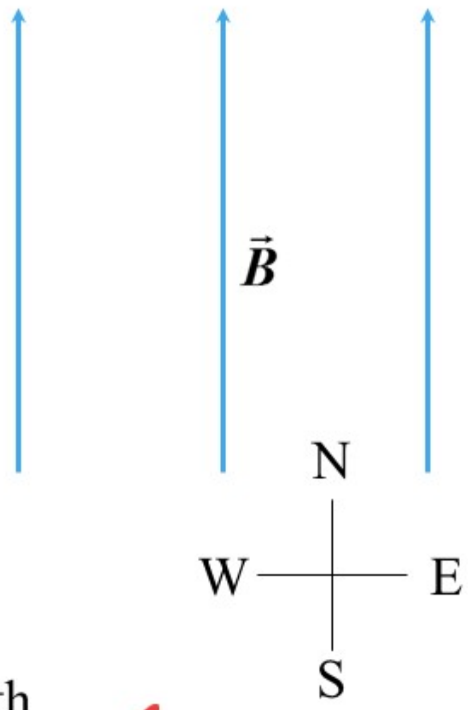
B. particle with $q = +2.00$ nC moving due east

C. particle with $q = -3.00$ nC moving 30° east of north

D. particle with $q = +1.50$ nC moving 30° south of west

A-RT27.1

A uniform, horizontal 1.00-mT magnetic field points north. Four charged particles, A, B, C, and D, each move in a horizontal plane in the presence of this field. Each particle moves at 3.00×10^3 m/s. ***Rank*** the four particles in order of the magnitude of the force exerted on them by this field, from largest to smallest.



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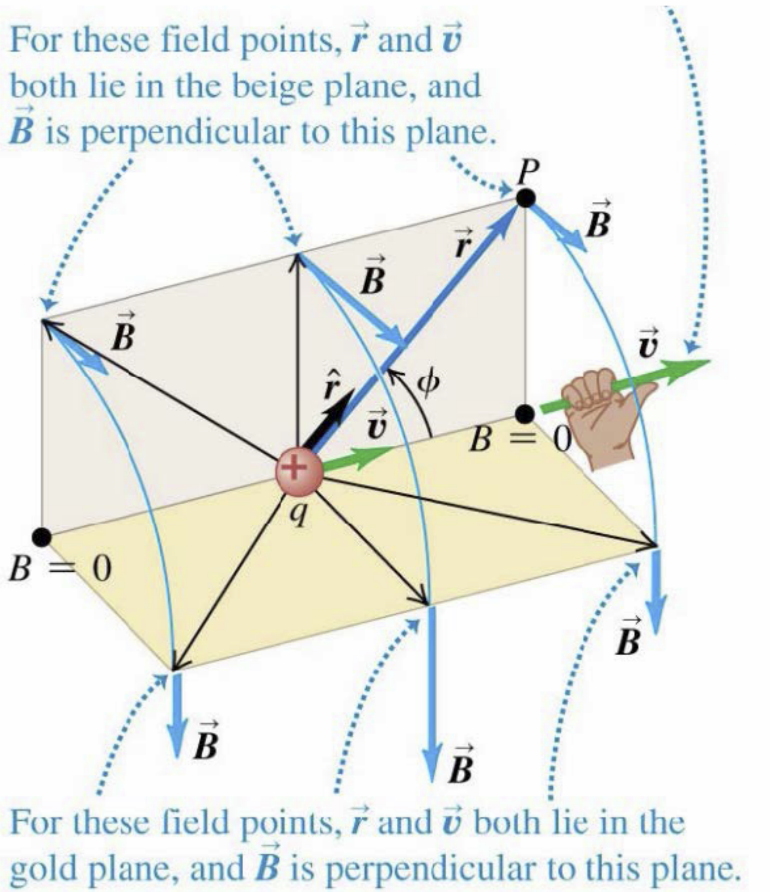
C. particle with $q = -3.00$ nC moving 30° east of north

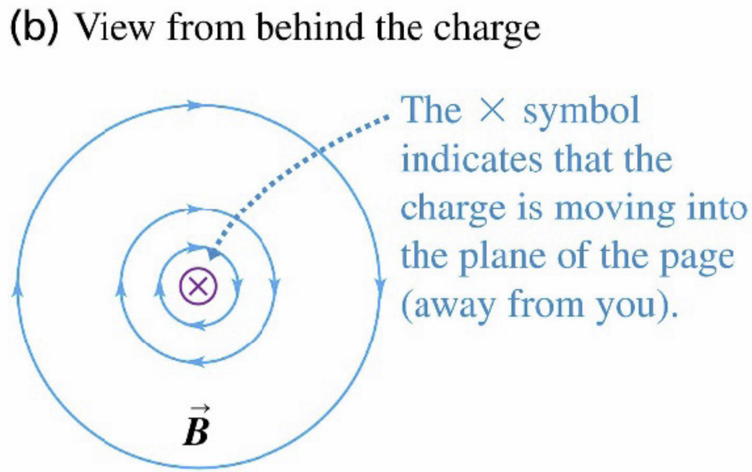
D. particle with $q = +1.50$ nC moving 30° south of west

✔ Answer: BCDA

Sources of magnetic field

Magnetic Field due to a Moving Charge





Instantaneous magnetic field due to the moving charge is

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2} \quad \text{or} \quad B = \frac{\mu_0}{4\pi}\frac{|q|v}{r^2}\sin\phi$$

⚠ Inverse square law! Like $\vec{E}$ due to a point charge

⚠ Ignore effect of the acceleration of the charge, which will cause *radiation*

μ_0 is a proportionality constant called the **vacuum permeability**, a measure of how easy/difficult to create a magnetic field in vacuum, *c.f.* ϵ_0 is the vacuum permittivity

SI unit: $\mathrm{T{\cdot}m^2/C{\cdot}ms^{-1}} = \mathrm{Tm/A}$

Value of μ_0 is exactly $4\pi \times 10^{-7}$ $\mathrm{Tm/A}$, *not* experimentally determined, *c.f.* ϵ_0

An example involving two protons

Two protons flying parallel to each other in opposite directions. At the instant when their perpendicular distance is r

Electric force on upper proton

$$F_E = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2}$$

Magnetic field felt by upper proton

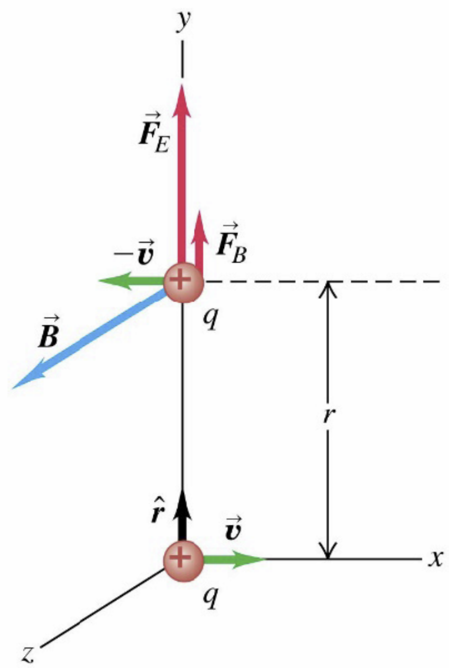
$$B = \frac{\mu_0}{4\pi}\frac{qv}{r^2}$$

Magnetic force on upper proton

$$F_B = qvB = \frac{\mu_0}{4\pi}\left(\frac{qv}{r}\right)^2$$

In later chapters we will show that the speed of light $\boxed{c = 1/\sqrt{\epsilon_0\mu_0}}$

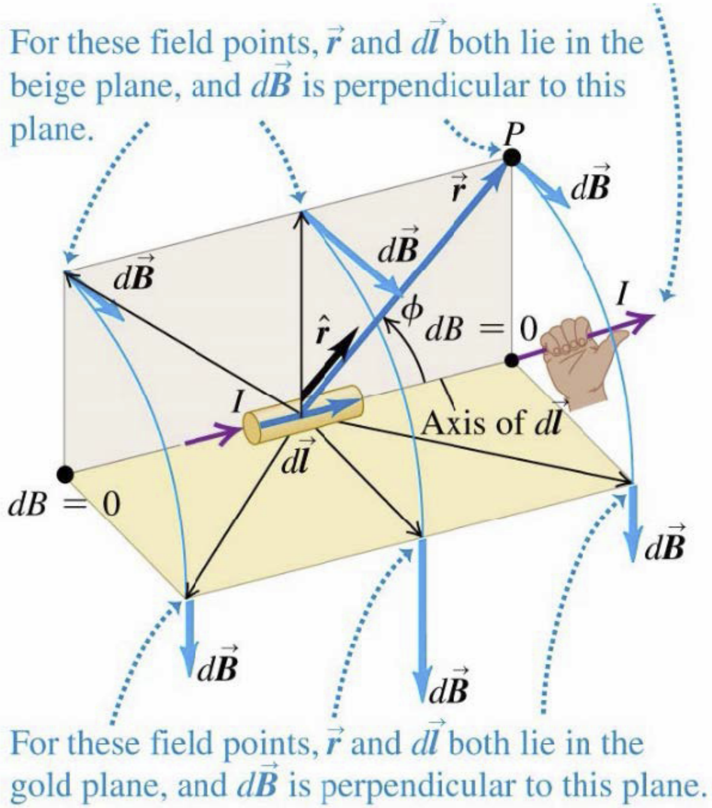
$$\frac{F_B}{F_E} = \mu_0\epsilon_0 v^2 = \left(\frac{v}{c}\right)^2$$



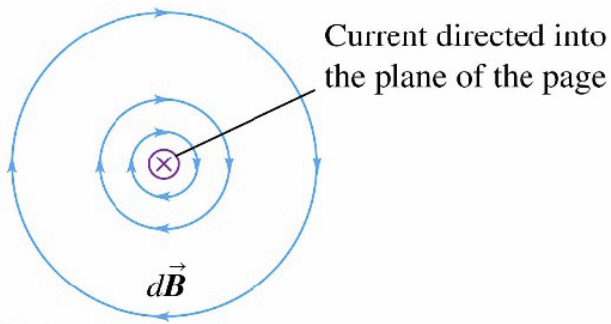
⚠ $F_B \ll F_E$ in the non-relativistic limit $v \ll c$

⚠ True only in the lab frame. If in a frame moving with lower proton, lower proton not moving, no B, and no F_B on upper proton? Need special relativity to resolve this paradox.

Magnetic Field due to a Current Element



(b) View along the axis of the current element



Total charge in the current element $d\vec{l}$

$$dQ = nq(A\,dl)$$

$$dB = \frac{\mu_0}{4\pi}\frac{|dQ|v_d}{r^2}\sin\phi = \frac{\mu_0}{4\pi}\frac{nqv_dA\,dl}{r^2}\sin\phi$$

Since $(nqv_d)A = JA = I$

$$\Rightarrow \boxed{dB = \frac{\mu_0}{4\pi}\frac{I\,dl}{r^2}\sin\phi} \quad \text{or} \quad \boxed{d\vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{\boldsymbol{l}}\times\hat{\boldsymbol{r}}}{r^2}}$$

This is called the **Biot-Savart law**. To sum up the total magnetic field

$$\vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi} \int \frac{I\, d\vec{\boldsymbol{l}} \times \hat{\boldsymbol{r}}}{r^2}$$

⚠ Given the Biot-Savart law, we can derive the previous formula of the magnetic field due to a moving charge: if the current is due to a single charge q, the current element traced out in time Δt is $dl = v\Delta t$, i.e., $I\ dl = (q/\Delta t)v\Delta t = qv$. *The Biot-Savart law and the formula of a single moving charge are equivalent to each other.*

The Biot-Savart law is *fundamental* (i.e., cannot be derived), just like the Newton's law of gravity in gravitation, and Coulomb's law in electrostatics.

Magnetic Field due to a Straight Current-carrying Conductor

$$dB = \frac{\mu_0 I}{4\pi}\frac{dl}{r^2}\sin\phi = \frac{\mu_0 I}{4\pi}\frac{dy}{(x^2+y^2)}\frac{x}{\sqrt{x^2+y^2}}$$

Note that $d\vec{\boldsymbol{B}}$ in the same direction independent of $d\vec{\boldsymbol{l}}$

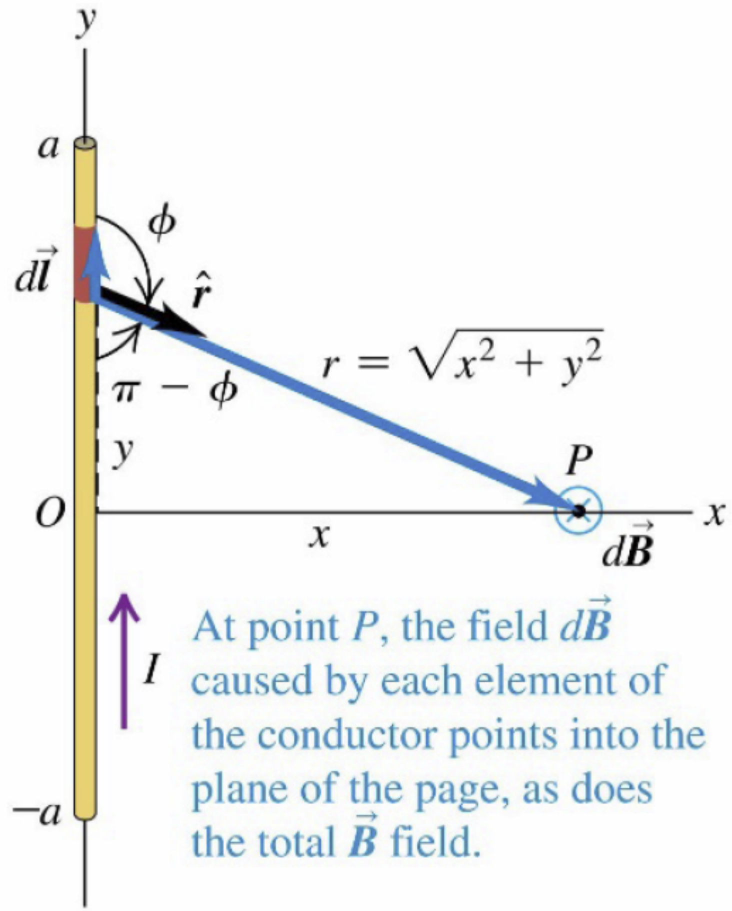
$$\Rightarrow \quad B = \frac{\mu_0 I}{4\pi}\int_{-a}^{a}\frac{x\,dy}{(x^2+y^2)^{3/2}} = \frac{\mu_0 I}{4\pi}\frac{2a}{x\sqrt{x^2+a^2}}$$

For a long conductor, $a \to \infty$, and

$$\frac{a}{\sqrt{x^2+a^2}} = \frac{1}{\sqrt{(x/a)^2+1}} \to 1$$

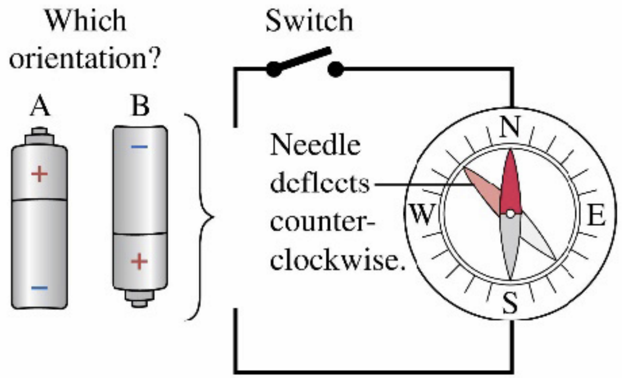
Therefore

$$\boxed{B = \frac{\mu_0 I}{2\pi x}}$$



⚠ *c.f.* $\vec{\boldsymbol{E}}$ due to a long line of charge is $E = \lambda/2\pi\epsilon_0 x$

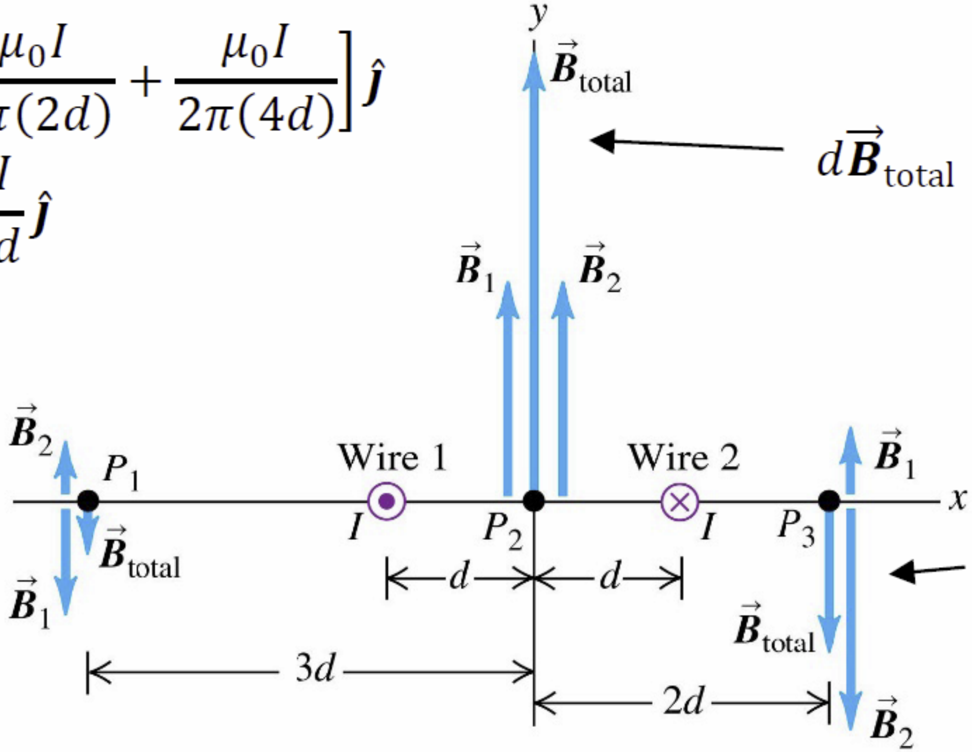
Question: if switching on the current causes the compass needle to deflected as shown, then the polarity of the cell is (A / B).



Example 28.4 Magnetic field of two wires

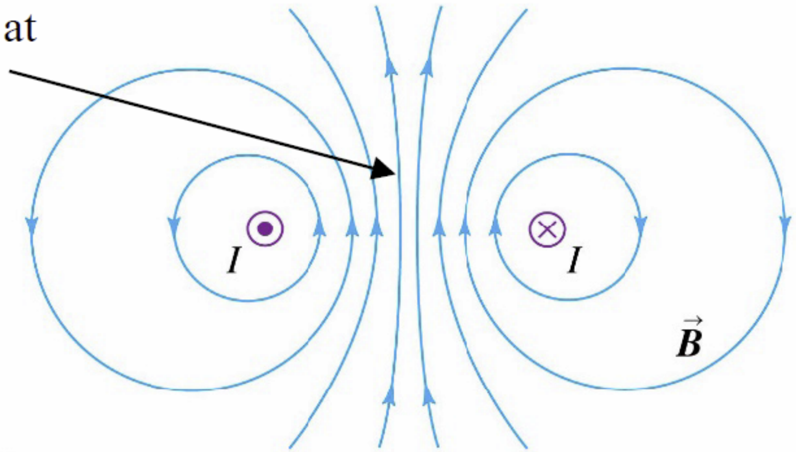
$$d\vec{\boldsymbol{B}}_{\text{total}} = \left[-\frac{\mu_0 I}{2\pi(2d)} + \frac{\mu_0 I}{2\pi(4d)}\right]\hat{\boldsymbol{j}}$$
$$= -\frac{\mu_0 I}{8\pi d}\hat{\boldsymbol{j}}$$

$$d\vec{\boldsymbol{B}}_{\text{total}} = \left[\frac{\mu_0 I}{2\pi d} + \frac{\mu_0 I}{2\pi d}\right]\hat{\boldsymbol{j}} = \frac{\mu_0 I}{\pi d}\boldsymbol{j}$$



$$d\vec{\boldsymbol{B}}_{\text{total}} = \left[\frac{\mu_0 I}{2\pi(3d)} - \frac{\mu_0 I}{2\pi d}\right]\hat{\boldsymbol{j}}$$
$$= -\frac{\mu_0 I}{3\pi d}\hat{\boldsymbol{j}}$$

Total field strongest at the middle



Question: Place a third straight wires at P_3 and parallel to wires 1 and 2. Suppose it is carrying a current flowing into the plane, what is the direction of the magnetic force on the third wire?

Force Between Parallel Conductors

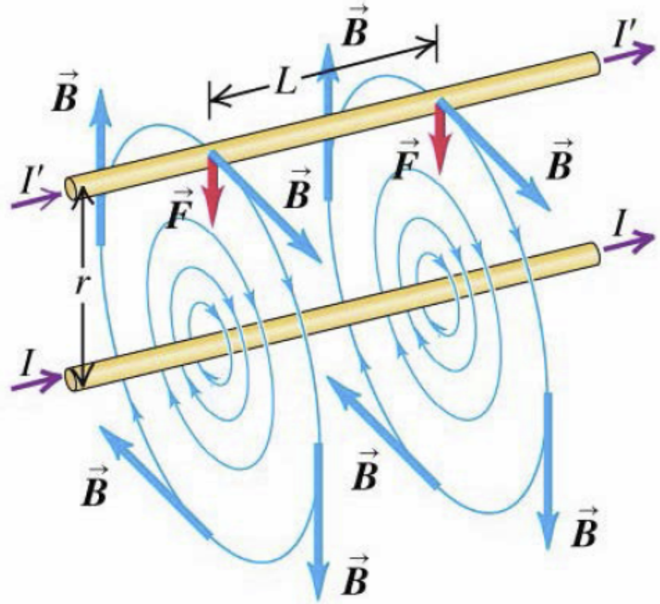
Suppose the currents I and I' are in the same direction.
Magnetic field felt by upper wire (due to lower wire)

$$B = \frac{\mu_0 I}{2\pi r}$$

Magnetic force on upper wire is $F = I'LB$, therefore force per unit length

$$\frac{F}{L} = \frac{\mu_0 I I'}{2\pi r}$$

- ⚠ Lower wire experience the same force, but in opposite direction, due to the magnetic field of the upper wire, leading to **attraction** between the wires
- ⚠ If currents are in opposite direction, leads to repulsion



To conclude: **parallel currents attract, anti-parallel currents repel**.

Definition of Ampere in SI unit:

One ampere is that unvarying current that, if present in each of two parallel conductors of infinite length and one meter apart in empty space, causes each conductor to experience a force of exactly 2×10^{-7} newtons per meter of length.

$$\frac{F}{L} = \frac{\mu_0 I I'}{2\pi r} \Rightarrow \frac{2 \times 10^{-7}\ \mathrm{N}}{1\ \mathrm{m}} = \mu_0 \frac{(1\ \mathrm{A})(1\ \mathrm{A})}{2\pi(1\ \mathrm{m})}$$

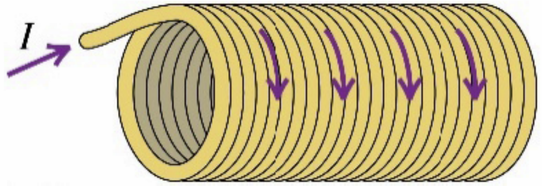
Therefore **in SI units $\boldsymbol{\mu_0}$ is defined to be exactly** $4\pi \times 10^{-7}\ \mathrm{Tm/A}$

Digression: **Why can't ϵ_0 be chosen to be an exact value just like μ_0?**
Through the definition of ampere and the chosen value of μ_0, coulomb is defined. And through Coulomb's law, $F = q_1 q_2/4\pi\epsilon_0 r^2$, ϵ_0 is defined. Once μ_0 is chosen to be exact, ϵ_0 must be determined by experiment. We don't have the freedom to choose a value for ϵ_0.
The value of ϵ_0 defined in the above way *depends* on the value of μ_0 chosen, i.e., they are not independent. Later we will show that they are related by a universal constant – the speed of light in vacuum, $c = 1/\sqrt{\epsilon_0\mu_0}$. This is a hint that electricity and magnetism are the same thing.

Ampere -> Coulomb -> ε_0

Question: A solenoid carrying a current:

1. The magnetic force that one turn of the coil exerts on an adjacent turn is (attractive / repulsive / zero).
2. The electric force that one turn of the coil exerts on an adjacent turn is (attractive / repulsive / zero).
3. The magnetic force between opposite sides of the same turn of the coil is (attractive / repulsive / zero).

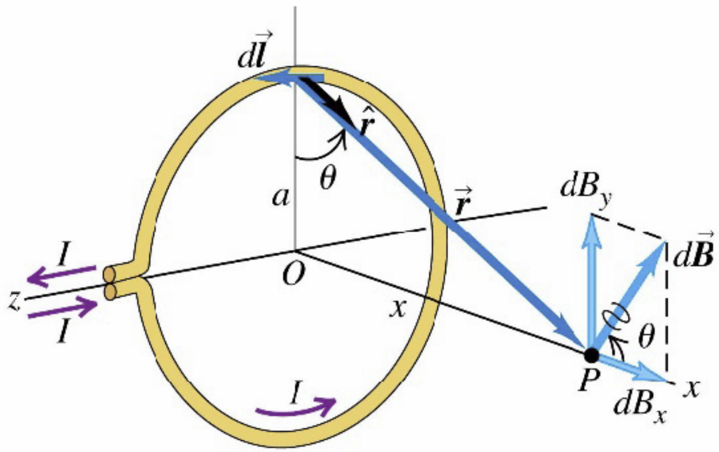


Magnetic Field of a Circular Current Loop

By symmetry dB_y adds up to zero

$$dB_x = \frac{\mu_0 I}{4\pi}\frac{dl}{r^2}\cos\theta = \frac{\mu_0 I}{4\pi}\frac{dl}{x^2+a^2}\frac{a}{\sqrt{x^2+a^2}}$$

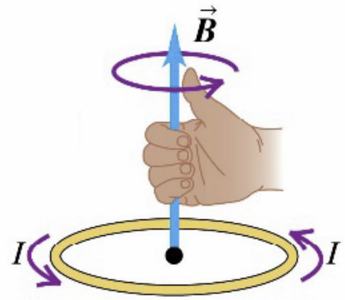
$$\Rightarrow \quad B_x = \frac{\mu_0 I}{4\pi}\frac{a}{(x^2+a^2)^{3/2}}\underbrace{\int dl}_{2\pi a} = \frac{\mu_0 I}{2}\frac{a^2}{(x^2+a^2)^{3/2}}$$

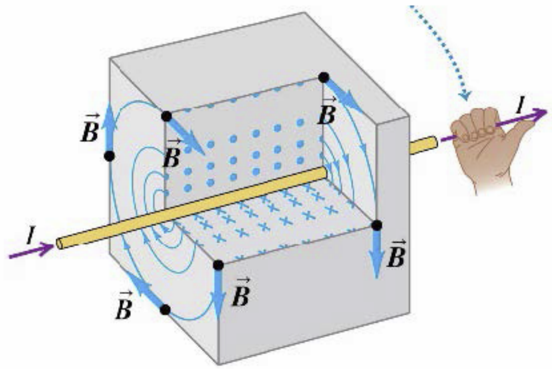


⚠ Direction of magnetic field given by the right-hand-rule

Note that we have encountered **two right-hand rules**:

1. Curl your fingers along *I* and your thumb points along the B field	2. Point your thumb along the current and your fingers give you the B field



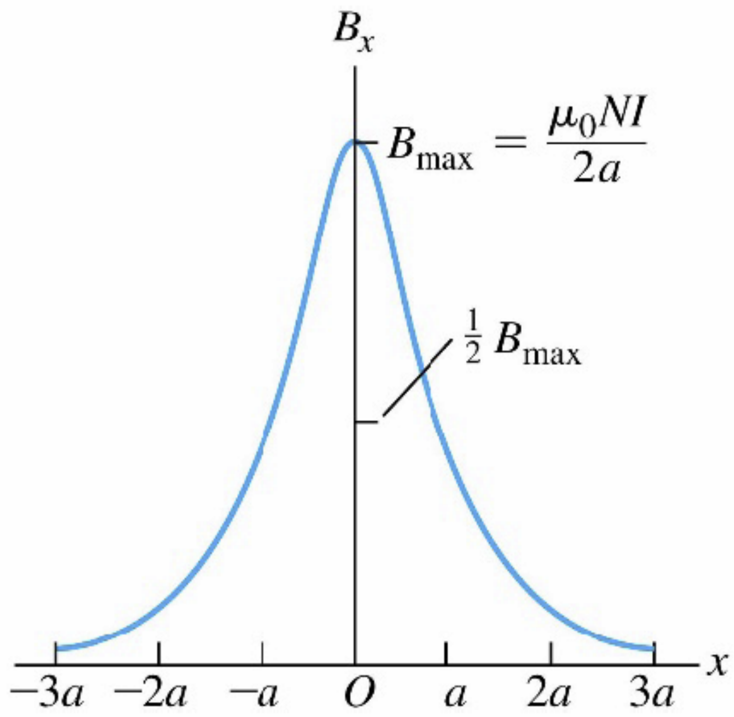


⚠ If coil has N loops, magnetic field adds up

$$B_x = \frac{\mu_0 N I}{2} \frac{a^2}{(x^2 + a^2)^{3/2}}$$

Symmetric and decreasing function of x, maximum at the center of the loop ($x = 0$) with value

$$\boxed{B_m = \frac{\mu_0 N I}{2a}}$$



⚠ Magnetic moment of current loop is $\mu = NIA$, magnetic field produced by a magnetic moment $\vec{\boldsymbol{\mu}}$ along its direction is

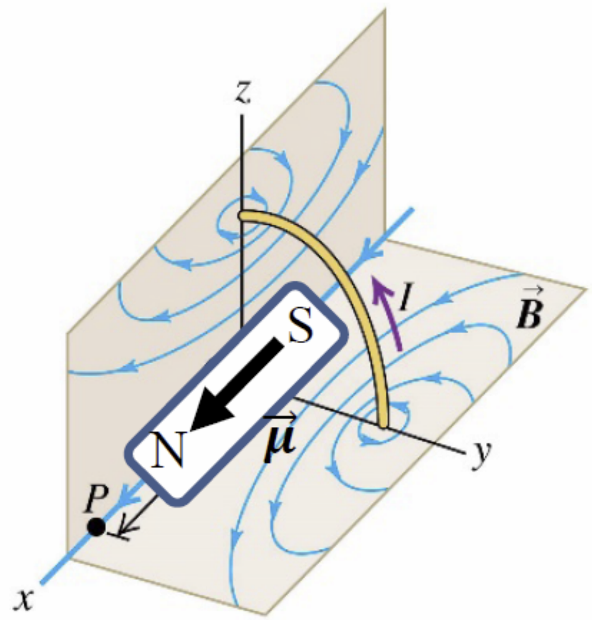
$$B_x = \frac{\mu_0 \mu}{2\pi(x^2 + a^2)^{3/2}}$$

If $x \gg a$,

$$B_x \cong \frac{\mu_0}{2\pi} \frac{\mu}{x^3}$$

c.f. for electric dipole of dipole moment p

$$E_x \cong \frac{1}{2\pi\epsilon_0} \frac{p}{x^3}$$



⚠ $\vec{\boldsymbol{\mu}}$ can be viewed as a current-carrying loop with current defined by the right-hand-rule, or as a bar magnet where $\vec{\boldsymbol{\mu}}$ points from S to N

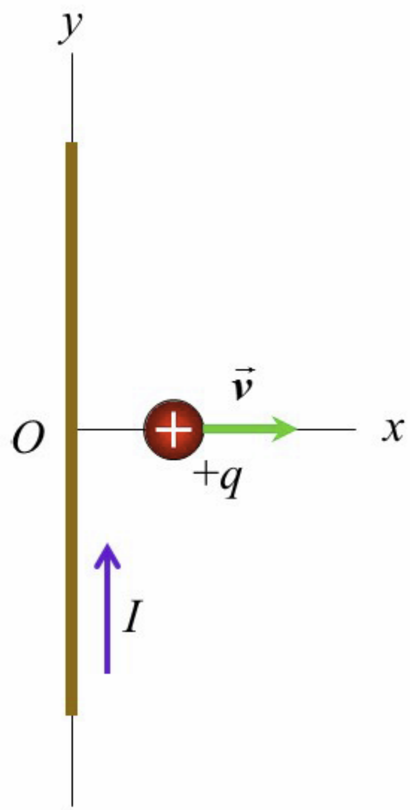
Question: charge has centripetal acceleration while goes through a loop, $\vec{\boldsymbol{v}}$ not constant. Does this lead to radiation?

In principle yes, but normally v_d is so small that this is not important.

Clicker Questions

Q28.3

A long, straight wire lies along the y-axis and carries current in the positive y-direction. A positive point charge moves along the x-axis in the positive x-direction. The magnetic force that the wire exerts on the point charge is in



A. the positive x-direction.

B. the negative x-direction.

C. the positive y-direction.

D. the negative y-direction.

E. none of the above.

A28.3

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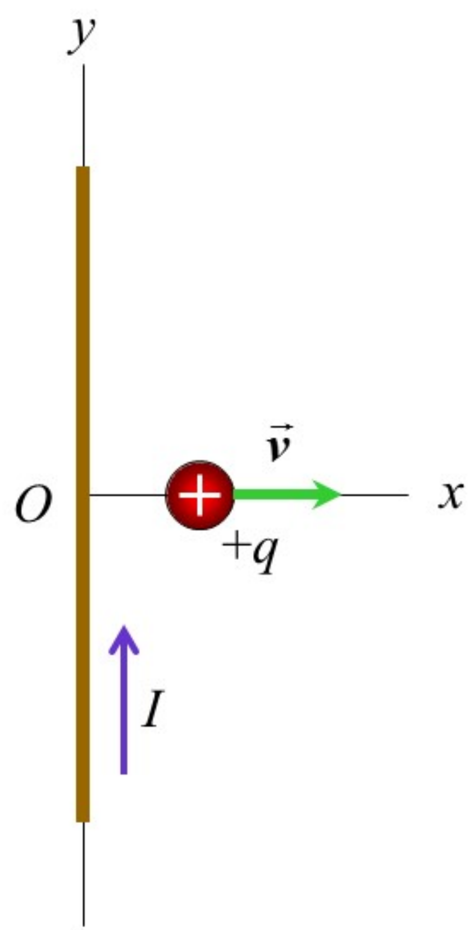


A. the positive x-direction.

B. the negative x-direction.

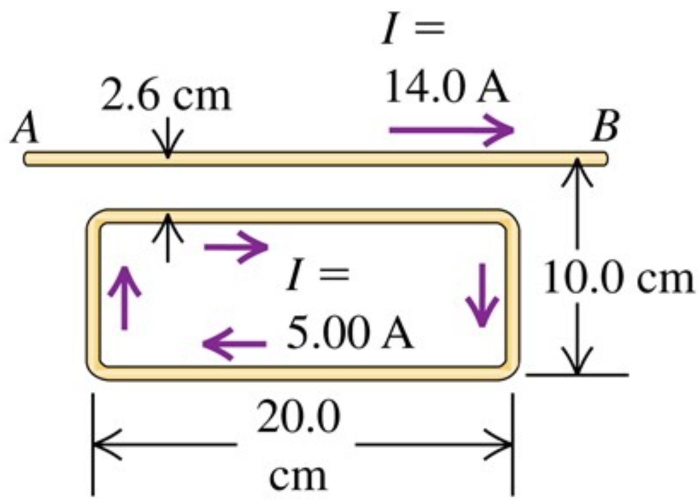
✔ C. the positive y-direction.

D. the negative y-direction.

E. none of the above.

Q28.5

The long, straight wire *AB* carries a 14.0-A current as shown. The rectangular loop has long edges parallel to *AB* and carries a clockwise 5.00-A current. What is the direction of the net magnetic force that the straight wire *AB* exerts on the loop?



A. to the right

B. to the left

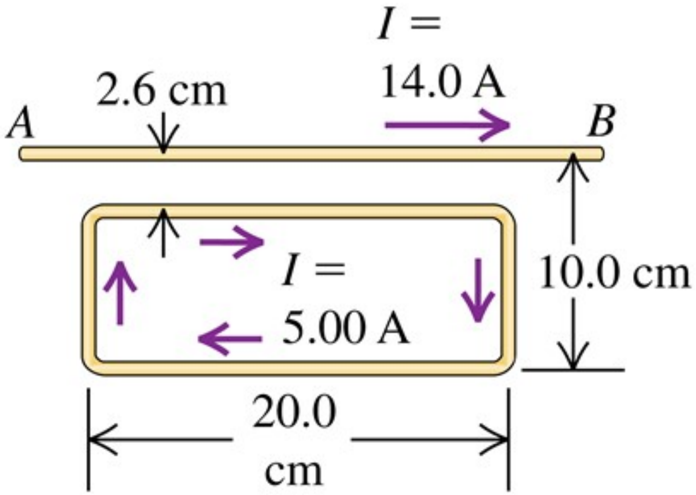
C. upward (toward *AB*)

D. downward (away from *AB*)

E. Misleading question—the net magnetic force is zero.

A28.5

The long, straight wire *AB* carries a 14.0-A current as shown. The rectangular loop has long edges parallel to *AB* and carries a clockwise 5.00-A current. What is the direction of the net magnetic force that the straight wire *AB* exerts on the loop?



A. to the right

B. to the left

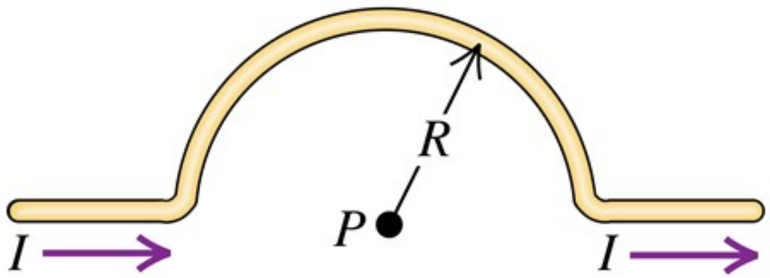
✔ C. upward (toward *AB*)

D. downward (away from *AB*)

E. Misleading question—the net magnetic force is zero.

Q28.6

A wire consists of two straight sections with a semicircular section between them. If current flows in the wire as shown, what is the direction of the magnetic field at *P* due to the current?



A. to the right

B. to the left

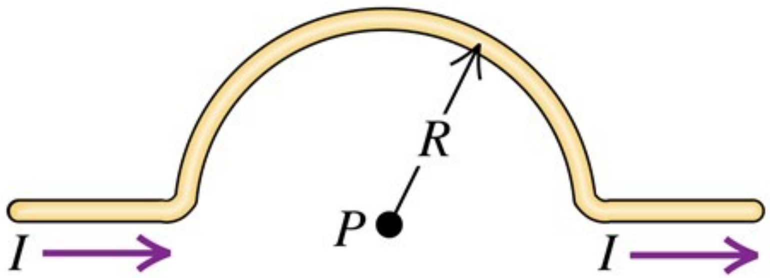
C. out of the plane of the figure

D. into the plane of the figure

E. Misleading question—the magnetic field at *P* is zero.

A28.6

A wire consists of two straight sections with a semicircular section between them. If current flows in the wire as shown, what is the direction of the magnetic field at *P* due to the current?



A. to the right

B. to the left

C. out of the plane of the figure

✔ D. into the plane of the figure

E. Misleading question—the magnetic field at *P* is zero.

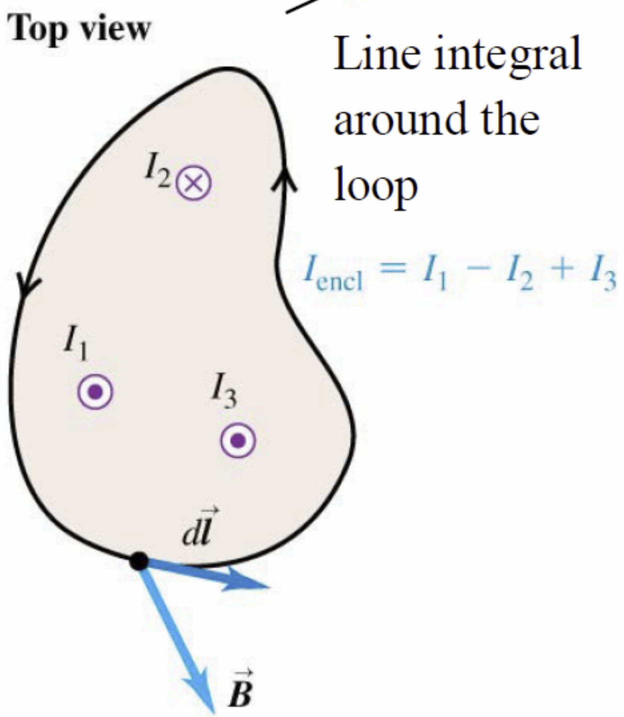
Ampere's Law

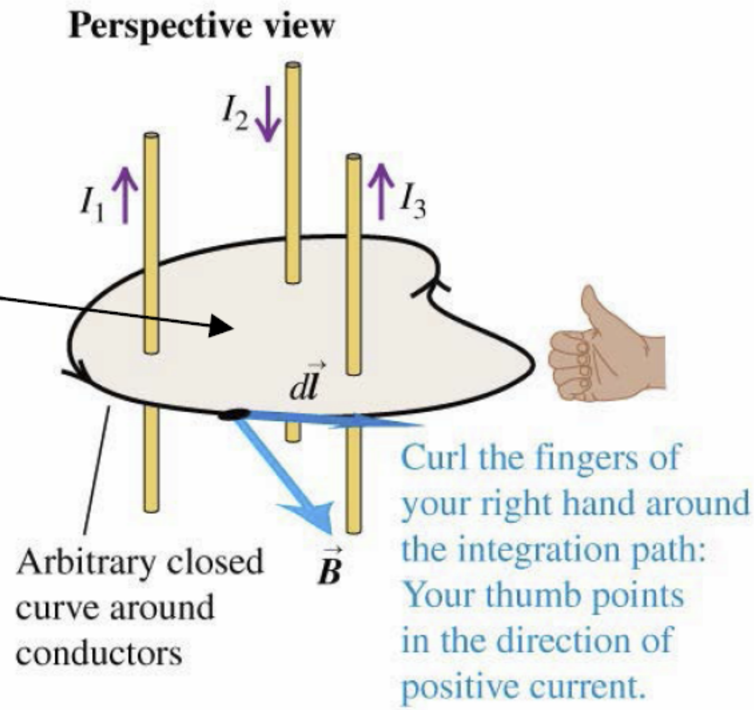
For an arbitrary loop (called Amperian loop)

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$$

Line integral around the loop

Algebraic sum of current crossing a surface with the loop as boundary





- ⚠ Currents not enclosed by the loop do not contribute to the line integral $\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{l}}$
- ⚠ Hold for any steady current (not limited to long straight wires), and any surface (not just flat ones) with the loop as boundary
- ⚠ $\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{l}} = 0$ does not imply $\vec{\boldsymbol{B}} = 0$ along the path
- ⚠ Just like Gauss's law, Ampere's law tell us $\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{l}}$ only. Can be used to find $\vec{\boldsymbol{B}}$ only when the problem has high symmetry
- ⚠ It can be proved that $\vec{\boldsymbol{B}}$ given by the Biot-Savart law always satisfy the Ampere's law
- ⚠ This form of Ampere's law holds only if the current is steady and continuous, and when no magnetic material nor time-varying electric fields are present.

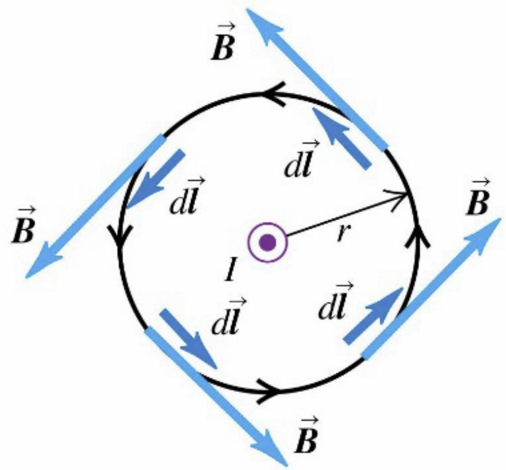
Magnetic field of a long straight conductor

Due to cylindrical symmetry, B constant along a circular path with the conductor as the center

Choose to go through the loop in counter-clockwise sense

$$\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{l}} = B(2\pi r) = \mu_0 I$$

$$\Rightarrow \quad B = \frac{\mu_o I}{2\pi r}$$



Magnetic field of a long cylindrical conductor (Example 28.8)

Cylindrical symmetry, choose concentric circles as Amperian loop

Outside conductor: $r > R$

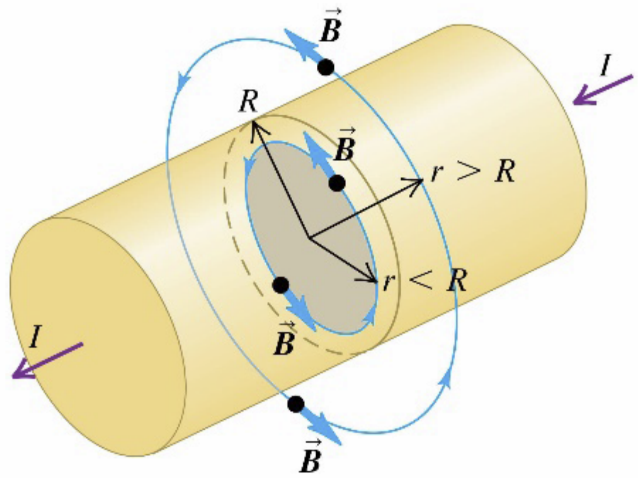
$$\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{l}} = B(2\pi r) = \mu_0 I \quad \Rightarrow \; B = \frac{\mu_o I}{2\pi r}$$

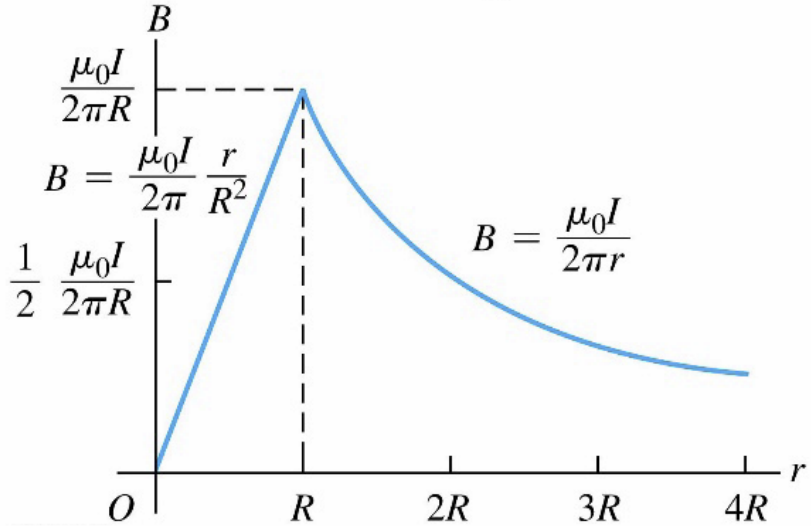
Inside conductor: $r < R$

$$\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{l}} = B(2\pi r) = \mu_0 \frac{\pi r^2}{\pi R^2} I = \mu_0 \frac{r^2}{R^2} I$$

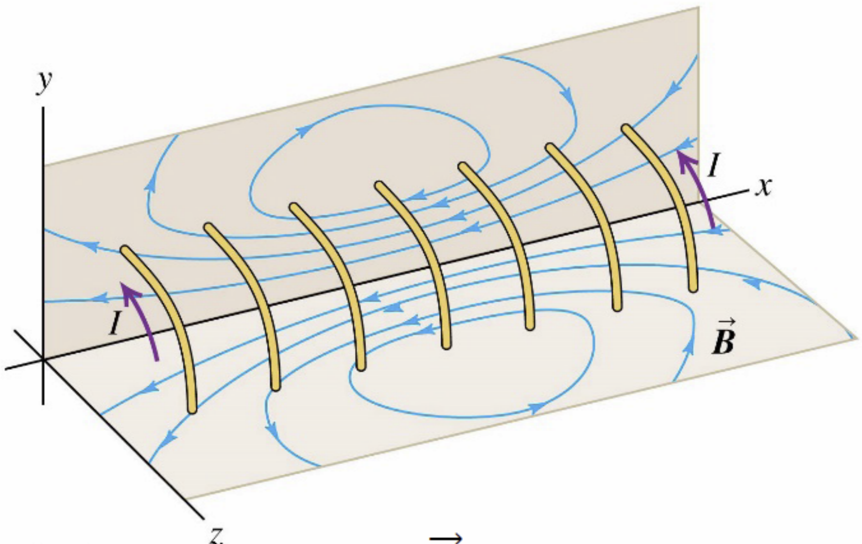
$$\Rightarrow \quad B = \frac{\mu_0 I}{2\pi R^2} r$$

Look familiar? Where have you seen similar behavior?

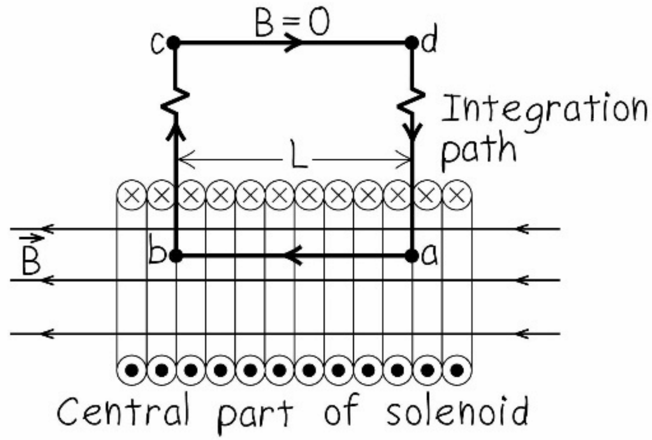




Magnetic field of a solenoid



For a short solenoid, $\vec{B}$ not uniform inside



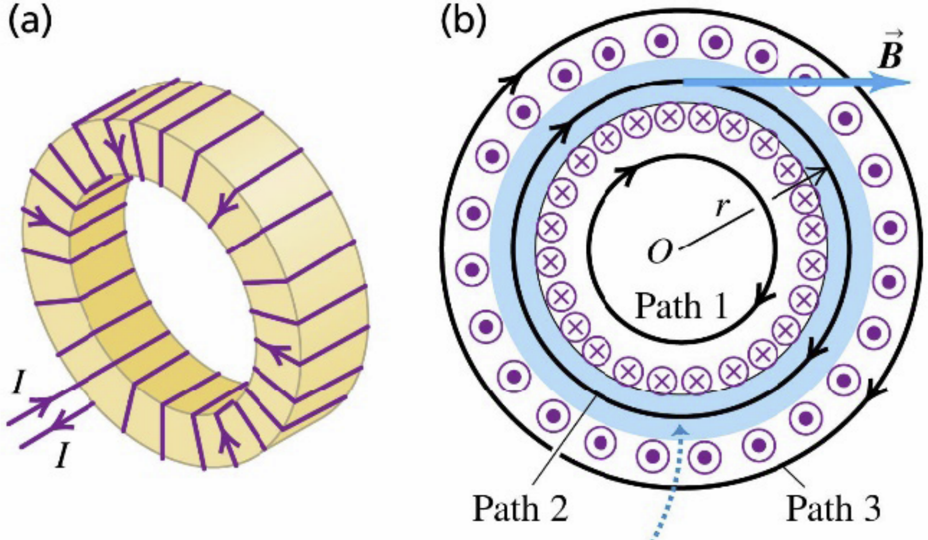
Near the central part of a long solenoid, assume B uniform inside and zero outside

$$\oint \vec{B} \cdot d\vec{l} = BL = \mu_0 NI \;\Rightarrow\; B = \frac{\mu_o NI}{L} = \mu_0 nI$$

where $n \equiv N/L$ number of turns per unit length

- **By symmetry analysis + Gauss's law, *B* is parallel to the solenoid, and *B* uniform inside the solenoid**
- **By taking path CD to infinity, one can proof that *B* is zero outside**

Magnetic field of a toroidal solenoid



The magnetic field is confined almost entirely to the space enclosed by the windings (in blue).

Take a long solenoid and bend into a toroid

B constant along a concentric circular path

Loops outside the toroid (paths 1 and 3) enclose no net current, $B = 0$

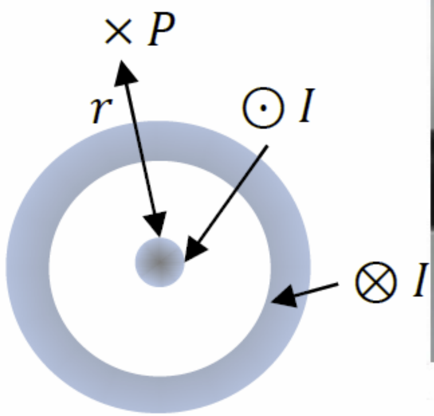
Inside toroid (path 2)

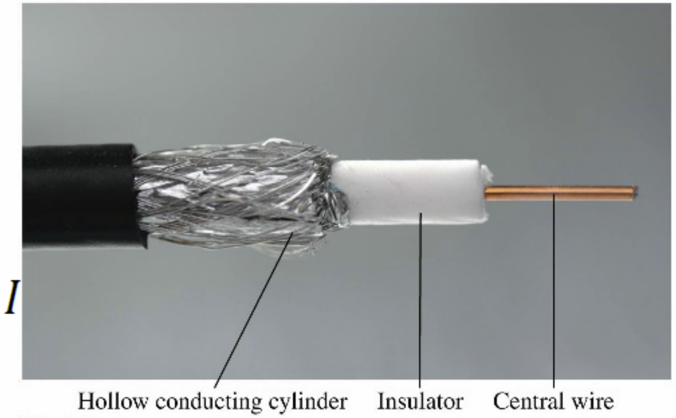
$$\oint \vec{B} \cdot d\vec{l} = B(2\pi r) = \mu_0 NI \Rightarrow B = \frac{\mu_o NI}{2\pi r}$$

⚠ B *not* uniform inside toroid (depends on r). If the "thickness" of toroid ≪ its radius R, then B inside toroid is approximately uniform, $B = \mu_o NI/2\pi R = \mu_0 nI$, same as a long straight solenoid

Question:

A coaxial cable has inner and outer concentric conductors carrying equal and opposite currents. At a point outside the cable and at a perpendicular distance r from its axis, B is ($\propto 1/r$ / $\propto 1/r^2$ / zero).





Magnetic Materials

An electron in an atom can be considered to create microscopic current loops (not totally correct, but not totally off either)

Classical picture: suppose electron goes in circular loop

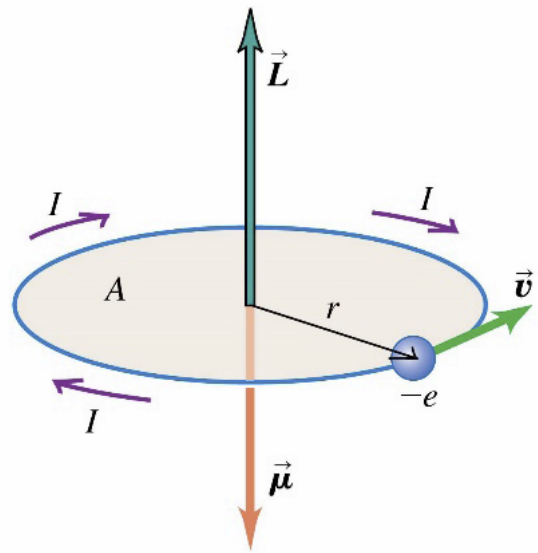
$$I = \frac{e}{T} = \frac{e}{2\pi r/v}$$

$$\mu = IA = \frac{e}{2\pi r/v}\pi r^2 = \frac{evr}{2} = \frac{e}{2m}L$$

where $L = mvr$ is the angular momentum. In vector form

$$\vec{\boldsymbol{\mu}} = -\frac{e}{2m}\vec{\boldsymbol{L}}$$

Due to –ve charge of electron



In the microscopic world, quantum mechanics introduces the following complications

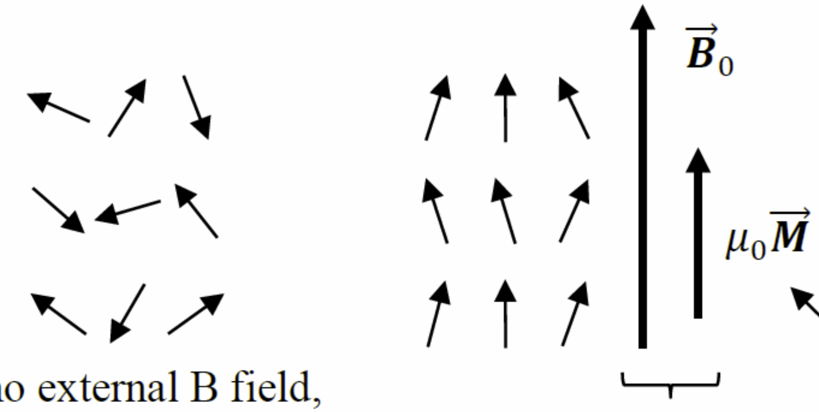
- ⚠ L must be integral multiples of $h/2\pi$, $L = n(h/2\pi)$, where $h = 6.62 \times 10^{-34}$ Js is the Planck's constant. Consequently $\mu = n(eh/4\pi m) \equiv n\mu_B$.
  $\mu_B \equiv eh/4\pi m = 9.274 \times 10^{-24}$ Am^2 $= 9.274 \times 10^{-24}$ J/T, called the **Bohr magneton**
- ⚠ There is intrinsic uncertainty in the direction of $\vec{\boldsymbol{L}}$, and therefore $\vec{\boldsymbol{\mu}}$. But we need not concern this here
- ⚠ There is another source of angular momentum coming from the electron called the spin angular momentum. We need not concern this here

Some atoms have an intrinsic magnetic moment μ_{atom}

Paramagnetism – (weak) attraction in an external field, e.g, aluminum, platinum

$\vec{\boldsymbol{\mu}}$ from different electrons in the same atom may/may not add up to zero.

In paramagnetic material, $\vec{\boldsymbol{\mu}}_{\text{atom}} \neq 0$, in an external field $\vec{\boldsymbol{B}}_0$, tends to align in order to minimize potential energy $U = -\vec{\boldsymbol{\mu}}_{\text{atom}} \cdot \vec{\boldsymbol{B}}_0$



under external field $\vec{\boldsymbol{B}}_0$, $\vec{\boldsymbol{\mu}}_{\text{total}} = \sum \vec{\boldsymbol{\mu}}_{\text{atom}} \neq 0$, define **magnetization** of material as $\vec{\boldsymbol{\mu}}_{\text{total}}$ per unit volume

$$\vec{\boldsymbol{M}} = \frac{\vec{\boldsymbol{\mu}}_{\text{total}}}{V}$$

no external B field, orientation of $\vec{\boldsymbol{\mu}}_{\text{atom}}$ random, add up to zero

$\vec{\boldsymbol{\mu}}_{\text{total}}$ causes a magnetic field $\mu_0\vec{\boldsymbol{M}}$ (not proved here) in same direction as $\vec{\boldsymbol{B}}_0$, leads to *attraction*

Unit: $[\text{Tm/A}][\text{Am}^2/\text{m}^3] = \text{T}$

Total field inside material *increases*

$$\vec{\boldsymbol{B}} = \vec{\boldsymbol{B}}_0 + \mu_0\vec{\boldsymbol{M}} = K_m\vec{\boldsymbol{B}}_0$$

A factor $K_m > 1$ (dimensionless, called **relative permeability**) larger than in vacuum, i.e., replace $\mu_0 \to K_m\mu_0 \equiv \mu$, the **permeability** of the material

- ⚠ μ again, but here not a magnetic moment!
- ⚠ *c.f.* dielectric constant of a dielectric with external E field
- ⚠ for most paramagnetic material $K_m \gtrsim 1$, define the **magnetic susceptibility** by $\chi_m = K_m - 1$, and $\chi_m \sim 10^{-5}$, see Table 28.1 in your textbook

Thermal agitation prevents $\vec{\boldsymbol{\mu}}_{\text{atom}}$ to align completely with $\vec{\boldsymbol{B}}_0$, higher temperature, more difficult to align

The Curie's law

$$M \propto \frac{B}{T} = C\frac{B}{T}$$

Example 28.11

Nitric oxide (NO) is paramagnetic. Each NO molecule has magnetic moment $\approx \mu_B$.

Maximum potential energy of a NO molecule in a 1.5 T magnetic field (if fully aligned)

$$U_m \approx \mu_B B = (9.27 \times 10^{-24}\ \mathrm{J/T})(1.5\ \mathrm{T}) = 1.4 \times 10^{-23}\ \mathrm{J}$$

c.f. average translational kinetic energy at room temperature

$$K = \frac{3}{2}k_B T = \frac{3}{2}(1.38 \times 10^{-23}\ \mathrm{J/K})(300\ \mathrm{K}) = 6.2 \times 10^{-21}\ \mathrm{J}$$

⚠ $K \gg U_m$, difficult for molecule to align with magnetic field, therefore paramagnetism is weak

Diamagnetism – repulsion in an external field, e.g., copper, water

$\vec{\boldsymbol{\mu}}_{\text{atom}} = 0$, but under external field $\vec{\boldsymbol{B}}_0$, induce effective $\vec{\boldsymbol{\mu}}$ (and as a result a magnetic field) that *oppose* $\vec{\boldsymbol{B}}_0$ (recall Lenz's law in electromagnetic induction, which you learned in high school), leading to *repulsion* (unlike paramagnetic materials).

⚠ $K_m < 1$, i.e., $\chi_m < 0$. Field inside the material *decreases*.

⚠ Like in paramagnetic material, $|\chi_m| \sim 10^{-5}$

⚠ χ_m usually not sensitive to temperature in diamagnetic materials, no Curie's law

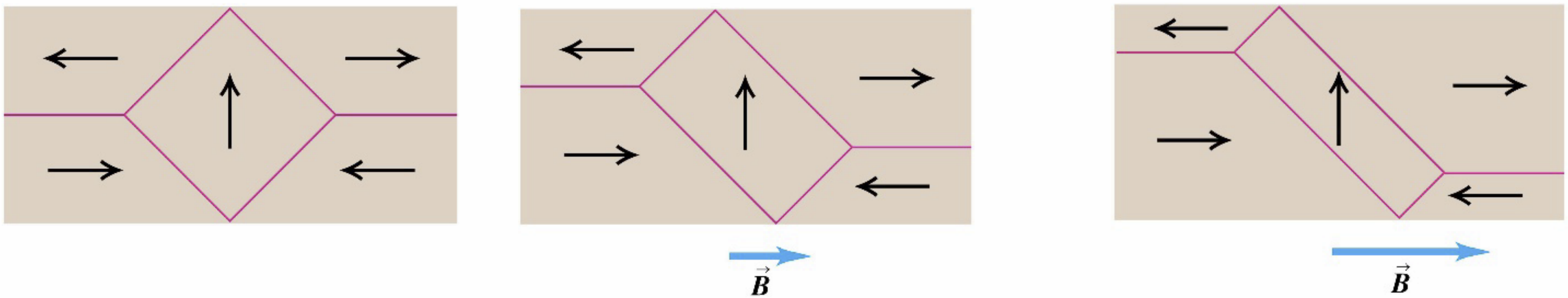
⚠ Water is diamagnetic, leading to magnetic levitation in a strong field, e.g., of a frog, see demo here

Ferromagnetism – strong attraction in an external field

e.g. iron, nickel, cobalt, and their alloys

Neigboring $\vec{\boldsymbol{\mu}}_{\text{atom}}$ like to align, forming **magnetic domains** (large region in which all $\vec{\boldsymbol{\mu}}_{\text{atom}}$ are parallel, adding up to a non-zero magnetization).

(a) No field (b) Weak field (c) Stronger field

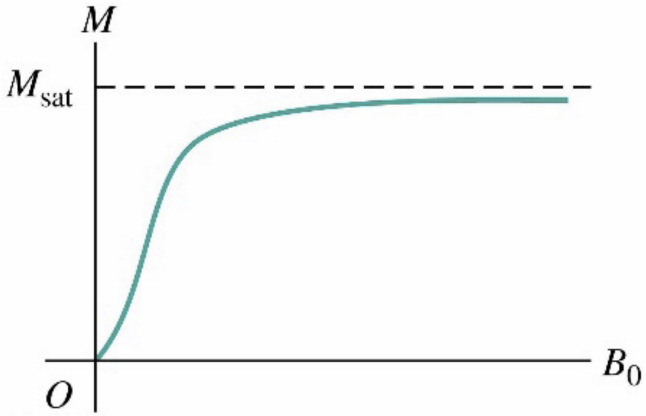


(a) Magnetizations of different domains random, adding up to zero

(b) Domains with magnetization along $\vec{\boldsymbol{B}}$ expand by moving domain walls (boundary between different domains), leading to non-zero total magnetization

(c) Domains aligned with $\vec{\boldsymbol{B}}$ further expand, larger total magnetization

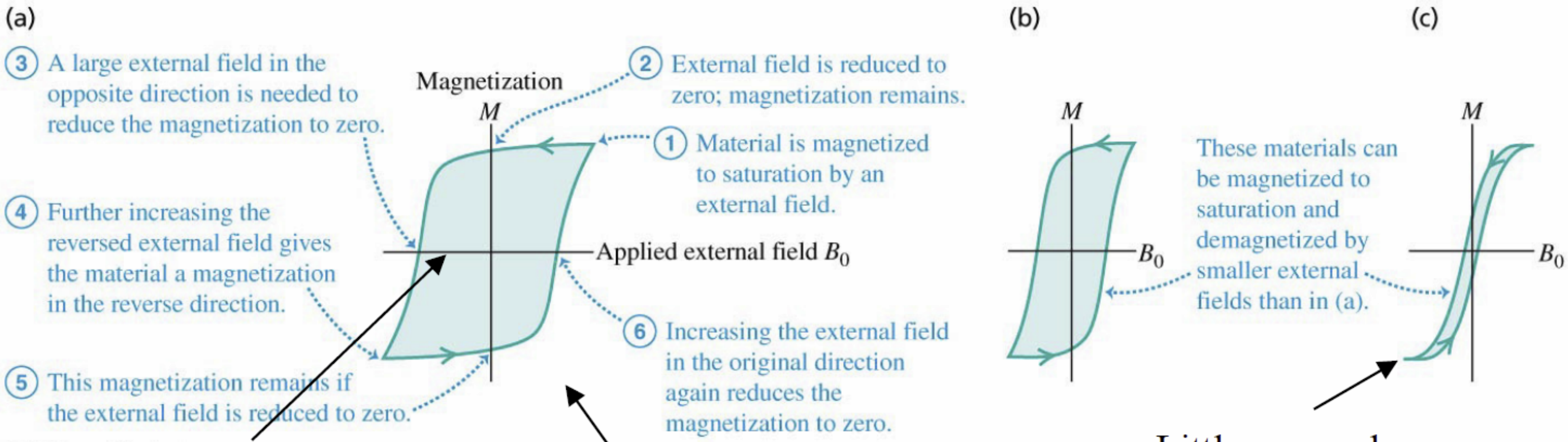
⚠ For ferromagnetic materials, $K_m \sim 10^3 - 10^5$, field inside material *increased tremendously*, used as cores in electromagnetics, transformer, motor, generators, etc to increase the magnetic field



When B_0 strong enough, most domains align, leading to *saturation*

To magnetize a ferromagnetic material, apply external field B_0

To *de-magnetize*, reducing B_0 to zero is usually not enough, need to apply an opposite field, i.e., need energy! Called **hysteresis** (meaning a delay of an effect behind its cause)



Enclosed area is the energy loss in the magnetizing-demagnetizing process

Difficult to demagnetized, suitable for making memory

Little energy loss, suitable for making electromagnets, motors, etc

Example 28.12

A magnet in the form of a cube of sides 2 cm, has a magnetization $M = 8 \times 10^5$ A/m. The magnetic dipole moment is

$$\mu_{\text{total}} = MV = (8 \times 10^5 \text{ A/m})(2 \times 10^{-2} \text{ m})^3 = 6 \text{ Am}^2$$

Magnetic field due to the magnet at a point 10 cm from the magnet along its axis

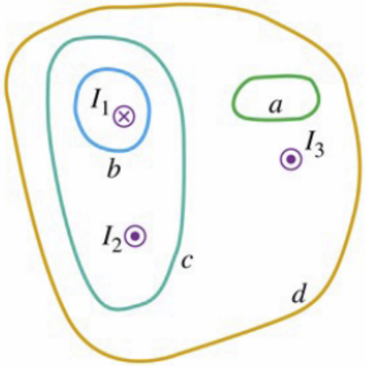
$$B \approx \frac{\mu_0 \mu_{\text{total}}}{2\pi x^3} = \frac{(4\pi \times 10^{-7} \text{ Tm/A})(6 \text{ Am}^2)}{2\pi (0.1 \text{ m})^3} = 10^{-3} \text{ T}$$

⚠ The point is outside the magnet, i.e., in vacuum (or air), therefore you should use the vacuum permeability μ_0. If the point is inside a material, should use the permeability $\mu = K_m \mu_0$

Clicker Questions

Q28.8

The figure shows, in cross section, three conductors that carry currents perpendicular to the plane of the figure. If the currents I_1, I_2, and I_3 all have the same magnitude, for which path(s) is/are the line integral of the magnetic field equal to zero?



A. path *a* only

B. paths *a* and *c*

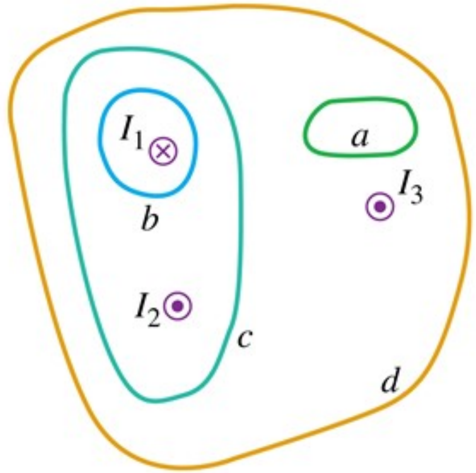
C. paths *b* and *d*

D. paths *a*, *b*, *c*, and *d*

E. depends on whether the integral goes clockwise or counterclockwise around the path

A28.8

The figure shows, in cross section, three conductors that carry currents perpendicular to the plane of the figure. If the currents I_1, I_2, and I_3 all have the same magnitude, for which path(s) is/are the line integral of the magnetic field equal to zero?



A. path *a* only

✔ B. paths *a* and *c*

C. paths *b* and *d*

D. paths *a*, *b*, *c*, and *d*

E. depends on whether the integral goes clockwise or counterclockwise around the path

Q-RT28.1

An infinitesimal current element located at the origin ($x = y = z = 0$) carries current I in the positive x-direction. ***Rank*** the following locations in order of the magnitude of the magnetic field that the current element produces at that location, from largest to smallest value.

A. $x = L,\ y = 0,\ z = 0$

B. $x = 0,\ y = L,\ z = 0$

C. $x = L/\sqrt{2},\ y = L/\sqrt{2},\ z = 0$

A-RT28.1

An infinitesimal current element located at the origin ($x = y = z = 0$) carries current I in the positive x-direction. ***Rank*** the following locations in order of the magnitude of the magnetic field that the current element produces at that location, from largest to smallest value.

A. $x = L,\ y = 0,\ z = 0$

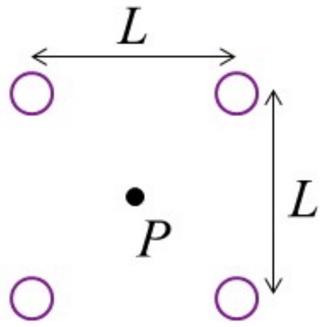
B. $x = 0,\ y = L,\ z = 0$

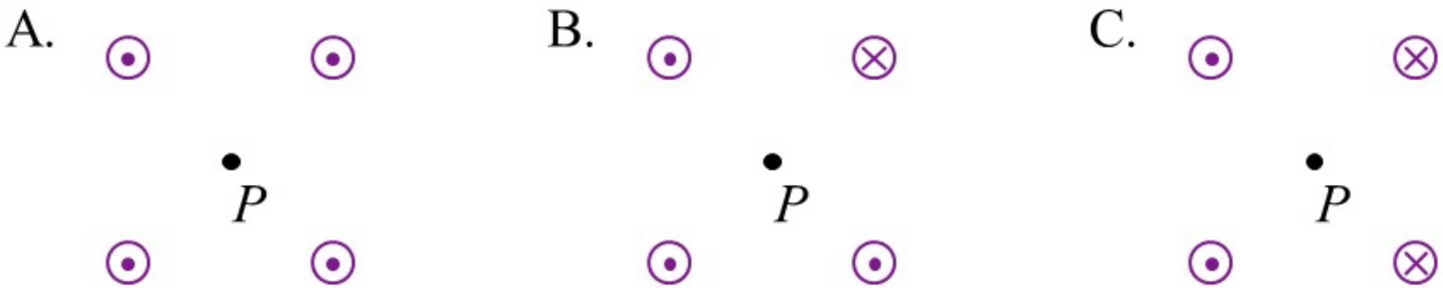
C. $x = L/\sqrt{2},\ y = L/\sqrt{2},\ z = 0$

✔ Answer: BCA

Q-RT28.2

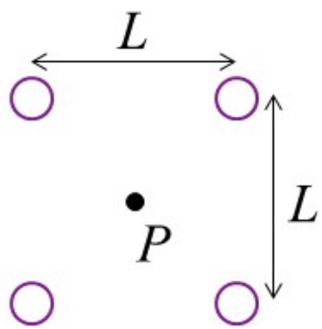
Four long parallel wires are oriented perpendicular to the plane of the figure. In cross section they form a square of side L. Point P is at the center of the square. In situations A, B, and C each wire carries the same current I but in different directions. ***Rank*** these three situations in order of the magnitude of the total magnetic field that the four currents produce at P, from largest to smallest value.

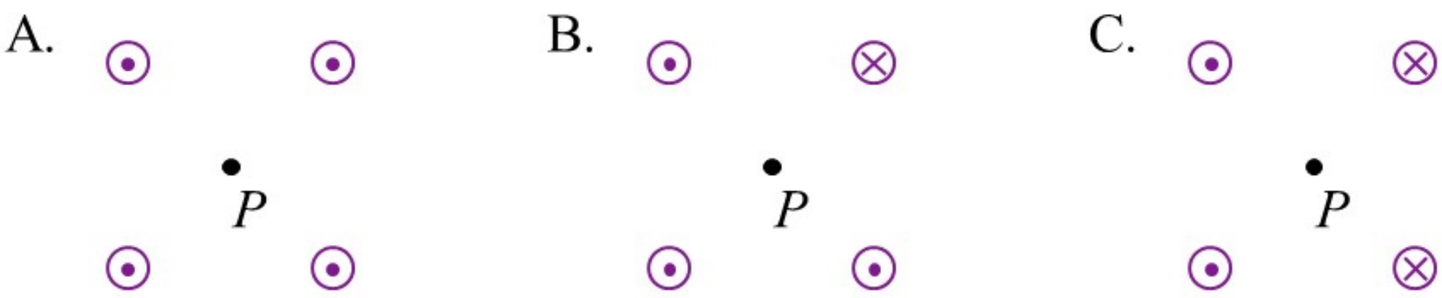




A-RT28.2

Four long parallel wires are oriented perpendicular to the plane of the figure. In cross section they form a square of side L. Point P is at the center of the square. In situations A, B, and C each wire carries the same current I but in different directions. ***Rank*** these three situations in order of the magnitude of the total magnetic field that the four currents produce at P, from largest to smallest value.





Answer: CBA