OPTICS III

Derivations for Fresnel equations

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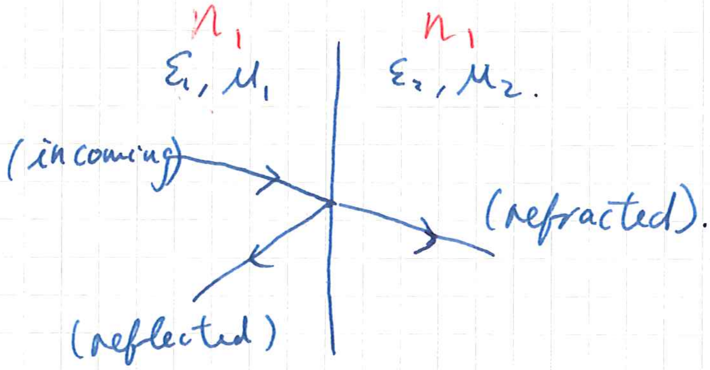
Fresnel Equations :

Boundary Conditions :

Maxwell's equations must be satisfied at the interface as well as in the bulk materials.



~~$\oint \varepsilon \vec{E} \cdot d\vec{A} = \Sigma q$~~

~~$\oiint \vec{E} \cdot d\vec{S}$~~

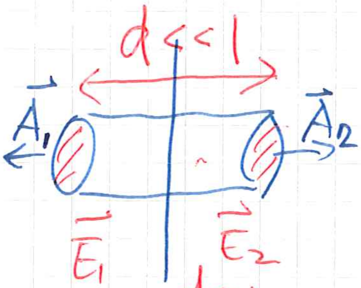
$$\oiint \varepsilon \vec{E} \cdot d\vec{S} = \Sigma q = Q_{encl.} \qquad \text{(Gauss)}$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint \vec{B} \cdot d\vec{S} \qquad \text{(Faraday).}$$

$$\oiint \vec{B} \cdot d\vec{S} = 0. \qquad \text{(Gauss)}$$

$$\oint \frac{\vec{B}}{\mu} \cdot d\vec{l} = \iint \vec{J} \cdot d\vec{S} + \frac{d}{dt} \iint \vec{E} \cdot d\vec{S} \qquad \text{(Ampere).}$$

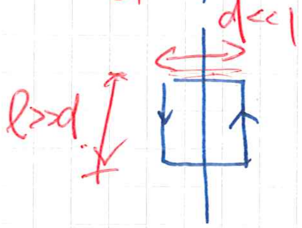
① .



$(\varepsilon_2 \vec{E}_2 - \varepsilon_1 \vec{E}_1) \cdot \vec{A} = 0$ $\quad \vec{E}_2 \cdot \vec{A}_2 + \vec{E}_1 \cdot \vec{A}_1$

$\Rightarrow \boxed{\varepsilon_1 E_{1\perp} = \varepsilon_2 E_{2\perp}}$ — (1)

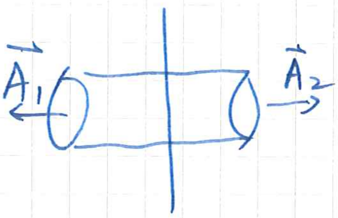
② .



$$\oint \vec{E} \cdot d\vec{l} = (\vec{E}_2 - \vec{E}_1) \vec{l} = -\frac{d}{dt} \iint \vec{B} \cdot \underset{\downarrow 0}{d\vec{S}} = 0.$$

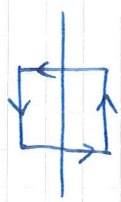
$$\boxed{E_{2\parallel} = E_{1\parallel}}$$

③



$\boxed{B_{1\perp} = B_{2\perp}}$ — (3)

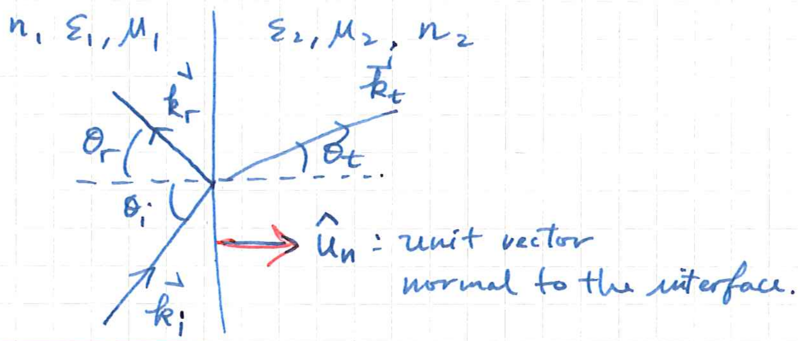
④



$$\left(\frac{\vec{B}_2}{\mu_2} - \frac{\vec{B}_1}{\mu_1} \right) \cdot \vec{l} = 0.$$

$$\Rightarrow \boxed{\frac{B_{1,\parallel}}{\mu_1} = \frac{B_{2,\parallel}}{\mu_2}}$$

Electromagnetic wave approach



$$\boxed{\vec{E}_i(\vec{r},t) = \vec{E}_{i0}\cos(\vec{k}_i\cdot\vec{r} - \omega_i t).} \quad \frac{c}{n_1} = v_1 = \frac{\omega_i}{k_i}, \quad n = \frac{c}{v}.$$

$$\vec{E}_r(\vec{r},t) = \vec{E}_{r0}\cos(\vec{k}_r\cdot\vec{r} - \omega_r t) \quad \text{(reflected)}$$

$$\vec{E}_t(\vec{r},t) = \vec{E}_{t0}\cos(\vec{k}_t\cdot\vec{r} - \omega_t t). \quad \text{(transmitted)}.$$

① $E_{1,\parallel} = E_{2,\parallel}$

$$(\vec{E}_i + \vec{E}_r)\times\hat{u}_n = \vec{E}_t\times\hat{u}_n.$$

$$\Rightarrow (\vec{E}_{i0}\times\hat{u}_n)\cos(\vec{k}_i\cdot\vec{r} - \omega_i t) + (\vec{E}_{r0}\times\hat{u}_n)\cos(\vec{k}_r\cdot\vec{r} - \omega_r t).$$

$$= (\vec{E}_{t0}\times\hat{u}_n)\cos(\vec{k}_t\cdot\vec{r} - \omega_t t)$$

* This equation holds for all time t and any pt. $\vec{r}$ on the interface.

T Remark: For a eqtn :

$$a_2x^2 + a_1x + a_0 + b_2y^2 + b_1y = 0.$$

is valid for all x and y, then $a_2 = a_1 = a_0 = b_2 = b_1 = 0.$ ⊥

$$\Rightarrow \cos(\vec{k}_i\cdot\vec{r} - \omega_i t) = \cos(\vec{k}_r\cdot\vec{r} - \omega_r t). = \cos(\vec{k}_t\cdot\vec{r} - \omega_t t).$$

and $\vec{E}_{i0}\times\hat{u}_n + \vec{E}_{r0}\times\hat{u}_n = \vec{E}_{t0}\times\hat{u}_n.$

$$\Rightarrow \omega_i = \omega_r = \omega_t. \quad \& \quad \begin{cases} (\vec{k}_i - \vec{k}_r)\cdot\vec{r}\,\big|_{\text{interface}} = 0 \\ (\vec{k}_i - \vec{k}_t)\cdot\vec{r}\,\big|_{\text{interface}} = 0. \end{cases}$$

(N.B. $v = \frac{\omega}{k}$).

$\Rightarrow$ $\vec{k}_i - \vec{k}_r$ and $\vec{k}_i - \vec{k}_t$ are parallel to $\hat{u}_n$.

$$\Rightarrow (\vec{k}_i - \vec{k}_r)\times\hat{u}_n = (\vec{k}_i - \vec{k}_t)\times\hat{u}_n = 0$$

$$\Rightarrow k_1\sin\theta_i = k_r\sin\theta_r.$$

$$\Rightarrow \frac{\omega_i}{v_1}\sin\theta_i = \frac{\omega_r}{v_1}\sin\theta_r \Rightarrow \boxed{\theta_i = \theta_r} \quad \text{(Law of Reflection)}$$

On the other hand, $(\vec{k_i} - \vec{k_t}) \times \hat{u}_n = 0$ ③

$$\Rightarrow k_i \sin\theta_i = k_t \sin\theta_t .$$

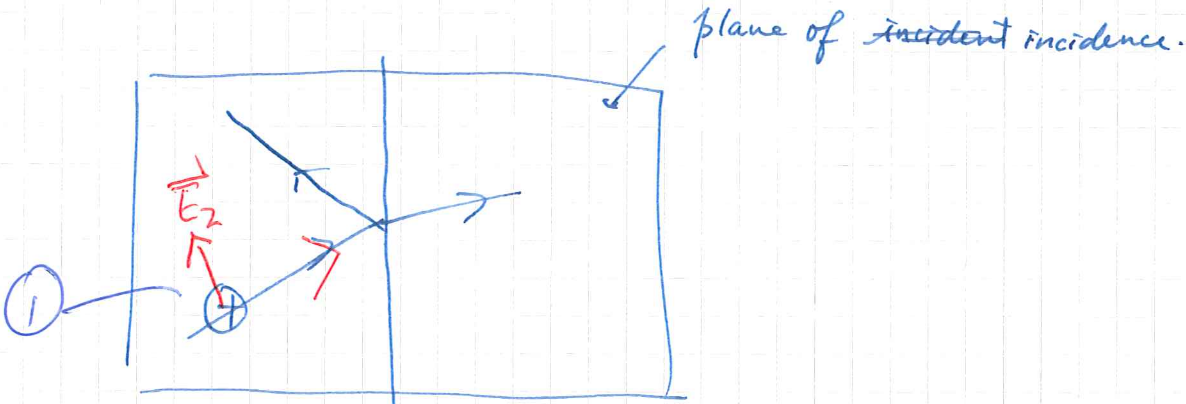
$$\Rightarrow \frac{w_i}{V_1} \sin\theta_i = \frac{w_t}{V_2} \sin\theta_t .$$

$$\Rightarrow \frac{c}{V_1} \sin\theta_i = \frac{c}{V_2} \sin\theta_t$$

$$\Rightarrow \boxed{n_1 \sin\theta_i = n_2 \sin\theta_t} . \quad \text{Snell's law.}$$

* We can derive the laws by matching the B.C. at the interface.

Next, I want to solve $\vec{E_r}, \vec{E_t}$ from $\vec{E_i}$. by the B.Cs.
But they depend on the polarization of the incident light.



1) $\vec{E_i}$ $\perp$ to the plance of incidence.
2) $\vec{E_i}$ // to the plane of incidence

④

Case 1 : $\vec{E}_i \perp$ to the plane-of-incidence.

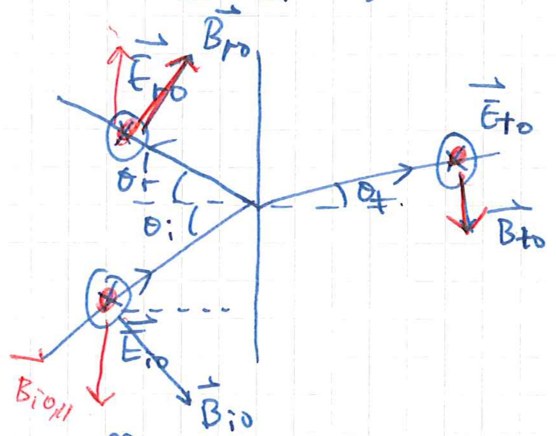
Recall that $E_0 = v B_0$ and $\vec{E}, \vec{B}, \vec{k}$ are othogonal to each other.

$$\Rightarrow \left.\begin{aligned} \hat{k} \times \vec{E}_0 &= v \vec{B} \\ \hat{k} \cdot \vec{E} &= 0 \end{aligned}\right\}$$

Now, from B.C. ① : ~~$\vec{E}_{0i} + \vec{E}_{ri}$~~ $\vec{E}_{i0} + \vec{E}_{r0} = \vec{E}_{t0}$.

We know that $\vec{E}_{r0}$, $\vec{E}_{t0}$ are perpendicular to the plane-of-incident by symmetry. We now assume they are parallel to $\vec{E}_{i0}$ i.e.

$$\boxed{E_{i0} + E_{r0} = E_{t0}} \quad \text{— ⓐ}$$



Next, we apply B.C. ④.

$$\frac{B_{\parallel}}{\mu_1} = \frac{B_{2,\parallel}}{\mu_2} \Rightarrow \frac{-B_{i0}\cos\theta_i}{\mu_1} + \frac{B_{r0}\cos\theta_r}{\mu_1} = -\frac{B_t \cos\theta_t}{\mu_2}.$$

$$\Rightarrow \cancel{\frac{1}{v_1\mu_1}\left(E_{t0} - E_{i0}\right) =}$$

$$\frac{1}{v_1\mu_1}\left(E_{i0} - E_{r0}\right)\cos\theta_i = \frac{1}{v_2\mu_2} E_{t0} \cos\theta_t$$

$$\Rightarrow \frac{n_1}{\mu_1}\left(E_{i0} - E_{r0}\right)\cos\theta_i = \frac{n_2}{\mu_2} E_{t0} \cos\theta_t . \quad \text{— ⓑ}$$

ⓐ + ⓑ $\Rightarrow$ ~~$E_0/$~~

amplitude of reflection coeff. $= r_\perp =$

$$\left(\frac{E_{r0}}{E_{i0}}\right)_\perp = \frac{\frac{n_1}{\mu_1}\cos\theta_i - \frac{n_2}{\mu_2}\cos\theta_t}{\cancel{\frac{n_2}{\mu_2}}\ \frac{n_1}{\mu_1}\cos\theta_i + \frac{n_2}{\mu_2}\cos\theta_t}.$$

amplitude transmission coefficient. $= t_\perp =$

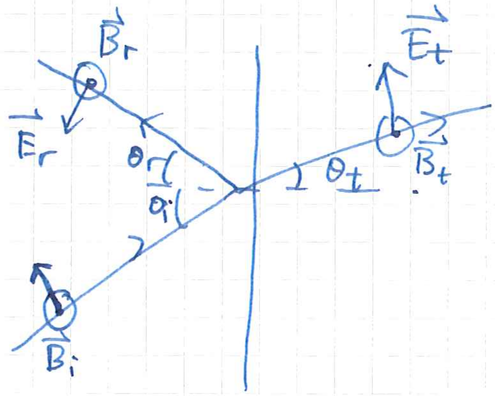
$$\left(\frac{E_{t0}}{E_{i0}}\right)_\perp = \frac{2\frac{n_1}{\mu_1}\cos\theta_i}{\frac{n_1}{\mu_1}\cos\theta_i + \frac{n_2}{\mu_2}\cos\theta_t}.$$

$\vec{E}$-field $\perp$ to the plane-of-incidence.

Case 2 : $\vec{E}$ parallel to the plane-of-incidence.

$\boxed{B\text{-}C\text{-}2}$ $\Rightarrow$ $E_{1\parallel} = E_{2\parallel}$

$$\Rightarrow E_{io}\cos\theta_i - E_{ro}\cos\theta_r = E_{to}\cos\theta_t \quad \text{— (c)}$$



$\boxed{B\text{-}C\text{-}4}$ $\Rightarrow$ $\dfrac{B_{1\parallel}}{\mu_1} = \dfrac{B_{2\parallel}}{\mu_2}$.

$$\Rightarrow \frac{\vec{B}_{io}}{\mu_1} + \frac{\vec{B}_{ro}}{\mu_1} = \frac{\vec{B}_{to}}{\mu_2}.$$

$$\Rightarrow \frac{1}{v_1\mu_1}(E_{io} + E_{or}) = \frac{1}{v_2\mu_2} E_{to} \quad \text{— d}$$

$$\Rightarrow \boxed{\begin{aligned} r_\parallel &\equiv \left(\frac{E_{ro}}{E_{io}}\right)_\parallel = \frac{\frac{n_2}{\mu_2}\cos\theta_i - \frac{n_1}{\mu_1}\cos\theta_t}{\frac{n_1}{\mu_1}\cos\theta_t + \frac{n_2}{\mu_2}\cos\theta_i} \\ t_\parallel &= \left(\frac{E_{to}}{E_{io}}\right)_\parallel = \frac{2n_1\cos\theta_i}{\frac{n_1}{\mu_1}\cos\theta_t + \frac{n_2}{\mu_2}\cos\theta_i}. \end{aligned}}$$

In most cases, we can assume $\mu_1 \cong \mu_2 \simeq \mu_0$. , and Snell's law.

$$r_\perp = -\frac{\sin(\theta_i-\theta_t)}{\sin(\theta_i+\theta_t)}$$

$$t_\perp = \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i+\theta_t)}$$

$$r_\parallel = \frac{\tan(\theta_i-\theta_t)}{\tan(\theta_i+\theta_t)} \leftarrow$$

$$t_\parallel = \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i+\theta_t)\cos(\theta_i-\theta_t)}.$$

Fresnel Equations :

if $\theta_i + \theta_t = \frac{\pi}{2}$

$r_\parallel = 0$

- These eqtns are related to the specific field directions from which they were derived.

Some ~~Appl~~ Application.

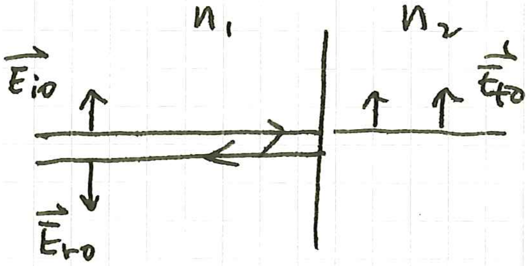
1) Normal incidence $(\theta_i = 0)$.

Snell's law $\Rightarrow \theta_t = 0$.

$$r_\perp = \frac{n_1 - n_2}{n_1 + n_2}, \quad t_\perp = \frac{2n_1}{n_1 + n_2}.$$

$$r_\parallel = \frac{n_2 - n_1}{n_2 + n_1}, \quad t_\parallel = \frac{2n_1}{n_2 + n_1}.$$

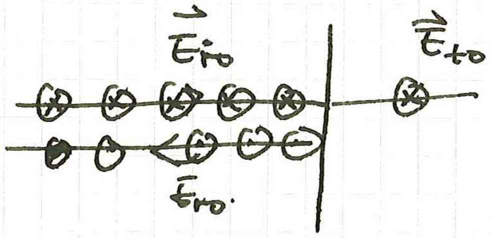
$\Big\}\ r_\perp = -r_\parallel$.



($E \parallel$ to plane of incidence)

$n_2 > n_1$

$(r_\parallel > 0)$ $\quad E_{r,\parallel} > 0$



($E \perp$ to plane of incidence).

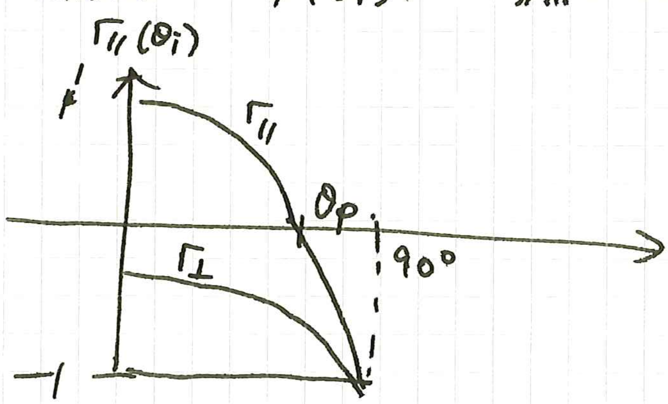
$(r_\perp < 0)$

$r_\perp < 0$.

$\Rightarrow$ The electric field is <u>flipped</u> after reflection if $n_2 > n_1$.

2) Brewster's law.

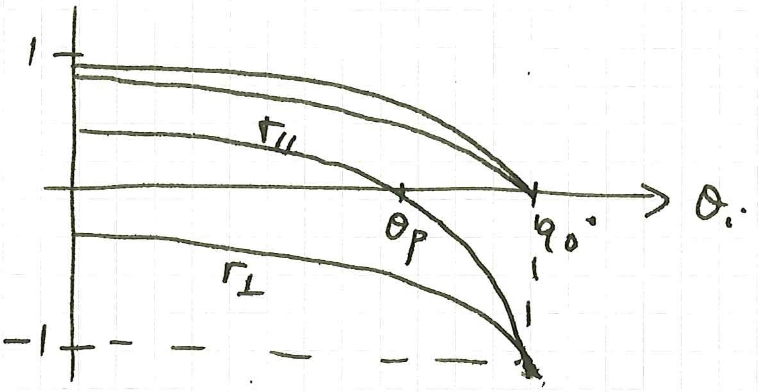
We consider $r_\parallel(\theta_i)$. $\cdot$ ~~At~~ when $\theta_i + \theta_t = 0 \Rightarrow r_\parallel = 0$.



$(n_2 > n_1)$.

$\Rightarrow$ There is NO reflection for EM which parallel to the incident plane.

$\Rightarrow$ All EM wave ~~can~~ will pass through the interface without any reflection (Brewster's law)



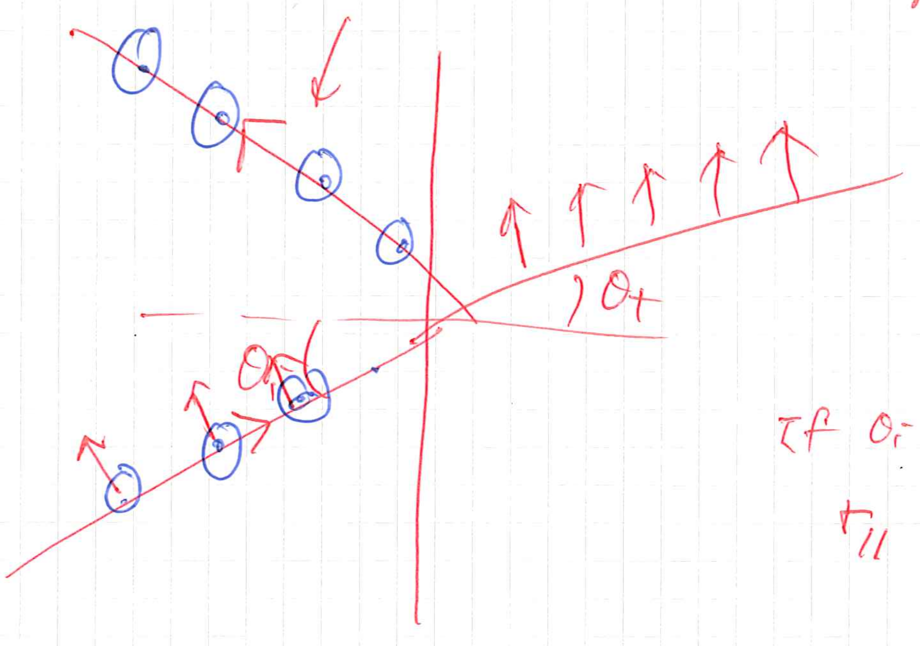
$(n_2 > n_1)$.

③. When $\theta_i \simeq 90°$, ~~$|r_{\parallel}|, |r_{\perp}| \simeq 1$~~. $\Rightarrow$ All lights are reflected..

$r_{\parallel}, r_{\perp} \to -1$

· Hold the ~~book~~ slide near your eye and increasing θ_i, the amount of light reflected will increase and the slide will look like a perfect mirror as $\theta_i \to 90°$.

NO Reflection at this angle.



If $\theta_i + \theta_t = 90°$

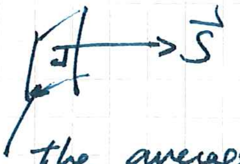
$r_{\parallel} = 0$

Reflectance and Transmittance

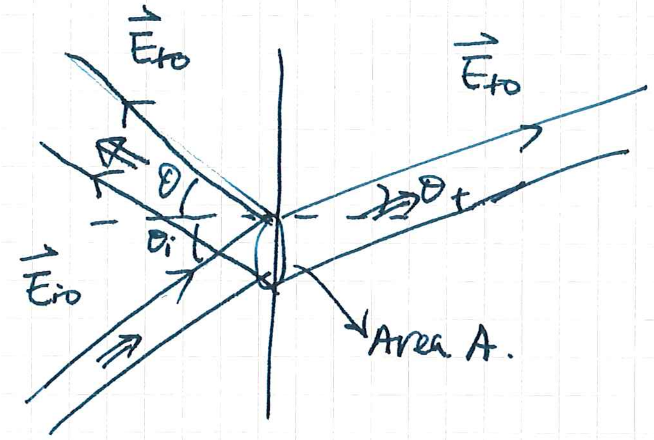
Recall that for a EM wave, the Poynting vector

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

And the intensity $I = \langle S \rangle_{\text{time average}} = \cancel{\frac{1}{2\mu_0} E_0^2} \cdot \frac{c\epsilon_0}{2} E_0^2$



the average energy per unit time crossing a unit area normal to $\vec{S}$.



$\Rightarrow$ ~~Area~~ Cross-sectional area (normal to the poynting vector) are $A\cos\theta_i$, $A\cos\theta_i$, $A\cos\theta_t$.

$\Rightarrow$ Incident Power for incident beam $= I_i A\cos\theta_i =$ ~~E_0~~

for reflected beam $= I_r A\cos\theta_i$

for transmitted beam $= I_t A\cos\theta_t$.

Reflectance $R \equiv \dfrac{I_r A\cos\theta_i}{I_i A\cos\theta_i} = \dfrac{I_r}{I_i} = \dfrac{E_{r0}^2}{E_{i0}^2} = r^2$

Transmittance $T = \dfrac{I_t \cos\theta_t}{I_i \cos\theta_i} = \dfrac{n_2 \cos\theta_t \, E_{t0}^2}{n_1 \cos\theta_i \, E_{i0}^2} = \left(\dfrac{n_2\cos\theta_t}{n_1\cos\theta_i}\right) t^2$

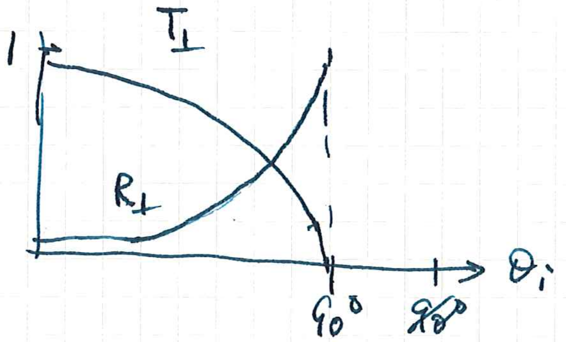
Energy conservation : Energy flowing into area A = Energy flowing outward from it.

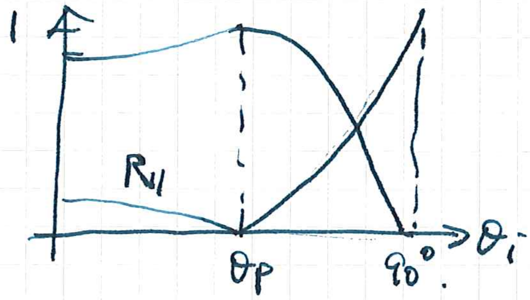
$$I_i A\cos\theta_i = I_r A\cos\theta_r + I_t A\cos\theta_t$$

$$\Rightarrow n_i E_{i0}^2\cos\theta_i = n_i E_{r0}^2\cos\theta_i + n_t E_{t0}^2\cos\theta_t.$$

$$\Rightarrow 1 = \frac{E_{r0}^2}{E_{i0}^2} + \frac{\cancel{E}\cdot n_t E_{t0}^2\cos\theta_t}{n_i E_{i0}^2\cos\theta_i}$$

$$\Rightarrow \boxed{1 = R + T}.$$





- At normal incidence, the reflectance is very small