Interference

Learning Goals

In this lecture, you will learn:

- **What happens when two waves meet in space.**
- **How to understand the interference pattern formed by the interference of two coherent light waves.**
- **How to calculate the light intensity at various points in an interference pattern.**
- **How interference occurs when light reflects from the two surfaces of a thin film.**

Soapy water is colorless, but when blown into bubbles it shows vibrant colors. How does the thickness of the bubble walls determine the particular colors that appear?

Interference

- **Interference refers to any situation in which two or more waves meet in space.**
- **The principle of superposition: When there are two or more waves in the same place in space, the resultant displacement at that place and at any time is found by adding the instantaneous “displacements” that would be produced at the point by individual waves if each were present alone.**
- **Reason: suppose both $E_1(r,t)$ and $E_2(r,t)$ satisfy the Maxwell’s equation because they are two EM waves, then it is easy to show that $E(r,t)=E_1(r,t)+E_2(r,t)$ also satisfies the Maxwell’s equation.**

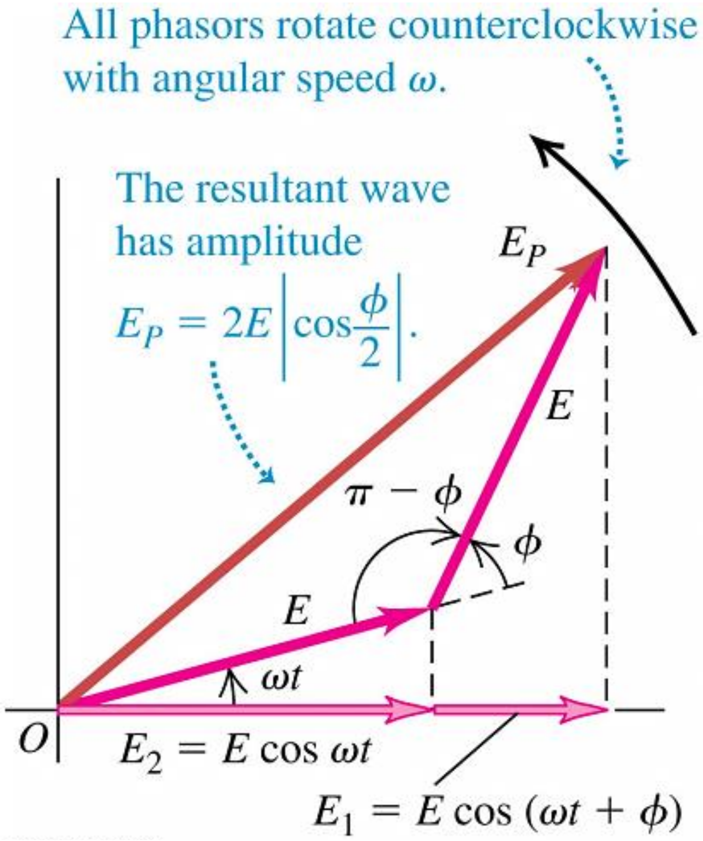
Interference

- **"Displacement" means:**
  - **Actual displacement of liquid surface above or below its normal level for waves on liquid surface.**
  - **Air pressure variation around its stationary value in sound waves.**
  - **Electric or magnetic field in EM waves.**
  - **…….**

Amplitude in Two-Source Interference

- **E_1 and E_2 are the horizontal components of the phasors representing the wave from sources S_1 and S_2, respectively.**
- **The sum of the projections on the horizontal axis at any time gives the instantaneous value of the total *E* field at point *P*.**

$$E_1(t) = E\cos(\omega t + \phi),$$
$$E_2(t) = E\cos\omega t.$$



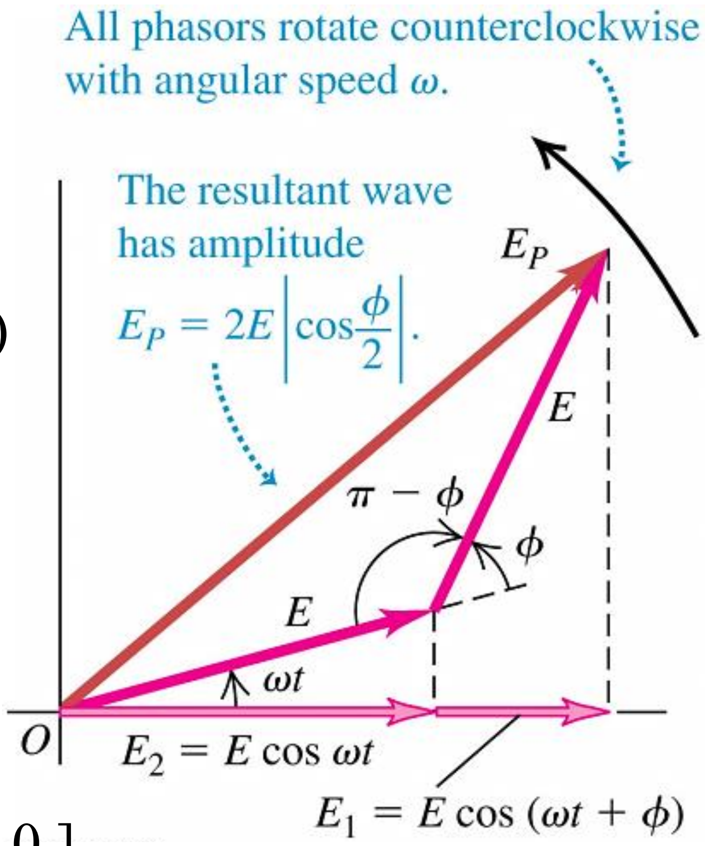
All phasors rotate counterclockwise with angular speed ω.

The resultant wave has amplitude $E_P = 2E\left|\cos\frac{\phi}{2}\right|$.

Amplitude in Two-Source Interference

- **The amplitude E_P of the resultant sinusoidal wave at P is the vector sum of the other two phasors:**

$$E_P^2 = E^2 + E^2 - 2E^2\cos(\pi-\phi)$$
$$= E^2 + E^2 + 2E^2\cos\phi$$
$$= 2E^2(1+\cos\phi)$$
$$= 4E^2\cos^2\left(\phi/2\right).$$
$$E_P = 2E\left|\cos\left(\phi/2\right)\right|.$$
$$[\phi=0,\ E_P=2E;\ \phi=\pi,\ E_P=0.]$$



Intensity in Two-Source Interference

$$I_0 = \frac{1}{2}\varepsilon_0 c E^2$$

$$I = \frac{1}{2}\varepsilon_0 c E_P^2 = \frac{1}{2}\varepsilon_0 c\left(4E^2\cos^2\frac{\phi}{2}\right)$$

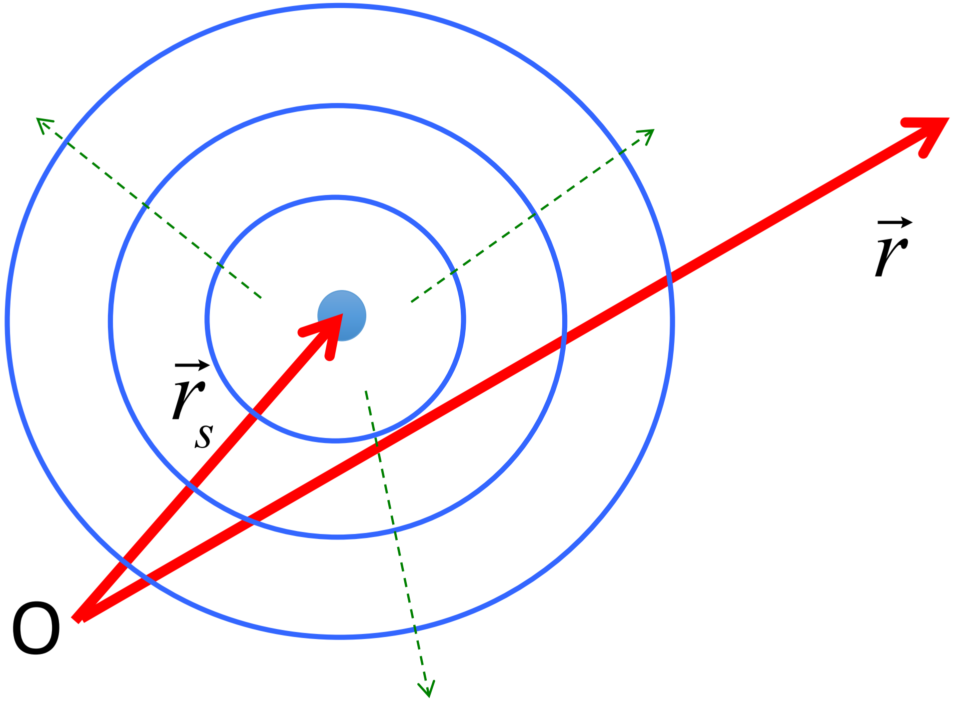
$$= \left(2\varepsilon_0 c E^2\right)\cos^2\frac{\phi}{2} = 4I_0\cos^2\frac{\phi}{2}.$$

$$I_{\text{av}} = \left(4I_0\cos^2\frac{\phi}{2}\right)_{\text{av}} = 4I_0\left(\frac{1}{2}\right) = 2I_0$$

- **The total energy output from the two sources remains unchanged by the interference effects, but the energy is redistributed in space.**

The (spherical)EM wave generated by a point source

$$\boxed{\vec{E}(\vec{r}) = \vec{E}_0 \cos(\vec{k} \cdot (\vec{r} - \vec{r}_s) - \omega t)}$$



Phase difference and Path difference

If two sources are in phase, then the waves that arrive at b differ in phase by an amount ϕ that is proportional to the difference in their path lengths, r_2-r_1. The electric fields superposed at b become:

$$E_1 \sim \cos(kr_1 - \omega t)$$

$$E_2 \sim \cos(kr_2 - \omega t)$$

$$E_b = E_1 + E_2$$

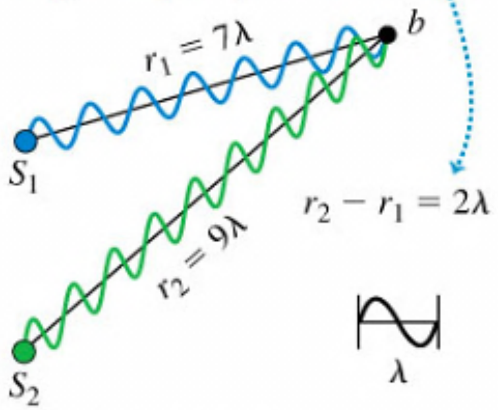
where the phase difference can be written as:

$$\phi = k\left(r_2 - r_1\right) = \frac{2\pi}{\lambda}(r_2 - r_1)$$

$$E_P \propto \left|\cos\left(\phi / 2\right)\right| = \left|\cos\left(\frac{\pi}{\lambda}(r_2 - r_1)\right)\right|$$

(b) Conditions for constructive interference:

Waves interfere constructively if their path lengths differ by an integral number of wavelengths: $r_2 - r_1 = m\lambda$.



Constructive and Destructive Interference

- **Constructive (for sources: same wavelength & in phase):**

$$\text{Path difference} \quad r_2 - r_1 = m\lambda$$

$$(m = 0, \ \pm 1, \ \pm 2, \ \pm 3, \ ...)$$

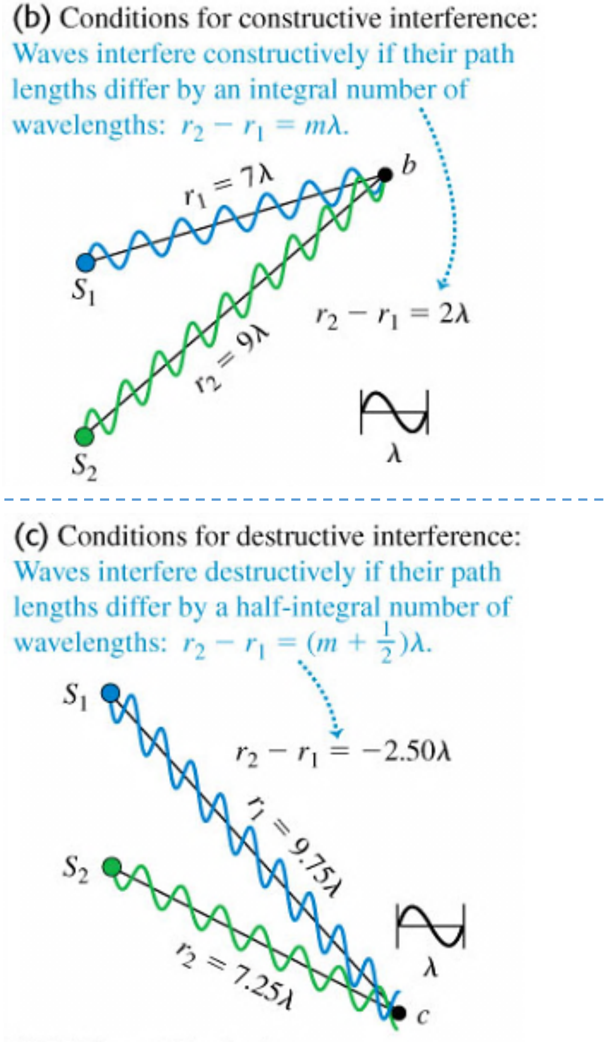
***e.g.*, at point *b*, *m* = +2.**

- **Destructive (for sources: same wavelength & in phase):**

$$\text{Path difference} \quad r_2 - r_1 = (m + \frac{1}{2})\lambda$$

$$(m = 0, \ \pm 1, \ \pm 2, \ \pm 3, \ ...)$$

***e.g.*, at point *c*, *m* = -3.**



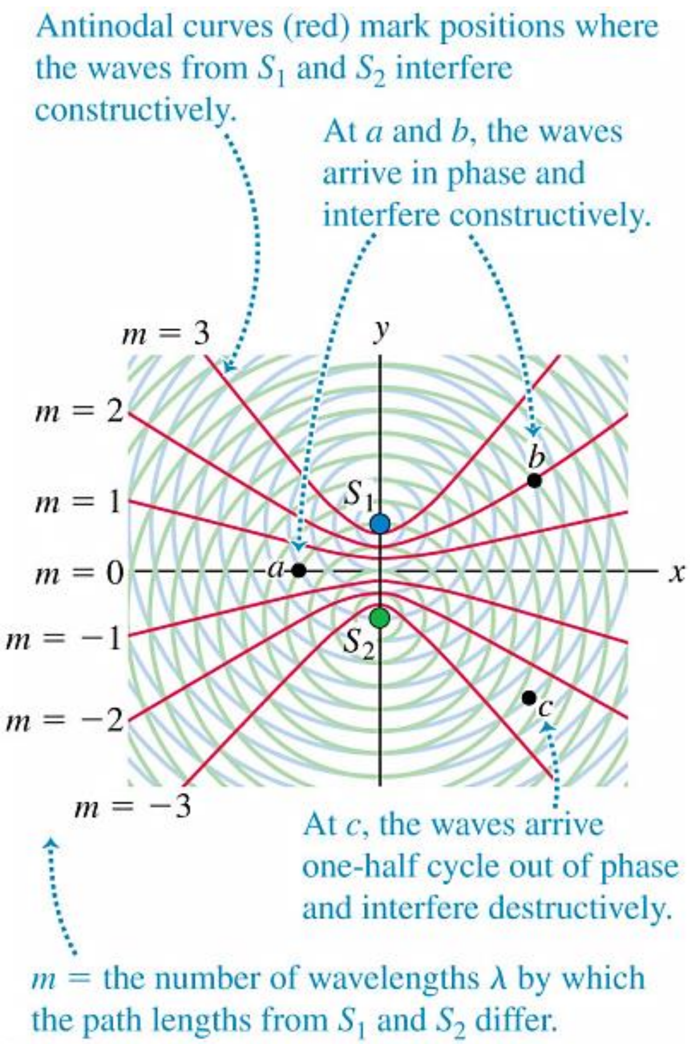
Constructive and Destructive Interference

- **Anti-nodal curves (red) show all positions where constructive interference occurs.**

$$r_2 - r_1 = m\lambda.$$

- **Nodal curves (not shown) show all positions where destructive interference occurs.**

$$r_2 - r_1 = (m + \frac{1}{2})\lambda.$$



Antinodal curves (red) mark positions where the waves from S_1 and S_2 interfere constructively.

At a and b, the waves arrive in phase and interfere constructively.

At c, the waves arrive one-half cycle out of phase and interfere destructively.

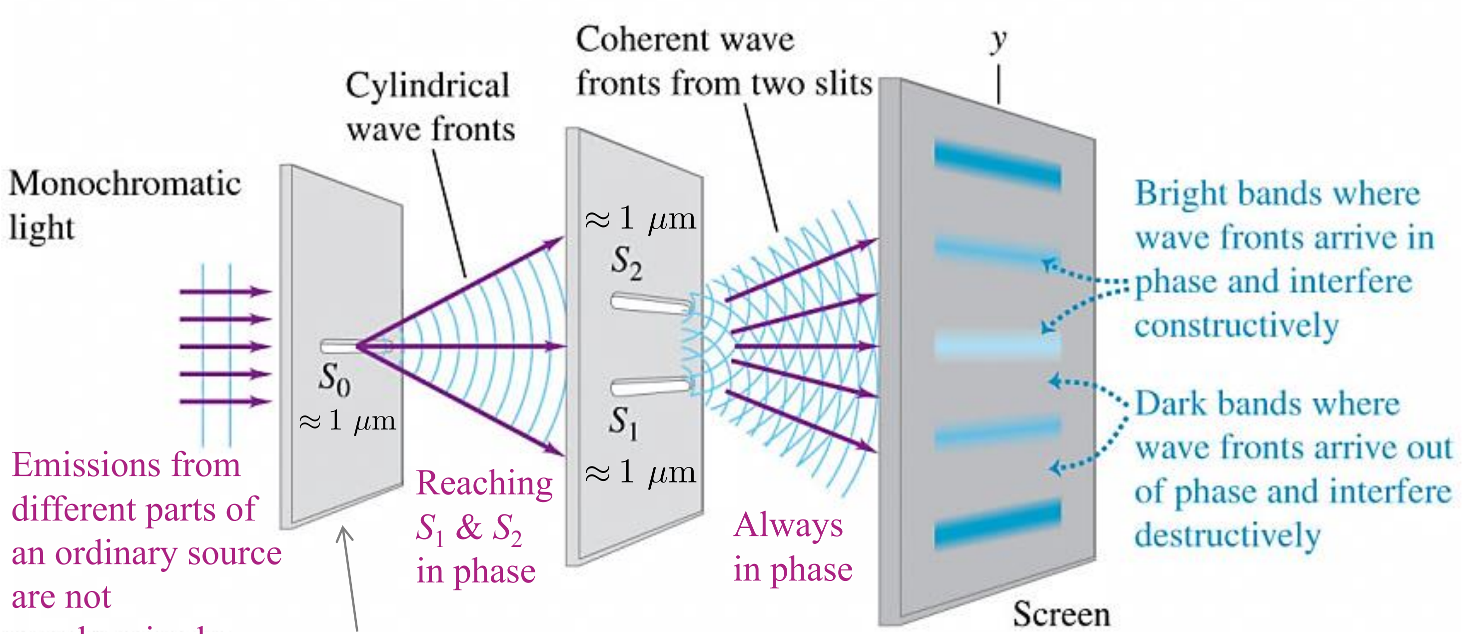
m = the number of wavelengths λ by which the path lengths from S_1 and S_2 differ.

Two-Source Interference of Light

The concepts of constructive interference and destructive interference apply to water waves as well as to light waves and sound waves, in fact to waves of any kind.

Two-Source Interference of Light: Young's Experiment

(a) Interference of light waves passing through two slits



Bright bands where wave fronts arrive in phase and interfere constructively

Dark bands where wave fronts arrive out of phase and interfere destructively

Emissions from different parts of an ordinary source are not synchronized

Reaching S_1 & S_2 in phase

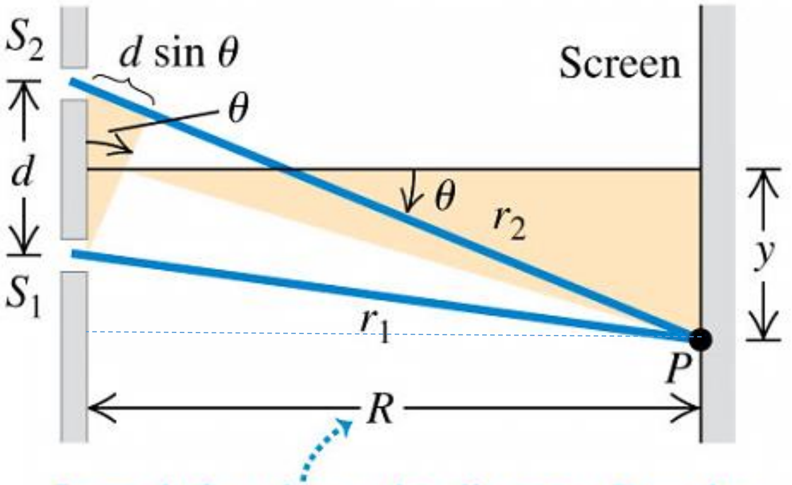
Always in phase

When a laser is used as a **coherent** light source, S_0 isn't needed

Two-Source Interference of Light: Young's Experiment

Assume that the distance *R* from the slits to the screen is much larger than the distance *d* between the slits (R>>d), then the lines from S_1 and S_2 to *P* are very nearly parallel.

(b) Actual geometry (seen from the side)



In real situations, the distance *R* to the screen is usually very much greater than the distance *d* between the slits ...

$$\Delta r = \sqrt{R^2 + (y+\frac{d}{2})^2} - \sqrt{R^2 + (y-\frac{d}{2})^2}$$

$$\approx R[1+\frac{1}{2R^2}(y+\frac{d}{2})^2] - R[1+\frac{1}{2R^2}(y-\frac{d}{2})^2]$$

$$= \frac{yd}{R} \approx d\sin\theta$$

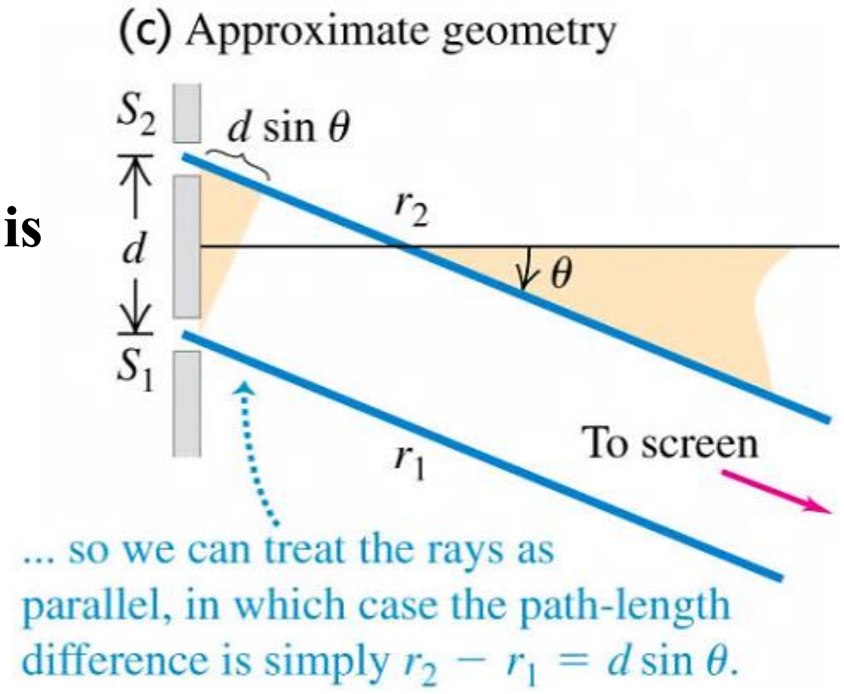
$$\boxed{\text{Path difference} = \Delta r = r_2 - r_1 = d\sin\theta.}$$

Constructive Two-Slit Interference

Constructive interference (reinforcement, bright regions) occurs at points where the path difference is

$$d\sin\theta = m\lambda.$$

$$(m = 0, \pm 1, \pm 2, \pm 3, \ldots)$$



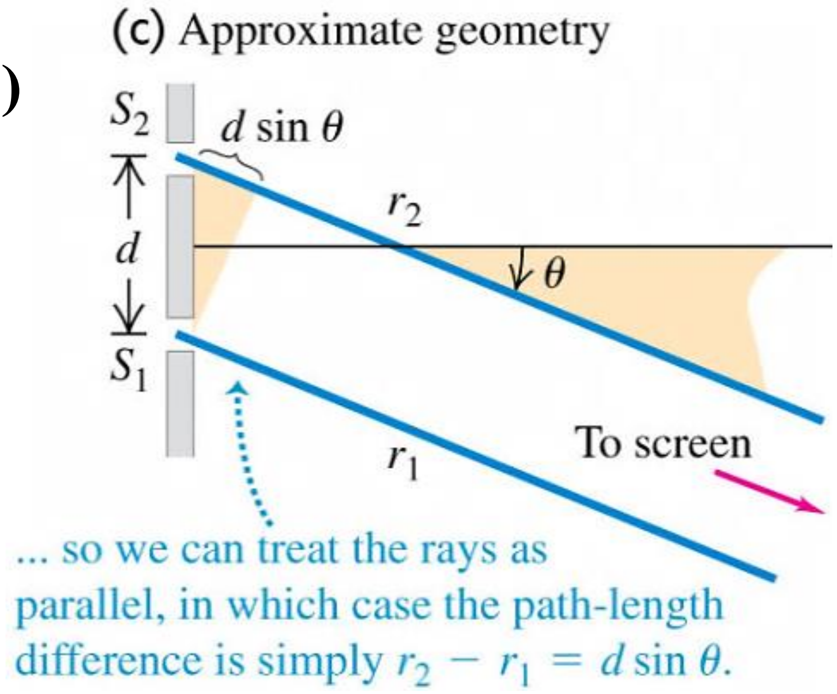
... so we can treat the rays as parallel, in which case the path-length difference is simply $r_2 - r_1 = d\sin\theta$.

Destructive Two-Slit Interference

Destructive interference (cancellation, dark regions) occurs at points where the path difference is

$$d\sin\theta = (m + \frac{1}{2})\lambda.$$

$$(m = 0, \pm 1, \pm 2, \pm 3, \ldots)$$



... so we can treat the rays as parallel, in which case the path-length difference is simply $r_2 - r_1 = d\sin\theta$.

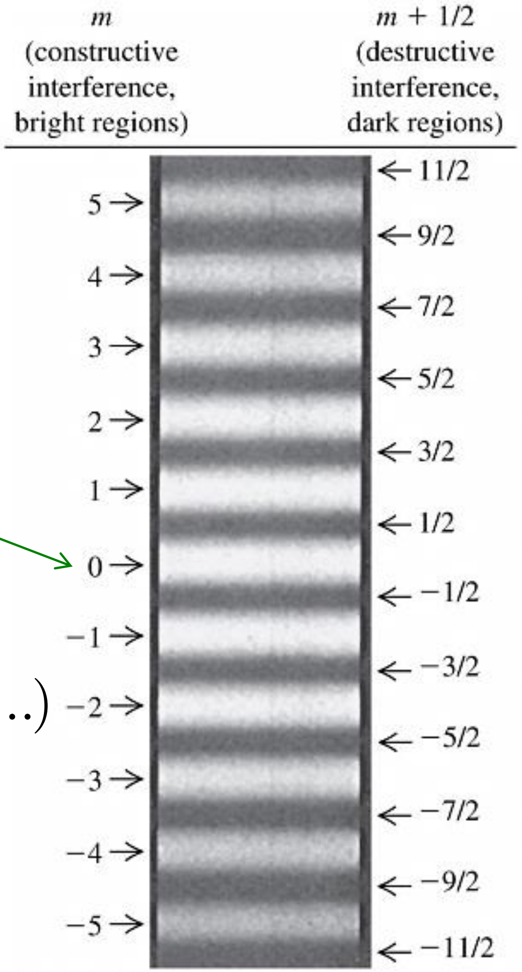
Young's experiment was the first direct measurement of wavelengths of light.

Constructive and Destructive Two-Slit Interference

Center of pattern is a bright band: *m* = 0.

$$d \sin\theta = m\lambda.$$

$$(m = 0, \pm 1, \pm 2, \pm 3, \ldots)$$



🡐 Photograph of interference fringes produced on a screen in Young's double-slit experiment.

$$d \sin\theta = (m + \frac{1}{2})\lambda.$$

$$(m = 0, \pm 1, \pm 2, \pm 3, \ldots)$$

Example 35.1
Two-slit interference

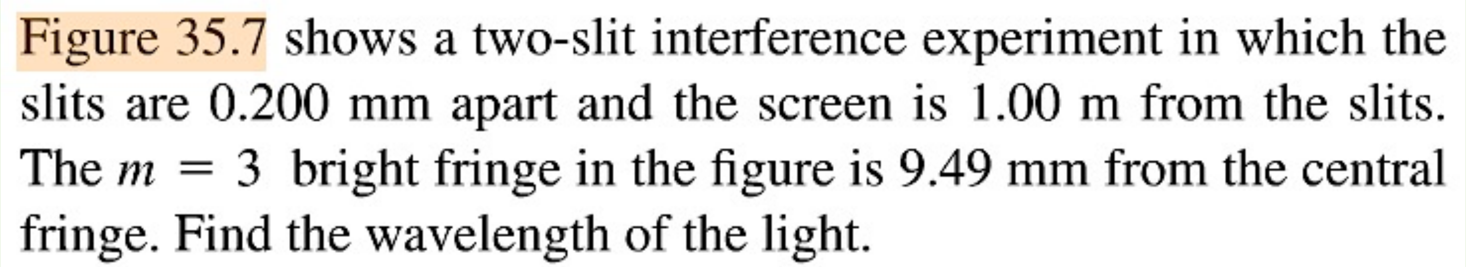
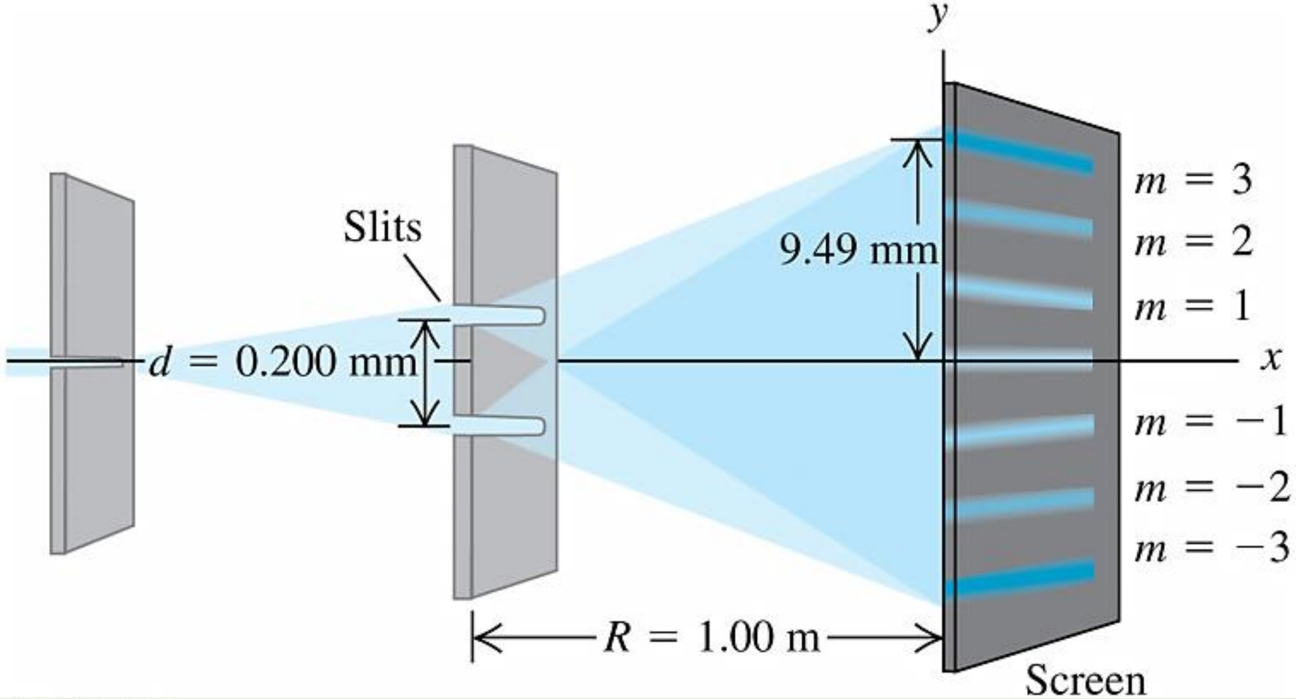
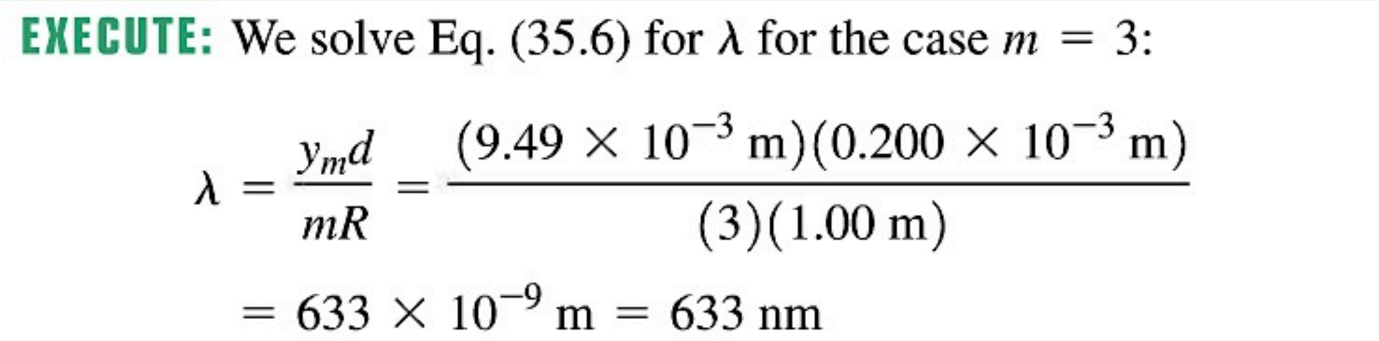


Figure 35.7 shows a two-slit interference experiment in which the slits are 0.200 mm apart and the screen is 1.00 m from the slits. The $m = 3$ bright fringe in the figure is 9.49 mm from the central fringe. Find the wavelength of the light.



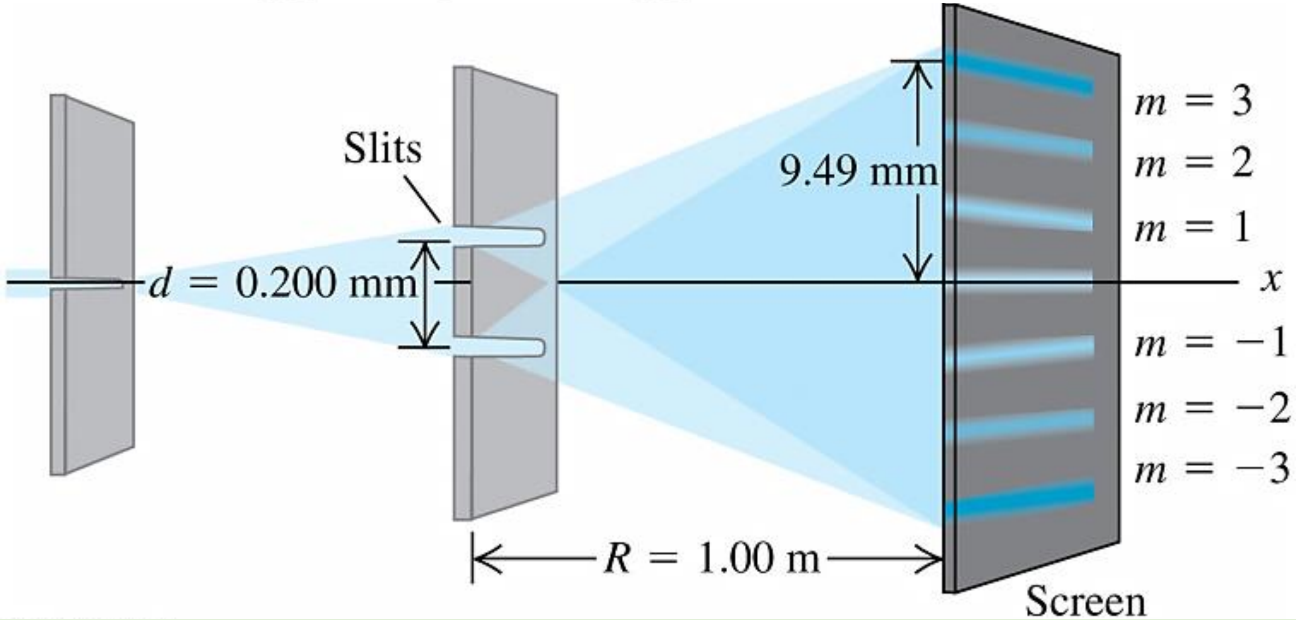
Example 35.1 Two-slit interference



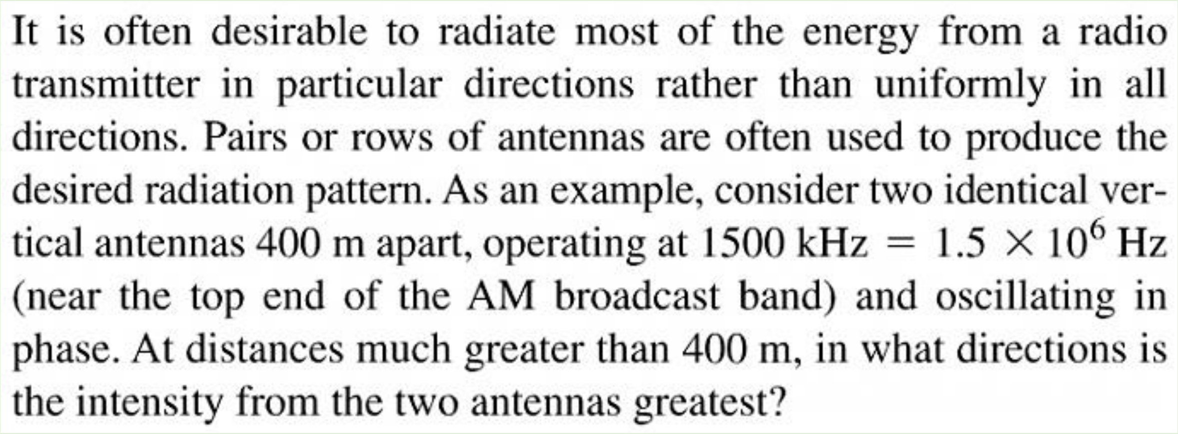
EXECUTE: We solve Eq. (35.6) for λ for the case $m = 3$:

$$\lambda = \frac{y_m d}{mR} = \frac{(9.49 \times 10^{-3}\ \text{m})(0.200 \times 10^{-3}\ \text{m})}{(3)(1.00\ \text{m})}$$

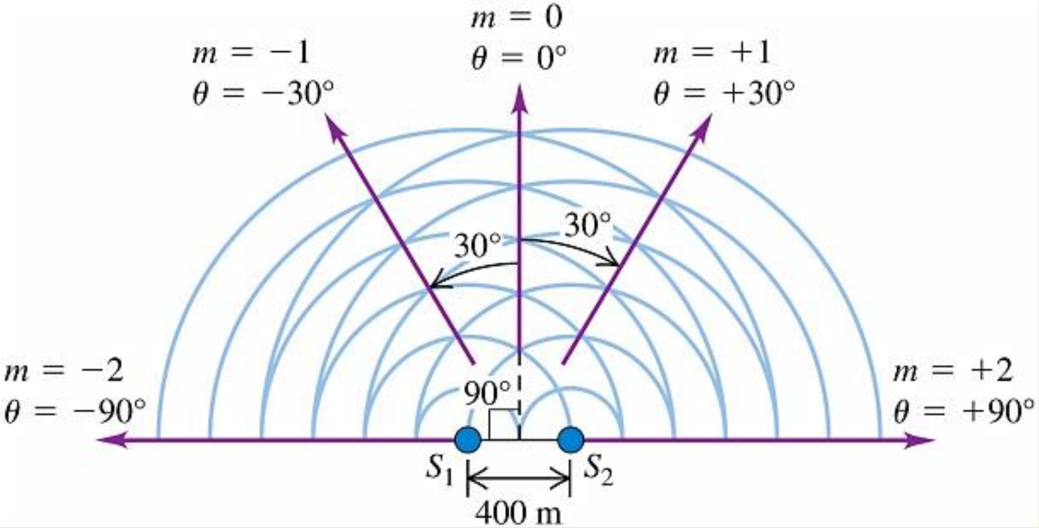
$$= 633 \times 10^{-9}\ \text{m} = 633\ \text{nm}$$



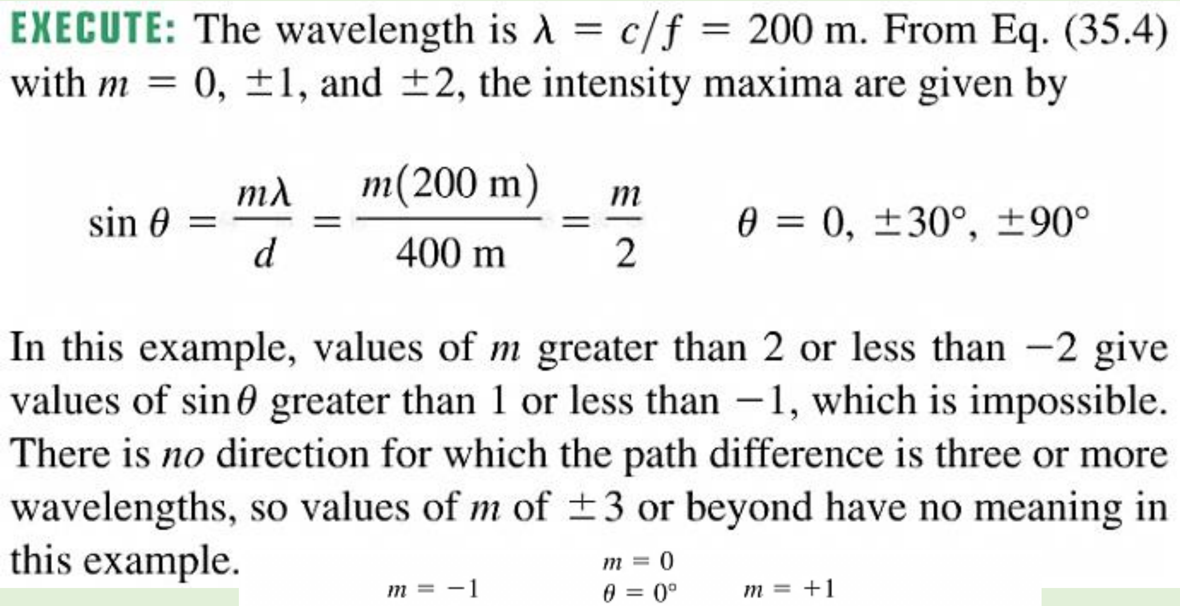
Example 35.2
Broadcast pattern of a radio station



It is often desirable to radiate most of the energy from a radio transmitter in particular directions rather than uniformly in all directions. Pairs or rows of antennas are often used to produce the desired radiation pattern. As an example, consider two identical vertical antennas 400 m apart, operating at 1500 kHz $= 1.5 \times 10^6$ Hz (near the top end of the AM broadcast band) and oscillating in phase. At distances much greater than 400 m, in what directions is the intensity from the two antennas greatest?



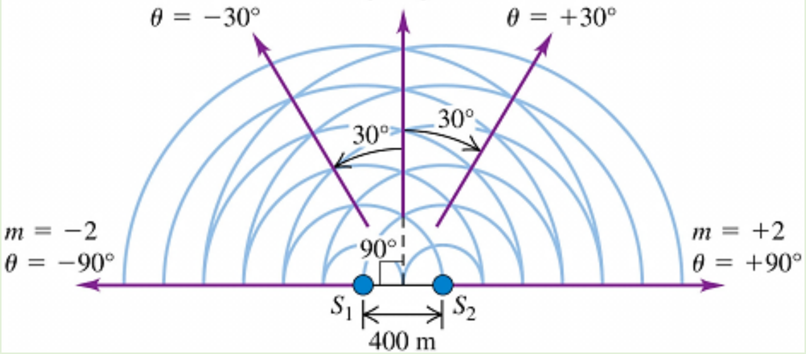
Example 35.2
Broadcast pattern of a radio station



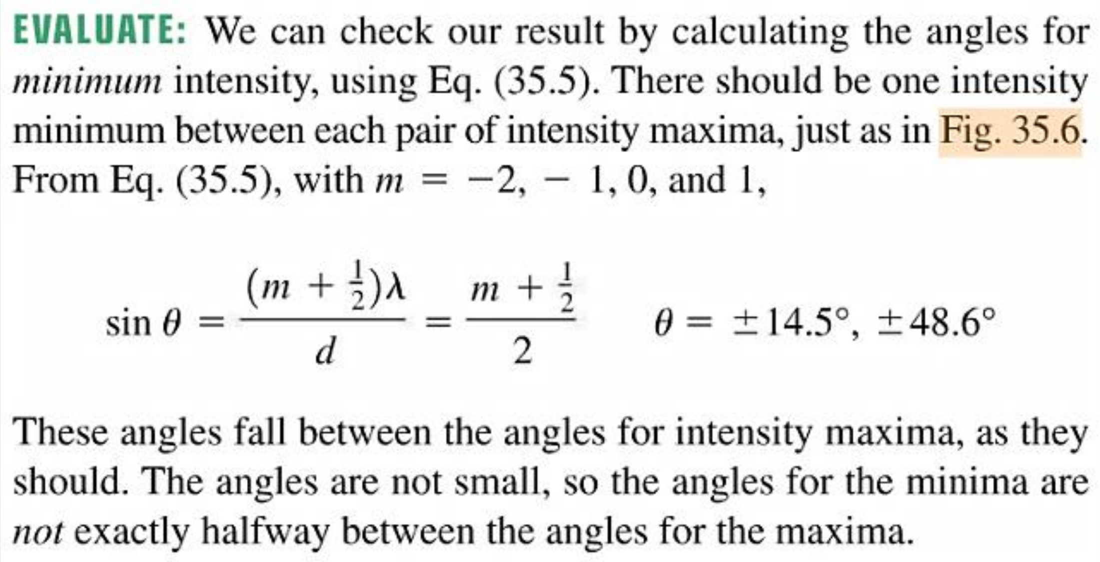
EXECUTE: The wavelength is $\lambda = c/f = 200\text{ m}$. From Eq. (35.4) with $m = 0, \pm1$, and ±2, the intensity maxima are given by

$$\sin\theta = \frac{m\lambda}{d} = \frac{m(200\text{ m})}{400\text{ m}} = \frac{m}{2} \qquad \theta = 0, \pm30°, \pm90°$$

In this example, values of m greater than 2 or less than -2 give values of $\sin\theta$ greater than 1 or less than -1, which is impossible. There is *no* direction for which the path difference is three or more wavelengths, so values of m of ±3 or beyond have no meaning in this example.



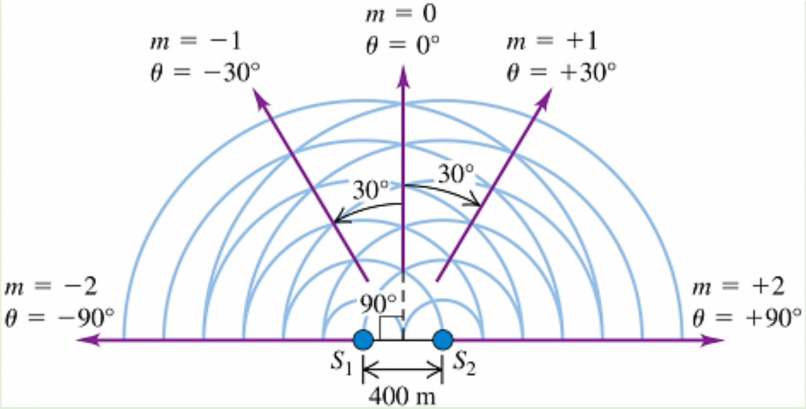
Example 35.2
Broadcast pattern of a radio station



EVALUATE: We can check our result by calculating the angles for *minimum* intensity, using Eq. (35.5). There should be one intensity minimum between each pair of intensity maxima, just as in Fig. 35.6. From Eq. (35.5), with $m = -2, -1, 0$, and 1,

$$\sin\theta = \frac{(m+\frac{1}{2})\lambda}{d} = \frac{m+\frac{1}{2}}{2} \qquad \theta = \pm 14.5^\circ, \pm 48.6^\circ$$

These angles fall between the angles for intensity maxima, as they should. The angles are not small, so the angles for the minima are *not* exactly halfway between the angles for the maxima.

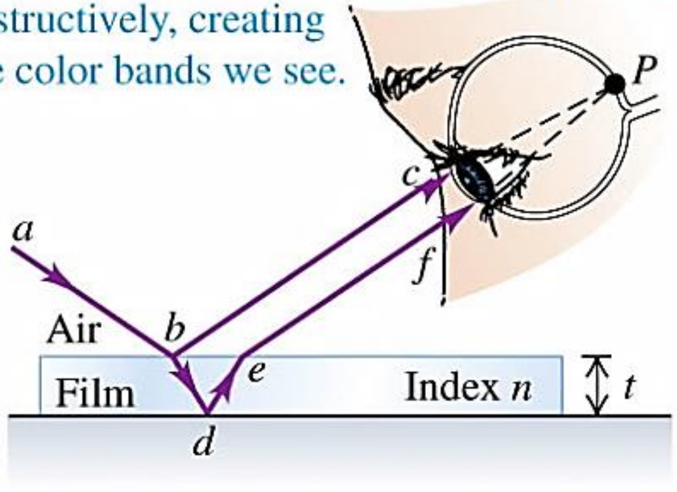


Interference in Thin Films

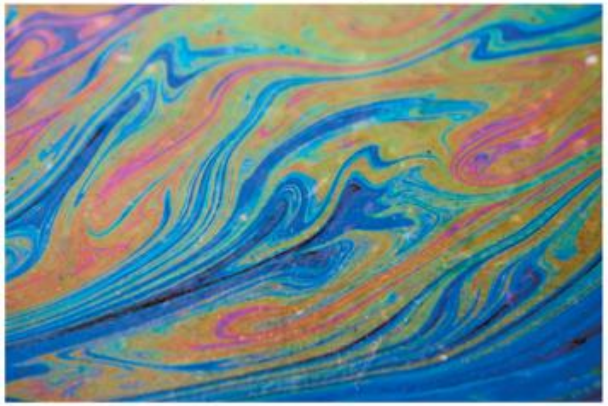
(a) Interference between rays reflected from the two surfaces of a thin film

Light reflected from the upper and lower surfaces of the film comes together in the eye at *P* and undergoes interference.

Some colors interfere constructively and others destructively, creating the color bands we see.



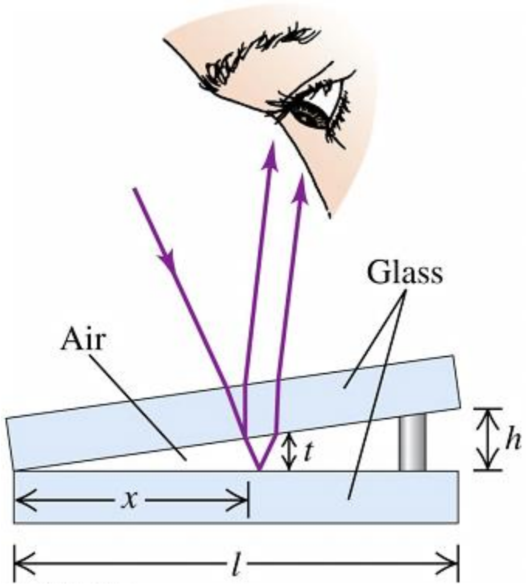
(b) The rainbow fringes of an oil slick on water



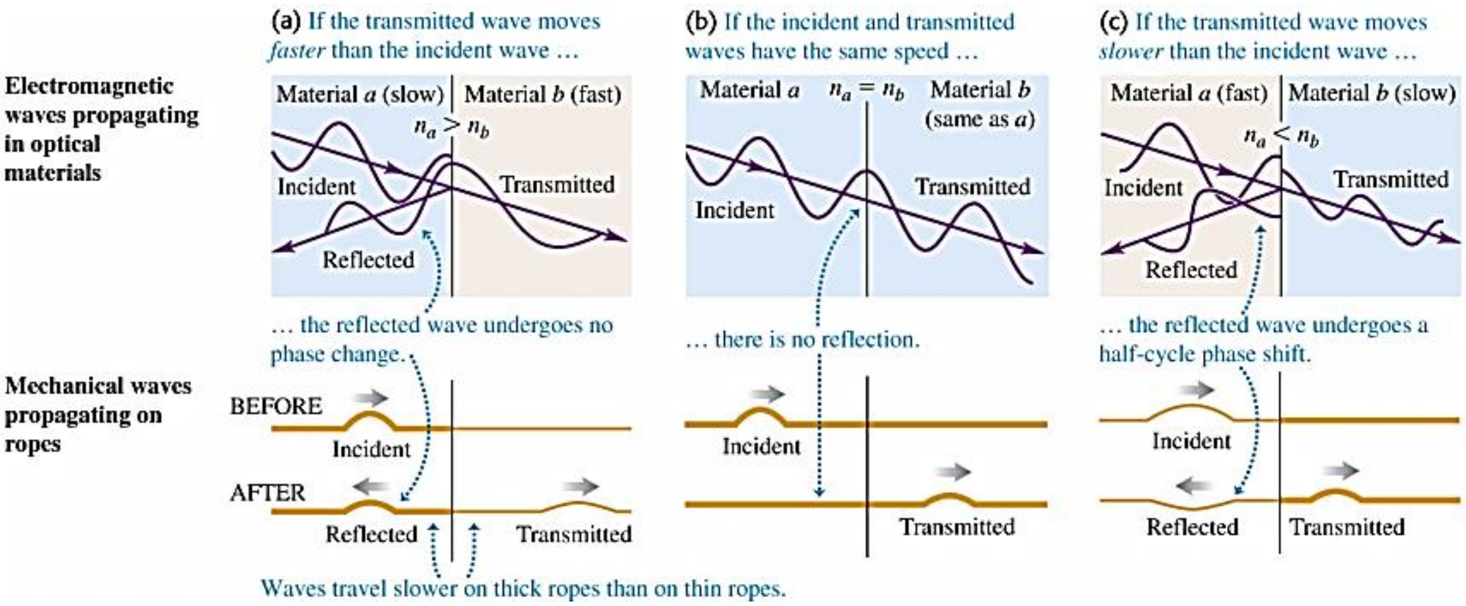
The complex shapes of the colored rings result from variations in the thickness of the film.

Thin-Film Interference and Phase Shifts at Reflection

- Monochromatic light reflects from two nearly parallel surfaces at nearly normal incidence.
- Along the line where the plates are in contact (with no path difference), we find a dark fringe, not a bright one.
- This suggests that one of the reflected waves has undergone a half-cycle phase shift during reflection.



Thin-Film Interference and Phase Shifts at Reflection



(a) If the transmitted wave moves *faster* than the incident wave …

… the reflected wave undergoes no phase change.

(b) If the incident and transmitted waves have the same speed …

… there is no reflection.

(c) If the transmitted wave moves *slower* than the incident wave …

… the reflected wave undergoes a half-cycle phase shift.

Waves travel slower on thick ropes than on thin ropes.

Predicted from Maxwell's equations

$n_a > n_b$,

$$E_{\text{ref}} = \frac{n_a - n_b}{n_a + n_b} E_{\text{inc}}$$

$\Rightarrow E_{\text{ref}}$ & E_{inc} :

same sign.

$n_a = n_b$,

$$E_{\text{ref}} = \frac{n_a - n_b}{n_a + n_b} E_{\text{inc}}$$

$\Rightarrow E_{\text{ref}} = 0.$

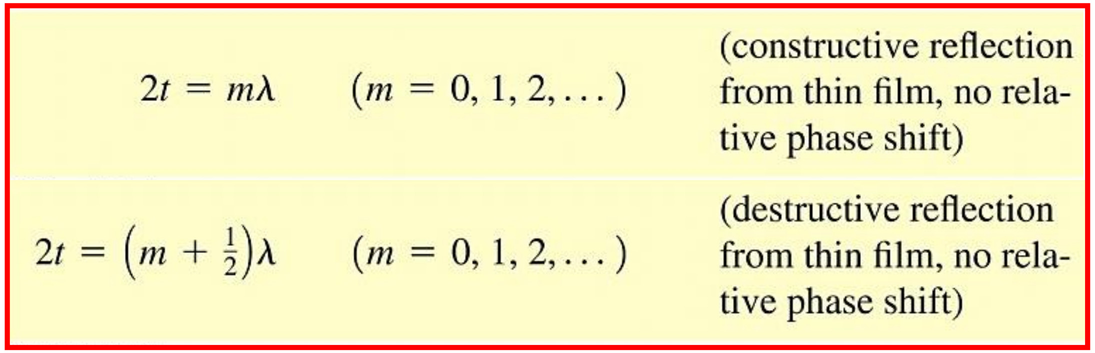
$n_a < n_b$,

$$E_{\text{ref}} = \frac{n_a - n_b}{n_a + n_b} E_{\text{inc}}$$

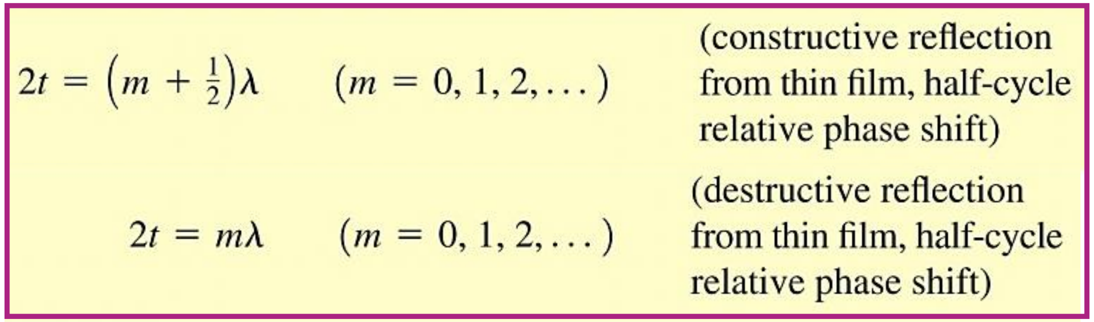
$\Rightarrow E_{\text{ref}}$ and E_{inc} :

opposite signs.

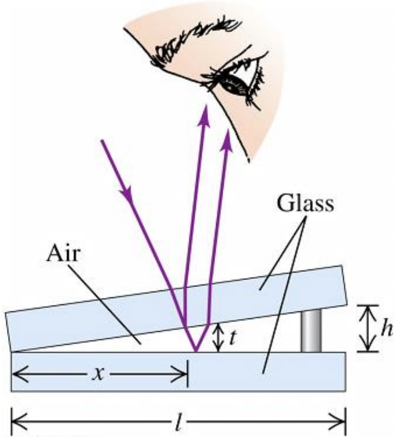
Thin-Film Interference and Phase Shifts at Reflection



$2t = m\lambda$	$(m = 0, 1, 2, \dots)$	(constructive reflection from thin film, no relative phase shift)
$2t = \left(m + \frac{1}{2}\right)\lambda$	$(m = 0, 1, 2, \dots)$	(destructive reflection from thin film, no relative phase shift)

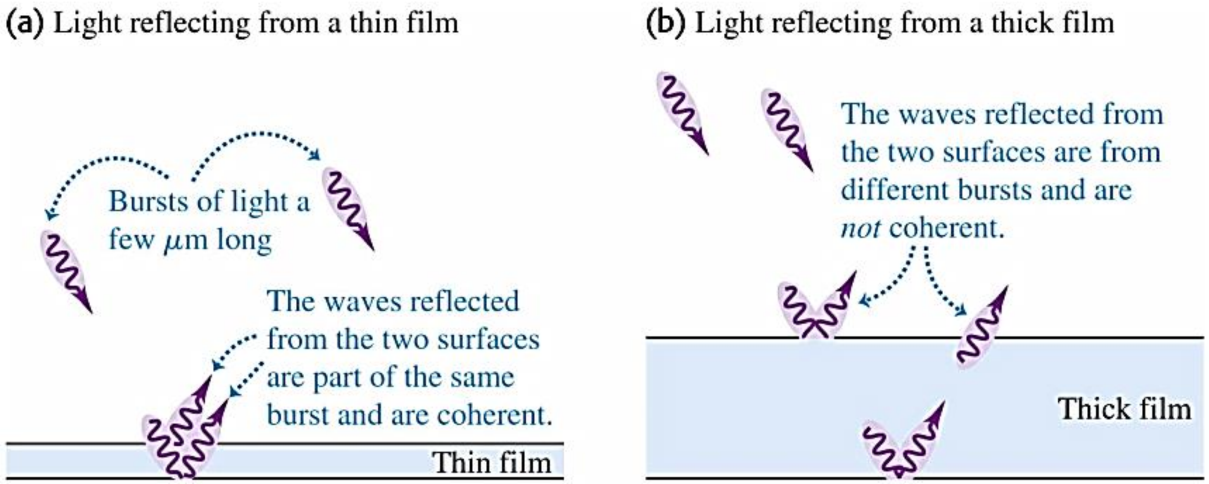


$2t = \left(m + \frac{1}{2}\right)\lambda$	$(m = 0, 1, 2, \dots)$	(constructive reflection from thin film, half-cycle relative phase shift)
$2t = m\lambda$	$(m = 0, 1, 2, \dots)$	(destructive reflection from thin film, half-cycle relative phase shift)



Thin-Film Interference and Wave Coherence

- **In order for two waves to cause a steady interference pattern, the waves must be coherent, with a definite and constant phase relationship ➔ Thin film**
- **The sun and light bulbs emit light in a stream of short bursts with only a few micrometers long**



(a) Light reflecting from a thin film

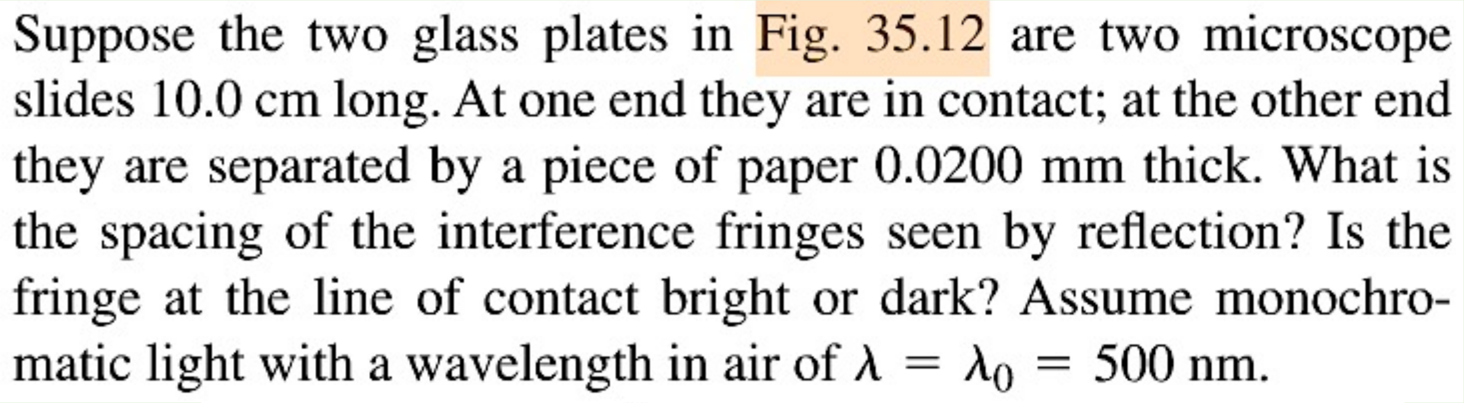
Bursts of light a few μm long

The waves reflected from the two surfaces are part of the same burst and are coherent.

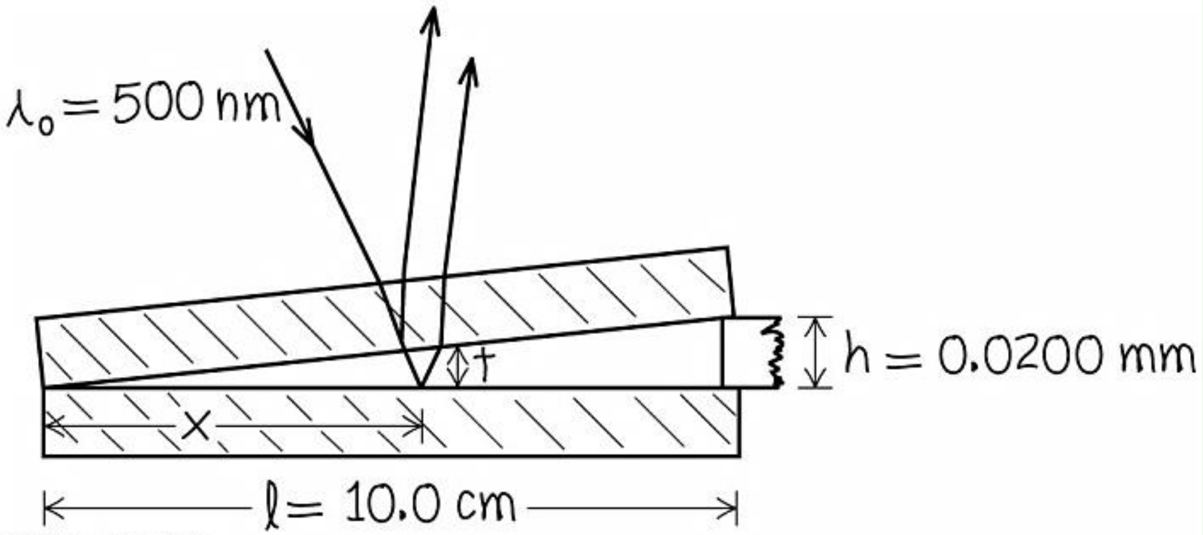
(b) Light reflecting from a thick film

The waves reflected from the two surfaces are from different bursts and are *not* coherent.

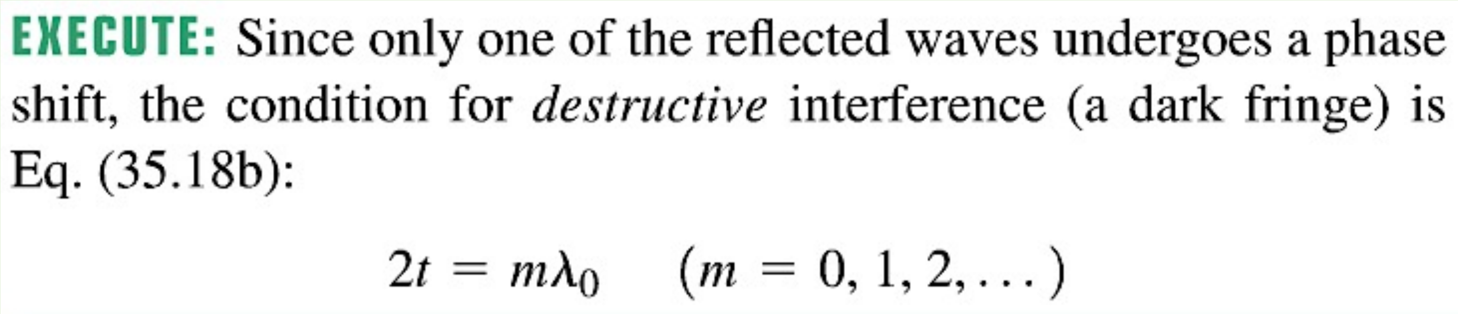
Example 35.4
Thin-film interference I



Suppose the two glass plates in Fig. 35.12 are two microscope slides 10.0 cm long. At one end they are in contact; at the other end they are separated by a piece of paper 0.0200 mm thick. What is the spacing of the interference fringes seen by reflection? Is the fringe at the line of contact bright or dark? Assume monochromatic light with a wavelength in air of $\lambda = \lambda_0 = 500$ nm.

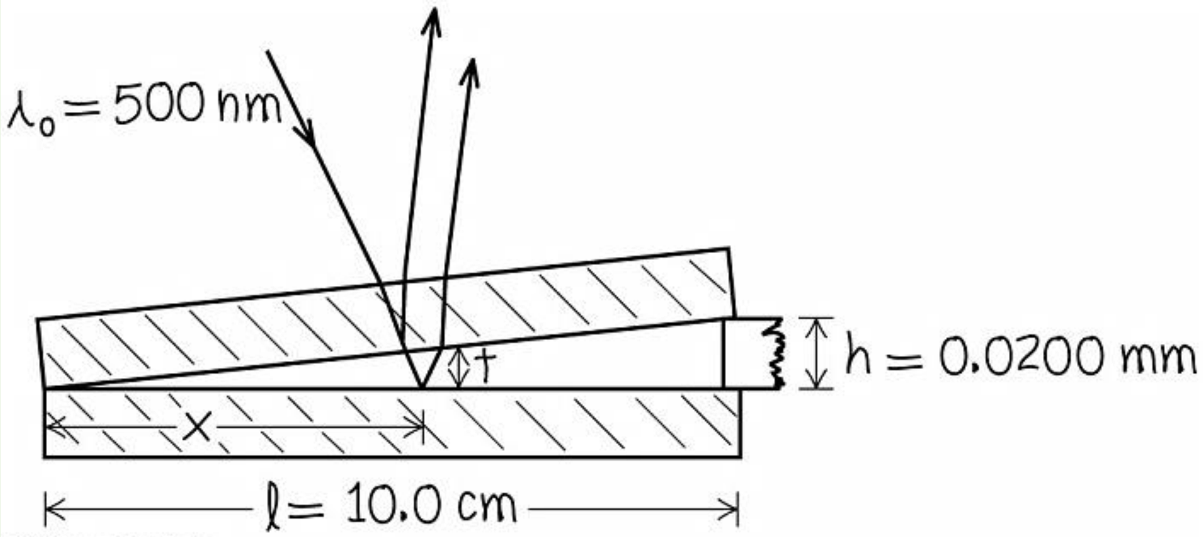


Example 35.4
Thin-film interference I



EXECUTE: Since only one of the reflected waves undergoes a phase shift, the condition for *destructive* interference (a dark fringe) is Eq. (35.18b):

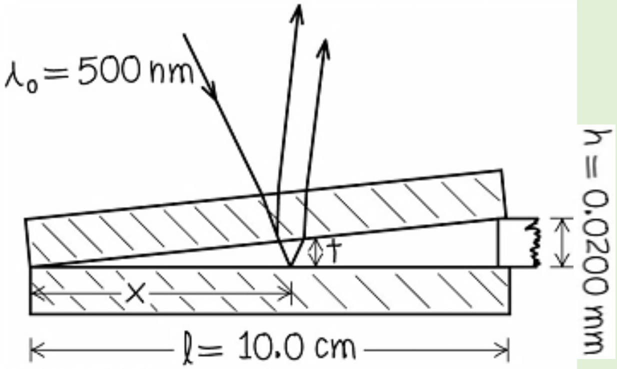
$$2t = m\lambda_0 \qquad (m = 0, 1, 2, \ldots)$$



Example 35.4
Thin-film interference I

From similar triangles in Fig. 35.15 the thickness t of the air wedge at each point is proportional to the distance x from the line of contact:

$$\frac{t}{x} = \frac{h}{l}$$



Combining this with Eq. (35.18b), we find

$$\frac{2xh}{l} = m\lambda_0$$

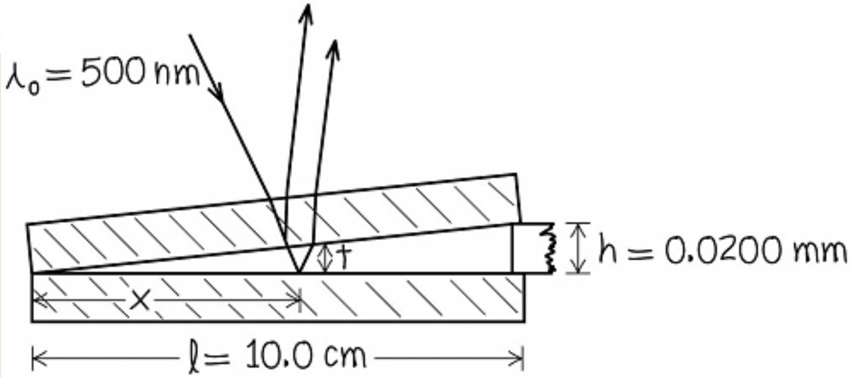
$$x = m\frac{l\lambda_0}{2h} = m\frac{(0.100\text{ m})(500 \times 10^{-9}\text{ m})}{(2)(0.0200 \times 10^{-3}\text{ m})} = m(1.25\text{ mm})$$

Successive dark fringes, corresponding to $m = 1, 2, 3, \ldots,$ are spaced 1.25 mm apart. Substituting $m = 0$ into this equation gives $x = 0$, which is where the two slides touch (at the left-hand side of Fig. 35.15). Hence there is a dark fringe at the line of contact.

Example 35.5
Thin-film interference II

Suppose the glass plates of Example 35.4 have $n = 1.52$ and the space between plates contains water ($n = 1.33$) instead of air. What happens now?

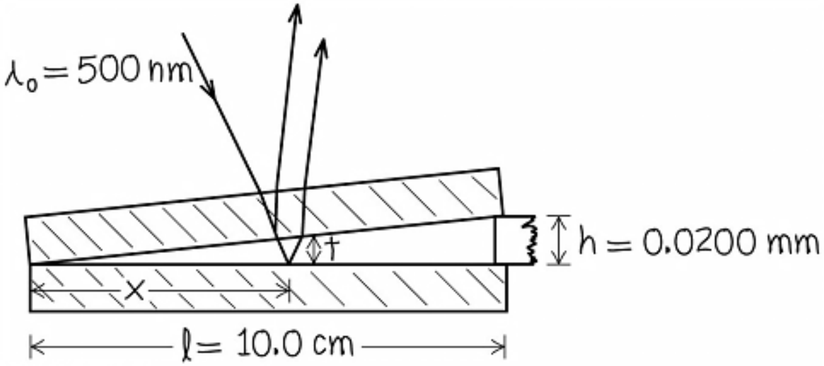
EXECUTE: In the film of water ($n = 1.33$), the wavelength is $\lambda = \lambda_0/n = (500\text{ nm})/(1.33) = 376\text{ nm}$. When we replace λ_0 by λ in the expression from Example 35.4 for the position x of the mth dark fringe, we find that the fringe spacing is reduced by the same factor of 1.33 and is equal to 0.940 mm. There is still a dark fringe at the line of contact.



Example 35.6
Thin-film interference III

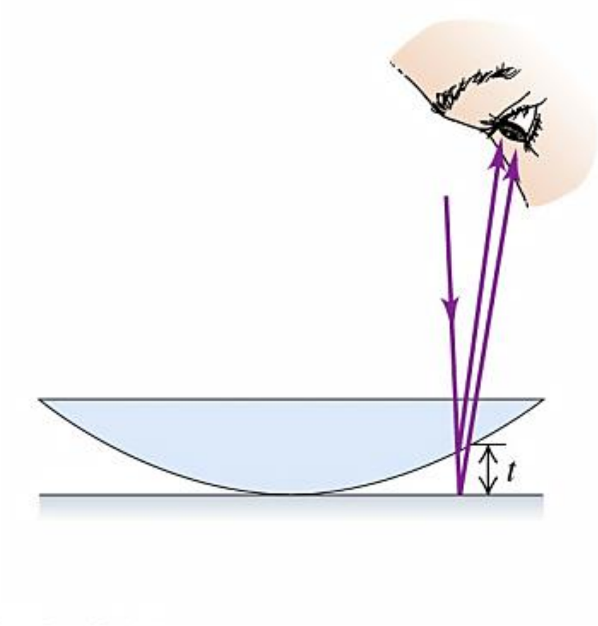
Suppose the upper of the two plates of Example 35.4 is a plastic with $n = 1.40$, the wedge is filled with a silicone grease with $n = 1.50$, and the bottom plate is a dense flint glass with $n = 1.60$. What happens now?

EXECUTE: The value of λ to use in Eq. (35.17b) is the wavelength in the silicone grease, $\lambda = \lambda_0/n = (500\text{ nm})/1.50 = 333\text{ nm}$. You can readily show that the fringe spacing is 0.833 mm. Note that the two reflected waves from the line of contact are in phase (they both undergo the same phase shift), so the line of contact is at a *bright* fringe.

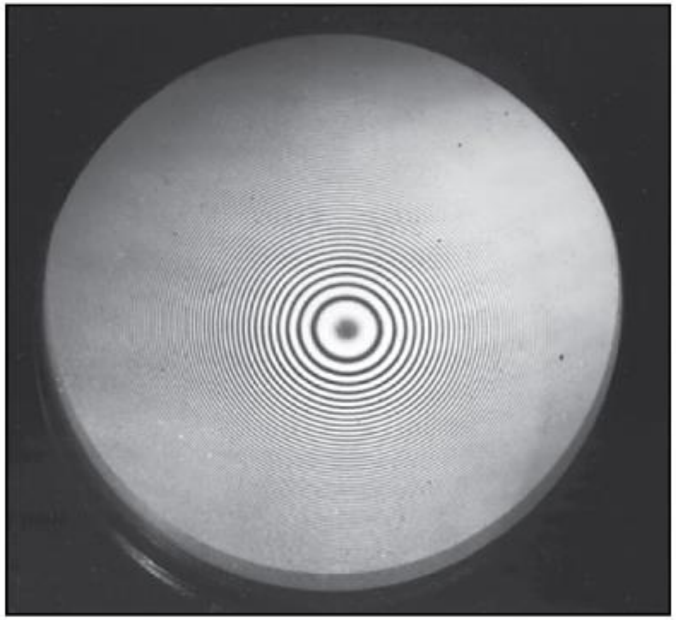


Newton’s Rings

- **Air gap between a convex lens and a plane surface. The thickness of the film *t* increases from zero as we move out from the center, giving a series of alternating dark and bright rings for monochromatic light:**



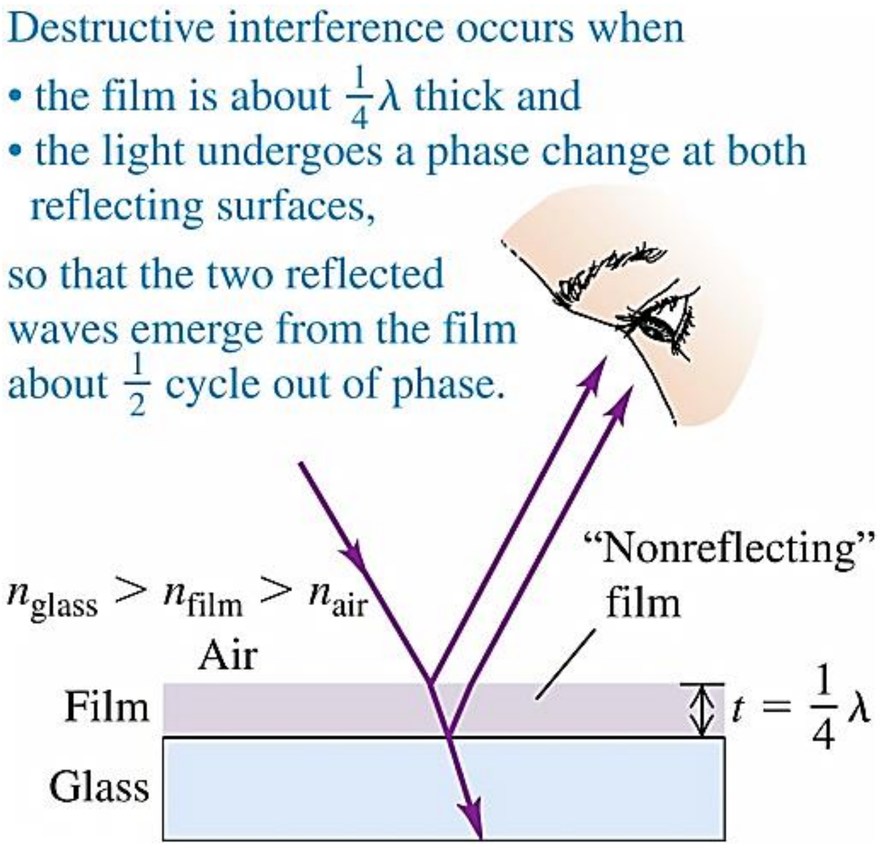
(a) A convex lens in contact with a glass plane



(b) Newton’s rings: circular interference fringes

Non-reflective and Reflective Coatings

In both reflections, the light is reflected from a medium of greater index than that in which it is traveling, so the same phase change occurs in both reflections.

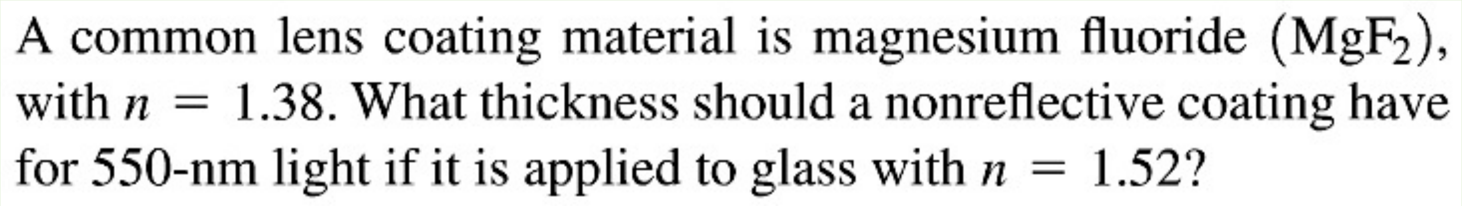


Destructive interference occurs when

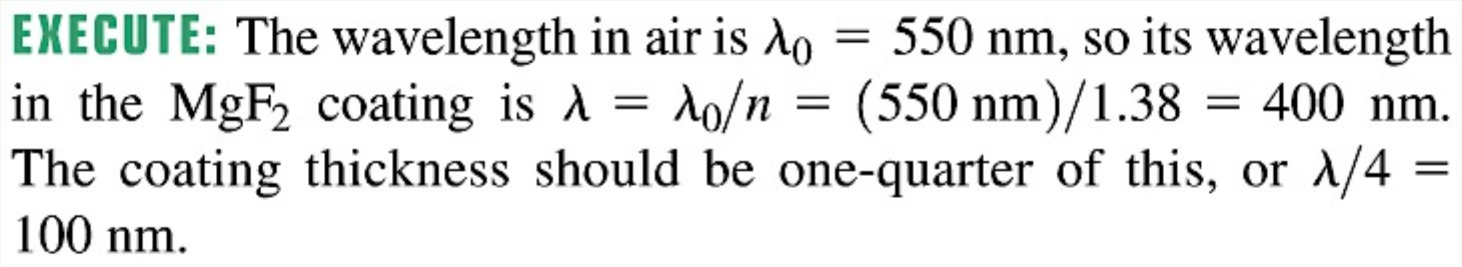
- the film is about $\frac{1}{4}\lambda$ thick and
- the light undergoes a phase change at both reflecting surfaces,

so that the two reflected waves emerge from the film about $\frac{1}{2}$ cycle out of phase.

Example 35.7
A non-reflective coating

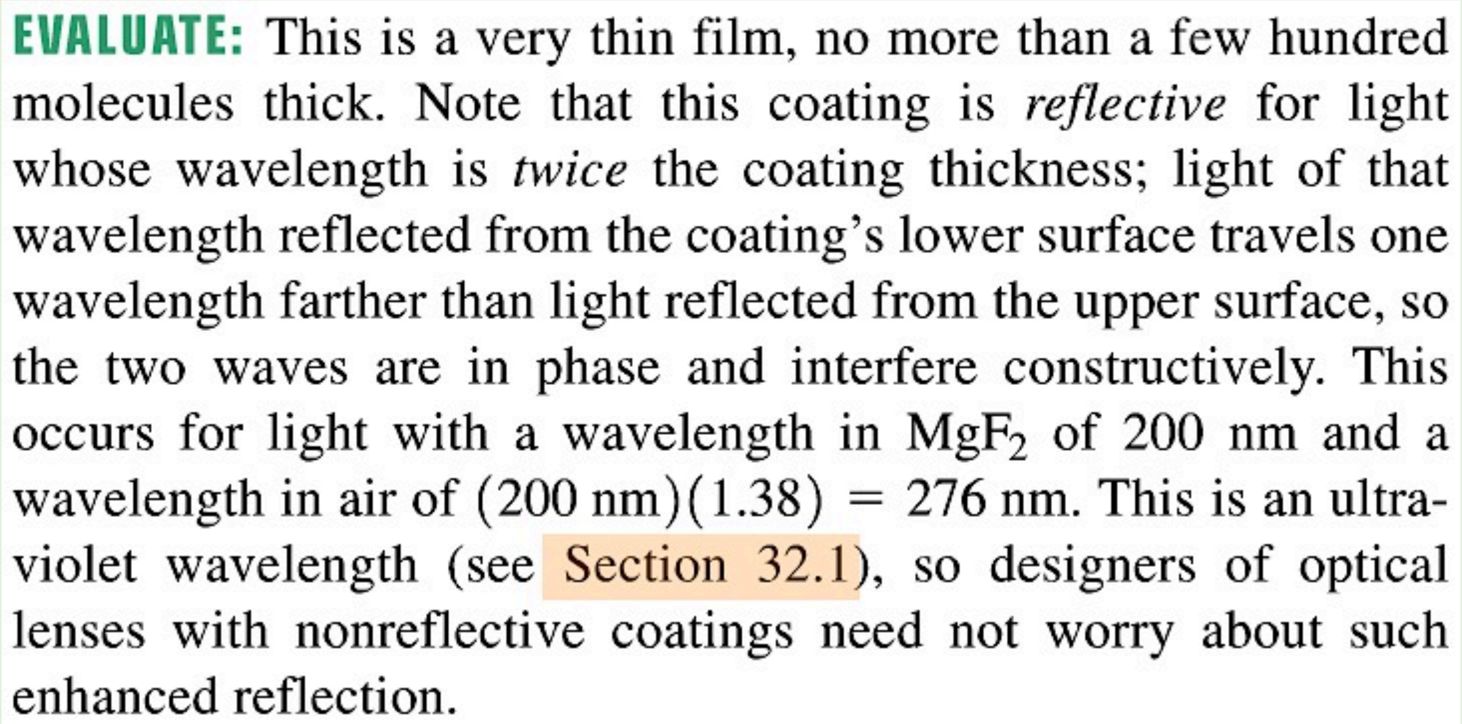


A common lens coating material is magnesium fluoride (MgF_2), with $n = 1.38$. What thickness should a nonreflective coating have for 550-nm light if it is applied to glass with $n = 1.52$?



EXECUTE: The wavelength in air is $\lambda_0 = 550$ nm, so its wavelength in the MgF_2 coating is $\lambda = \lambda_0/n = (550\text{ nm})/1.38 = 400$ nm. The coating thickness should be one-quarter of this, or $\lambda/4 = 100$ nm.

Example 35.7
A non-reflective coating



EVALUATE: This is a very thin film, no more than a few hundred molecules thick. Note that this coating is *reflective* for light whose wavelength is *twice* the coating thickness; light of that wavelength reflected from the coating's lower surface travels one wavelength farther than light reflected from the upper surface, so the two waves are in phase and interfere constructively. This occurs for light with a wavelength in MgF_2 of 200 nm and a wavelength in air of $(200 \text{ nm})(1.38) = 276 \text{ nm}$. This is an ultraviolet wavelength (see Section 32.1), so designers of optical lenses with nonreflective coatings need not worry about such enhanced reflection.

Diffraction

2-Slit Interference – slits width a -> 0

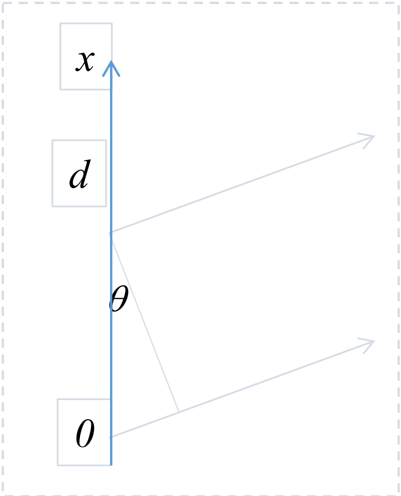
$$E_1(t) = E\cos(\omega t + \phi), \qquad \phi = kd\sin\vartheta$$
$$E_2(t) = E\cos\omega t.$$

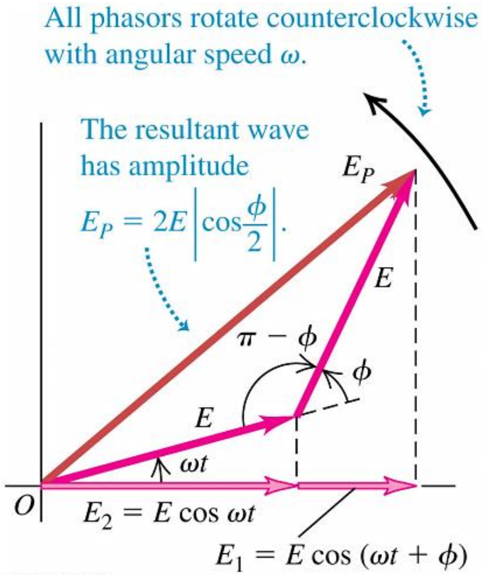
$$E_P^2 = E^2 + E^2 - 2E^2\cos(\pi - \phi)$$
$$= E^2 + E^2 + 2E^2\cos\phi$$
$$= 2E^2(1+\cos\phi)$$
$$= 4E^2\cos^2\left(\phi/2\right).$$
$$E_P = 2E\left|\cos\left(\phi/2\right)\right|.$$

$$I_0 = \frac{1}{2}\varepsilon_0 c E^2$$
$$I = \frac{1}{2}\varepsilon_0 c E_P^2 = \frac{1}{2}\varepsilon_0 c\left(4E^2\cos^2\frac{\phi}{2}\right)$$
$$= 4I_0\cos^2\frac{\phi}{2}.$$

Alternatively, we have

$$E_P = E(1+e^{i\phi})$$
$$\left|E_P\right| = E\left|1+e^{i\phi}\right| = E\sqrt{(2+2\cos\phi)} = 2E\left|\cos\frac{\phi}{2}\right|$$




All phasors rotate counterclockwise with angular speed ω.

The resultant wave has amplitude $E_P = 2E\left|\cos\frac{\phi}{2}\right|$.

Learning Goals

In this lecture, you will learn:

- **What happens when light shines on an object with an edge or aperture.**
- **How to understand the diffraction pattern formed when waves pass through a narrow slit.**
- **How to calculate the intensity at various points in a single-slit diffraction pattern.**
- **What happens when coherent light shines on an array of narrow, closely spaced slits.**

Learning Goals

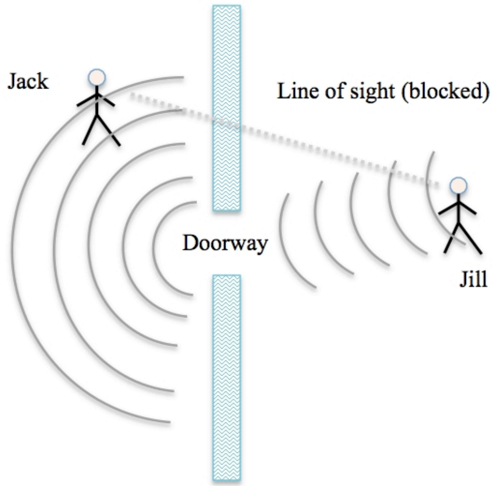
In this lecture, you will learn:

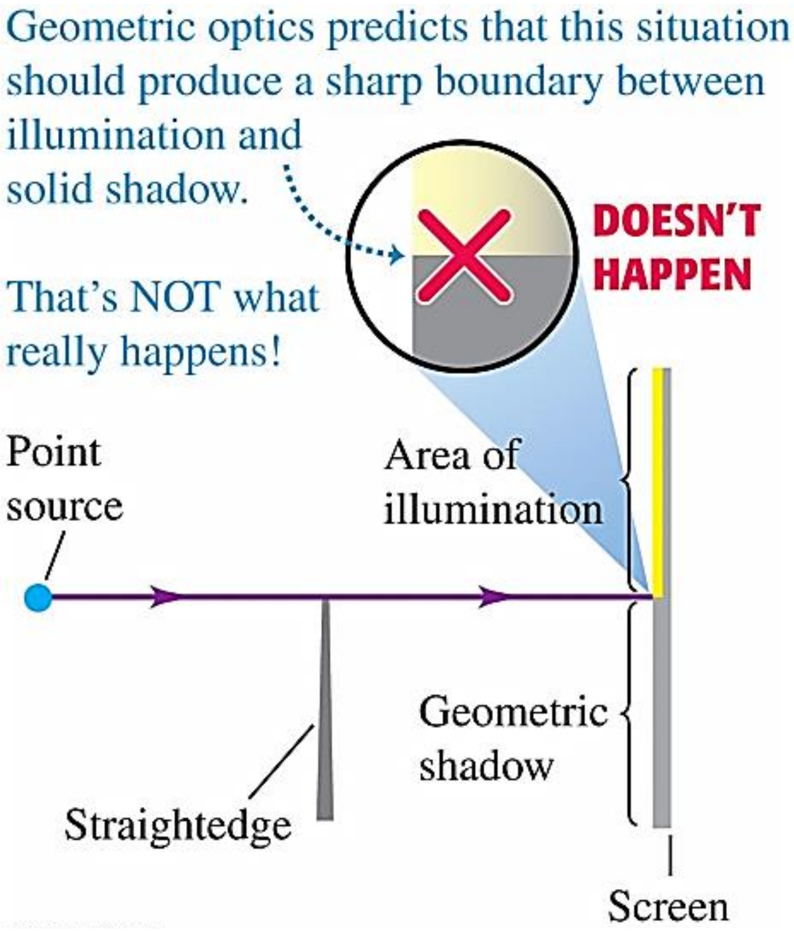
- **Multiple-slit diffraction**
- **How to use diffraction gratings for precise measurements of wavelength.**
- **How x-ray diffraction reveals the arrangement of atoms in a crystal.**
- **How diffraction sets limits on the smallest details that can be seen with a lens.**

The laser used to read a DVD has wavelength of 650 nm, while the laser used a Blu-ray disc has a shorter 405-nm wavelength. How does this make it possible for a Blu-ray disc to hold more information than a DVD?

Diffraction

- **Diffraction: Interference caused by bending of waves around an obstacle.**

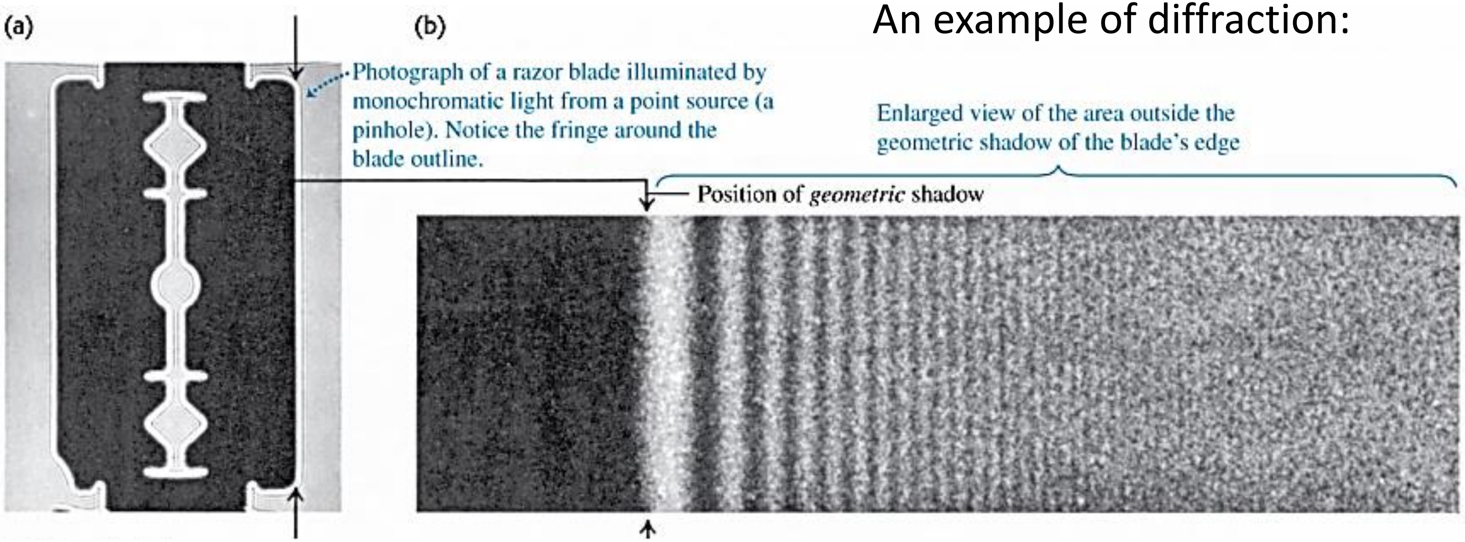




Diffraction

- **We don't often observe such diffraction patterns shown below in everyday life because most ordinary light sources are neither monochromatic nor point sources.**
- **If a white light bulb is used , each wavelength of the light from every point of the bulb forms its own diffraction pattern, but the patterns overlap so much that we can't see any individual pattern.**
- **Shadows of large buildings with fuzzy edges**

An example of diffraction:



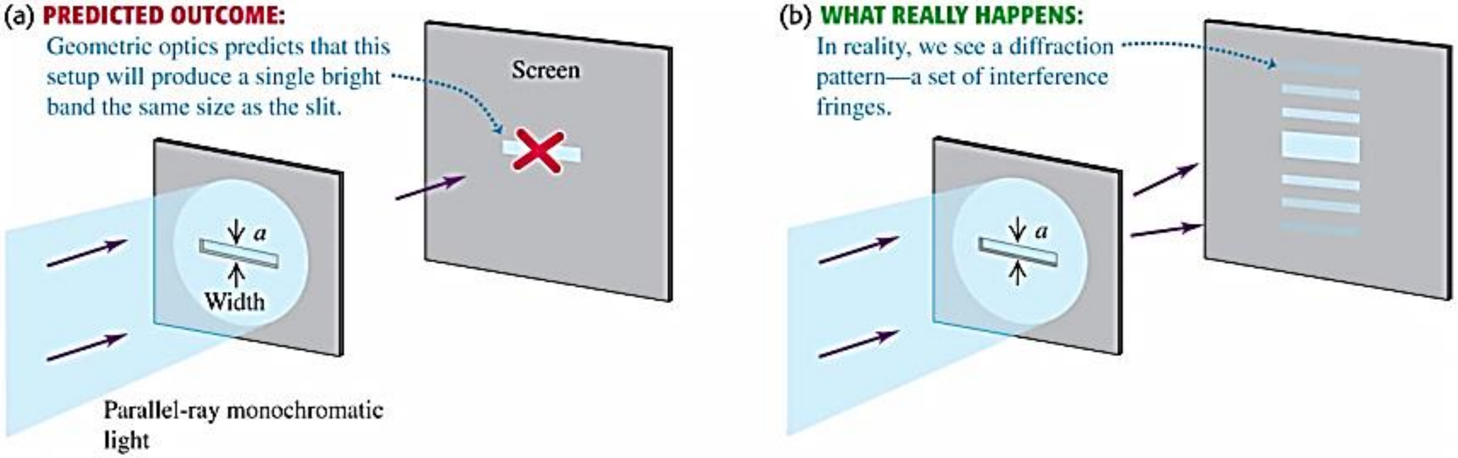
Fresnel and Fraunhofer Diffraction

- **Fresnel diffraction: Near-field diffraction. Both point source and screen are relatively close to obstacle**

- **Fraunhofer diffraction: Far-field diffraction. Point source, screen, and obstacle are far enough apart. (easier to solve)**

Diffraction from a Single Slit

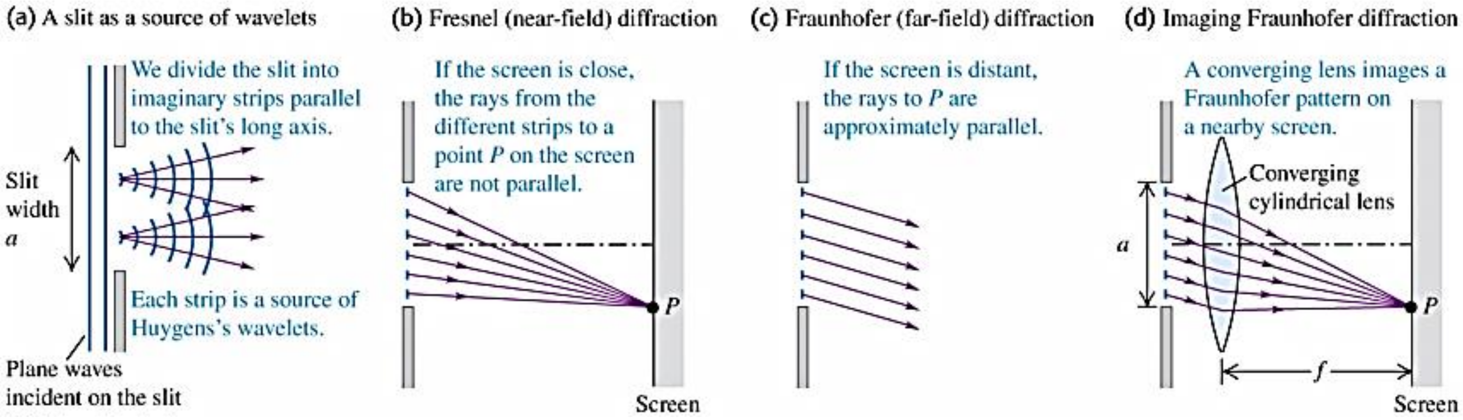
- **Diffraction pattern formed by plane-wave (parallel-ray) monochromatic light when it emerges from a long, narrow slit:**



- **The beam spreads out vertically; horizontal spreading is negligible because horizontal dimension of slit is relatively large.**

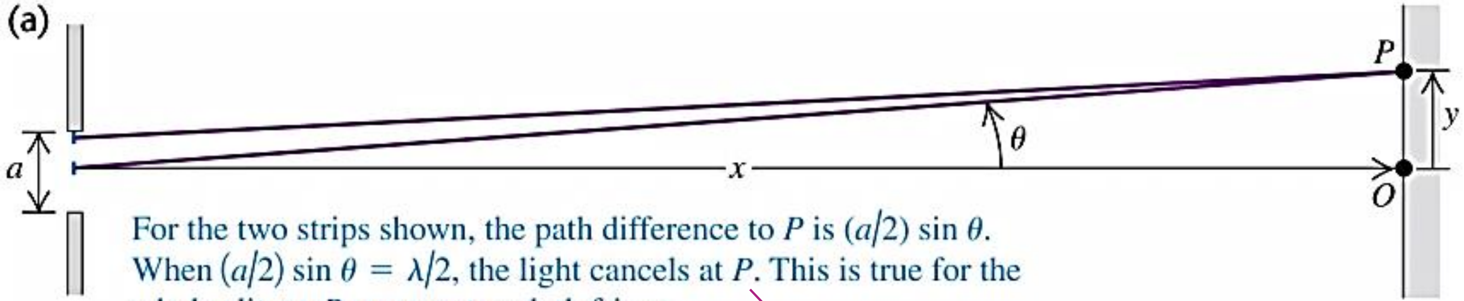
Single Slit Diffraction: Locating Dark Fringes

- **According to Huygens's principle, each element of area of the slit opening can be considered as a source of secondary waves.**
- **Divide the slit into several narrow strips of equal width, and cylindrical secondary wavelets spread out from each strip.**
- **The resultant intensity at point *P* can be calculated by adding the contributions from individual wavelets of various phases and amplitudes.**
- ***Can be shown rigorously by Kirchhoff's theorem***



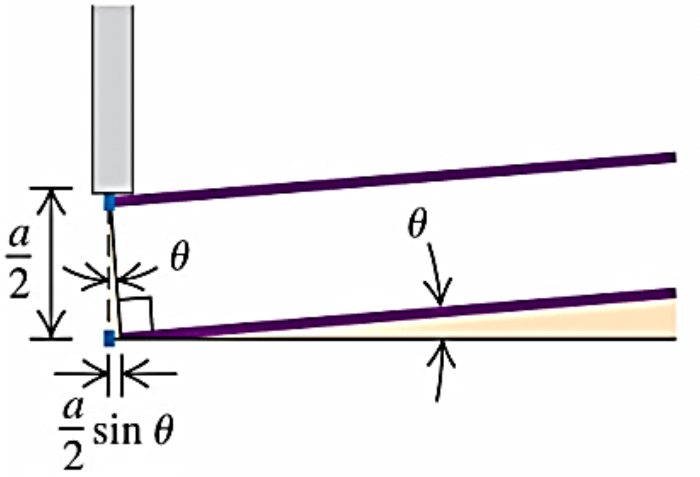
Single Slit Diffraction: Locating Dark Fringes

(a)



For the two strips shown, the path difference to P is $(a/2)\sin\theta$. When $(a/2)\sin\theta = \lambda/2$, the light cancels at P. This is true for the whole slit, so P represents a dark fringe.

(b) Enlarged view of the top half of the slit



Similarly, light from two strips immediately below the two in the figure also arrives at P a half-cycle out of phase.

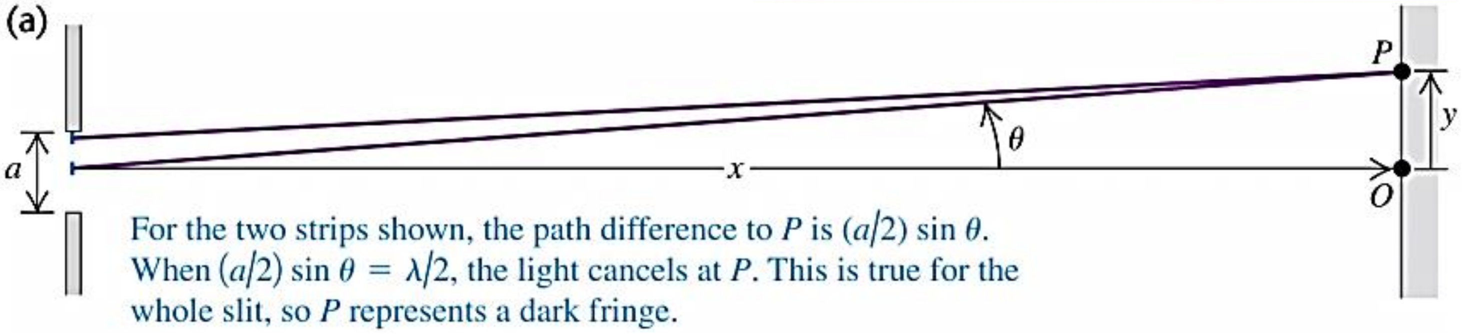
Single Slit Diffraction: Locating Dark Fringes

A dark fringe occurs whenever:

- **Divide slit into halves:** $(a\,/\,2)\sin\theta = \pm(\lambda\,/\,2)$ or $\sin\theta = \pm(\lambda\,/\,a)$;
- **Divide slit into quarters:** $(a\,/\,4)\sin\theta = \pm(\lambda\,/\,2)$ or $\sin\theta = \pm(2\lambda\,/\,a)$;
- **Divide slit into sixths:** $(a\,/\,6)\sin\theta = \pm(\lambda\,/\,2)$ or $\sin\theta = \pm(3\lambda\,/\,a)$.

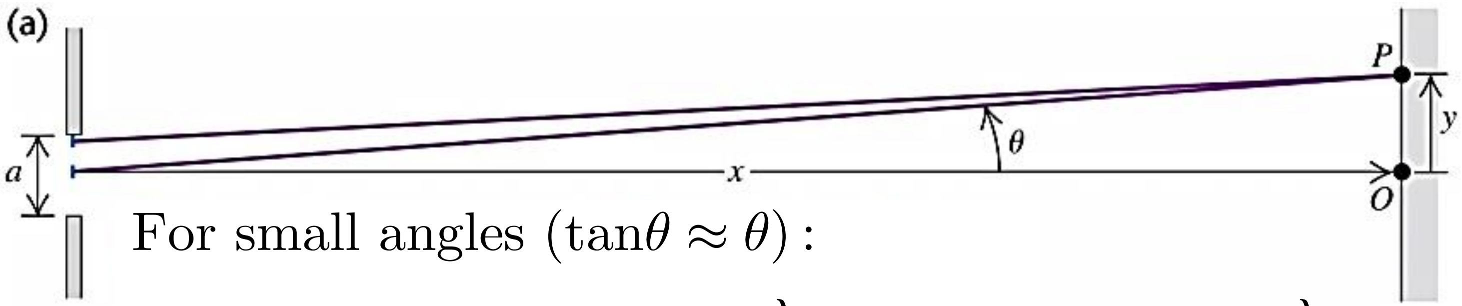
$$\sin\theta = \frac{m\lambda}{a} \qquad (m = \pm1, \pm2, \pm3, \ldots)$$ (dark fringes in single-slit diffraction)

For small angles $(\sin\theta \approx \theta)$: $$\theta = \frac{m\lambda}{a} \qquad (m = \pm1, \pm2, \pm3, \ldots)$$



For the two strips shown, the path difference to P is $(a/2)\sin\theta$. When $(a/2)\sin\theta = \lambda/2$, the light cancels at P. This is true for the whole slit, so P represents a dark fringe.

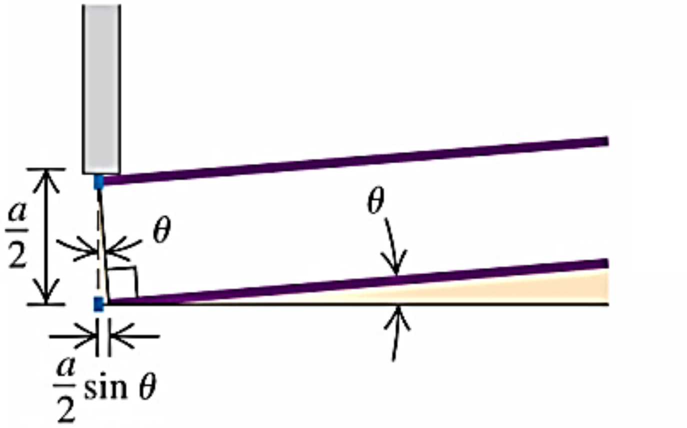
Single Slit Diffraction: Locating Dark Fringes



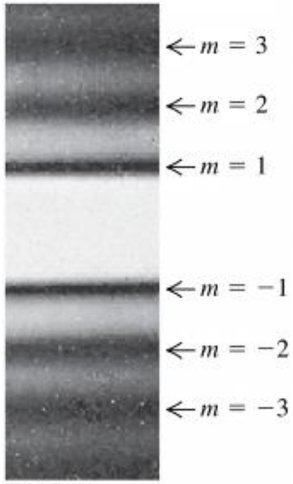
For small angles $(\tan\theta \approx \theta)$:

$$\tan\theta = \frac{y_m}{x} \Rightarrow \theta = \frac{y_m}{x} \Rightarrow \frac{m\lambda}{a} = \frac{y_m}{x} \quad \text{or} \quad y_m = x\frac{m\lambda}{a}.$$

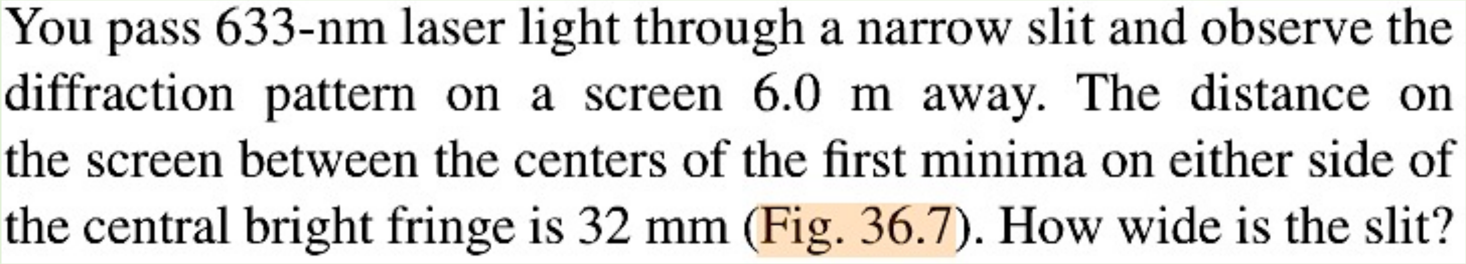
(b) Enlarged view of the top half of the slit



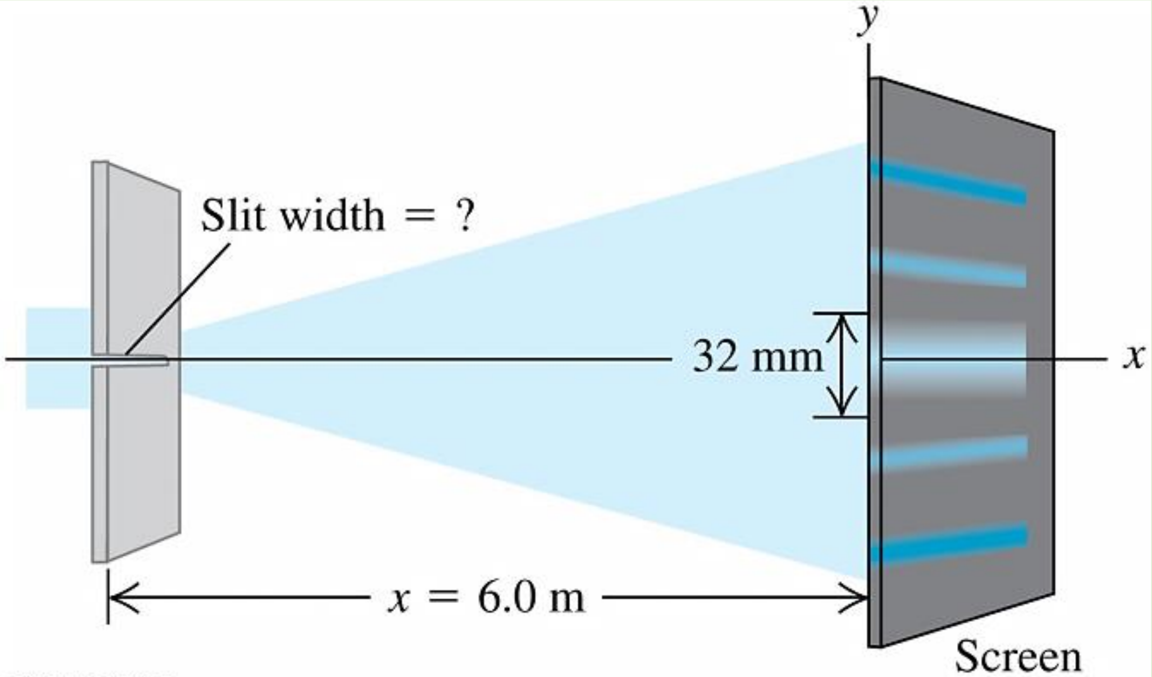
Photograph of the Fraunhofer diffraction pattern of a single slit →



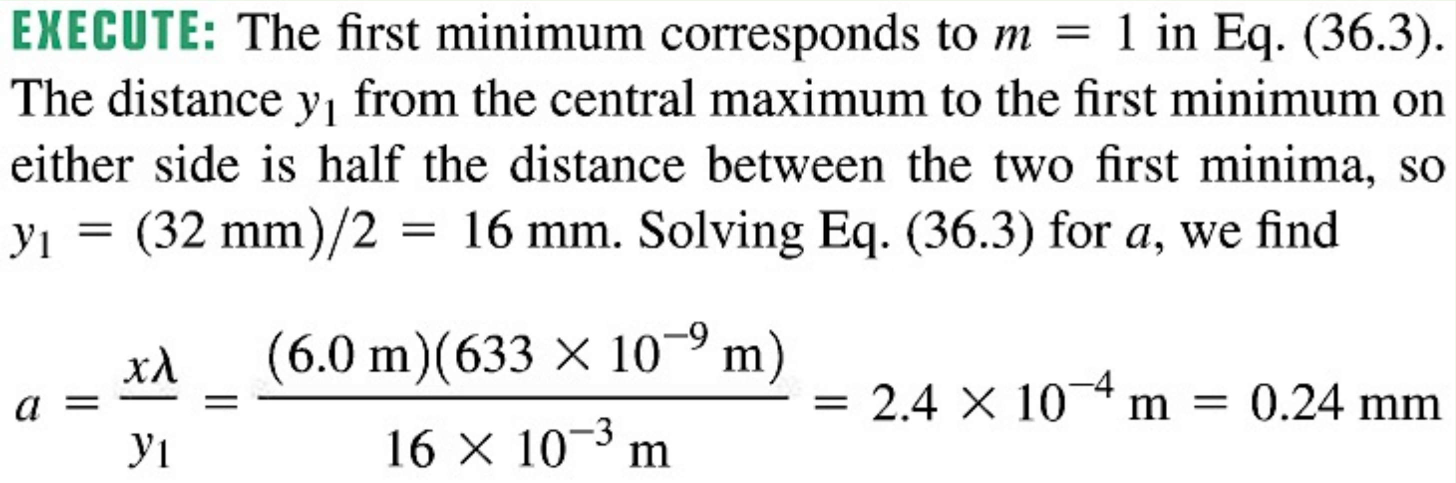
Example 36.1
Single-slit diffraction



You pass 633-nm laser light through a narrow slit and observe the diffraction pattern on a screen 6.0 m away. The distance on the screen between the centers of the first minima on either side of the central bright fringe is 32 mm (Fig. 36.7). How wide is the slit?

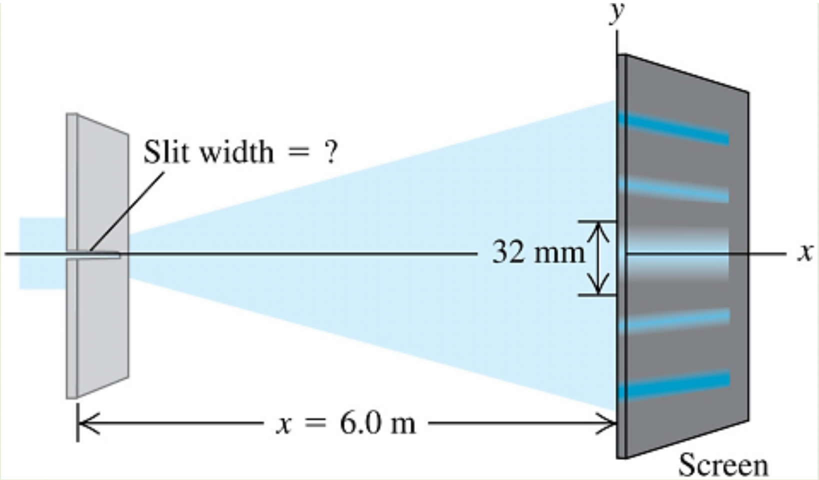


Example 36.1
Single-slit diffraction

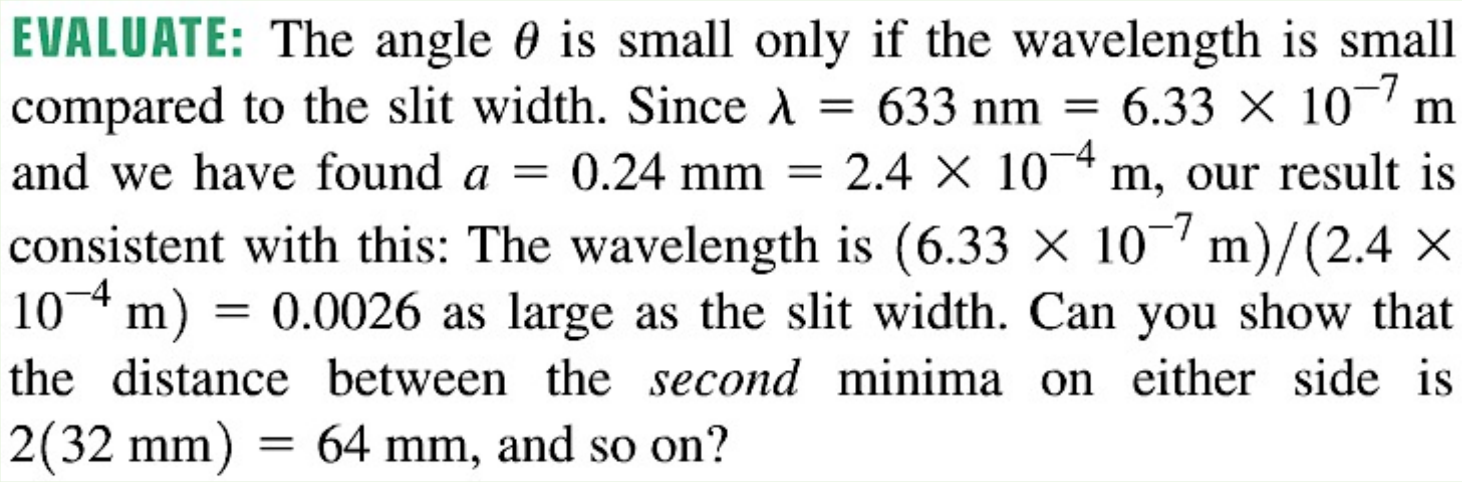


EXECUTE: The first minimum corresponds to $m = 1$ in Eq. (36.3). The distance y_1 from the central maximum to the first minimum on either side is half the distance between the two first minima, so $y_1 = (32\text{ mm})/2 = 16\text{ mm}$. Solving Eq. (36.3) for a, we find

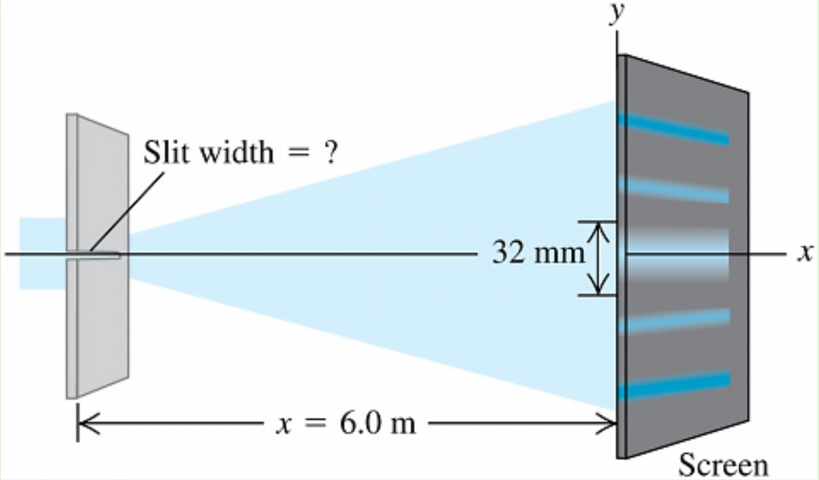
$$a = \frac{x\lambda}{y_1} = \frac{(6.0\text{ m})(633 \times 10^{-9}\text{ m})}{16 \times 10^{-3}\text{ m}} = 2.4 \times 10^{-4}\text{ m} = 0.24\text{ mm}$$



Example 36.1
Single-slit diffraction

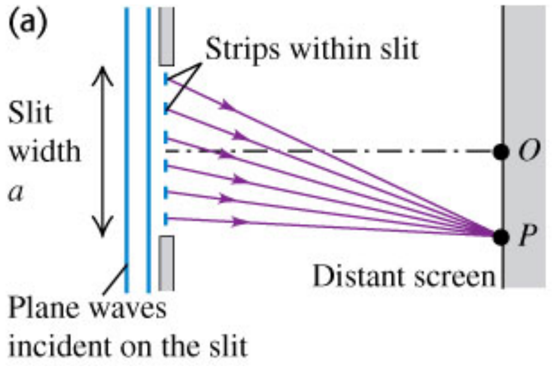


EVALUATE: The angle θ is small only if the wavelength is small compared to the slit width. Since $\lambda = 633\ \text{nm} = 6.33 \times 10^{-7}\ \text{m}$ and we have found $a = 0.24\ \text{mm} = 2.4 \times 10^{-4}\ \text{m}$, our result is consistent with this: The wavelength is $(6.33 \times 10^{-7}\ \text{m})/(2.4 \times 10^{-4}\ \text{m}) = 0.0026$ as large as the slit width. Can you show that the distance between the *second* minima on either side is $2(32\ \text{mm}) = 64\ \text{mm}$, and so on?



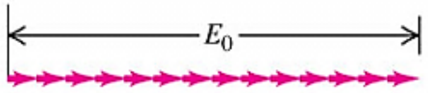
Intensity in the Single-Slit Pattern

- **Imagine a plane wave front at the slit subdivided into a large number of strips. The Huygens wavelets from all the strips are superposed at a point *P*.**
- **Using phasor diagrams to find the amplitude of the E field in single-slit diffraction. Each phasor represents the E field from a single strip within the slit:**

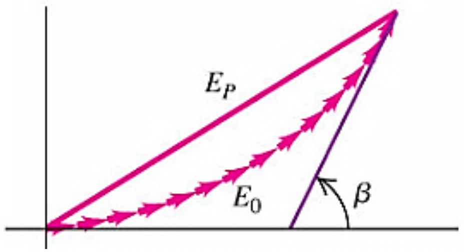


(b) At the center of the diffraction pattern (point O), the phasors from all strips within the slit are in phase.

(*e.g.*, 14 strips)



(c) Phasor diagram at a point slightly off the center of the pattern; β = total phase difference between the first and last phasors.



Intensity in the Single-Slit Pattern

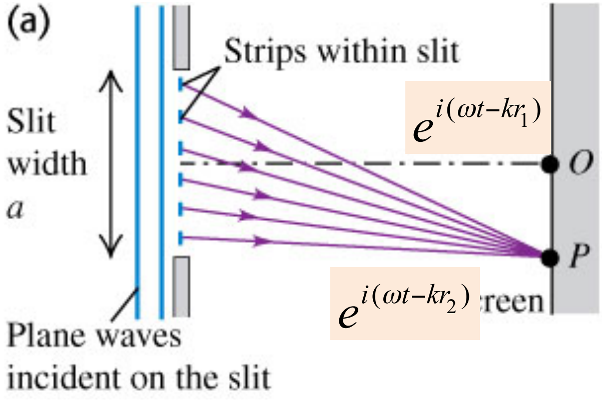
- **(b) At point *O*, the phasors are all in phase (*i.e.*, the same direction). E_O = resultant amplitude at *O*.**

- **(c) At point *P*, because of differences in path length, there are differences between wavelets from adjacent strips. E_P = resultant amplitude at *P*.**

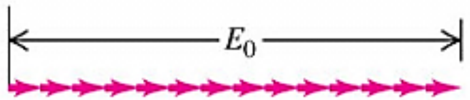
Phase difference between the wavelets from two edges

$\beta = kr_1 - kr_2 = (2\pi/\lambda) \times$ path difference

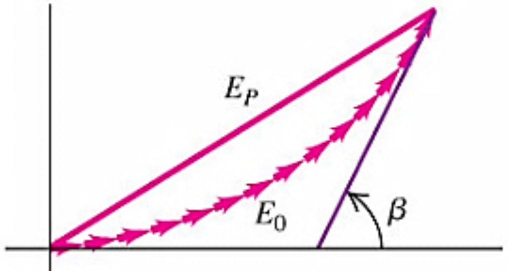
$$\Rightarrow \beta = \frac{2\pi a \sin\theta}{\lambda}.$$



(b) At the center of the diffraction pattern (point O), the phasors from all strips within the slit are in phase. (*e.g.*, 14 strips)



(c) Phasor diagram at a point slightly off the center of the pattern; β = total phase difference between the first and last phasors.



Intensity in the Single-Slit Pattern

- **(d) Dividing the slit into narrower and narrower strips:**

$$\overset{\frown}{AB} = \beta \cdot \overleftrightarrow{AC}$$

$$\Rightarrow \overleftrightarrow{AC} = \frac{\overset{\frown}{AB}}{\beta} = \frac{E_0}{\beta}$$

$$E_p = \overleftrightarrow{AB} = 2\overleftrightarrow{AC}\sin\left(\frac{\beta}{2}\right)$$

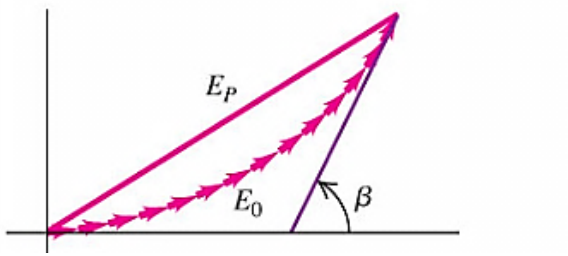
$$= E_0 \frac{\sin\left(\beta/2\right)}{\beta/2}$$

$$\boxed{I = I_0\left[\frac{\sin(\beta/2)}{(\beta/2)}\right]^2.}$$

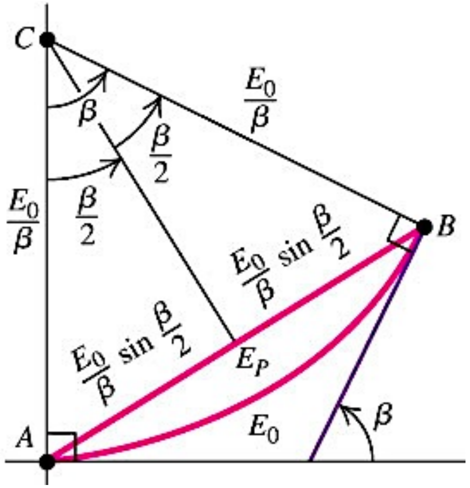
$$(\theta = \beta = 0,\ I = I_0)$$

$$\beta = \frac{2\pi a \sin\theta}{\lambda}$$

(c) Phasor diagram at a point slightly off the center of the pattern; β = total phase difference between the first and last phasors.



(d) As in (c), but in the limit that the slit is subdivided into infinitely many strips



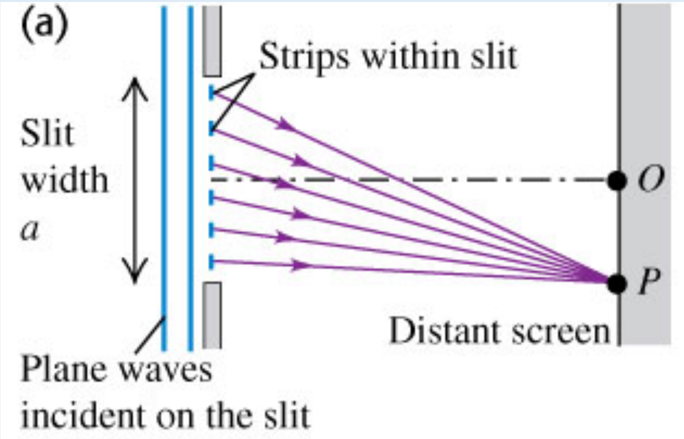
Single Slit Diffraction: Alternative Derivation

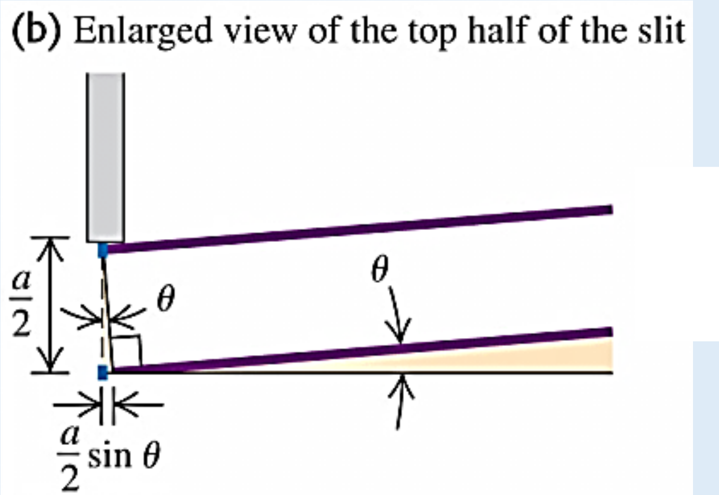
f **= The distance between slit and screen**

$$E(\theta)=\int_{-a/2}^{a/2}\frac{E_0}{f}e^{ikx\sin\theta}\,dx=\frac{E_0}{ikf\sin\theta}\left(e^{i\frac{ka\sin\theta}{2}}-e^{-i\frac{ka\sin\theta}{2}}\right)=\frac{2aE_0}{f}\left(\frac{\sin\alpha}{\alpha}\right)$$

$$I\left(\theta\right)=\frac{1}{2}\,|\,E\,|^2=\frac{2a^2E_0^2}{f^2}\left(\frac{\sin\alpha}{\alpha}\right)^2=I_0\left(\frac{\sin\alpha}{\alpha}\right)^2$$

$$\alpha\equiv\frac{1}{2}ka\sin\theta=\frac{\pi}{\lambda}a\sin\theta=\frac{\beta}{2}$$





Intensity in the Single-Slit Pattern

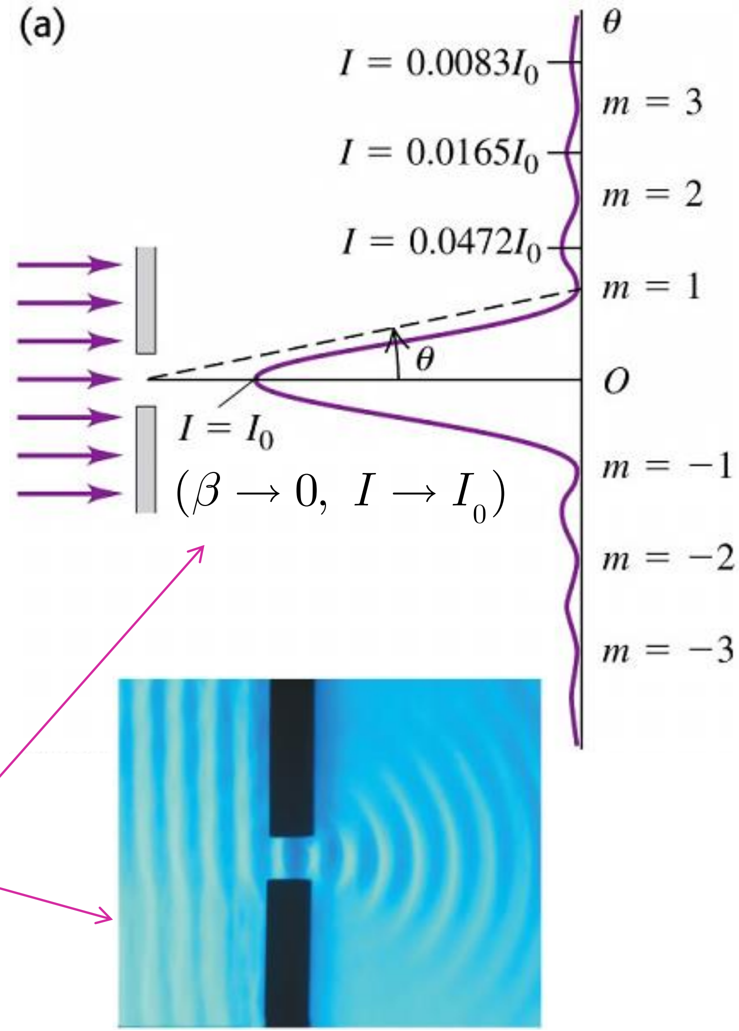
- **Dark fringes (*I* = 0):**

$$I = I_0 \left\{ \frac{\sin\left[\pi a\left(\sin\theta\right)/\lambda\right]}{\pi a\left(\sin\theta\right)/\lambda} \right\}^2 .$$

$$I = 0 \Rightarrow \frac{\beta}{2} = \pi a\left(\sin\theta\right)/\lambda = m\pi$$

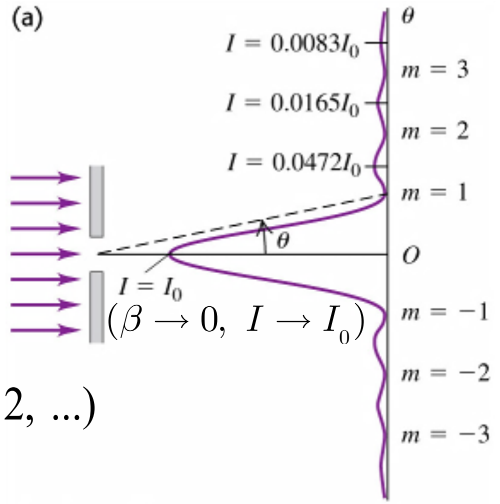
$$\text{or} \quad \sin\theta = \frac{m\lambda}{a}.$$

$$(m = \pm 1, \ \pm 2, \ \ldots)$$



Most of the wave power goes into the central intensity peak:

Intensity Maxima in the Single-Slit Pattern



- **Peak or intensity maxima:**

$$I = I_0\left[\frac{\sin(\beta/2)}{(\beta/2)}\right]^2.$$

$$I = I_\mathrm{m} \quad \text{(local maxima) when } \beta \approx \pm\left(2m+1\right)\pi. \quad (m = 0,\ 1,\ 2,\ \ldots)$$

- **Take derivative of *I* w.r.t. β to find maxima and minima ➔ transcendental equation ➔ solve numerically.**
- **Side maxima intensities:**

$$I_m \approx \frac{I_0}{\left(m+\frac{1}{2}\right)^2\pi^2}.$$

I_0 is the intensity of the central maximum

I_m is the intensity of the *m*-th side maximum

Width of the Single-Slit Pattern

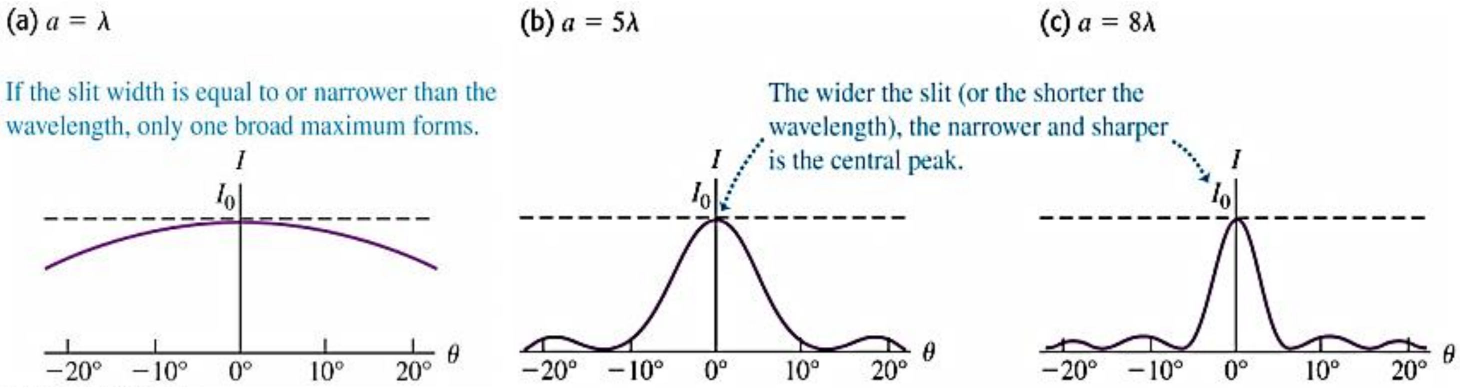
- **Width (angular spread) of the central maximum:**

$$I = I_0 \left[\frac{\sin(\beta/2)}{(\beta/2)}\right]^2 = I_0 \left\{\frac{\sin\left[\pi a\left(\sin\theta\right)/\lambda\right]}{\pi a\left(\sin\theta\right)/\lambda}\right\}^2.$$

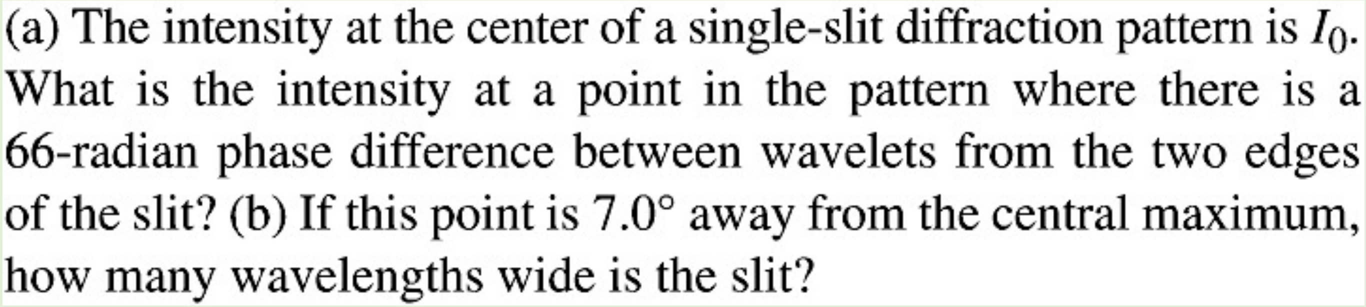
$$\beta/2 = \pi,\ \ \pi a\left(\sin\theta\right)/\lambda = \pi$$

First minimum beside central maximum

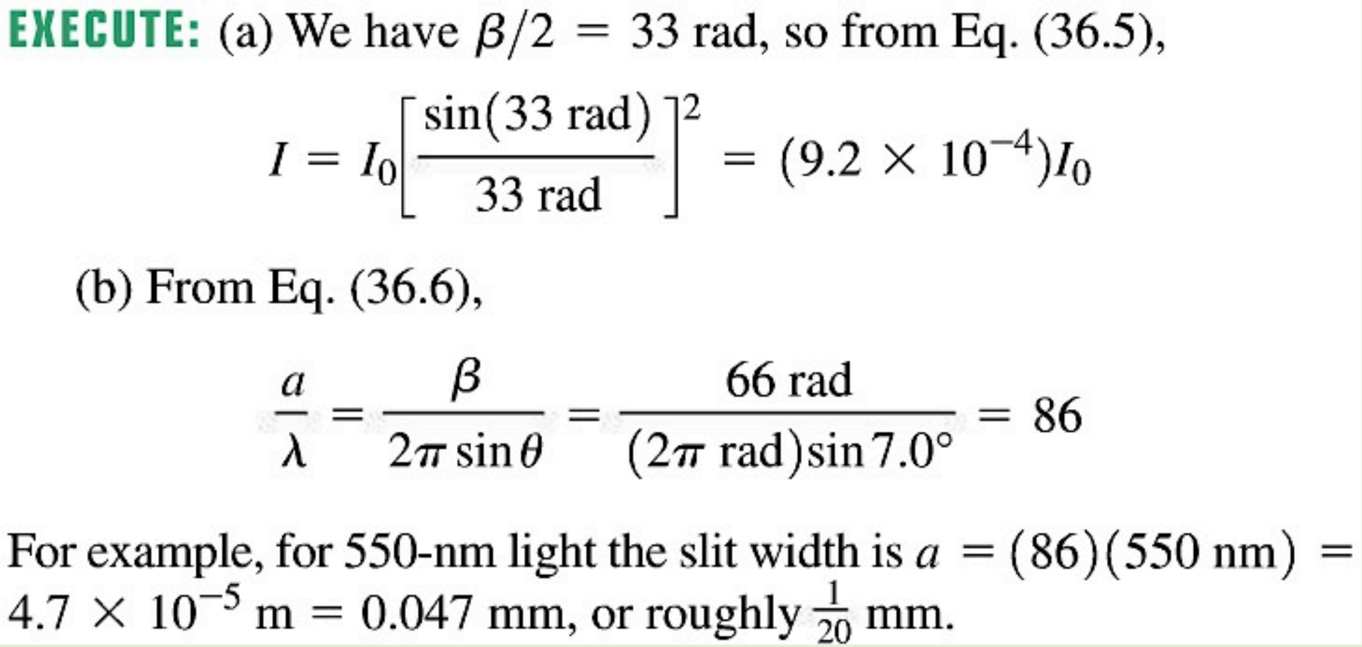
$$\text{or}\ \ \theta_1 \approx \sin\theta_1 = \lambda/a.$$



Example 36.2
Single-slit diffraction: Intensity I



(a) The intensity at the center of a single-slit diffraction pattern is I_0. What is the intensity at a point in the pattern where there is a 66-radian phase difference between wavelets from the two edges of the slit? (b) If this point is 7.0° away from the central maximum, how many wavelengths wide is the slit?



EXECUTE: (a) We have $\beta/2 = 33$ rad, so from Eq. (36.5),

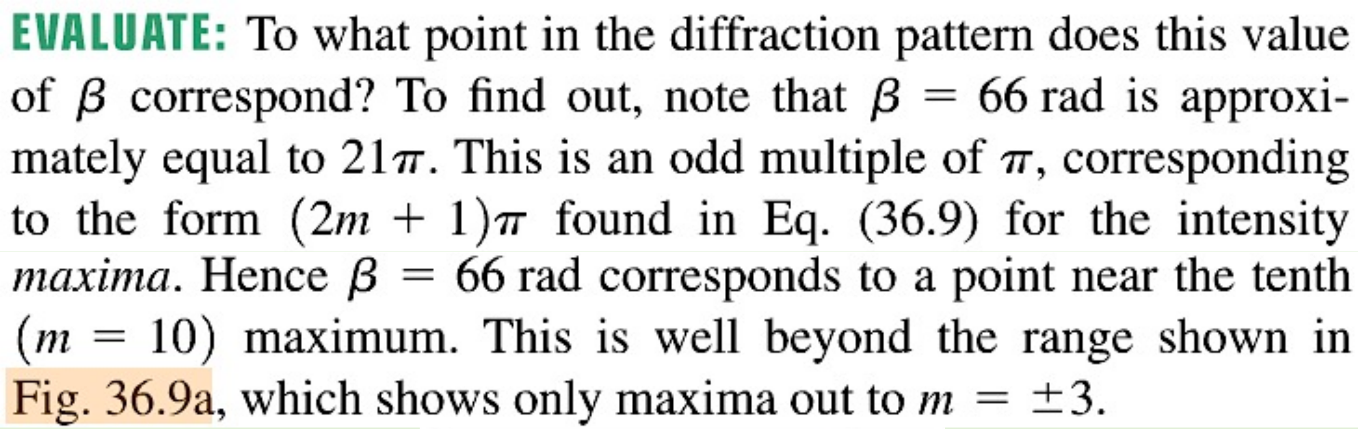
$$I = I_0\left[\frac{\sin(33\text{ rad})}{33\text{ rad}}\right]^2 = (9.2 \times 10^{-4})I_0$$

(b) From Eq. (36.6),

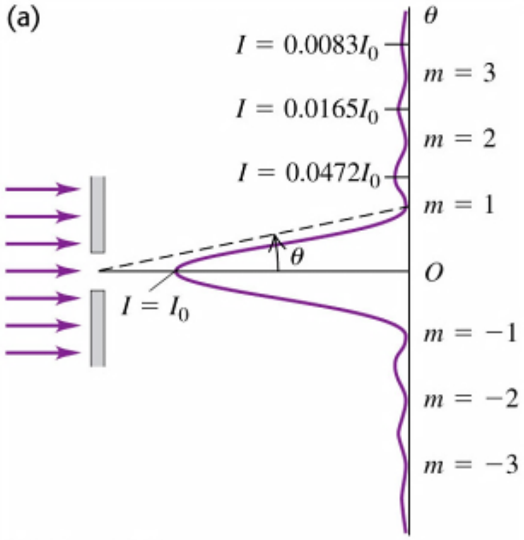
$$\frac{a}{\lambda} = \frac{\beta}{2\pi\sin\theta} = \frac{66\text{ rad}}{(2\pi\text{ rad})\sin 7.0°} = 86$$

For example, for 550-nm light the slit width is $a = (86)(550\text{ nm}) = 4.7 \times 10^{-5}\text{ m} = 0.047\text{ mm}$, or roughly $\frac{1}{20}$ mm.

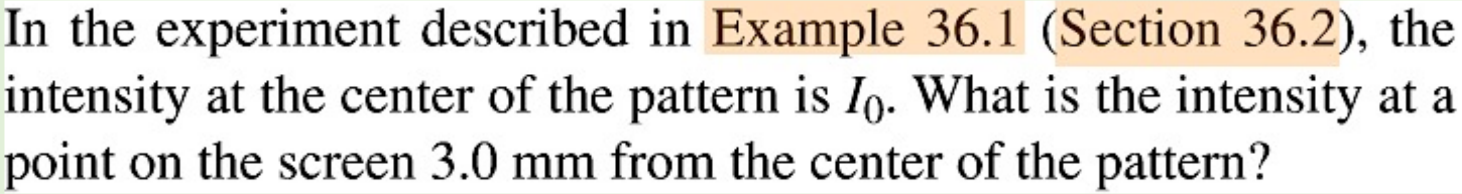
Example 36.2
Single-slit diffraction: Intensity I



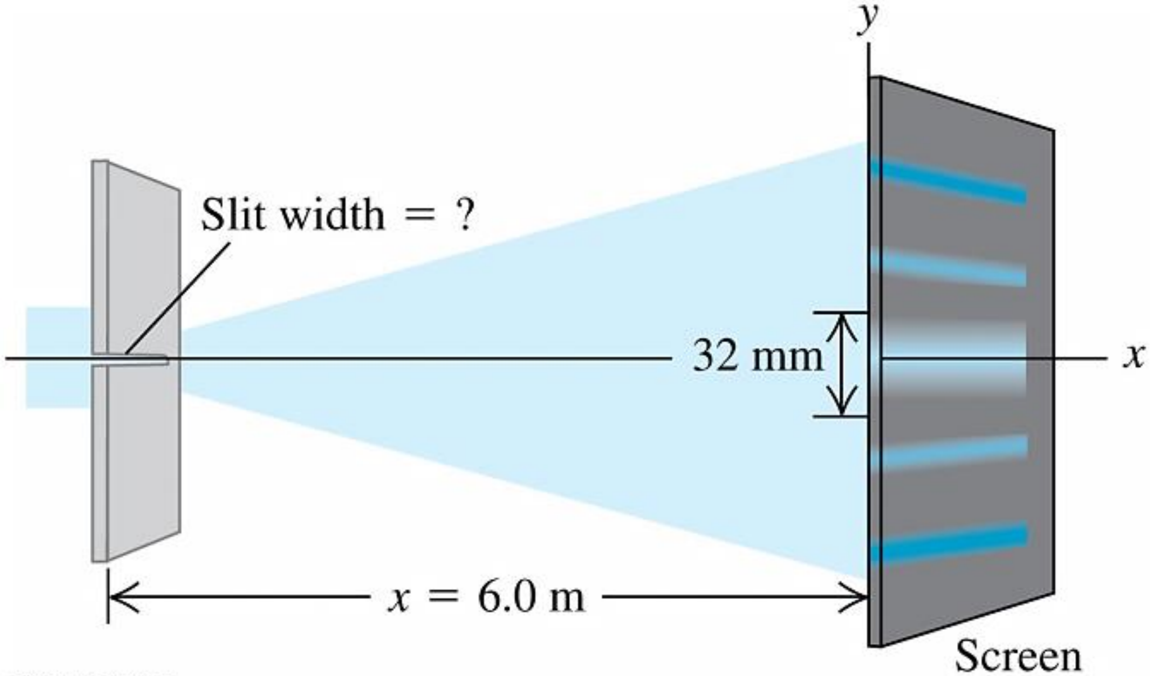
EVALUATE: To what point in the diffraction pattern does this value of β correspond? To find out, note that $\beta = 66$ rad is approximately equal to 21π. This is an odd multiple of π, corresponding to the form $(2m + 1)\pi$ found in Eq. (36.9) for the intensity *maxima*. Hence $\beta = 66$ rad corresponds to a point near the tenth $(m = 10)$ maximum. This is well beyond the range shown in Fig. 36.9a, which shows only maxima out to $m = \pm 3$.



Example 36.3
Single-slit diffraction: Intensity II



In the experiment described in Example 36.1 (Section 36.2), the intensity at the center of the pattern is I_0. What is the intensity at a point on the screen 3.0 mm from the center of the pattern?

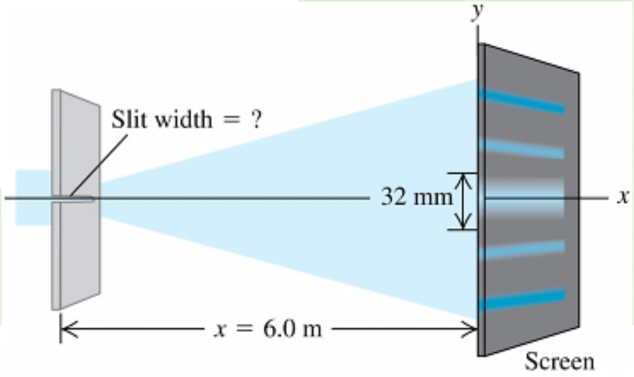


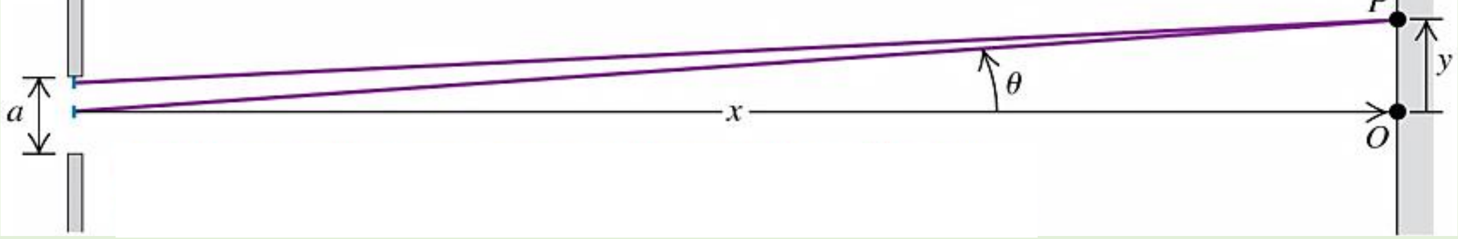
Example 36.3
Single-slit diffraction: Intensity II

EXECUTE: Referring to Fig. 36.5a, we have $y = 3.0\ \text{mm}$ and $x = 6.0\ \text{m}$, so $\tan\theta = y/x = (3.0 \times 10^{-3}\ \text{m})/(6.0\ \text{m}) = 5.0 \times 10^{-4}$. This is so small that the values of $\tan\theta$, $\sin\theta$, and θ (in radians) are all nearly the same. Then, using Eq. (36.7),

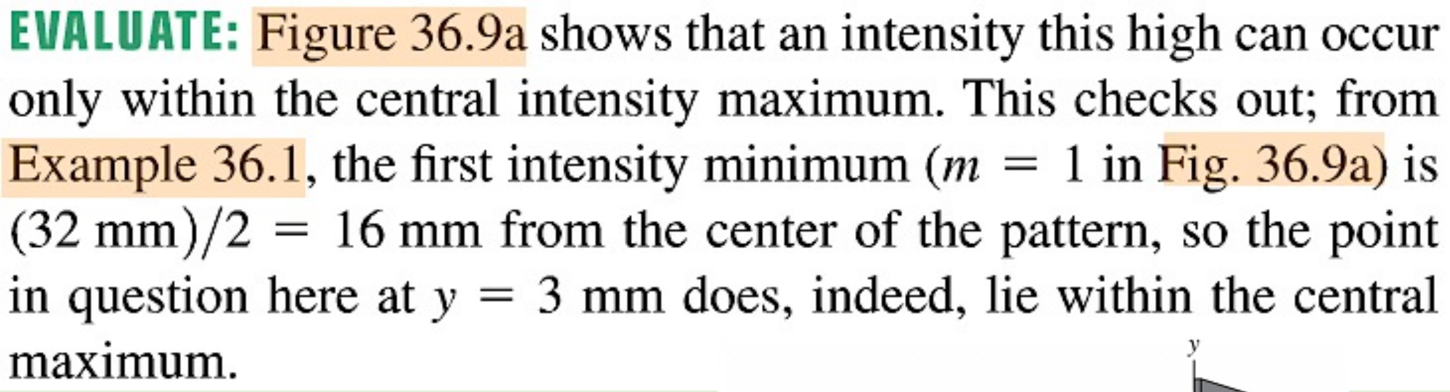
$$\frac{\pi a \sin\theta}{\lambda} = \frac{\pi(2.4 \times 10^{-4}\ \text{m})(5.0 \times 10^{-4})}{6.33 \times 10^{-7}\ \text{m}} = 0.60$$

$$I = I_0\left(\frac{\sin 0.60}{0.60}\right)^2 = 0.89 I_0$$

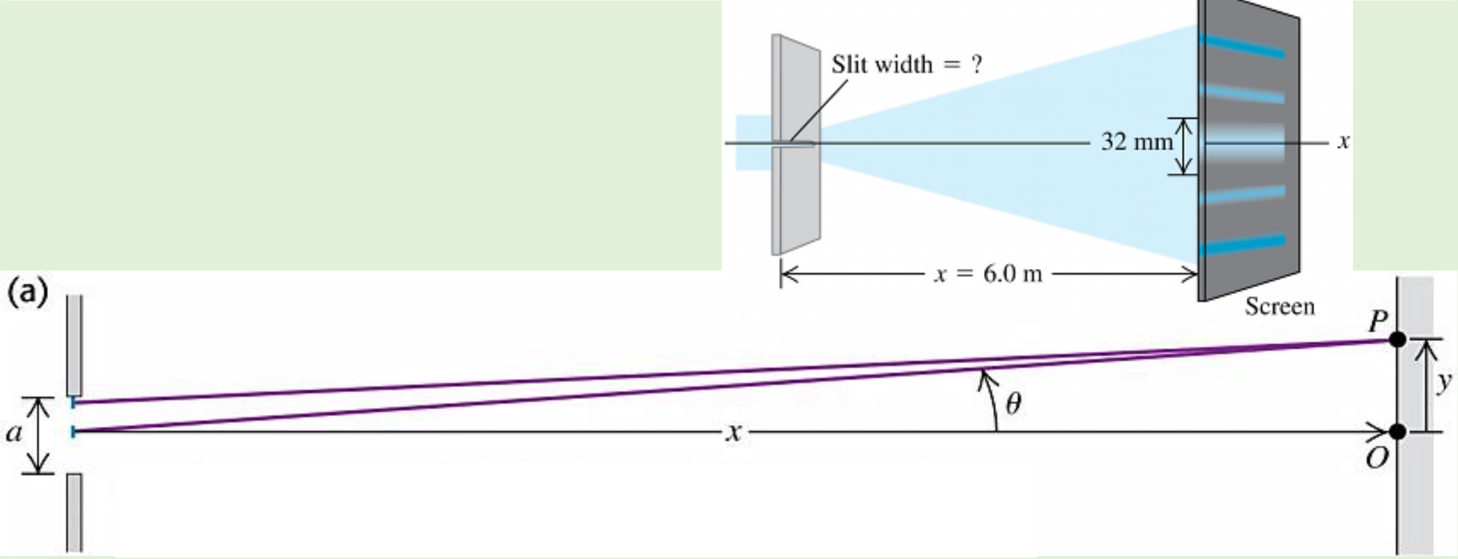




Example 36.3
Single-slit diffraction: Intensity II



EVALUATE: Figure 36.9a shows that an intensity this high can occur only within the central intensity maximum. This checks out; from Example 36.1, the first intensity minimum ($m = 1$ in Fig. 36.9a) is $(32\text{ mm})/2 = 16\text{ mm}$ from the center of the pattern, so the point in question here at $y = 3\text{ mm}$ does, indeed, lie within the central maximum.



2-Slit Interference – slits width a -> 0

$$E_1(t) = E\cos(\omega t + \phi), \qquad \phi = kd\sin\vartheta$$
$$E_2(t) = E\cos\omega t.$$

$$\begin{aligned} E_P^2 &= E^2 + E^2 - 2E^2\cos(\pi - \phi) \\ &= E^2 + E^2 + 2E^2\cos\phi \\ &= 2E^2(1+\cos\phi) \\ &= 4E^2\cos^2\left(\phi/2\right). \end{aligned}$$

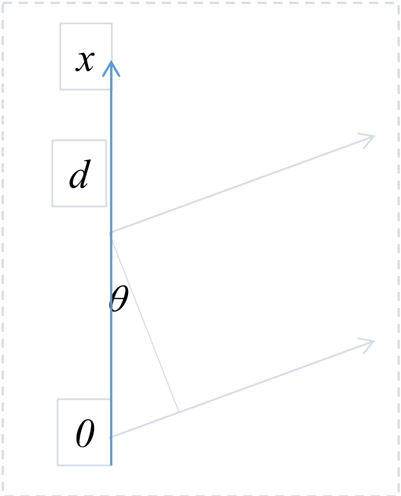
$$E_P = 2E\left|\cos\left(\phi/2\right)\right|.$$

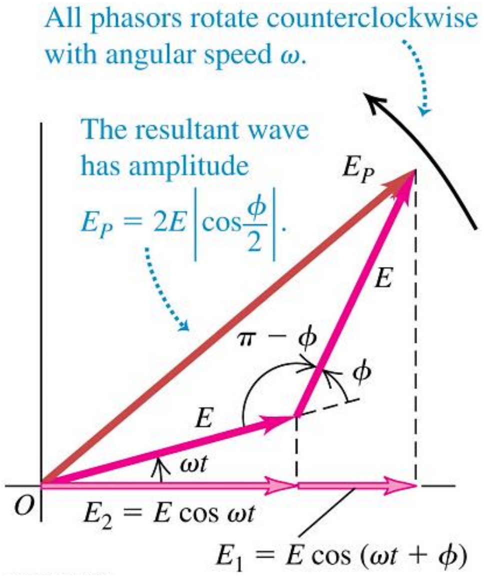
$$I_0 = \frac{1}{2}\varepsilon_0 c E^2$$
$$I = \frac{1}{2}\varepsilon_0 c E_P^2 = \frac{1}{2}\varepsilon_0 c\left(4E^2\cos^2\frac{\phi}{2}\right)$$
$$= 4I_0\cos^2\frac{\phi}{2}.$$

Alternatively, we have

$$E_P = E(1+e^{i\phi})$$

$$\left|E_P\right| = E\left|1+e^{i\phi}\right| = E\sqrt{(2+2\cos\phi)} = 2E\left|\cos\frac{\phi}{2}\right|$$



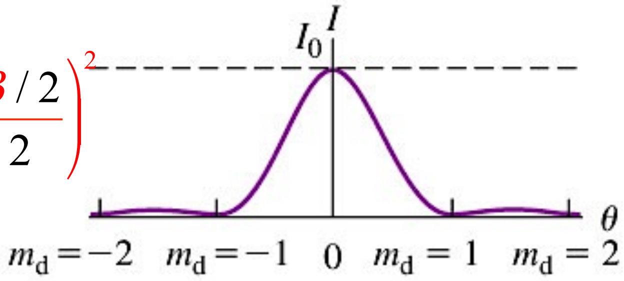

All phasors rotate counterclockwise with angular speed ω.

The resultant wave has amplitude $E_P = 2E\left|\cos\frac{\phi}{2}\right|$.

$$\beta = \frac{2\pi a \sin\theta}{\lambda}$$

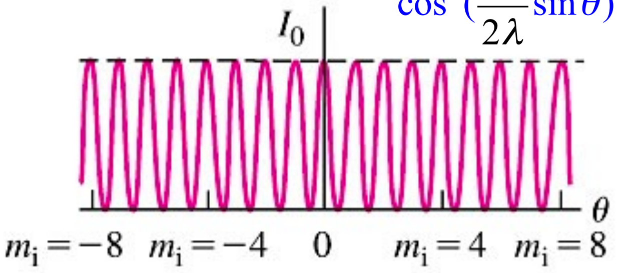
Two Slits of Finite Width

(a) Single-slit diffraction pattern for a slit width *a*



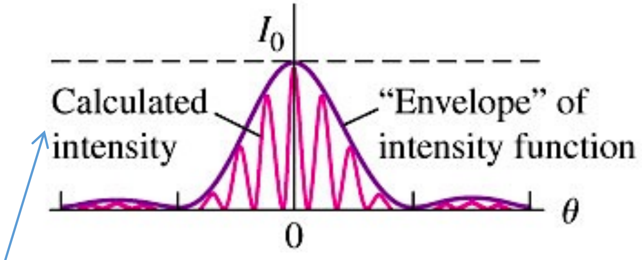
$$\left(\frac{\sin\beta/2}{\beta/2}\right)^2$$

(b) Two-slit interference pattern for narrow slits whose separation *d* is four times the width of the slit in (a)



$$\cos^2\left(\frac{\pi d}{2\lambda}\sin\theta\right)$$

(c) Calculated intensity pattern for two slits of width *a* and separation $d = 4a$, including both interference and diffraction effects



$$I = I_0 \cos^2\frac{\phi}{2} \cdot \left[\frac{\sin(\beta/2)}{\beta/2}\right]^2$$

where

$$\phi = \frac{2\pi d}{\lambda}\sin\theta, \quad \beta = \frac{2\pi a}{\lambda}\sin\theta.$$

$d = 2b =$ separation between slits

Two Slits of Finite Width

One slit centered at $x = b$, the other at $x = -b$.

Slit-1 **Slit-2**

$$E(\theta) = \int_{d/2-a/2}^{d/2+a/2} \frac{E_0}{f} e^{ikx\sin\theta}\, dx + \int_{-d/2-a/2}^{-d/2+a/2} \frac{E_0}{f} e^{ikx\sin\theta}\, dx$$

$$= \frac{2aE_0}{f}\left(\frac{\sin\beta/2}{\beta/2}\right)\left(e^{ik\frac{d}{2}\sin\theta} + e^{-ik\frac{d}{2}\sin\theta}\right) = \frac{4aE_0}{f}\left(\frac{\sin\beta/2}{\beta/2}\right)\cos(\frac{kb}{2}\sin\theta)$$

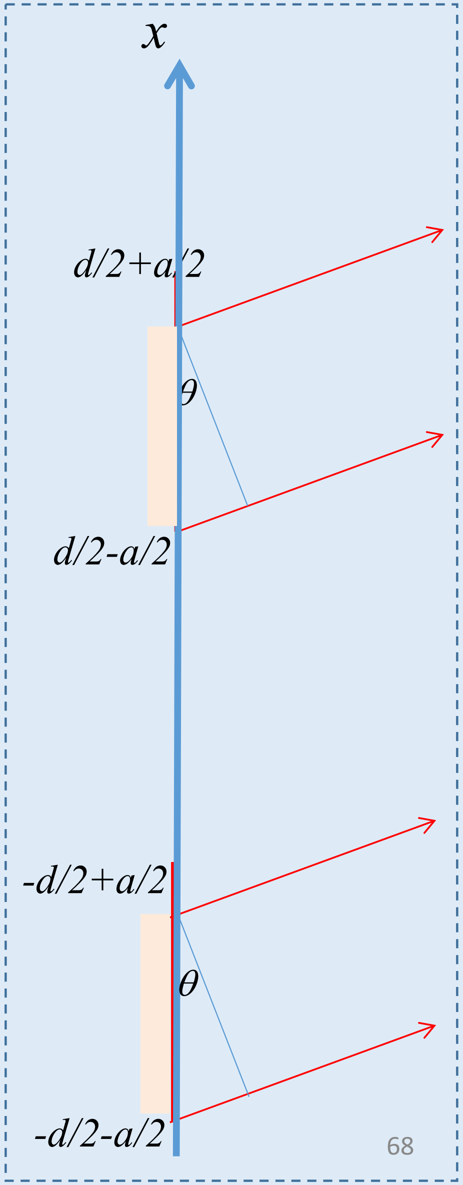
$$\boxed{I(\theta) = I_0\left(\frac{\sin\beta/2}{\beta/2}\right)^2 \cos^2(\frac{kd}{2}\sin\theta)}$$

$$\beta \equiv ka\sin\theta = \frac{2\pi}{\lambda} a\sin\theta$$

Single Slit Diffraction

Double-Slit Interference

d = separation between slits



Two Slits of Finite Width

$$\boxed{I(\theta) = I_0\left(\frac{\sin\beta/2}{\beta/2}\right)^2 \cos^2\left(\frac{\phi}{2}\right)}$$

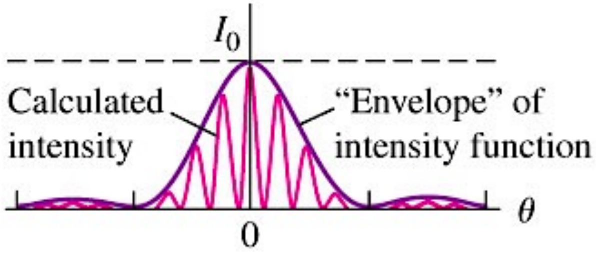
For *d* = 4*a*: $\boxed{4\beta = \phi = 2m_i\pi}$

- **Every 4th interference maximum at the sides is missing because these interference maxima**
  $$m_\mathrm{i} = \pm 4, \ \pm 8, \ \ldots$$

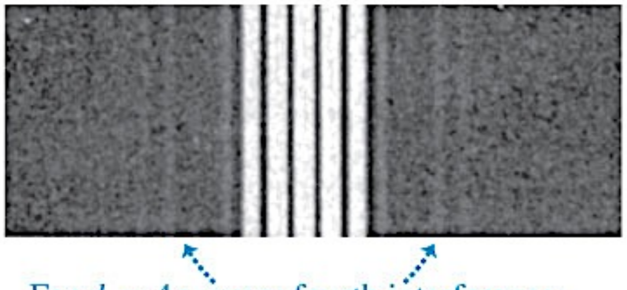
coincide with diffraction minima ($m_\mathrm{d} = \pm 1, \ \pm 2,$**)**…

- **There will be missing maxima whenever *d* is an integer multiple of *a*.**

(c) Calculated intensity pattern for two slits of width a and separation $d = 4a$, including both interference and diffraction effects



(d) Actual photograph of the pattern calculated in (c)



For $d = 4a$, every fourth interference maximum at the sides ($m_\mathrm{i} = \pm 4, \pm 8, \ldots$) is missing.

N-Slit Interference – slits width a -> 0

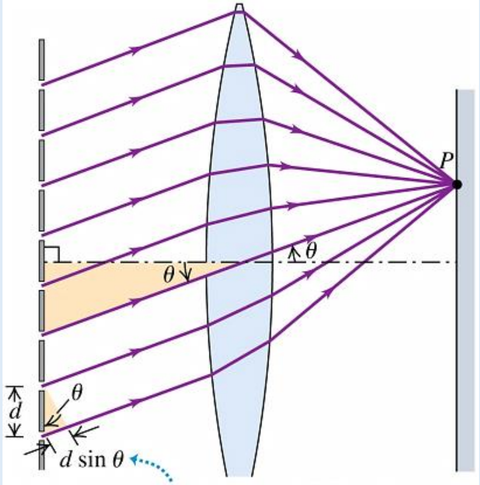
$$E(\theta) = E_0 + E_0 e^{ikd\sin\theta} + E_0 e^{ik2d\sin\theta} + \cdots + E_0 e^{ik(N-1)d\sin\vartheta}$$

$$= E_0\left(1 + e^{ikd\sin\vartheta} + e^{ik2d\sin\vartheta} + \cdots + e^{ik(N-1)d\sin\vartheta}\right)$$

$$= E_0\left(1 + p + p^2 + .. + p^{N-1}\right) \quad , \; p = e^{ikd\sin\vartheta} = e^{i\frac{2\pi}{\lambda}d\sin\vartheta} = e^{i2\alpha}$$

$$= E_0\left(\frac{1-p^N}{1-p}\right)$$

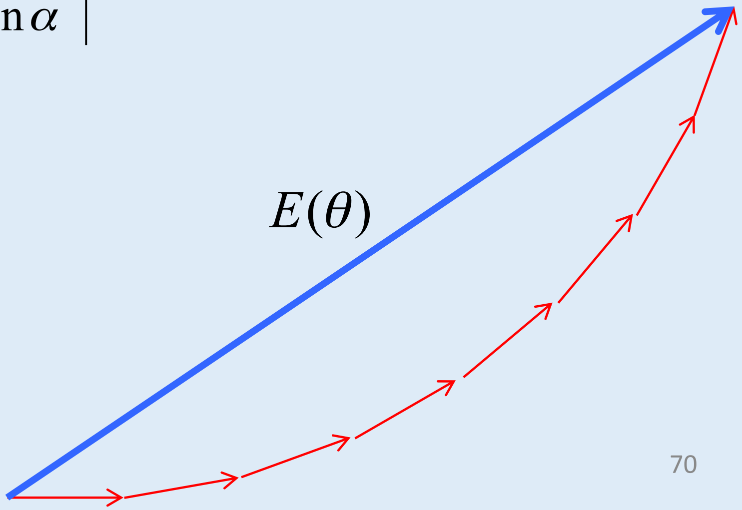
$$\left|E(\vartheta)\right| = E_0\left|\frac{1-p^N}{1-p}\right| = E_0\sqrt{\frac{1-\cos 2N\alpha}{1-\cos 2\alpha}} = E_0\left|\frac{\sin N\alpha}{\sin\alpha}\right|$$



Maxima occur where the path difference for adjacent slits is a whole number of wavelengths: $d \sin\theta = m\lambda$.

$$\boxed{I(\theta) = I_0\left(\frac{\sin(N\alpha)}{\sin\alpha}\right)^2}$$

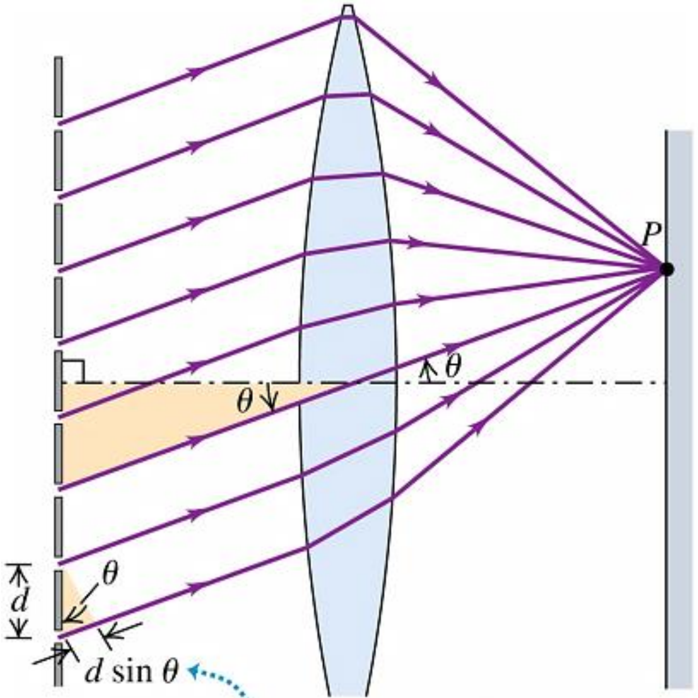
***N*-Slit Interference**



N-Slit Interference – slits width a -> 0

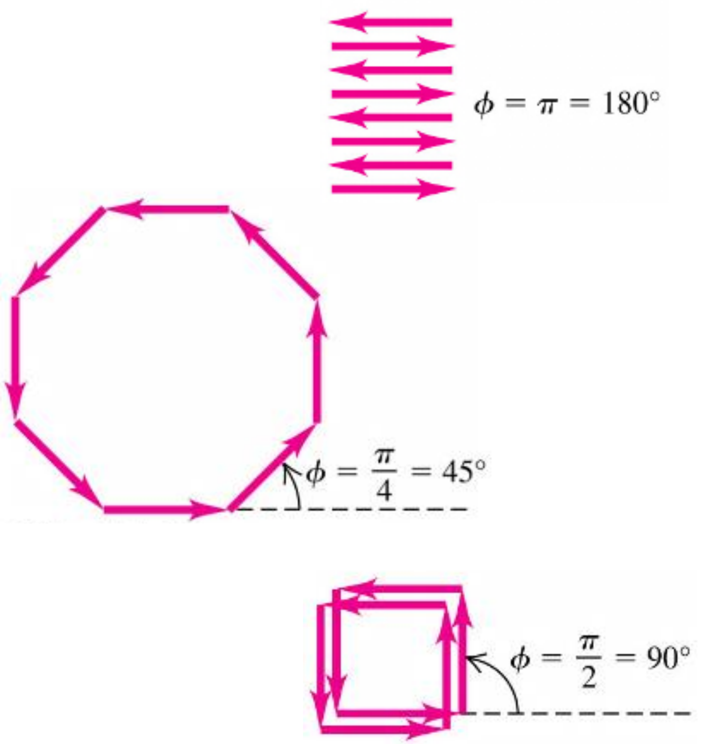
Generally, the intensity with 8 slits is 0 whenever ϕ is an integer multiple of $\pi/4$, except when ϕ is a multiple of 2π. Thus 7 minima for every maximum.

$$\phi = \frac{2\pi}{\lambda} d \sin\vartheta$$



Maxima occur where the path difference for adjacent slits is a whole number of wavelengths: $d \sin\theta = m\lambda$. $(m = 0, \ \pm 1, \ \pm 2, \ \ldots)$

Phase diagrams (for 8 narrow slits):



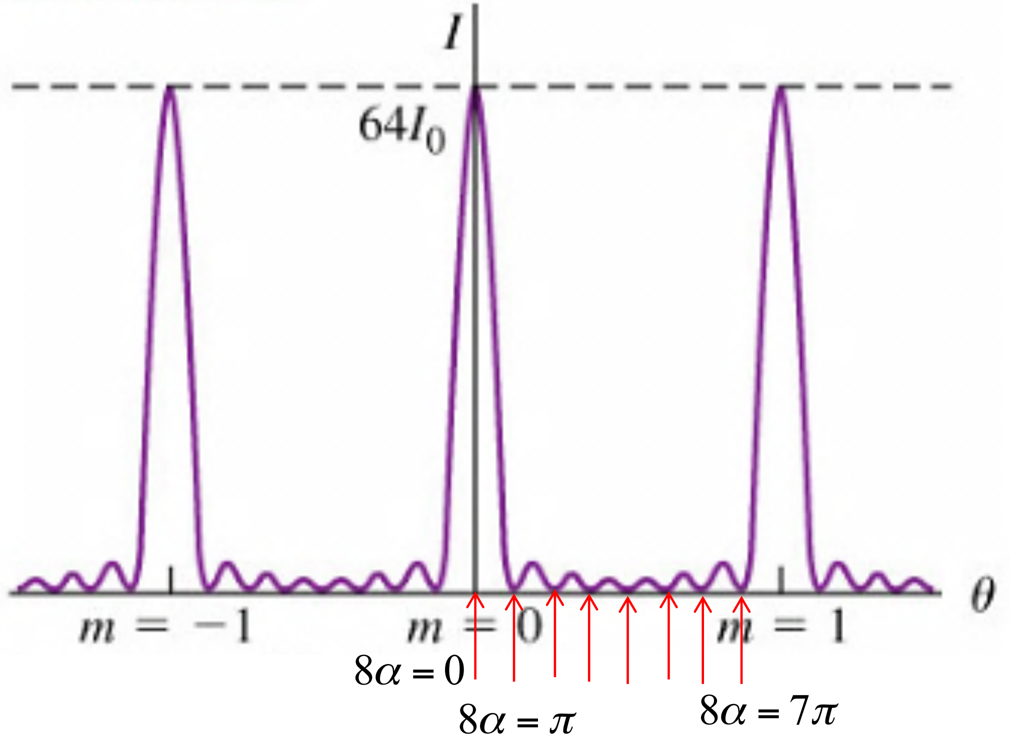
N=*8*-Slit Interference

$$I(\theta) = I_0\left(\frac{\sin(8\alpha)}{\sin\alpha}\right)^2$$

$$\alpha \equiv \frac{\pi}{\lambda} d\sin\theta$$

a: width of each slit
d: distance between two slits

(b) $N = 8$: eight slits produce taller, narrower maxima in the same locations, separated by seven minima.

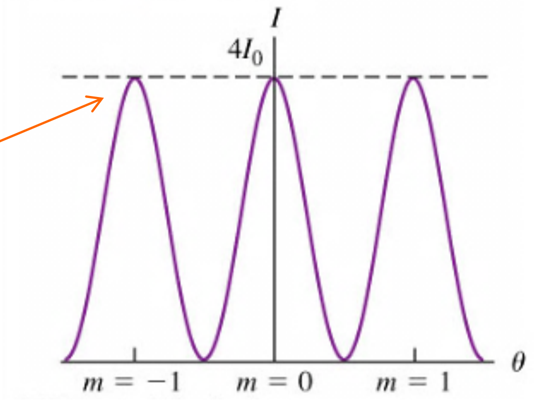


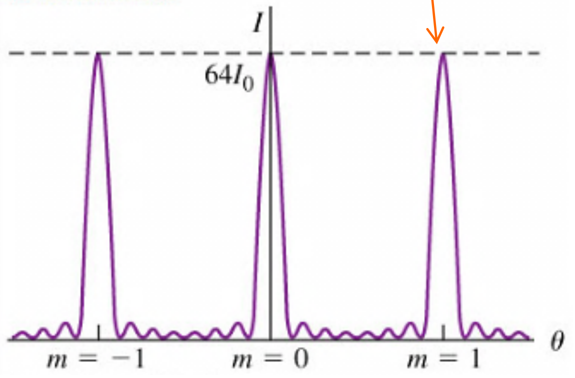
N-Slit Interference

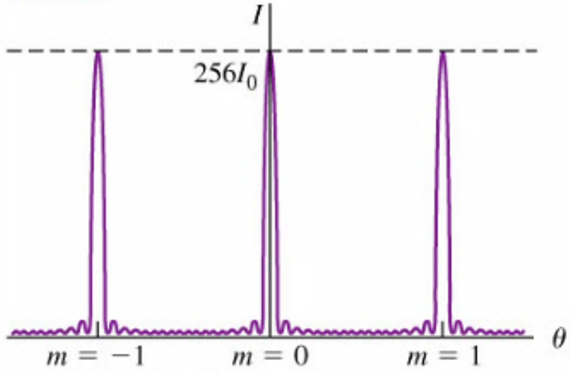
A minimum occurs whenever $N\alpha$ = an integral multiple of π (except when α is an integral multiple of π, which gives a principal maximum.)

$$I(\theta) = I_0 \left(\frac{\sin(N\alpha)}{\sin\alpha} \right)^2$$

$$\alpha \equiv \frac{\pi}{\lambda} d \sin\theta$$


(a) $N = 2$: two slits produce one minimum between adjacent maxima.


(b) $N = 8$: eight slits produce taller, narrower maxima in the same locations, separated by seven minima.


(c) $N = 16$: with 16 slits, the maxima are even taller and narrower, with more intervening minima.

$$\alpha = n\pi \Rightarrow d \sin\theta = n\lambda \quad \text{(constructive interference)}$$

N-Slit interference + Diffraction (slits have width a)

Slits are centered at *x* =0, *d*, 2*d*, 3*d*… *(N-1)d*.

$$E(\theta)=\int_{-a/2}^{a/2}\frac{E_0}{f}e^{ikx\sin\theta}\,dx+\int_{d-a/2}^{d+a/2}\frac{E_0}{f}e^{ikx\sin\theta}\,dx+\int_{2d-a/2}^{2d+a/2}\frac{E_0}{f}e^{ikx\sin\theta}\,dx+\ldots$$

$$=\frac{2aE_0}{f}\left(\frac{\sin\beta/2}{\beta/2}\right)\left(1+p+p^2+..+p^{N-1}\right)$$

$$=\frac{4aE_0}{f}\left(\frac{\sin\beta/2}{\beta/2}\right)\left(\frac{1-p^N}{1-p}\right)$$

$$p\equiv e^{ikd\sin\theta},$$

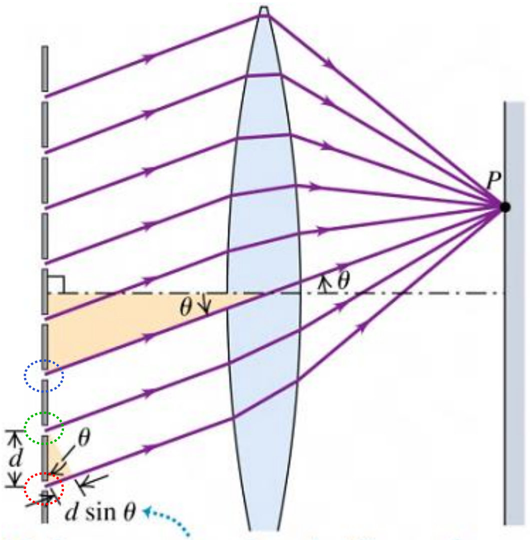
$$\beta\equiv\frac{2\pi}{\lambda}a\sin\theta,$$

$$\alpha\equiv\frac{\pi}{\lambda}d\sin\theta$$

$$\boxed{I(\theta)=I_0\left(\frac{\sin\beta/2}{\beta/2}\right)^2\left(\frac{\sin(N\alpha)}{\sin\alpha}\right)^2}$$

Single Slit Diffraction

***N*-Slit Interference**



Maxima occur where the path difference for adjacent slits is a whole number of wavelengths: $d\sin\theta = m\lambda$.

$$\left(\frac{\sin(N\alpha)}{\sin\alpha}\right)^2\xrightarrow{\alpha=m\pi}N^2\Rightarrow\boxed{\text{Maximum occurs when } d\sin\theta=m\lambda}$$

N-Slit interference + Diffraction (slits have width a)

N-Slit Interference

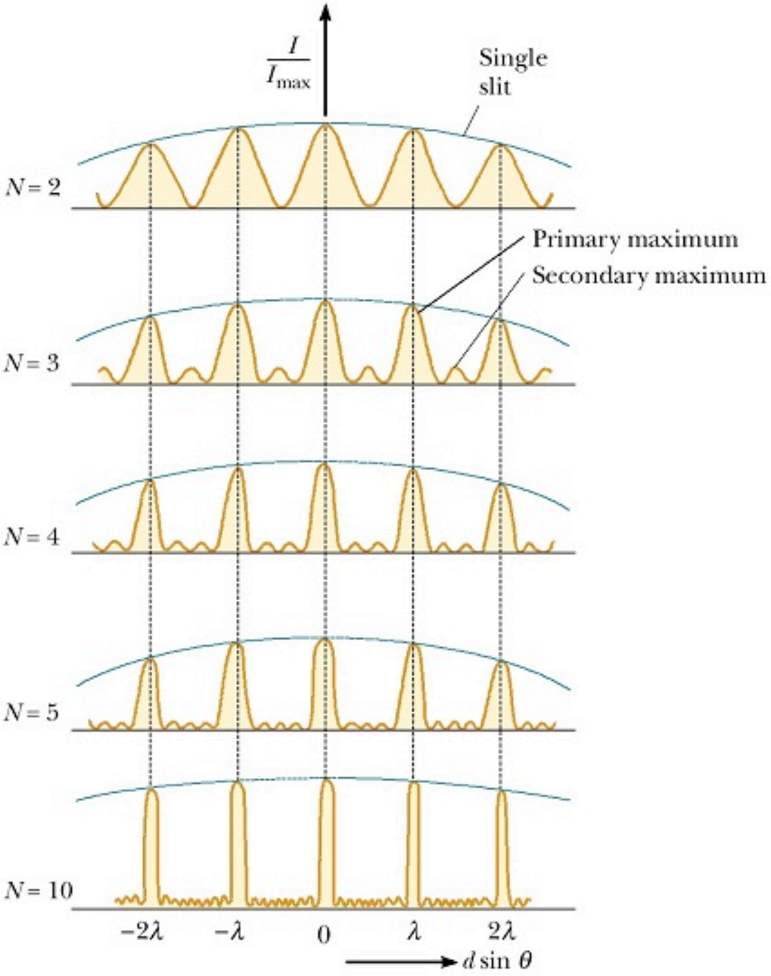
$$I(\theta)=I_0\left(\frac{\sin\beta/2}{\beta/2}\right)^2\left(\frac{\sin(N\alpha)}{\sin\alpha}\right)^2$$

Single Slit Diffraction

$$\beta\equiv\frac{2\pi}{\lambda}a\sin\theta,$$

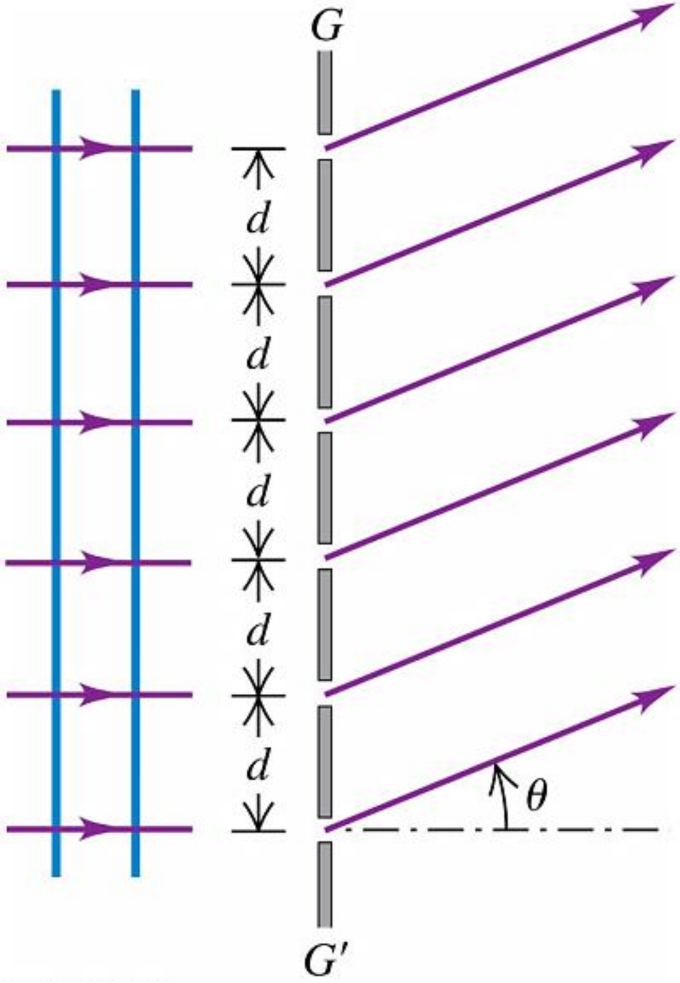
$$\alpha\equiv\frac{\pi}{\lambda}d\sin\theta$$

$$d>a$$



Diffraction Gratings

- **An array of a large number of parallel slits, all with the same width *a* and spaced at equal distances *d* between centers, is called a diffraction gratings.**
- ***e.g.*, a transmission diffraction gratings →**



Diffraction Grating

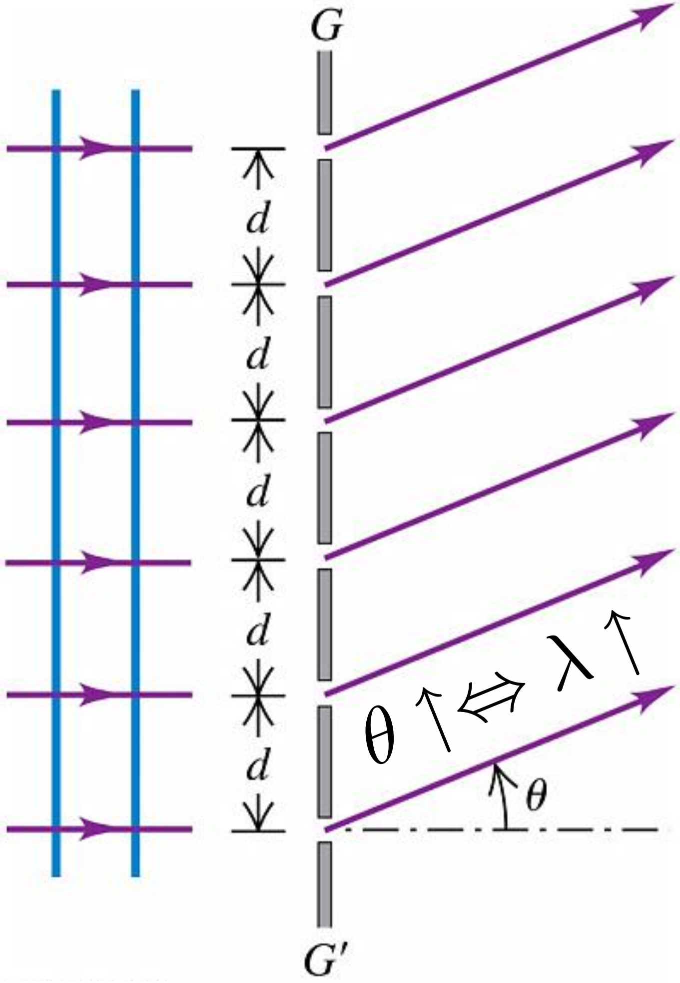
- Assume far-field conditions.
- Principal intensity maxima with multiple slits occur in the same directions as for the two-slit pattern.
- Positions of the maxima are:

$$d\sin\theta = m\lambda.$$

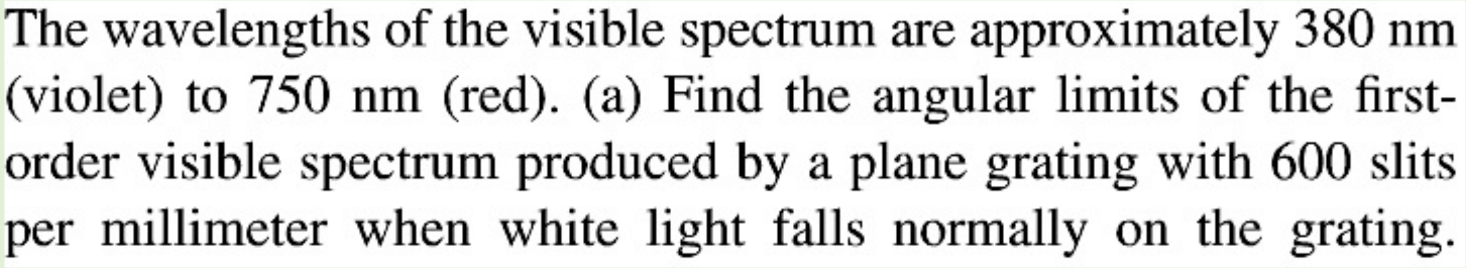
$$(m = 0, \pm1, \pm2, \pm3, \ \ldots)$$

First-order lines

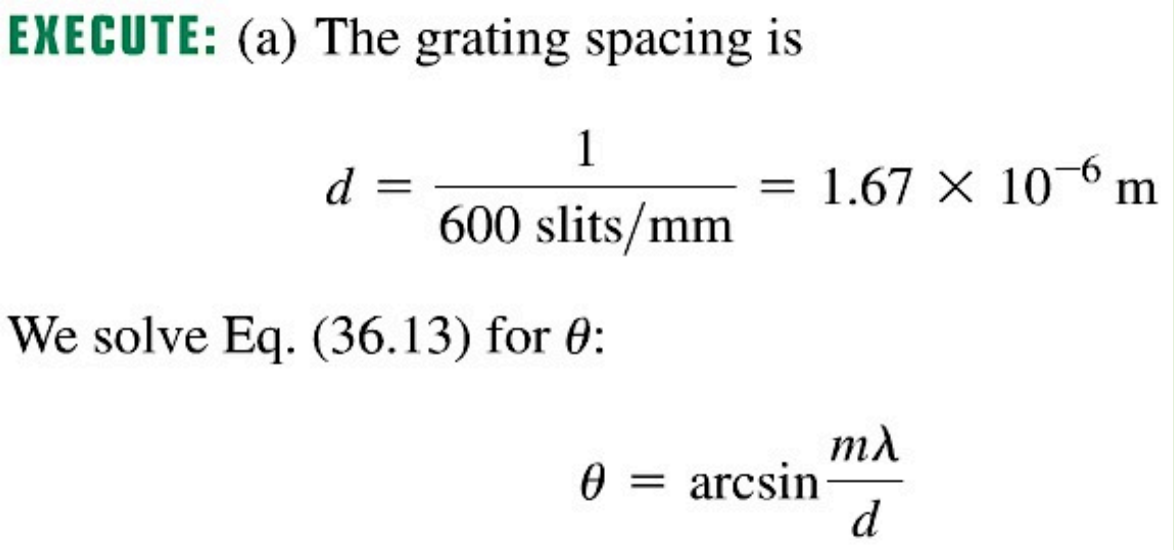
Second-order lines



Example 36.4
Width of a grating spectrum



The wavelengths of the visible spectrum are approximately 380 nm (violet) to 750 nm (red). (a) Find the angular limits of the first-order visible spectrum produced by a plane grating with 600 slits per millimeter when white light falls normally on the grating.



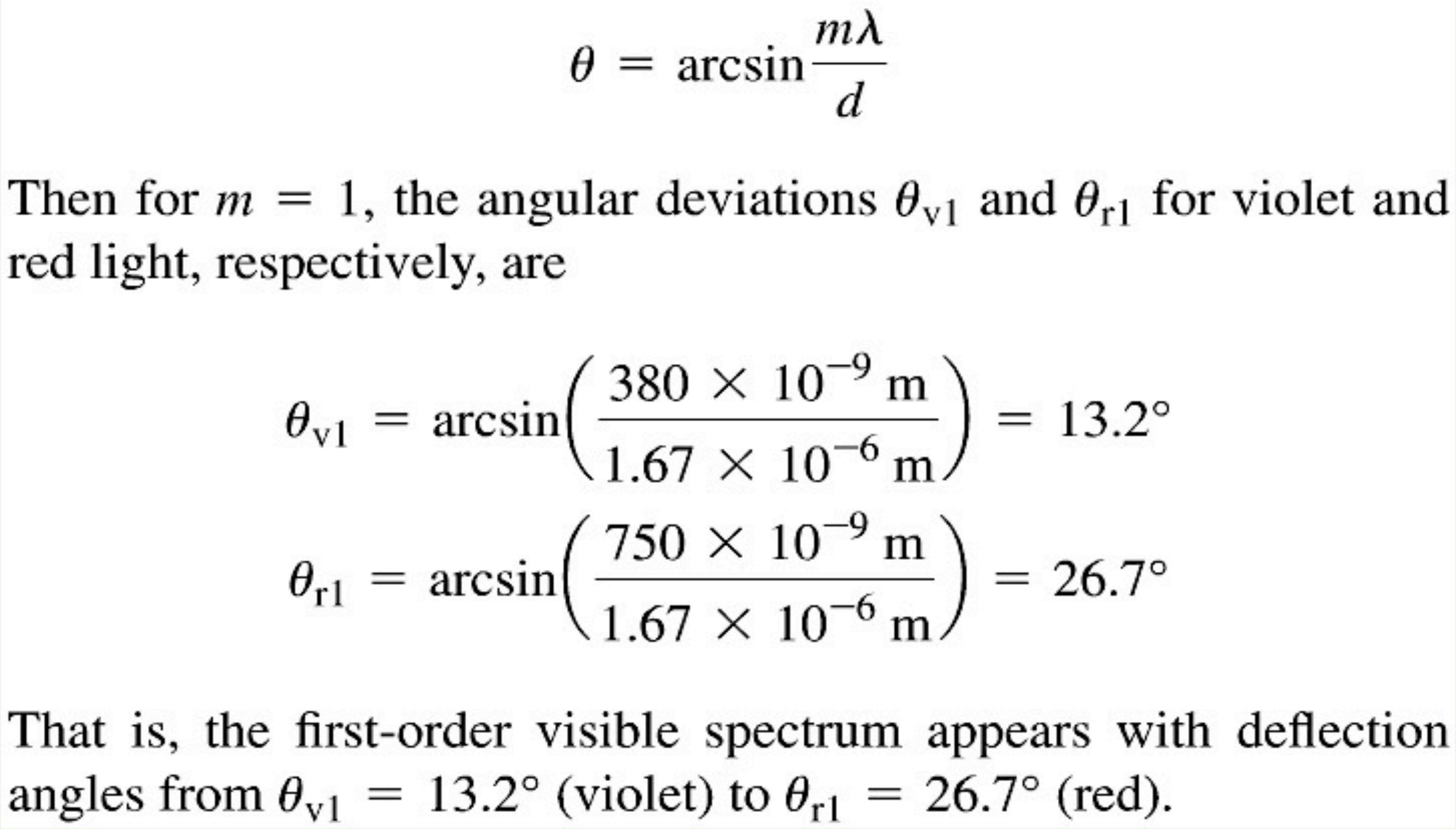
EXECUTE: (a) The grating spacing is

$$d = \frac{1}{600 \text{ slits/mm}} = 1.67 \times 10^{-6} \text{ m}$$

We solve Eq. (36.13) for θ:

$$\theta = \arcsin\frac{m\lambda}{d}$$

Example 36.4
Width of a grating spectrum



$$\theta = \arcsin\frac{m\lambda}{d}$$

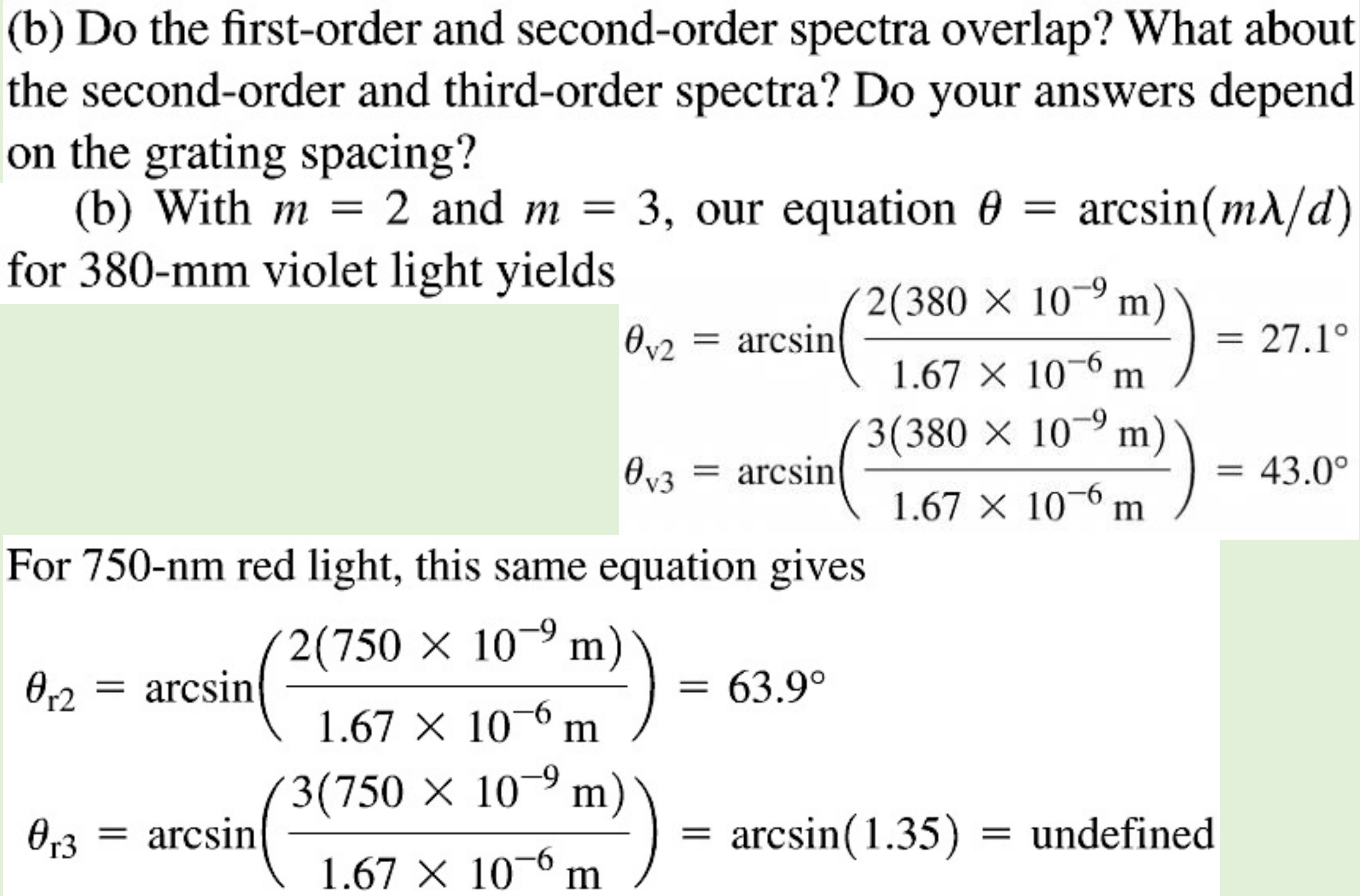
Then for $m = 1$, the angular deviations $\theta_{\mathrm{v}1}$ and $\theta_{\mathrm{r}1}$ for violet and red light, respectively, are

$$\theta_{\mathrm{v}1} = \arcsin\left(\frac{380 \times 10^{-9}\ \mathrm{m}}{1.67 \times 10^{-6}\ \mathrm{m}}\right) = 13.2^\circ$$

$$\theta_{\mathrm{r}1} = \arcsin\left(\frac{750 \times 10^{-9}\ \mathrm{m}}{1.67 \times 10^{-6}\ \mathrm{m}}\right) = 26.7^\circ$$

That is, the first-order visible spectrum appears with deflection angles from $\theta_{\mathrm{v}1} = 13.2^\circ$ (violet) to $\theta_{\mathrm{r}1} = 26.7^\circ$ (red).

Example 36.4
Width of a grating spectrum



(b) Do the first-order and second-order spectra overlap? What about the second-order and third-order spectra? Do your answers depend on the grating spacing?

(b) With $m = 2$ and $m = 3$, our equation $\theta = \arcsin(m\lambda/d)$ for 380-mm violet light yields

$$\theta_{v2} = \arcsin\left(\frac{2(380 \times 10^{-9}\ \text{m})}{1.67 \times 10^{-6}\ \text{m}}\right) = 27.1°$$

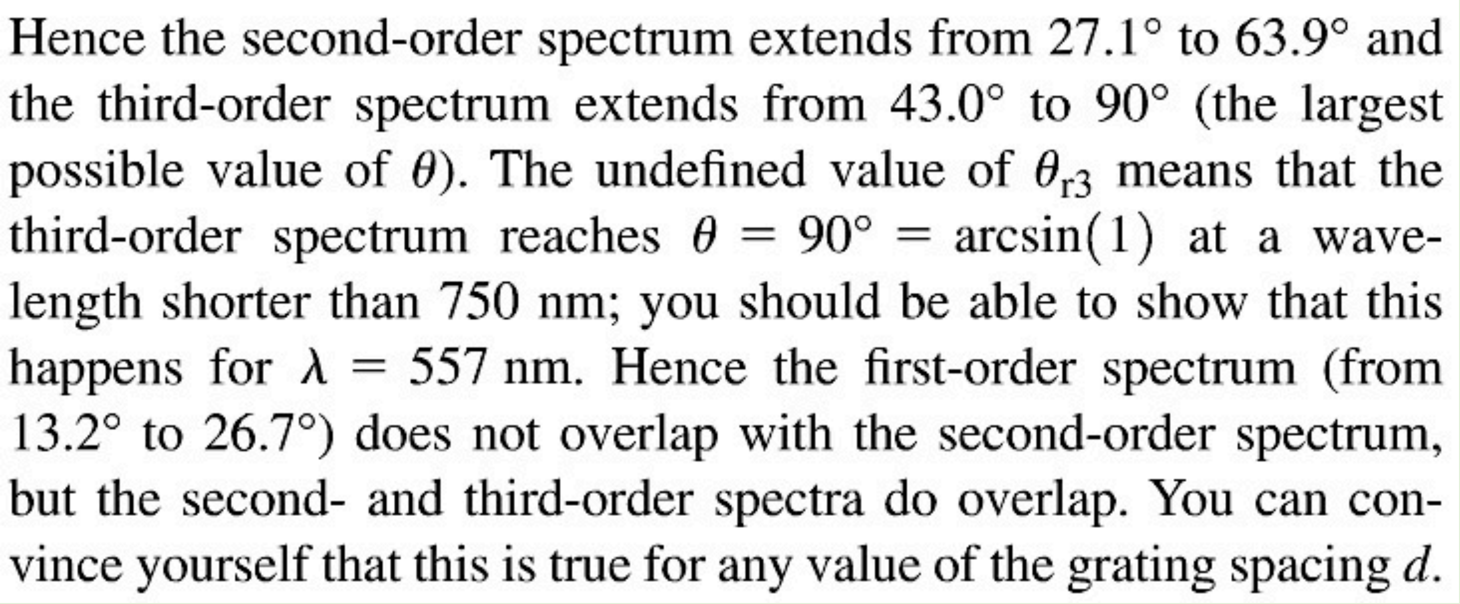
$$\theta_{v3} = \arcsin\left(\frac{3(380 \times 10^{-9}\ \text{m})}{1.67 \times 10^{-6}\ \text{m}}\right) = 43.0°$$

For 750-nm red light, this same equation gives

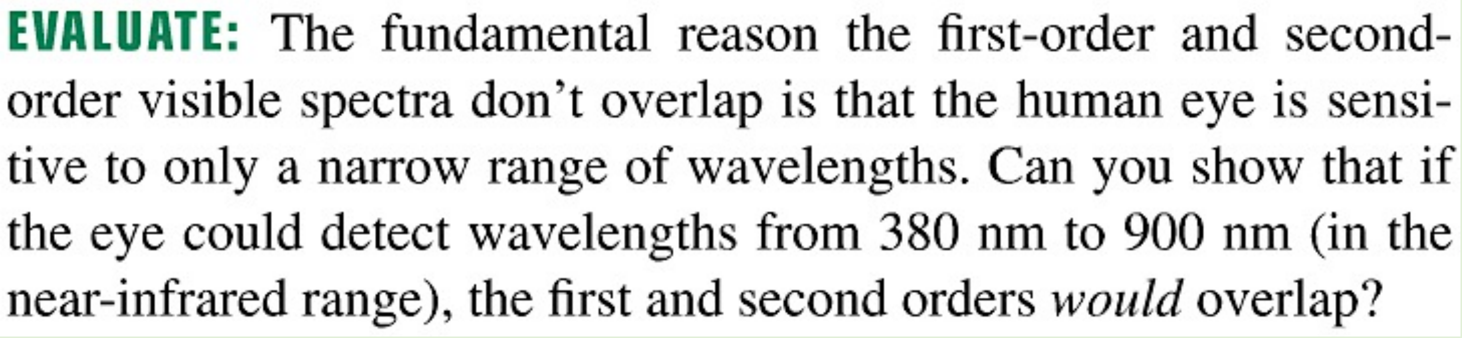
$$\theta_{r2} = \arcsin\left(\frac{2(750 \times 10^{-9}\ \text{m})}{1.67 \times 10^{-6}\ \text{m}}\right) = 63.9°$$

$$\theta_{r3} = \arcsin\left(\frac{3(750 \times 10^{-9}\ \text{m})}{1.67 \times 10^{-6}\ \text{m}}\right) = \arcsin(1.35) = \text{undefined}$$

Example 36.4
Width of a grating spectrum



Hence the second-order spectrum extends from 27.1° to 63.9° and the third-order spectrum extends from 43.0° to 90° (the largest possible value of θ). The undefined value of θ_{r3} means that the third-order spectrum reaches $\theta = 90° = \arcsin(1)$ at a wavelength shorter than 750 nm; you should be able to show that this happens for $\lambda = 557$ nm. Hence the first-order spectrum (from 13.2° to 26.7°) does not overlap with the second-order spectrum, but the second- and third-order spectra do overlap. You can convince yourself that this is true for any value of the grating spacing d.



EVALUATE: The fundamental reason the first-order and second-order visible spectra don't overlap is that the human eye is sensitive to only a narrow range of wavelengths. Can you show that if the eye could detect wavelengths from 380 nm to 900 nm (in the near-infrared range), the first and second orders *would* overlap?

Grating Spectrographs

- **Diffraction gratings are widely used to measure the spectrum of light emitted by a source, a process called spectroscopy or spectrometry.**
- **Light incident on a gratings of know spacing is dispersed into a spectrum. The angles of deviation of the maxima are then measured, and the wavelength is computed from the equation:**

$$d \sin \theta = m\lambda.$$

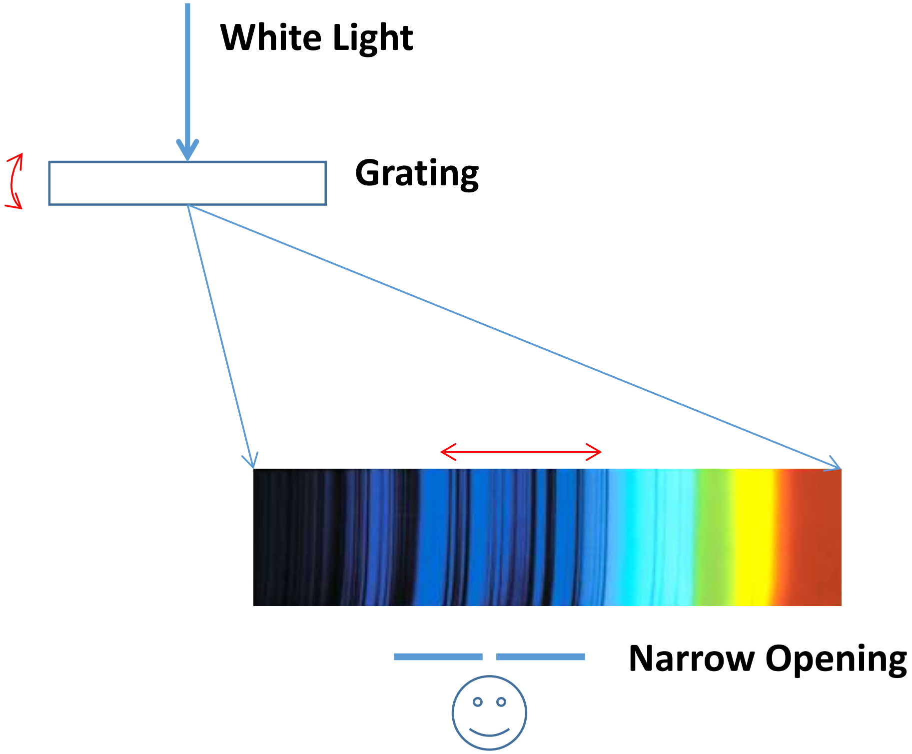
$$(m = 0,\ \ \pm 1,\ \ \pm 2,\ \ \pm 3,\ \ ...)$$

Grating Spectrographs

- **Sunlight is dispersed into a spectrum by a diffraction grating. Specific wavelengths are absorbed as sunlight passes through the sun’s atmosphere, leaving dark lines in the spectrum:**

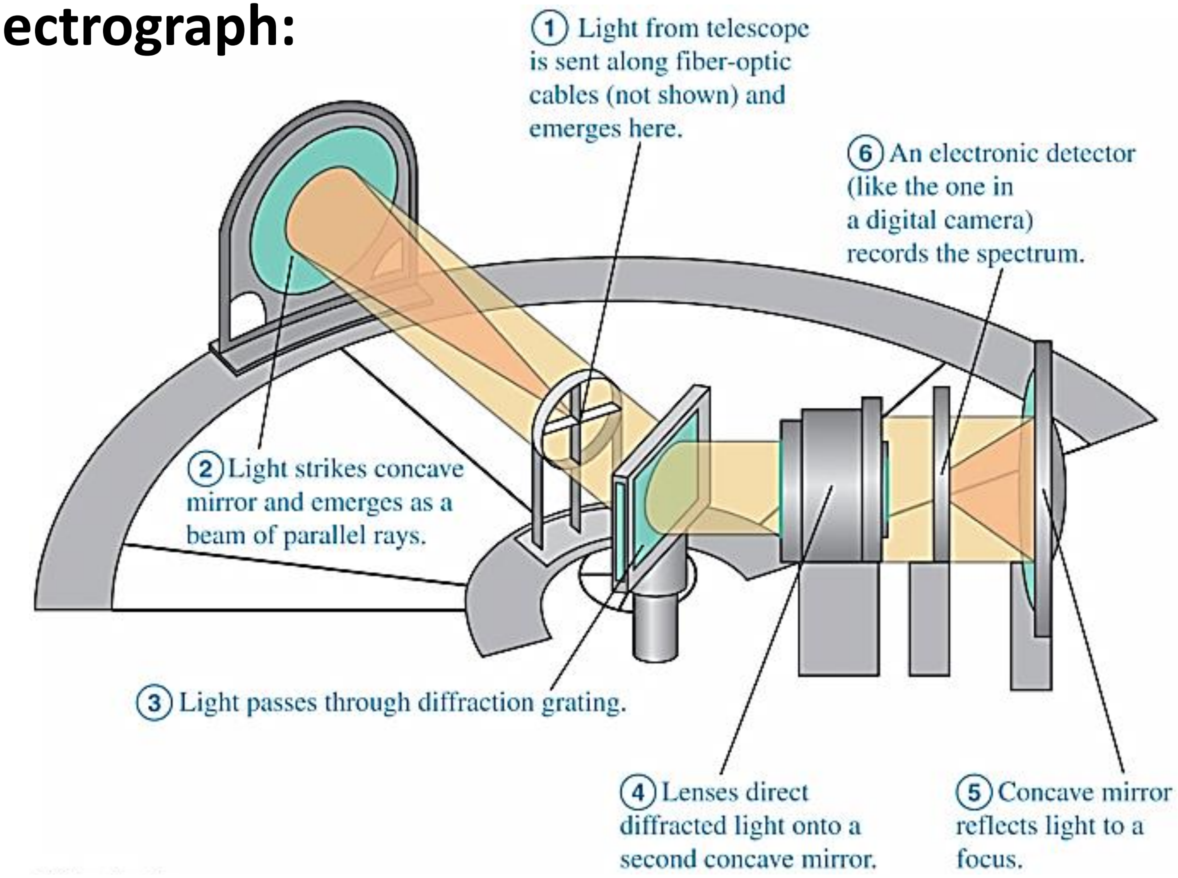
Grating Spectrographs

- **A schematic diagram of a diffraction-grating spectrograph:**



Grating Spectrographs

- **A schematic diagram of a diffraction-grating spectrograph:**



Resolution of a Grating Spectrograph

- **In spectroscopy it is often important to distinguish slightly different wavelengths.**
- **The minimum wavelength difference $\Delta\lambda$ that can be distinguished by a spectrograph is described by the chromatic resolving power *R*:**

$$R = \frac{\lambda}{\Delta\lambda}.$$

- **For example, when sodium atoms are heated, they emit strongly at the yellow wavelengths 589.00 nm and 589.59 nm. A spectrograph that can barely distinguish these two lines (sodium doublet) has *R* = (589.00 nm)/(0.59 nm) = 1000.**

Resolution of a Grating Spectrograph

- **The *m*-th order maximum occurs at:**

$$d\sin\theta = m\lambda.$$

- **For *N* slits, the first minimum beside the maximum occurs at:**

$$N\alpha = (mN+1)\lambda \Rightarrow d\sin(\theta+\Delta\theta) = m\lambda + \lambda / N$$

- **Angular width *Δθ* of maximum peak is**

$$(d\cos\theta)\Delta\theta = \lambda / N$$

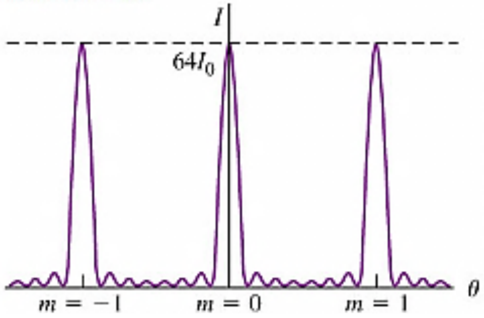
$$\boxed{\Delta\theta = \frac{\lambda}{Nd\cos\theta}}$$

- **Larger *N* leads to narrower peak width**

$$I(\theta) = I_0\left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin(N\alpha)}{\sin\alpha}\right)^2$$

$$\alpha \equiv \frac{\pi}{\lambda} d\sin\theta$$

(b) $N = 8$: eight slits produce taller, narrower maxima in the same locations, separated by seven minima.



Resolution of a Grating Spectrograph

- **Two different wavelengths give diffraction maxima at slightly different angles.**
- **Assume that we can distinguish them as two separate peaks if the maximum of one coincides with the first minimum of the other.**
- **Suppose the maximum for wavelength λ+$\Delta\lambda$ is at θ. Then**

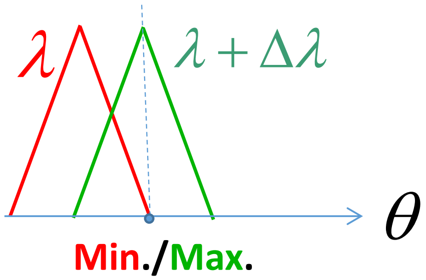
$$d\sin\theta = m(\lambda + \Delta\lambda)$$

- **If it is also the minimum for wavelength, then**

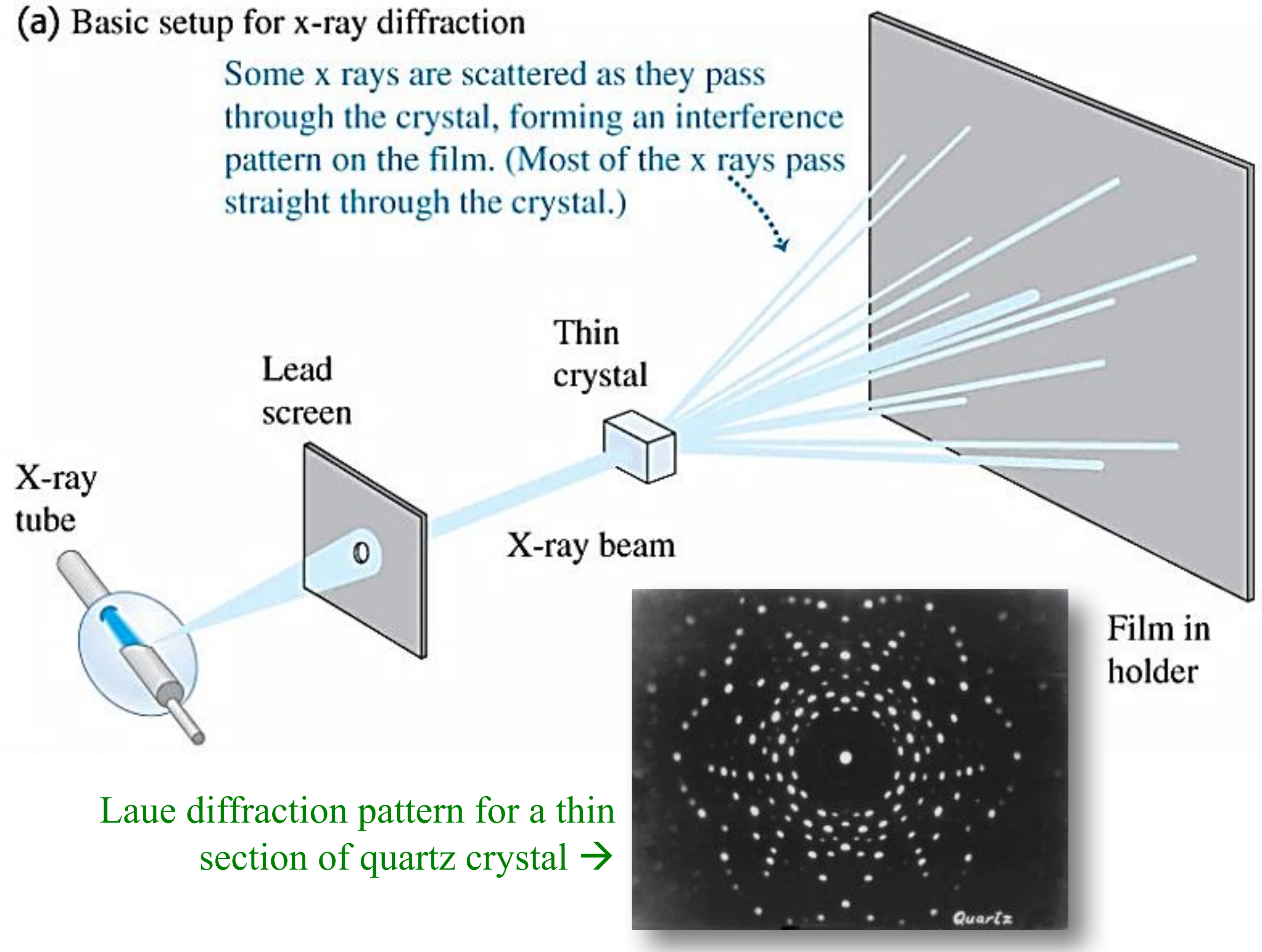
$$d\sin\theta = m\lambda + \lambda / N$$

- **Combining the two equations:**

$$\boxed{R \equiv \frac{\lambda}{\Delta\lambda} = mN.}$$



X-Ray Diffraction (3-Dimensional Diffraction)

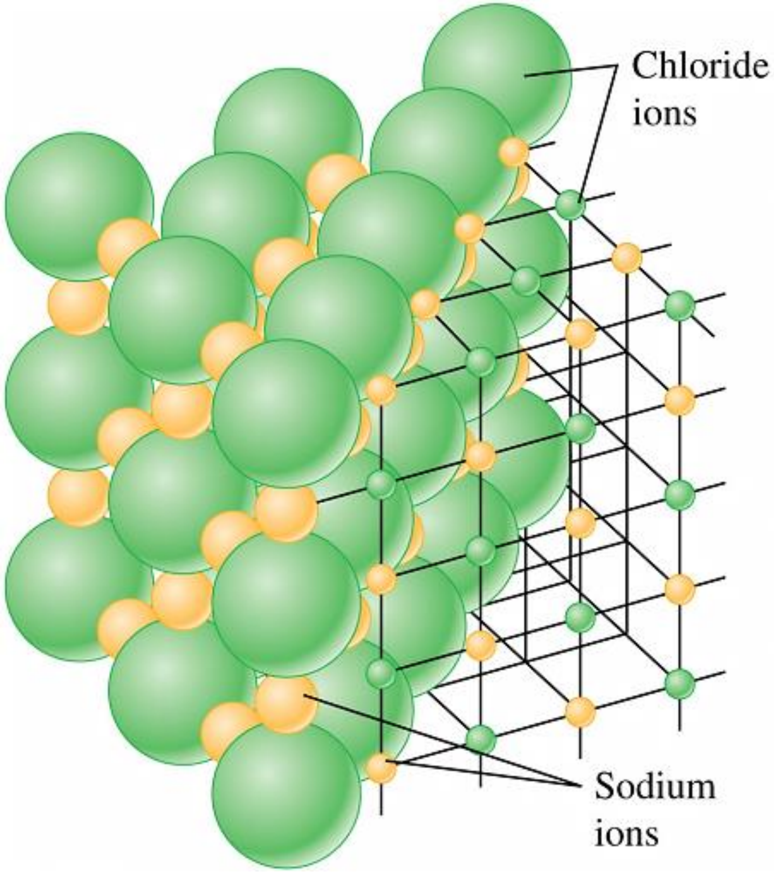

(a) Basic setup for x-ray diffraction

Some x rays are scattered as they pass through the crystal, forming an interference pattern on the film. (Most of the x rays pass straight through the crystal.)

Laue diffraction pattern for a thin section of quartz crystal →

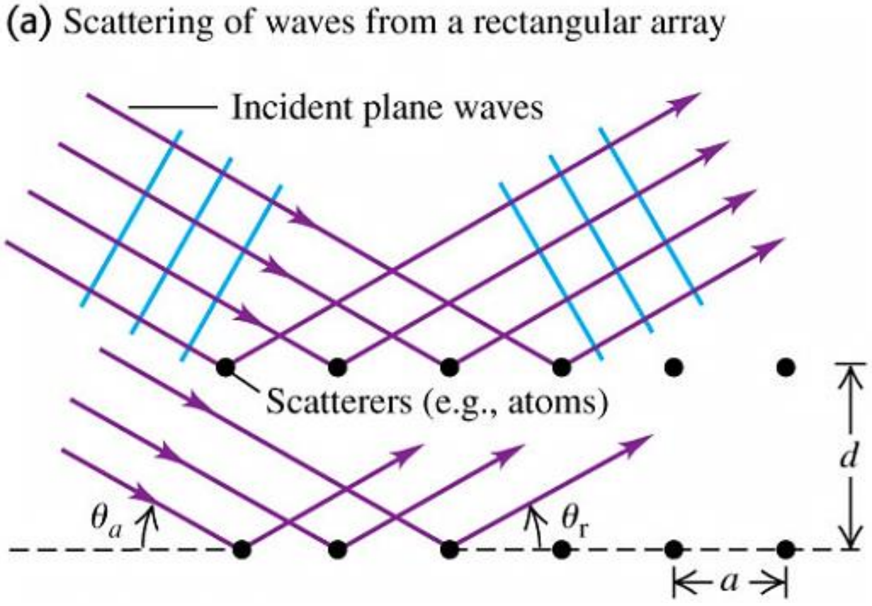
X-Ray Diffraction

- **These experiments verified that x rays are waves (have wavelike properties), and that the atoms in a crystal are arranged in a regular pattern.**

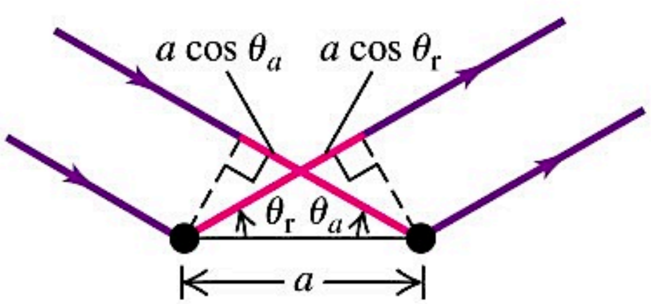


A Simple Model of X-Ray Diffraction

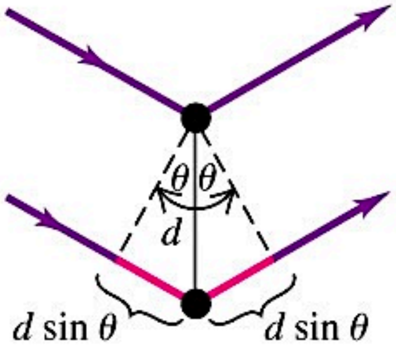
- **Each atom acts as a new point source of X-ray**
- **The resulting interference pattern is the superposition of all the scattered waves.**
- **The scattered waves are not all in phase because their distances from the source are different.**



A Simple Model of X-Ray Diffraction



(b) Scattering from adjacent atoms in a row
Interference from adjacent atoms in a row is constructive when the path lengths $a \cos \theta_a$ and $a \cos \theta_r$ are equal, so that the angle of incidence θ_a equals the angle of reflection (scattering) θ_r.



(c) Scattering from atoms in adjacent rows
Interference from atoms in adjacent rows is constructive when the path difference $2d \sin \theta$ is an integral number of wavelengths, as in Eq. (36.16).

- **Note the angles here are measured relative to the atomic plane (not from the surface of the sample).**

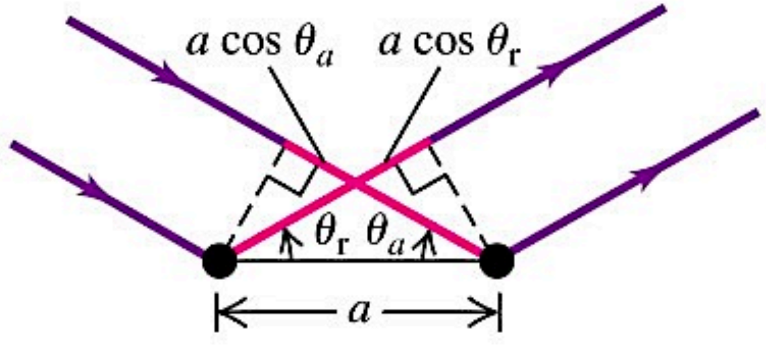
A Simple Model of X-Ray Diffraction

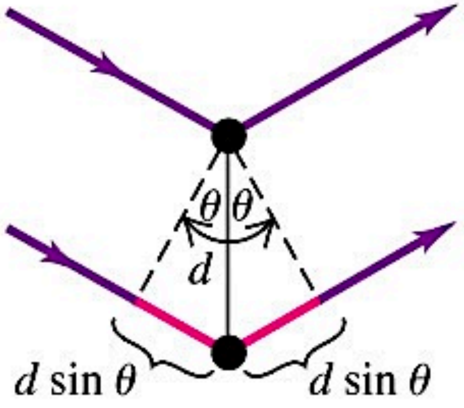
- **The conditions for radiation from the entire array to reach the observer in phase are:**

(1) **angle of incidence = angle of scattering, and**

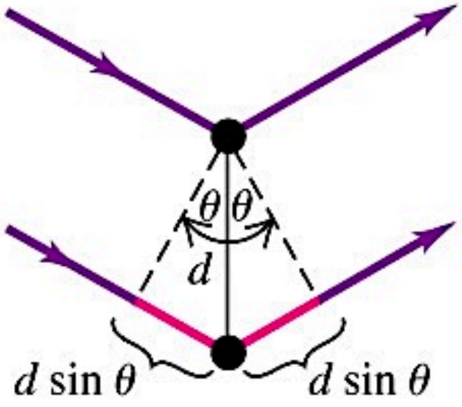
(2) **path difference for adjacent rows =** $m\lambda$**:**

$$\underbrace{2d\sin\theta = m\lambda \quad (m = 0,\ 1,\ 2,\ 3,\ \ldots)}_{\text{Bragg condition for constructive interference from an array.}}$$





A Simple Model of X-Ray Diffraction



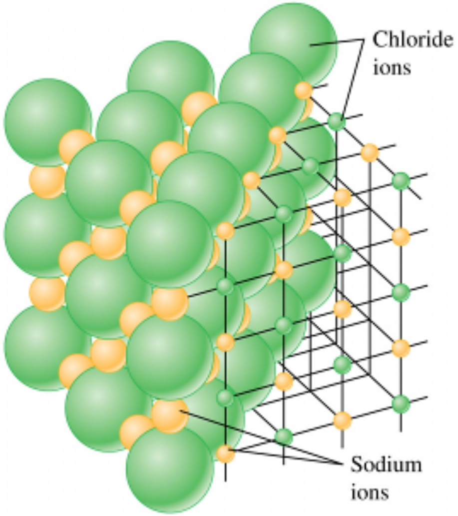
- **To have constructive interference:**

$$2d\sin\theta = m\lambda \Rightarrow \frac{m\lambda}{2d} = \sin\theta < 1$$

$$\Rightarrow m\lambda < 2d \Rightarrow \lambda < \frac{2d}{m}.$$

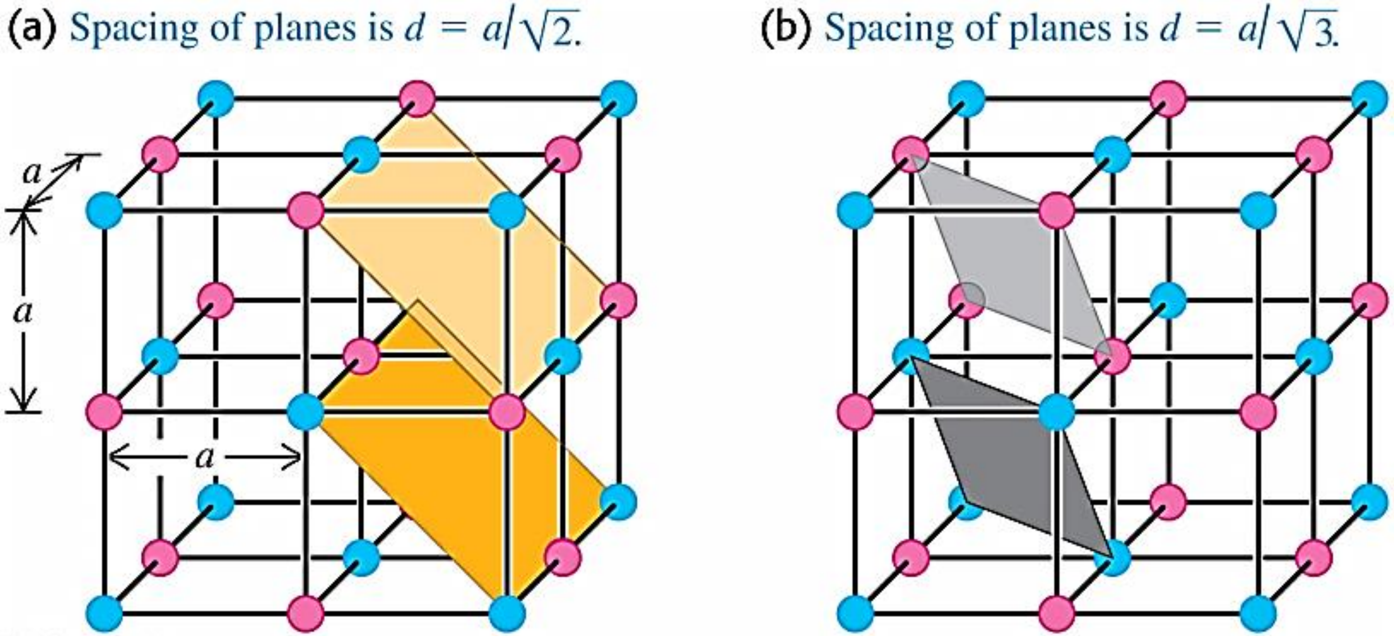
- ***e.g.*, for NaCl crystal:**

$$\underbrace{\begin{aligned} \lambda_{m=1} &< \frac{2d}{m} = \frac{2(0.282\ \text{nm})}{1} = 0.564\ \text{nm} \\ \lambda_{m=2} &< \frac{2d}{m} = \frac{2(0.282\ \text{nm})}{2} = 0.282\ \text{nm} \\ \lambda_{m=3} &< \frac{2d}{m} = \frac{2(0.282\ \text{nm})}{3} = 0.188\ \text{nm} \end{aligned}}_{\text{x ray wavelengths}}$$



A Simple Model of X-Ray Diffraction

- **2D → 3D (array): Rows → planes (of scatterers).**
- **(a) & (b) A cubic crystal and two different families of crystal planes. There are also three sets of planes parallel to the cube faces, with spacing *a*:**

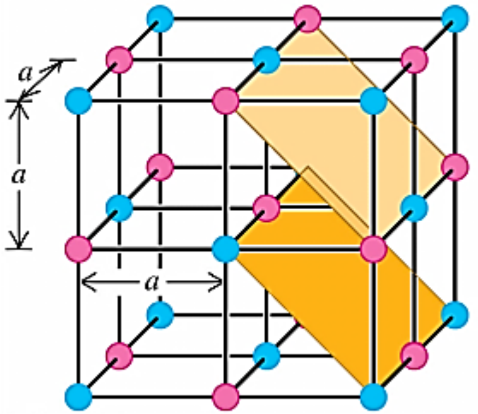

(a) Spacing of planes is $d = a/\sqrt{2}$.

(b) Spacing of planes is $d = a/\sqrt{3}$.

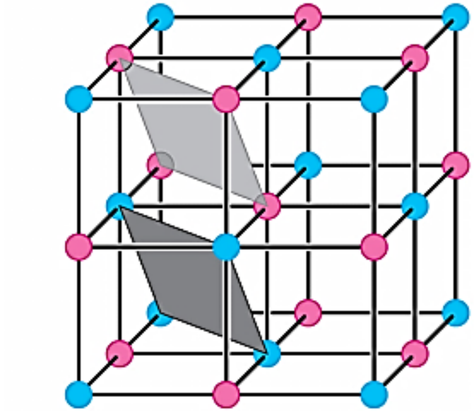
A Simple Model of X-Ray Diffraction

- **Bragg reflection: Because there are many different sets of parallel planes, there are also many values of *d* and many sets of angles that give constructive interference for the whole crystal lattice.**
- **Bragg condition:** $2d\sin\theta = m\lambda.$
- **X-ray diffraction is by far the most important experimental tool in the investigation of crystal structure of solids.**

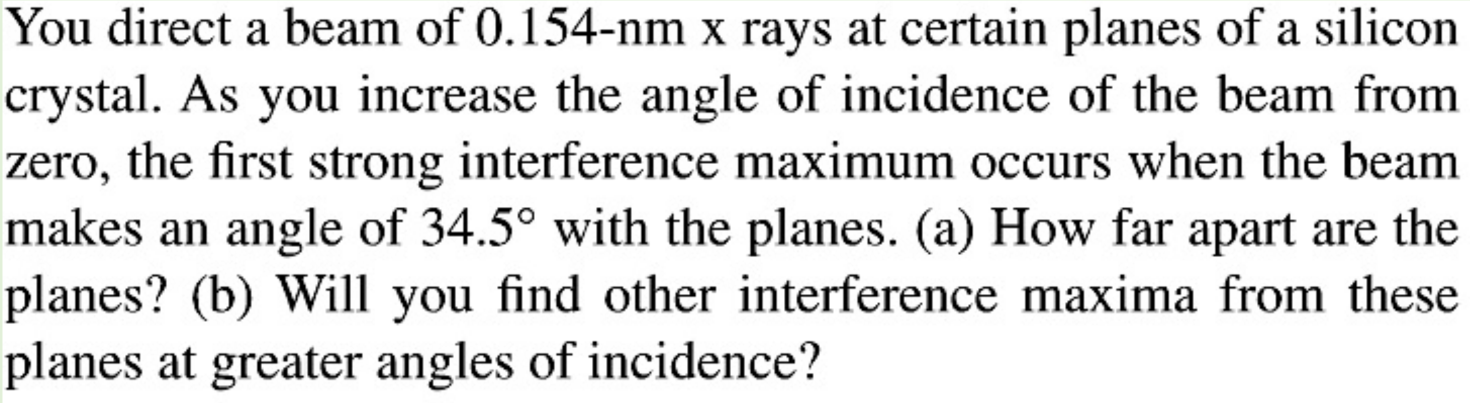
(a) Spacing of planes is $d = a/\sqrt{2}.$



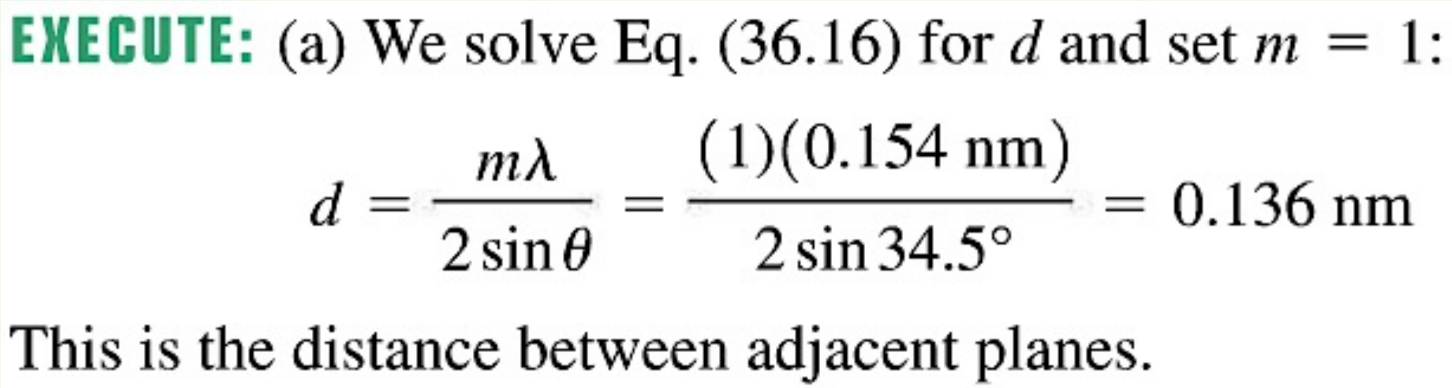
(b) Spacing of planes is $d = a/\sqrt{3}.$



Example 36.5
X-ray diffraction



You direct a beam of 0.154-nm x rays at certain planes of a silicon crystal. As you increase the angle of incidence of the beam from zero, the first strong interference maximum occurs when the beam makes an angle of 34.5° with the planes. (a) How far apart are the planes? (b) Will you find other interference maxima from these planes at greater angles of incidence?

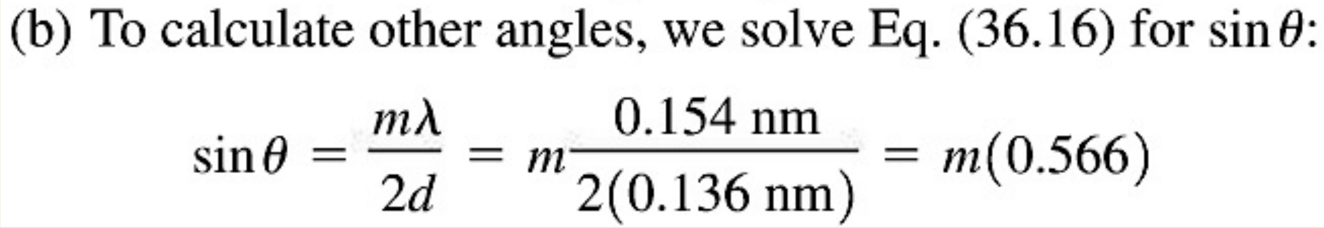


EXECUTE: (a) We solve Eq. (36.16) for d and set $m = 1$:

$$d = \frac{m\lambda}{2\sin\theta} = \frac{(1)(0.154\text{ nm})}{2\sin 34.5^\circ} = 0.136\text{ nm}$$

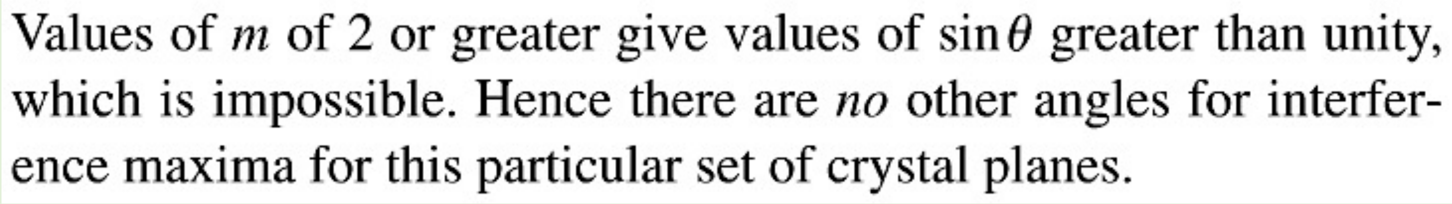
This is the distance between adjacent planes.

Example 36.5
X-ray diffraction

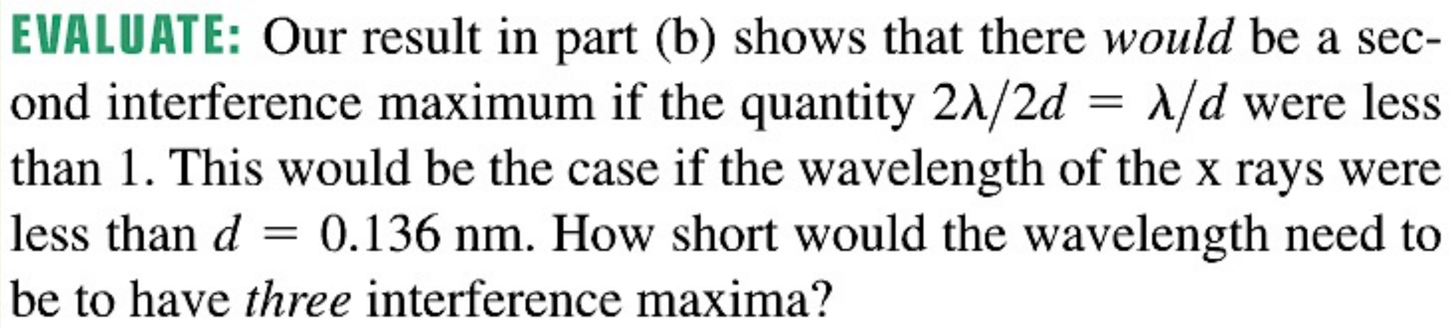


(b) To calculate other angles, we solve Eq. (36.16) for $\sin\theta$:

$$\sin\theta = \frac{m\lambda}{2d} = m\frac{0.154\ \text{nm}}{2(0.136\ \text{nm})} = m(0.566)$$



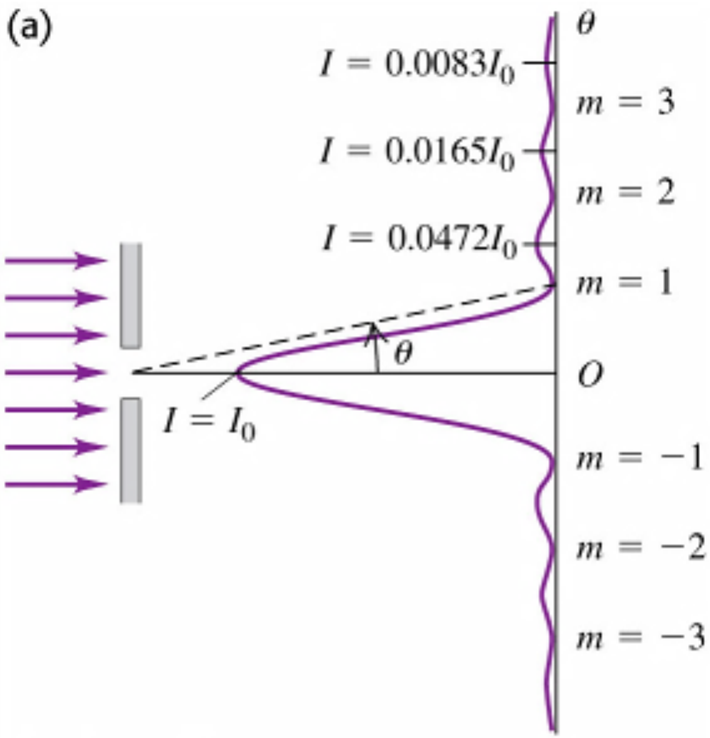
Values of m of 2 or greater give values of $\sin\theta$ greater than unity, which is impossible. Hence there are *no* other angles for interference maxima for this particular set of crystal planes.



EVALUATE: Our result in part (b) shows that there *would* be a second interference maximum if the quantity $2\lambda/2d = \lambda/d$ were less than 1. This would be the case if the wavelength of the x rays were less than $d = 0.136$ nm. How short would the wavelength need to be to have *three* interference maxima?

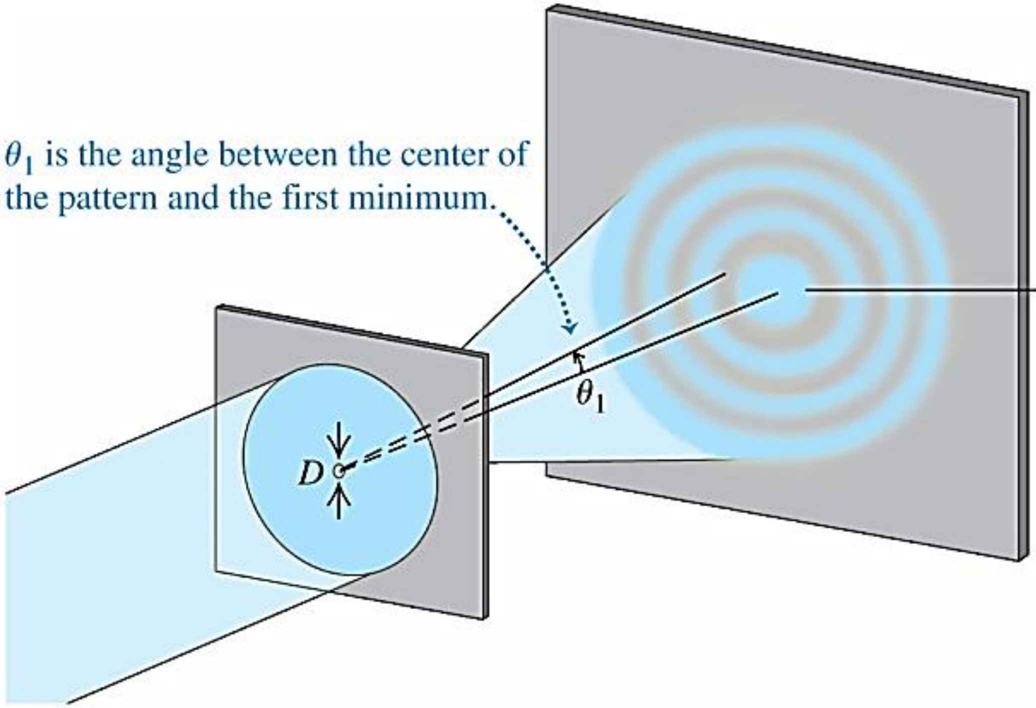
Diffraction from a single slit

$$I = I_0\left[\frac{\sin(\beta/2)}{(\beta/2)}\right]^2.$$

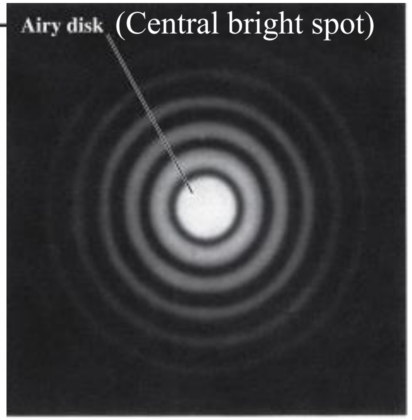


Circular Apertures and Resolving Power

- **The diffraction pattern formed by a circular aperture consists of a central bright spot surrounded by a series of bright and dark rings.**



The intensities of the bright rings drop off very quickly with increasing angle:



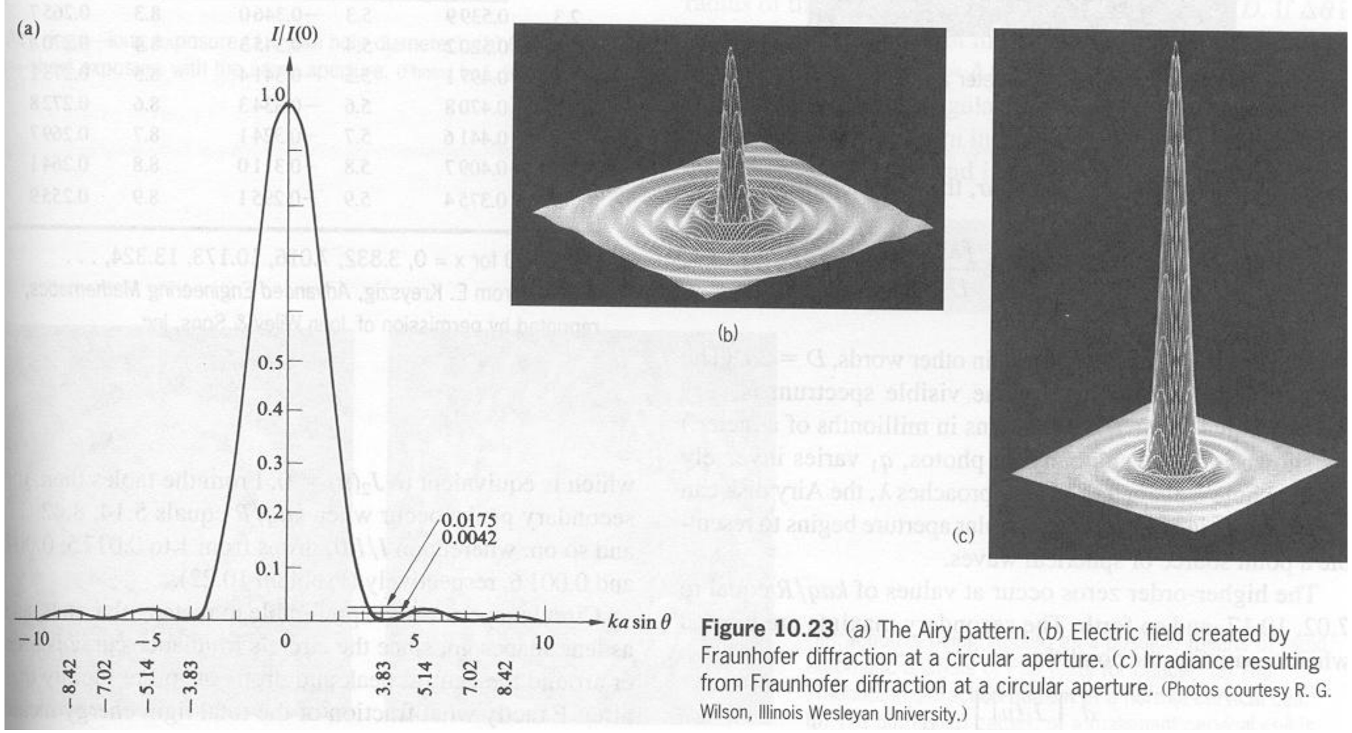


Figure 10.23 (*a*) The Airy pattern. (*b*) Electric field created by Fraunhofer diffraction at a circular aperture. (*c*) Irradiance resulting from Fraunhofer diffraction at a circular aperture. (Photos courtesy R. G. Wilson, Illinois Wesleyan University.)

Circular Apertures and Resolving Power

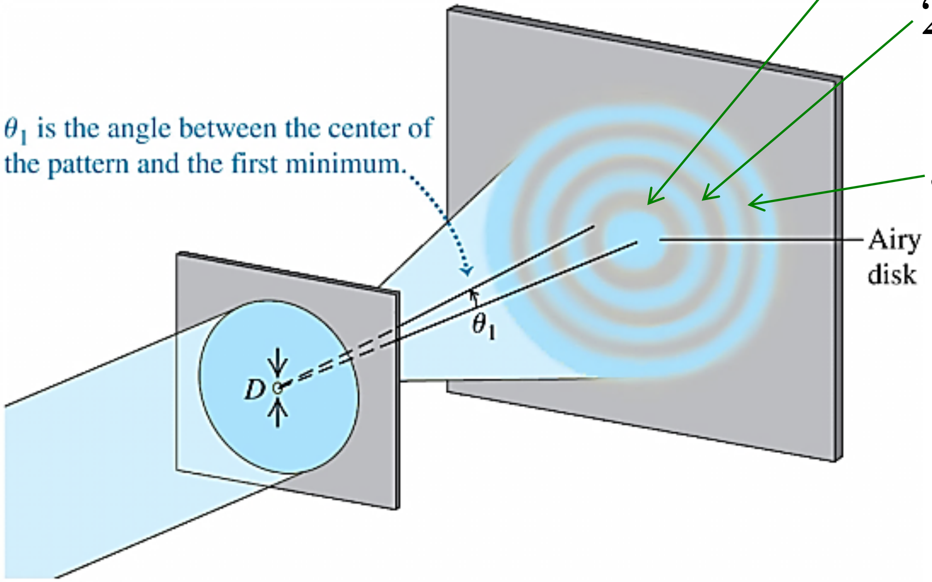
- **If the aperture diameter is *D* and the wavelength is λ, the angular radius θ_i of the *i*-th dark ring is:**

The angular radius of the Airy disk is that of the first dark ring: θ_1

$$\text{1st:} \quad \sin\theta_1 = 1.22\frac{\lambda}{D}$$

$$\text{2nd:} \quad \sin\theta_2 = 2.23\frac{\lambda}{D}$$

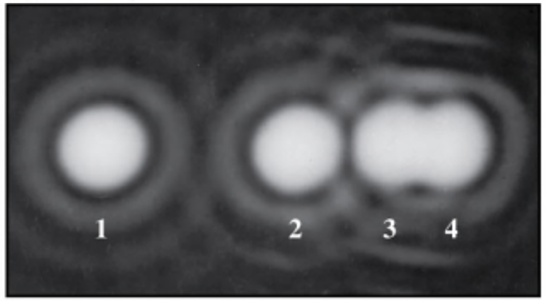
$$\text{3rd:} \quad \sin\theta_3 = 3.24\frac{\lambda}{D}$$



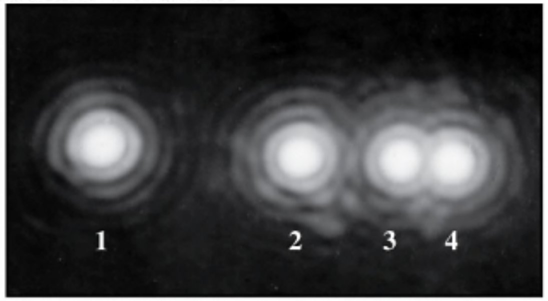
Diffraction and Image Resolution

- **If we have two point objects, their images are not two points but two diffraction patterns.**
- **When the objects are close enough, their diffraction patterns overlap, almost completely and cannot be distinguished.**
- **Larger aperture diameter ➔ smaller Airy disks ➔ better resolved.**

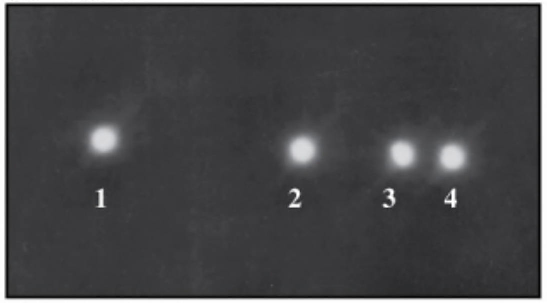
(a) Small aperture



(b) Medium aperture



(c) Large aperture

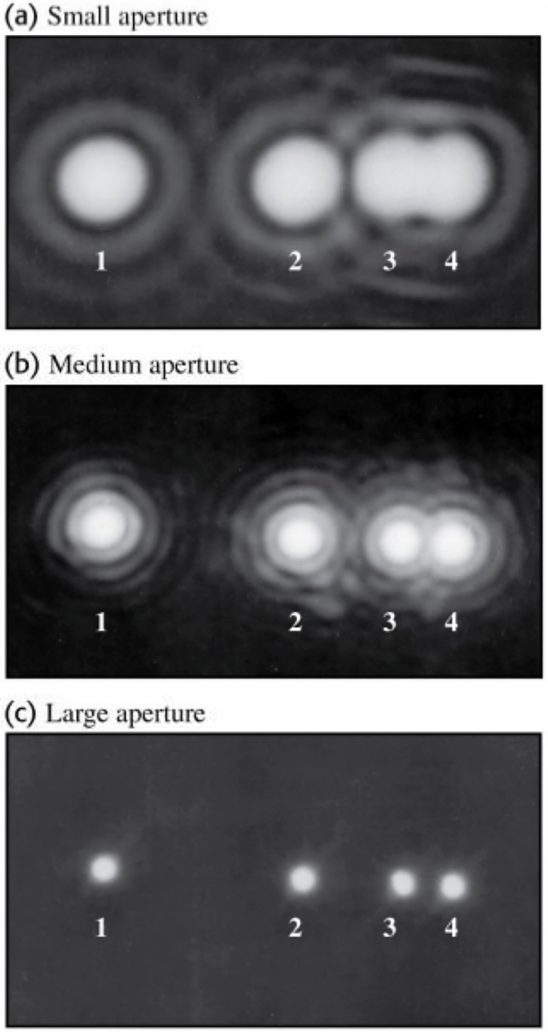


Diffraction and Image Resolution

- **Rayleigh's criterion: Two point objects are just barely resolved (*i.e.*, distinguishable) if the center of one diffraction pattern coincides with the first minimum of the other.**
- **In that case, the angular separation of the image centers is given by:**

$$\sin\theta_1 = 1.22\frac{\lambda}{D}$$

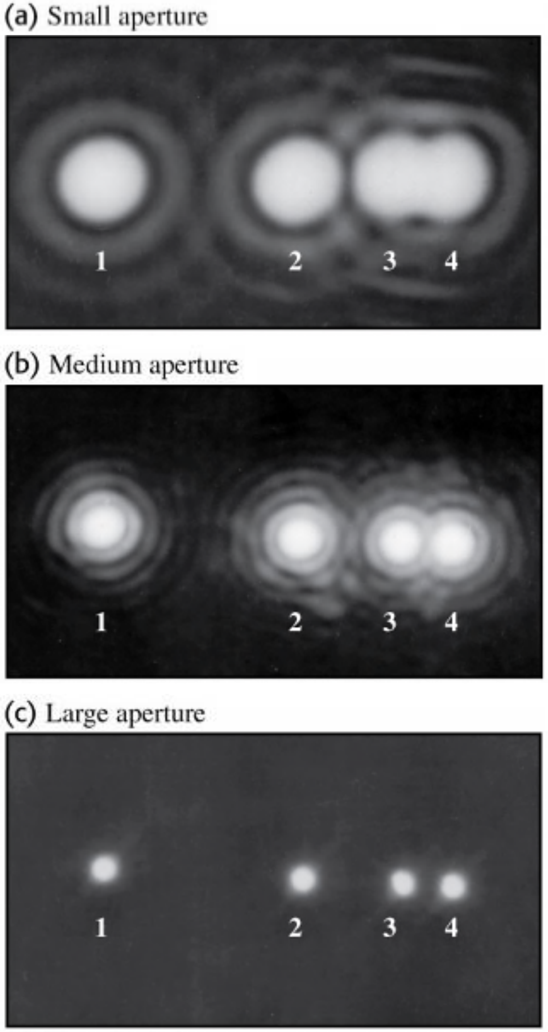
(diffraction by a circular aperture)



Diffraction and Image Resolution

- **The minimum separation of two objects that can just be resolved by an optical instrument is called the limit of resolution of the instrument.**
- **The smaller the limit of resolution, the greater the resolution, or resolving power, of the instrument.**
- **Resolution (resolving power)↑ improves with larger diameter and shorter wavelengths.**

$$\downarrow \sin\theta_1 = 1.22\frac{\lambda\downarrow}{D\uparrow}$$

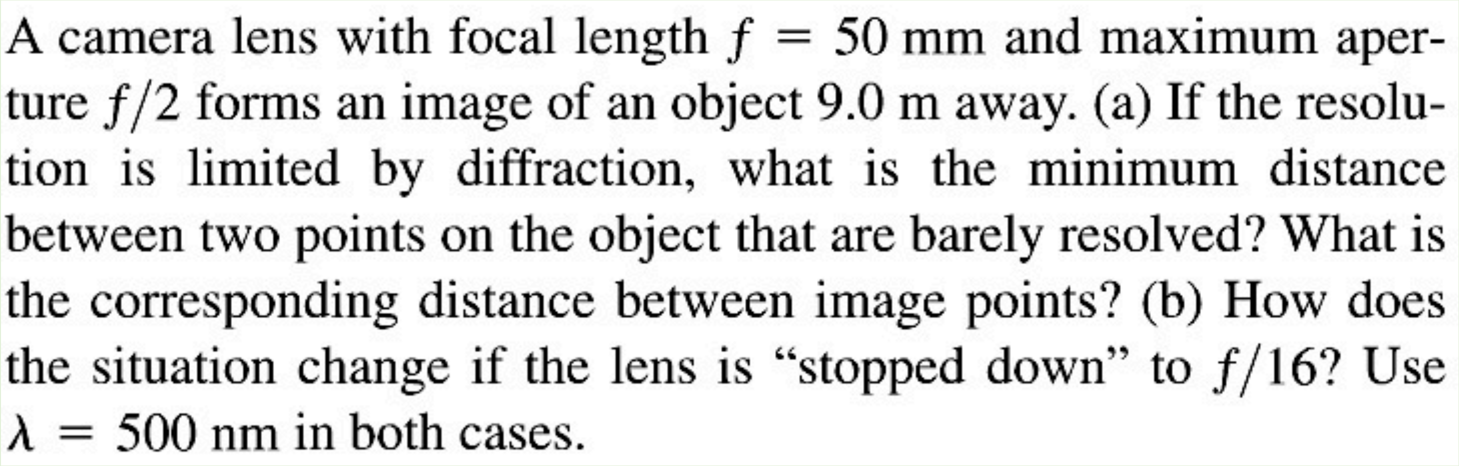


Diffraction and Image Resolution

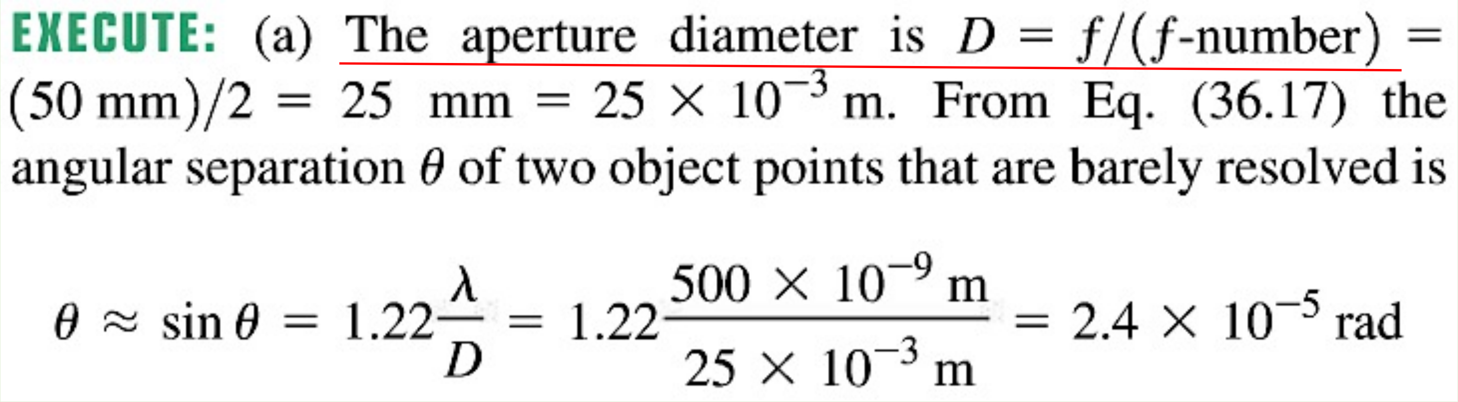
- **Ultraviolet microscopes have higher resolution than visible-light microscopes.**
- **In electron microscopes, the wavelengths associated with electrons can be made 100,000 times smaller than wavelengths of visible light ➔ gain in resolution.**
- **The blue scanning laser used in a Blu-ray player has a shorter wavelength (405 nm) than the 650-nm red laser in a DVD player ➔ better resolving power ➔ pits (information) in Blu-ray discs can be spaced closer together than in a DVD ➔ more information can be stored.**

Example 36.6

Resolving power of a camera lens



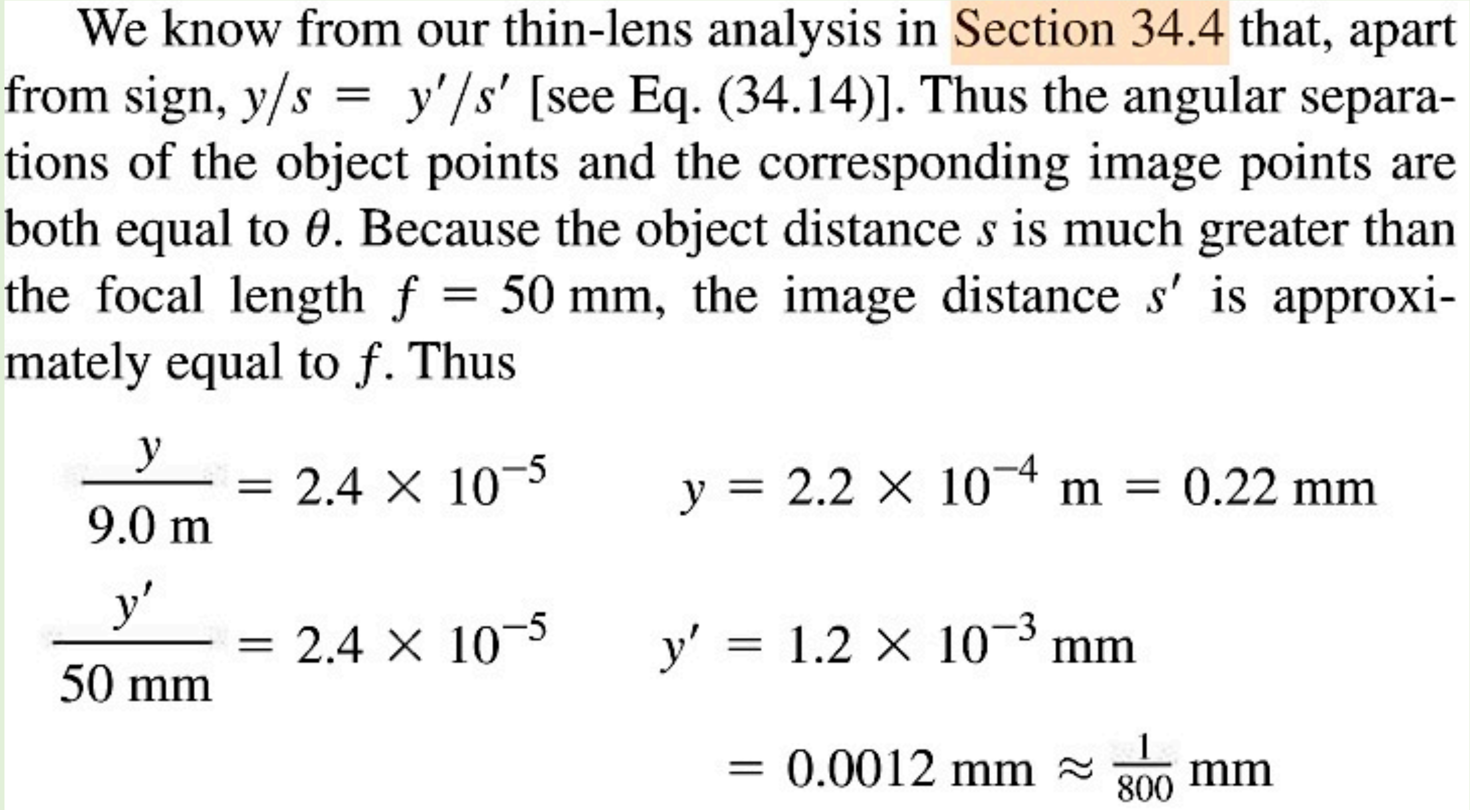
A camera lens with focal length $f = 50$ mm and maximum aperture $f/2$ forms an image of an object 9.0 m away. (a) If the resolution is limited by diffraction, what is the minimum distance between two points on the object that are barely resolved? What is the corresponding distance between image points? (b) How does the situation change if the lens is "stopped down" to $f/16$? Use $\lambda = 500$ nm in both cases.



EXECUTE: (a) The aperture diameter is $D = f/(f\text{-number}) = (50\ \text{mm})/2 = 25\ \text{mm} = 25 \times 10^{-3}\ \text{m}$. From Eq. (36.17) the angular separation θ of two object points that are barely resolved is

$$\theta \approx \sin\theta = 1.22\frac{\lambda}{D} = 1.22\frac{500 \times 10^{-9}\ \text{m}}{25 \times 10^{-3}\ \text{m}} = 2.4 \times 10^{-5}\ \text{rad}$$

Example 36.6
Resolving power of a camera lens



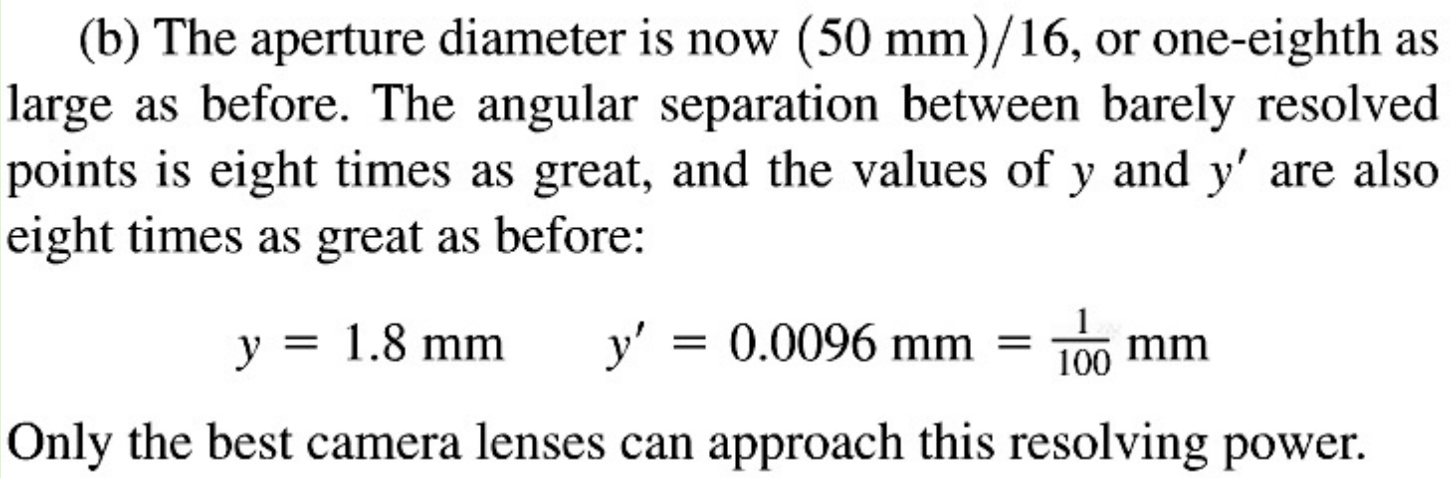
We know from our thin-lens analysis in Section 34.4 that, apart from sign, $y/s = y'/s'$ [see Eq. (34.14)]. Thus the angular separations of the object points and the corresponding image points are both equal to θ. Because the object distance s is much greater than the focal length $f = 50$ mm, the image distance s' is approximately equal to f. Thus

$$\frac{y}{9.0\text{ m}} = 2.4 \times 10^{-5} \qquad y = 2.2 \times 10^{-4}\text{ m} = 0.22\text{ mm}$$

$$\frac{y'}{50\text{ mm}} = 2.4 \times 10^{-5} \qquad y' = 1.2 \times 10^{-3}\text{ mm}$$

$$= 0.0012\text{ mm} \approx \tfrac{1}{800}\text{ mm}$$

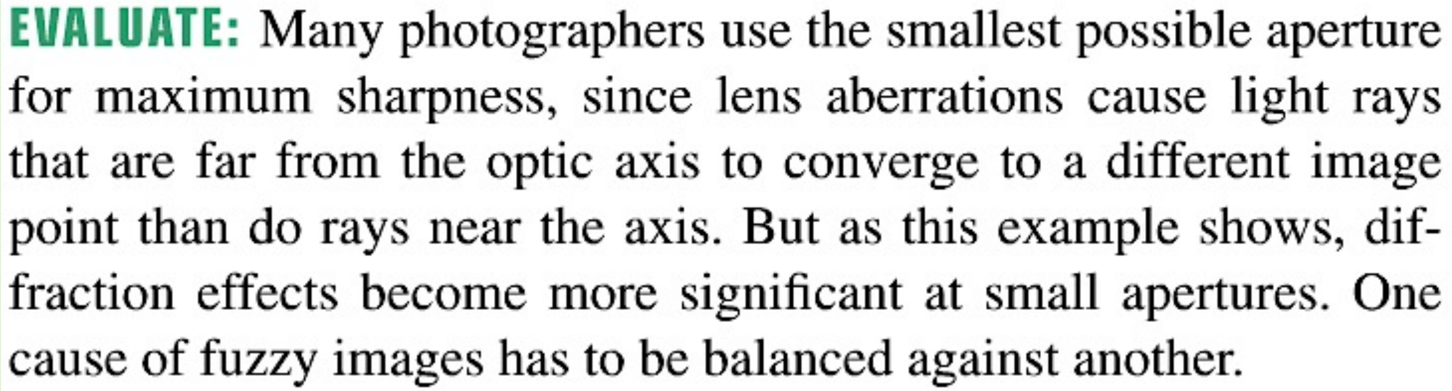
Example 36.6
Resolving power of a camera lens



(b) The aperture diameter is now (50 mm)/16, or one-eighth as large as before. The angular separation between barely resolved points is eight times as great, and the values of y and y' are also eight times as great as before:

$$y = 1.8\ \text{mm} \qquad y' = 0.0096\ \text{mm} = \tfrac{1}{100}\ \text{mm}$$

Only the best camera lenses can approach this resolving power.



EVALUATE: Many photographers use the smallest possible aperture for maximum sharpness, since lens aberrations cause light rays that are far from the optic axis to converge to a different image point than do rays near the axis. But as this example shows, diffraction effects become more significant at small apertures. One cause of fuzzy images has to be balanced against another.