Special Relativity

Learning Goals

Looking forward at …

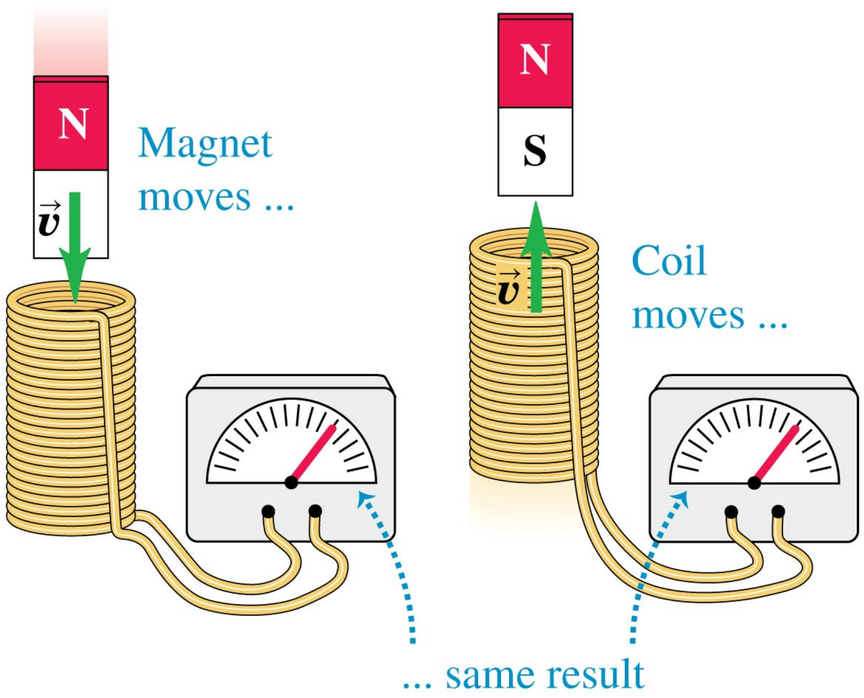
- why different observers can disagree about whether two events are simultaneous.
- how relativity predicts that moving clocks run slow, and what experimental evidence confirms this.
- how the length of an object changes due to the object's motion.
- how the theory of relativity modifies the relationship between velocity and momentum.
- some of the key concepts of Einstein's general theory of relativity.

Introduction

- At Brookhaven National Laboratory in New York, atomic nuclei are accelerated to 99.995% of the ultimate speed limit of the universe — the speed of light, *c*.
- It is impossible for any object to travel at or beyond *c*.
- We shall see some of the far-reaching implications of relativity, such as the effect of motion on time and length.
- We'll see that momentum and kinetic energy must be redefined.

Einstein’s first postulate

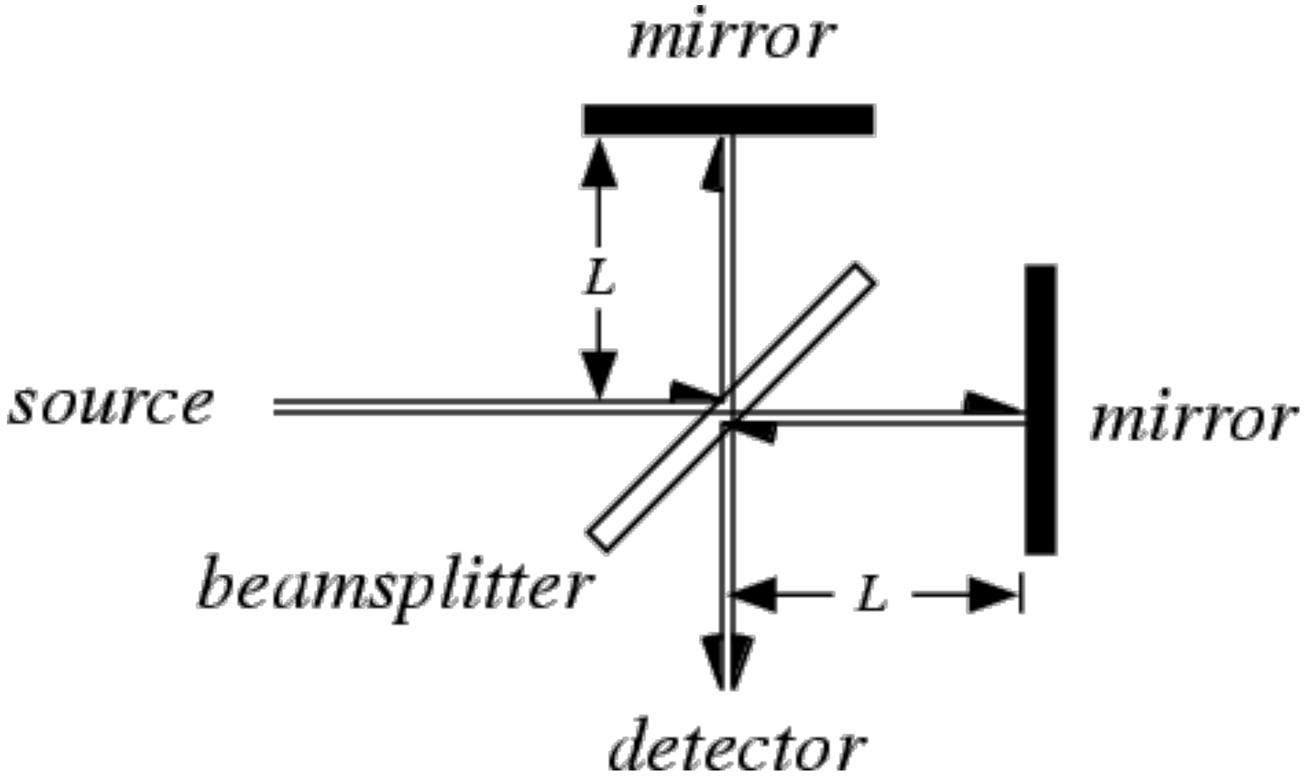
- Einstein’s first postulate, known as the *principle of relativity*, states that **the laws of physics are the same in every inertial reference frame**.
- For example, the same emf is induced in the coil whether the magnet moves relative to the coil, or the coil moves relative to the magnet.



Einstein's second postulate

- Einstein's second postulate is that **the speed of light in vacuum is the same in all inertial frames of reference and is independent of the motion of the source**.
- Suppose two observers measure the speed of light in vacuum.
- One is at rest with respect to the light source, and the other is moving away from it.
- According to the principle of relativity, the two observers must obtain the same result, despite the fact that one is moving with respect to the other.

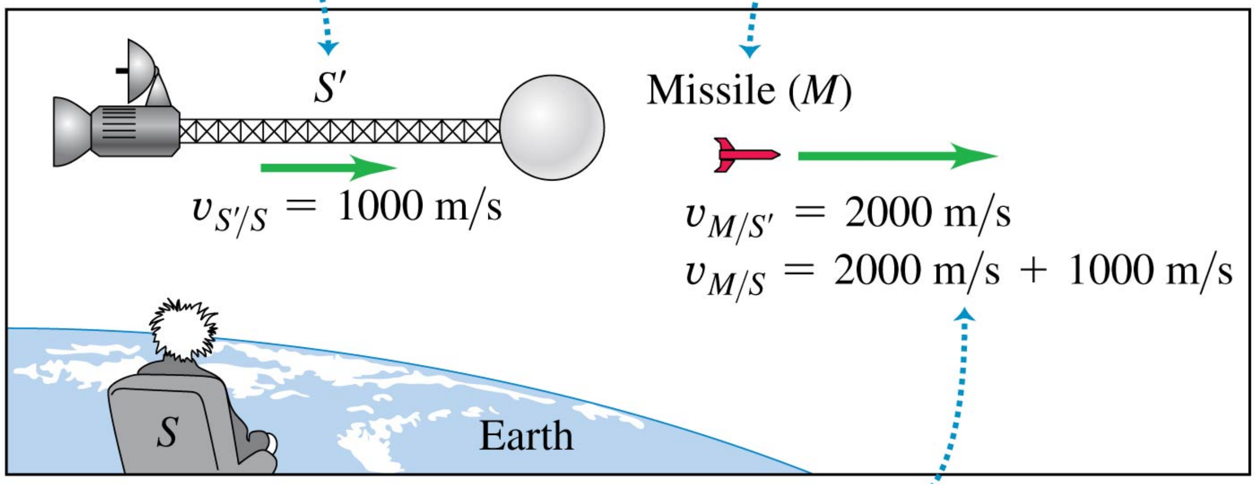
Michelson-Morley experiment



Relative velocity of slow-moving objects

A spaceship (S') moves with speed $v_{S'/S} = 1000$ m/s relative to an observer on earth (S).

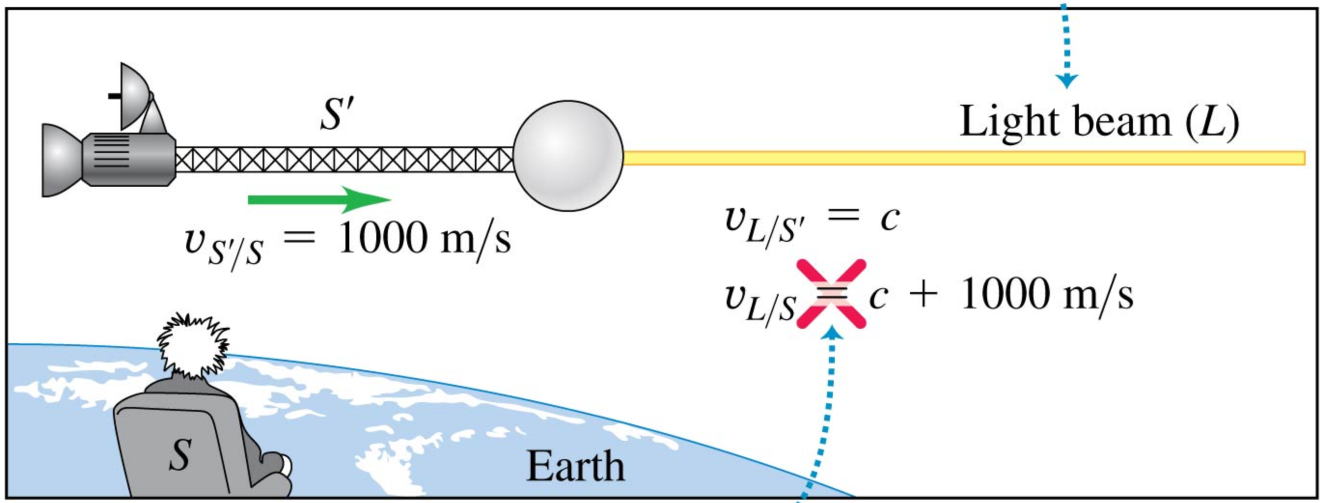
A missile (M) is fired with speed $v_{M/S'} = 2000$ m/s relative to the spaceship.



NEWTONIAN MECHANICS HOLDS: Newtonian mechanics tells us correctly that the missile moves with speed $v_{M/S} = 3000$ m/s relative to the observer on earth.

Relative velocity of light

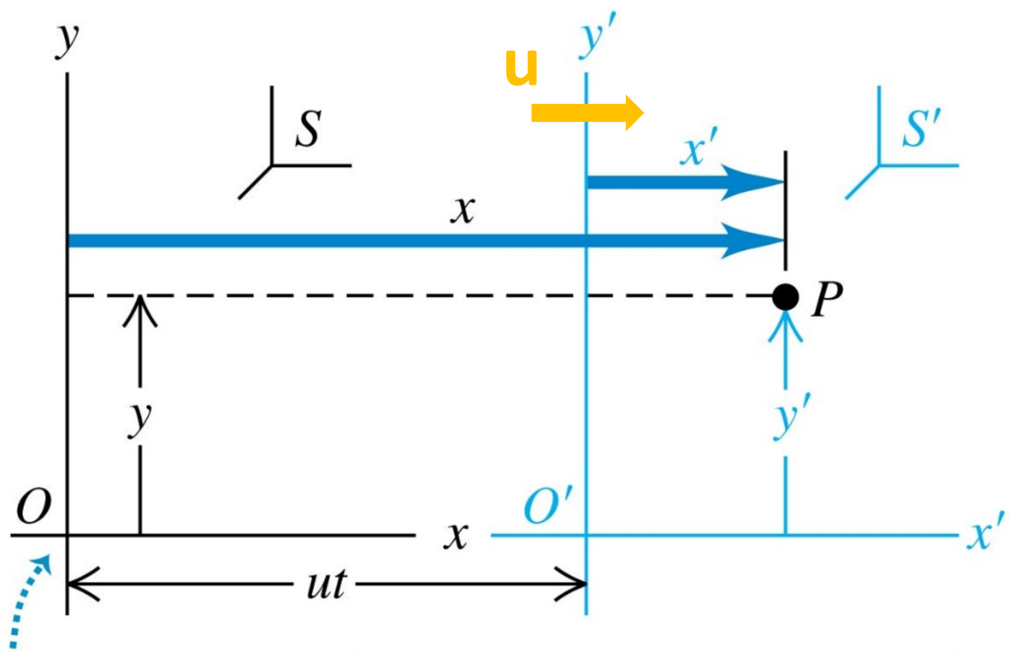
A light beam (L) is emitted
from the spaceship at speed c.



NEWTONIAN MECHANICS FAILS: Newtonian mechanics tells us *incorrectly* that the light moves at a speed greater than c relative to the observer on earth ... which would contradict Einstein's second postulate.

The Galilean transformation

Frame S' moves relative to frame S with constant velocity u along the common x-x'-axis.



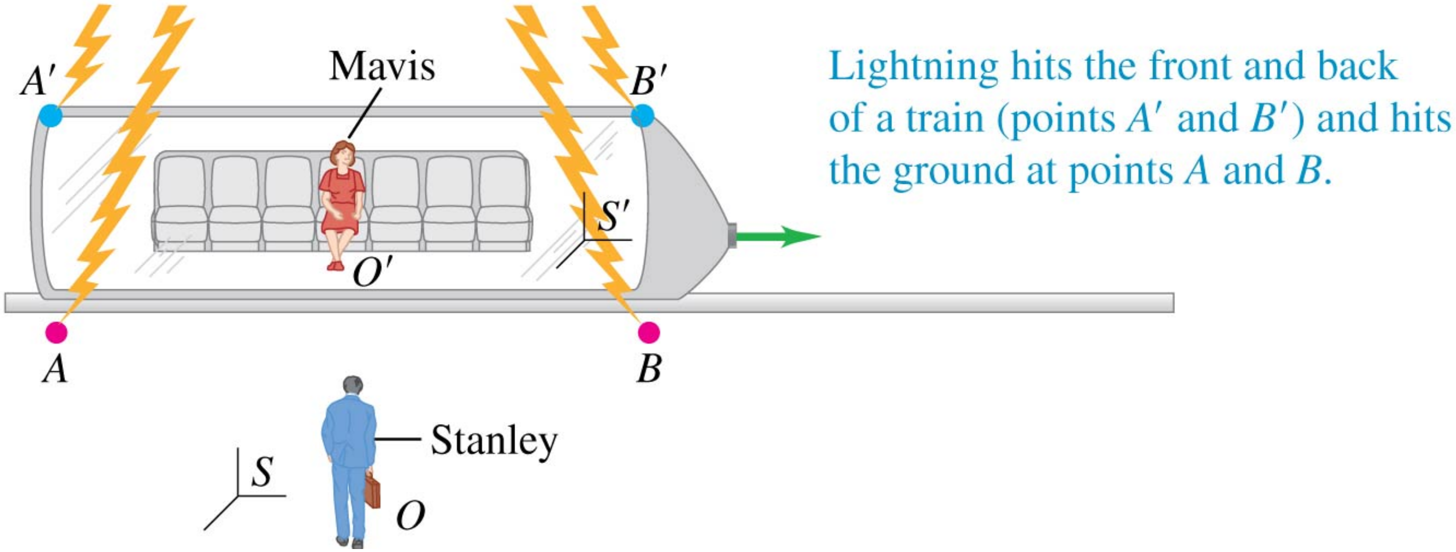
Origins O and O' coincide at time $t = 0 = t'$.

- The *Galilean transformation* is a transformation between two inertial frames of reference.
- In the figure, and the equations below, the position of particle *P* is described in two frames of reference.

$$x = x' + ut \qquad y = y' \qquad z = z' \qquad \text{(Galilean coordinate transformation)}$$

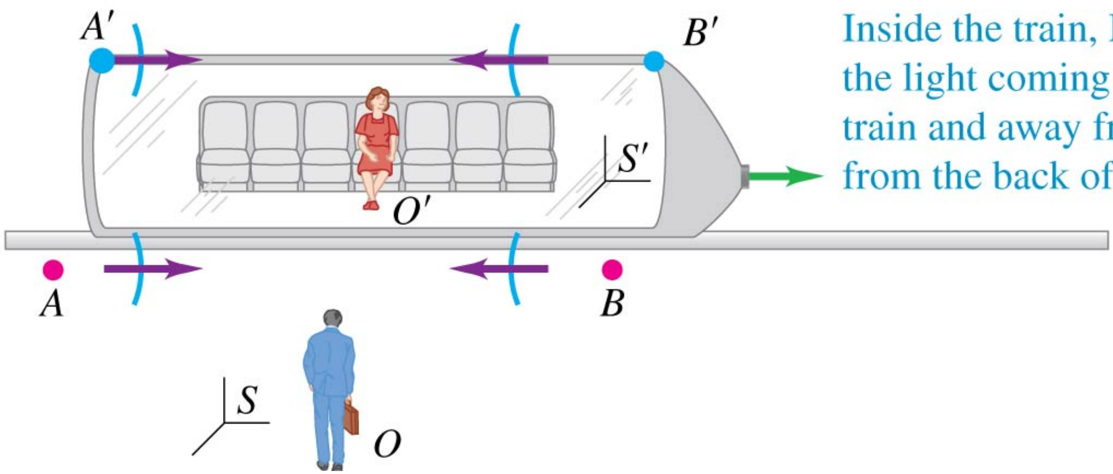
A thought experiment in simultaneity: Slide 1 of 4

- Imagine a train moving with a speed comparable to *c*, with uniform velocity.
- Two lightning bolts strike a passenger car, one near each end.



A thought experiment in simultaneity: Slide 2 of 4

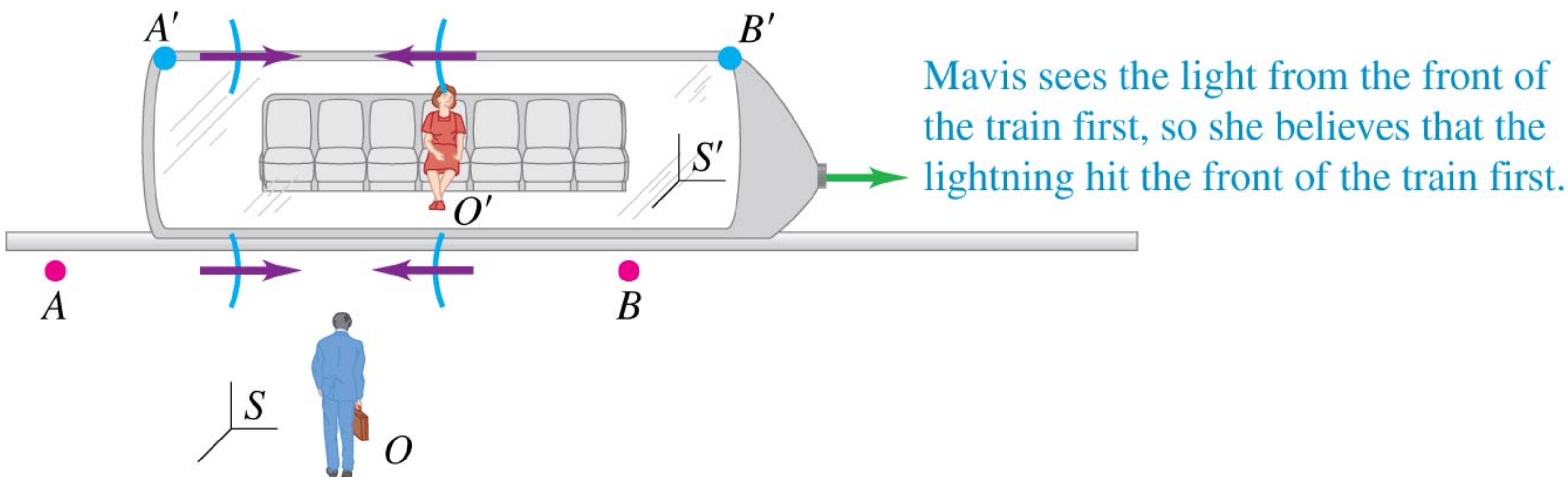
- Stanley is stationary on the ground at *O*, midway between *A* and *B*.
- Mavis is moving with the train at O’ in the middle of the passenger car, midway between A’ and B’.



Inside the train, Mavis moves toward the light coming from the front of the train and away from the light coming from the back of the train.

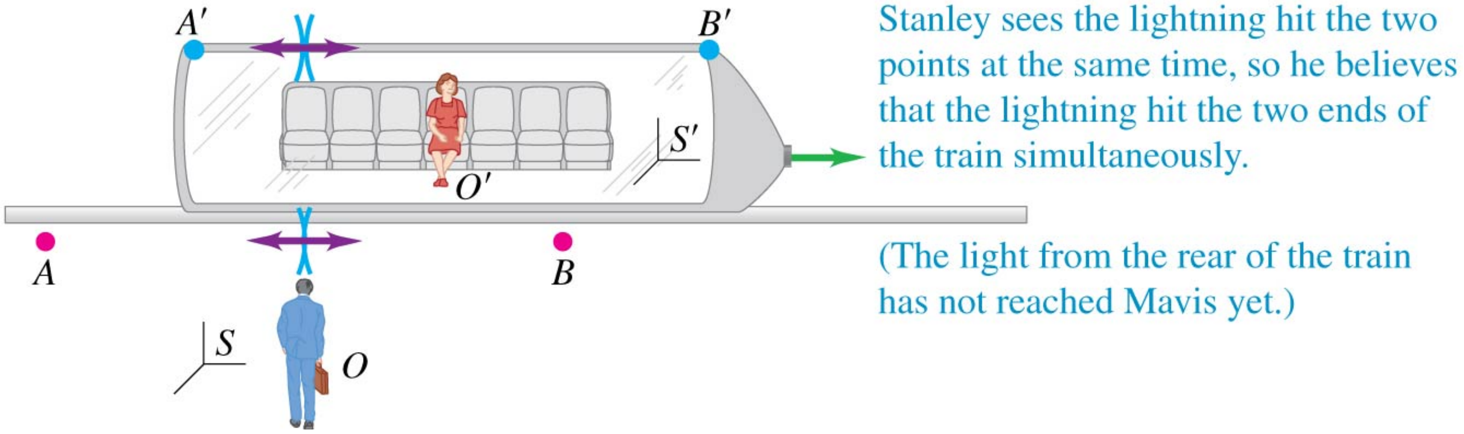
A thought experiment in simultaneity: Slide 3 of 4

- Mavis runs into the wave front from B’ before the wave front from A’ catches up to her.
- Thus she concludes that the lightning bolt at B’ struck *before* the one at A’



A thought experiment in simultaneity: Slide 4 of 4

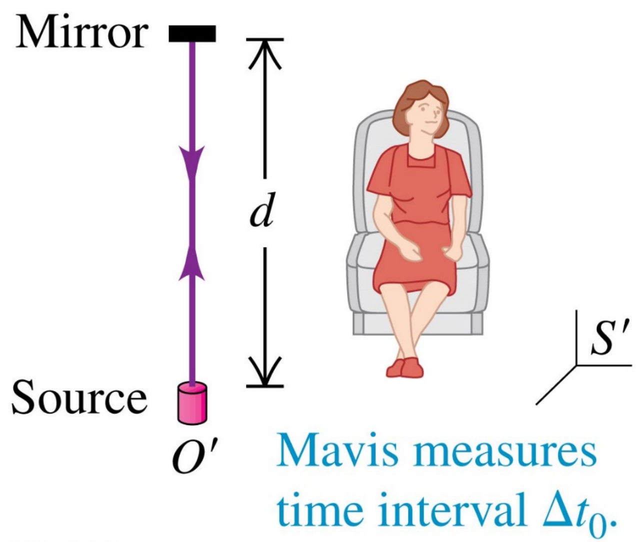
- The two wave fronts from the lightning strikes reach Stanley at *O* simultaneously, so Stanley concludes that the two bolts struck *B* and *A* simultaneously.
- Whether or not two events at different locations are simultaneous depends on the state of motion of the observer.



Relativity of time intervals

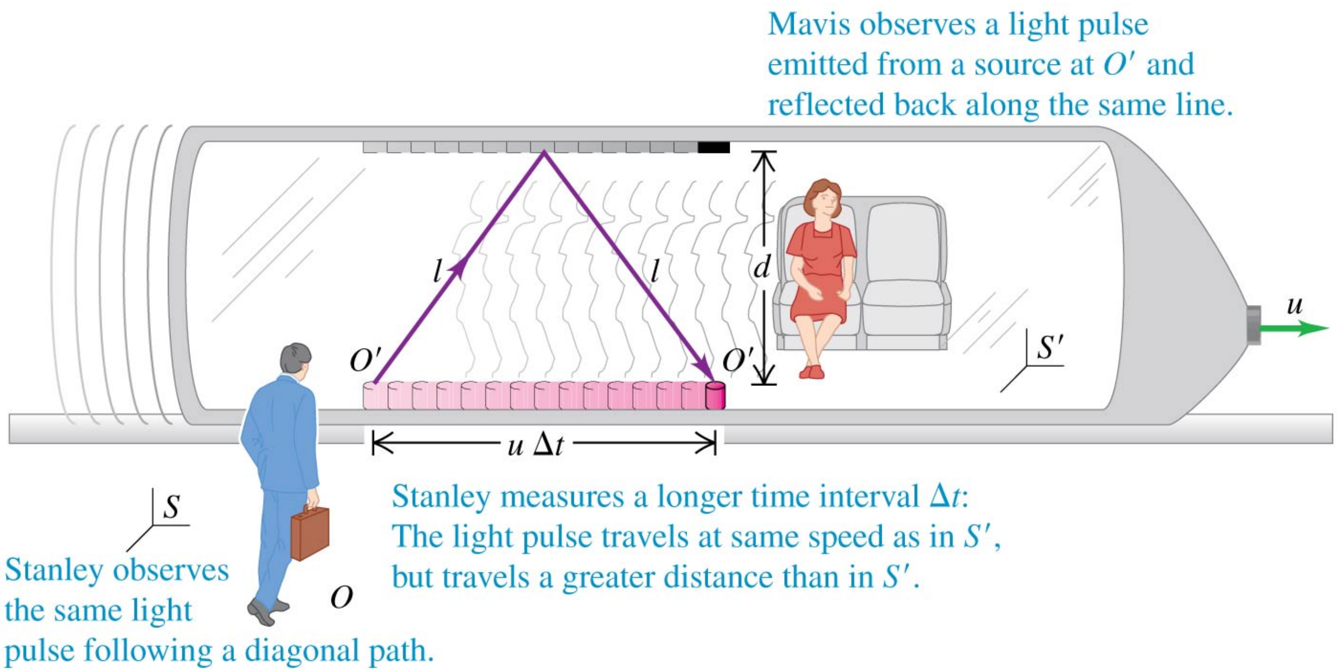
- Let’s consider another thought experiment.
- Mavis, in frame S', measures the time interval between two events.
- Event 1 is when a flash of light from a light source leaves O'.
- Event 2 is when the flash returns to O', having been reflected from a mirror a distance d away.
- The flash of light moves a total distance $2d$, so the time interval is:

$$\Delta t_0 = \frac{2d}{c}$$



Relativity of time intervals

- The round-trip time measured by Stanley in frame *S* is a longer interval Δ*t*; in his frame of reference the two events occur at **different points in space**.



Relativity of time intervals

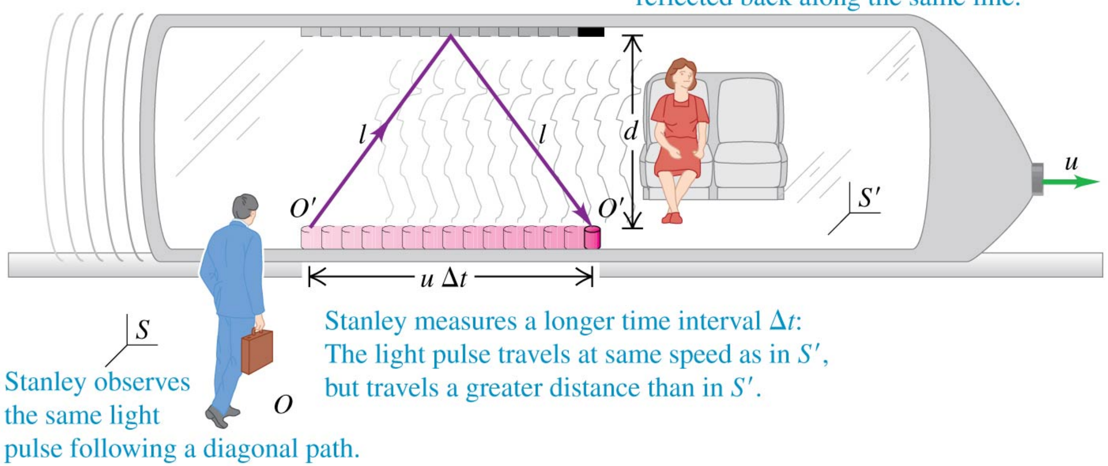
- Stanley in frame S will observe the light propagate along the diagonal with the same speed of light.

- the time for the light pulse return to the source is:

$$\Delta t = \frac{2l}{c} = \frac{2}{c}\sqrt{d^2 + \left(\frac{u\,\Delta t}{2}\right)^2}$$

$$\Delta t = \frac{2}{c}\sqrt{\left(\frac{c\,\Delta t_0}{2}\right)^2 + \left(\frac{u\,\Delta t}{2}\right)^2}$$

$$l = \sqrt{d^2 + \left(\frac{u\,\Delta t}{2}\right)^2}$$



Time dilation and proper time

- Let Δt_0 be the **proper time** between two events.
- An observer moving with constant speed u will measure the time interval to be Δt, where

Time dilation:

Proper time between two events (measured in rest frame)

$$\Delta t = \gamma \Delta t_0$$

Lorentz factor relating the two frames

Time interval between same events measured in second frame of reference

where the Lorentz factor $\boldsymbol{\gamma}$ is defined as:

Lorentz factor

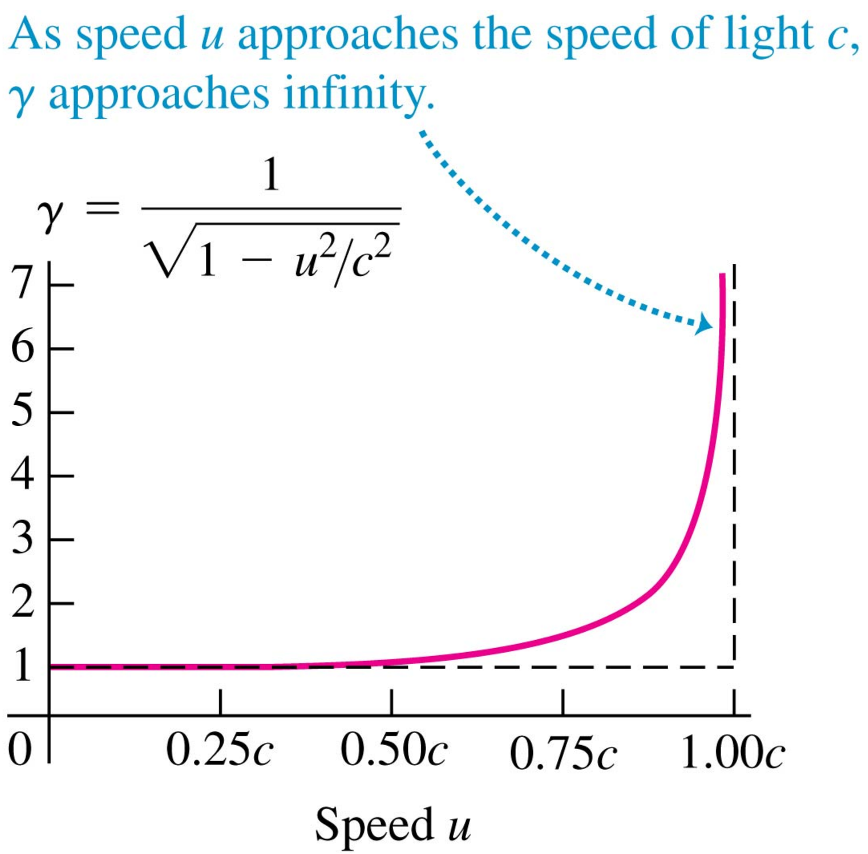
$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$$

Speed of light in vacuum

Speed of one frame of reference relative to another

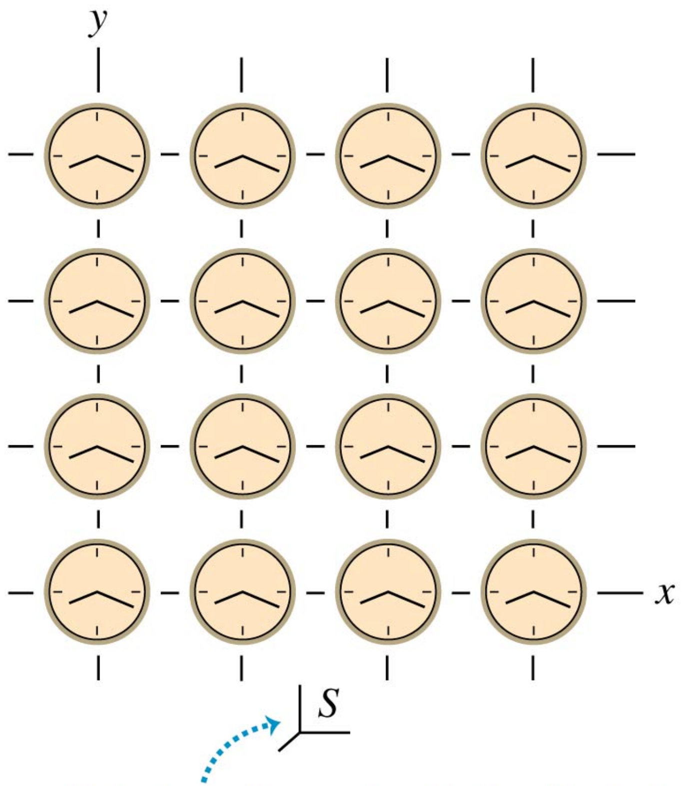
The Lorentz factor

- When u is very small compared to c, $\boldsymbol{\gamma}$ is very nearly equal to 1.
- If the relative speed u is great enough that $\boldsymbol{\gamma}$ is appreciably greater than 1, the speed is said to be relativistic.



Proper time

- *Proper time* is the time interval between two events that occur at the *same point*.
- A frame of reference can be pictured as a coordinate system with a grid of synchronized clocks, as in the figure at the right.



The grid is three dimensional; identical planes of clocks lie in front of and behind the page, connected by grid lines perpendicular to the page.

Example 37.1 Time dilation at 0.990c

High-energy subatomic particles coming from space interact with atoms in the earth's upper atmosphere, in some cases producing unstable particles called *muons*. A muon decays into other particles with a mean lifetime of $2.20\ \mu\text{s} = 2.20 \times 10^{-6}$ s as measured in a reference frame in which it is at rest. If a muon is moving at $0.990c$ relative to the earth, what will an observer on earth measure its mean lifetime to be?

SOLUTION

IDENTIFY and SET UP: The muon's lifetime is the time interval between two events: the production of the muon and its subsequent decay. Our target variable is the lifetime in your frame of reference on earth, which we call frame S. We are given the lifetime in a frame S' in which the muon is at rest; this is its *proper* lifetime, $\Delta t_0 = 2.20\ \mu\text{s}$. The relative speed of these two frames is $u = 0.990c$. We use Eq. (37.6) to relate the lifetimes in the two frames.

EXECUTE: The muon moves relative to the earth between the two events, so the two events occur at different positions as measured in S and the time interval in that frame is Δt (the target variable). From Eq. (37.6),

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - u^2/c^2}} = \frac{2.20\ \mu\text{s}}{\sqrt{1 - (0.990)^2}} = 15.6\ \mu\text{s}$$

EVALUATE: Our result predicts that the mean lifetime of the muon in the earth frame (Δt) is about seven times longer than in the muon's frame (Δt_0). This prediction has been verified experimentally; indeed, this was the first experimental confirmation of the time dilation formula, Eq. (37.6).

Example 37.2 Time dilation at airliner speeds

An airplane flies from San Francisco to New York (about 4800 km, or 4.80×10^6 m) at a steady speed of 300 m/s (about 670 mi/h). How much time does the trip take, as measured by an observer on the ground? By an observer in the plane?

SOLUTION

IDENTIFY and SET UP: Here we're interested in the time interval between the airplane departing from San Francisco and landing in New York. The target variables are the time intervals as measured in the frame of reference of the ground S and in the frame of reference of the airplane S'.

EXECUTE: As measured in S the two events occur at different positions (San Francisco and New York), so the time interval measured by ground observers corresponds to Δt in Eq. (37.6). To find it, we simply divide the distance by the speed $u = 300$ m/s:

$$\Delta t = \frac{4.80 \times 10^6 \text{ m}}{300 \text{ m/s}} = 1.60 \times 10^4 \text{ s} \quad (\text{about } 4\tfrac{1}{2} \text{ hours})$$

In the airplane's frame S', San Francisco and New York passing under the plane occur at the same point (the position of the plane). Hence the time interval in the airplane is a proper time, corresponding to Δt_0 in Eq. (37.6). We have

$$\frac{u^2}{c^2} = \frac{(300 \text{ m/s})^2}{(3.00 \times 10^8 \text{ m/s})^2} = 1.00 \times 10^{-12}$$

From Eq. (37.6),

$$\Delta t_0 = (1.60 \times 10^4 \text{ s})\sqrt{1 - 1.00 \times 10^{-12}}$$

The square root can't be evaluated with adequate precision with an ordinary calculator. But we can approximate it using the binomial theorem (see Appendix B):

$$(1 - 1.00 \times 10^{-12})^{1/2} = 1 - \left(\tfrac{1}{2}\right)(1.00 \times 10^{-12}) + \cdots$$

The remaining terms are of the order of 10^{-24} or smaller and can be discarded. The approximate result for Δt_0 is

$$\Delta t_0 = (1.60 \times 10^4 \text{ s})(1 - 0.50 \times 10^{-12})$$

The proper time Δt_0, measured in the airplane, is very slightly less (by less than one part in 10^{12}) than the time measured on the ground.

EVALUATE: We don't notice such effects in everyday life. But present-day atomic clocks (see Section 1.3) can attain a precision of about one part in 10^{13}. A cesium clock traveling a long distance in an airliner has been used to measure this effect and thereby verify Eq. (37.6) even at speeds much less than c.

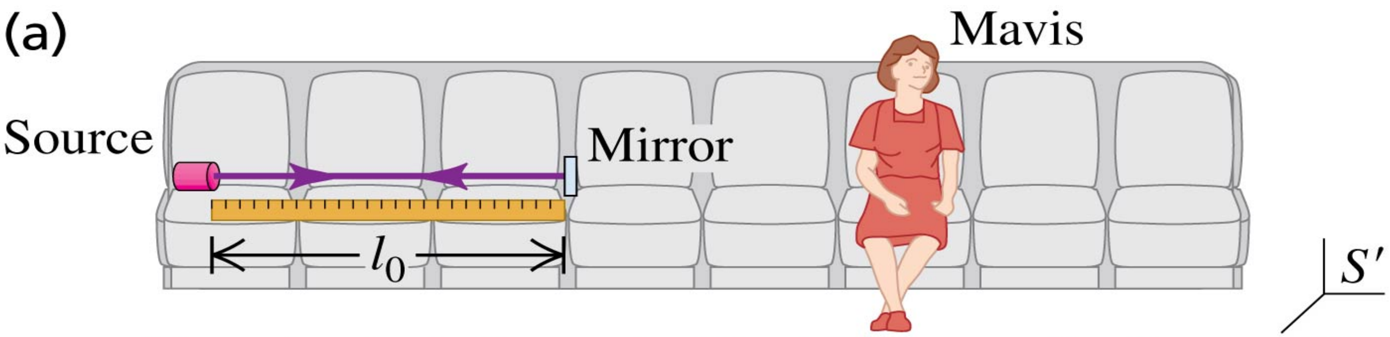
Twin paradox

- Consider identical twin astronauts names Eartha and Astrid.
- Eartha remains on earth while her twin Astrid takes off on a high-speed trip through the galaxy
- Because of time dilation, Earth observes Astrid’s ages more slowly and younger when Astrid returns to earth
- All inertia frames are equivalent.
- Astrid can make the same arguments to conclude that Eartha is younger

- Astrid must accelerate with respect to Earth and her reference frame is not inertial
- Correct answer: Astrid is younger than Eartha

Relativity of length

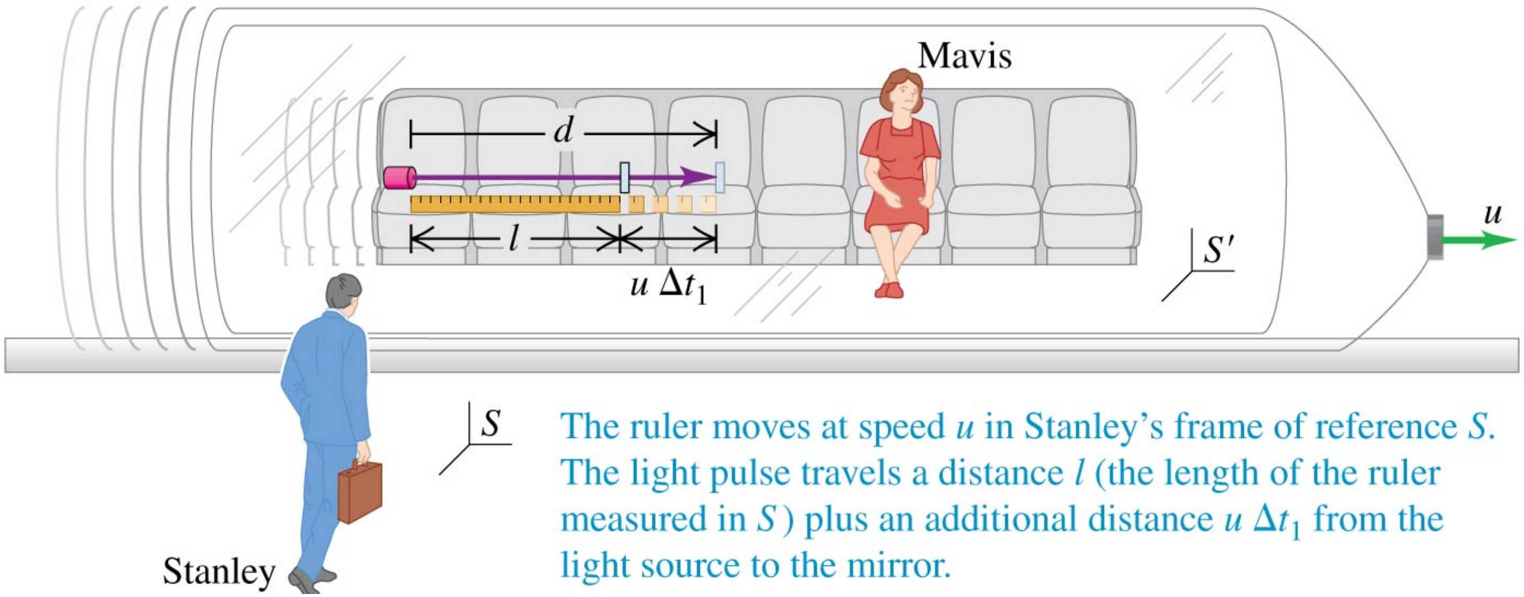
- We attach a light source to one end of a ruler and a mirror to the other end.
- The ruler is at rest in reference frame S', and its length in this frame is l_0.



The ruler is stationary in Mavis's frame of reference S'. The light pulse travels a distance l_0 from the light source to the mirror.

Relativity of length

- In reference frame *S* the ruler is moving to the right with speed *u*.
- The length of the ruler is shorter in *S*.

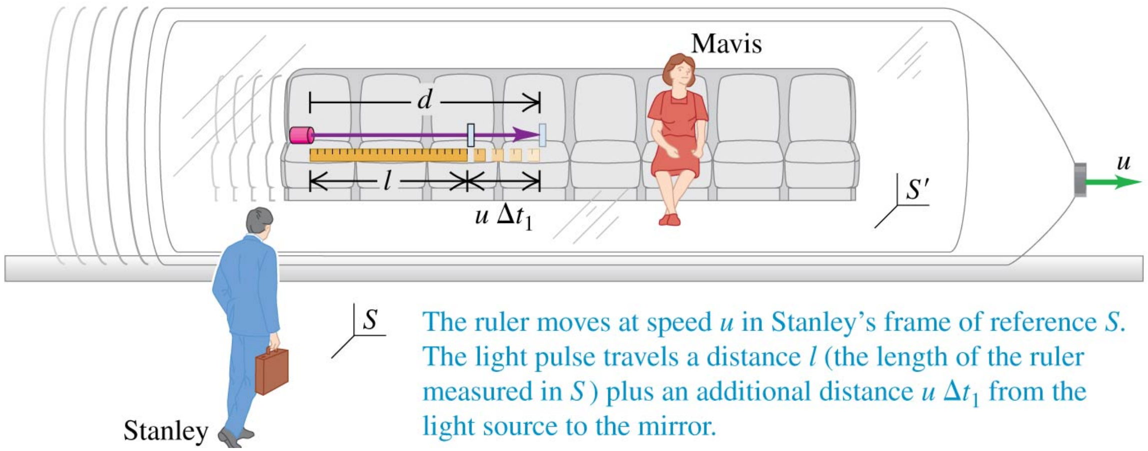


The ruler moves at speed u in Stanley's frame of reference S. The light pulse travels a distance l (the length of the ruler measured in S) plus an additional distance $u\,\Delta t_1$ from the light source to the mirror.

Relativity of length

The total length of path d from source to mirror is:

$$d = l + u\,\Delta t_1$$
$$d = c\,\Delta t_1$$



The ruler moves at speed u in Stanley's frame of reference S. The light pulse travels a distance l (the length of the ruler measured in S) plus an additional distance $u\,\Delta t_1$ from the light source to the mirror.

Eliminate d, we get

$$c\,\Delta t_1 = l + u\,\Delta t_1 \quad \text{or}$$

$$\Delta t_1 = \frac{l}{c - u}$$

Similarly, the return trip (mirror to source) takes the time

$$\Delta t_2 = \frac{l}{c + u}$$

The total time is: $$\Delta t = \frac{l}{c - u} + \frac{l}{c + u} = \frac{2l}{c(1 - u^2/c^2)}$$

We measure the distance *l* by measuring the time taken for light to make a round trip.

Length contraction and proper length

- A length measured in the frame in which the body is at rest (the rest frame of the body) is called a **proper length**.
- Thus l_0 is a proper length in S', and the length measured in any other frame moving relative to S is less than l_0.
- This effect is called length contraction.

Proper length of object (measured in rest frame)

Length contraction:

$$l = l_0\sqrt{1 - \frac{u^2}{c^2}} = \frac{l_0}{\gamma}$$

Speed of second frame relative to rest frame

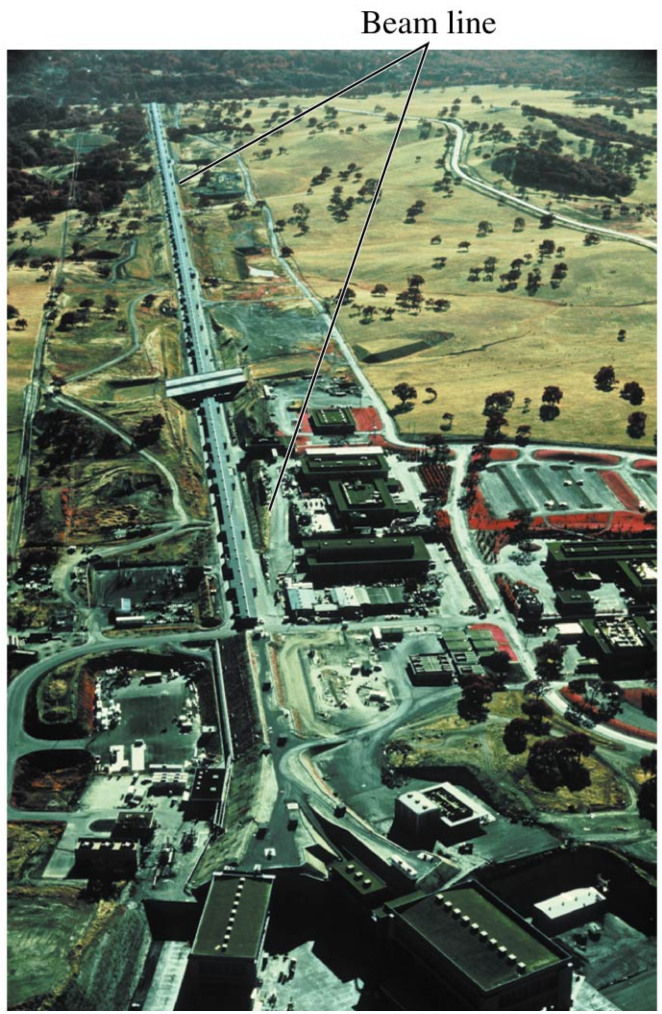
Lorentz factor relating the two frames

Length in second frame of reference moving parallel to object's length

Speed of light in vacuum

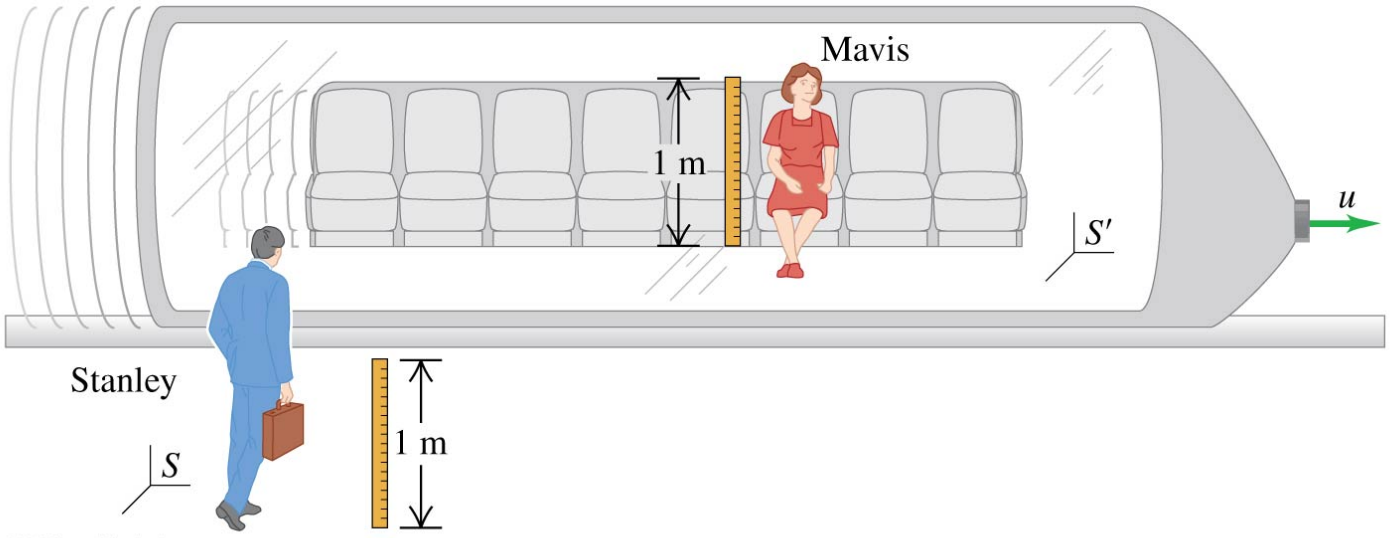
Example of length contraction

- The speed at which electrons traverse the 3-km beam line of the SLAC National Accelerator Laboratory is slower than *c* by less than 1 cm/s.
- As measured in the reference frame of such an electron, the beam line (which extends from the top to the bottom of this photograph) is only about 15 cm long!



Lengths perpendicular to the direction of motion

- There is no length contraction for lengths perpendicular to the direction of relative motion.



Example 37.4 How long is the spaceship?

A spaceship flies past earth at a speed of $0.990c$. A crew member on board the spaceship measures its length, obtaining the value 400 m. What length do observers measure on earth?

SOLUTION

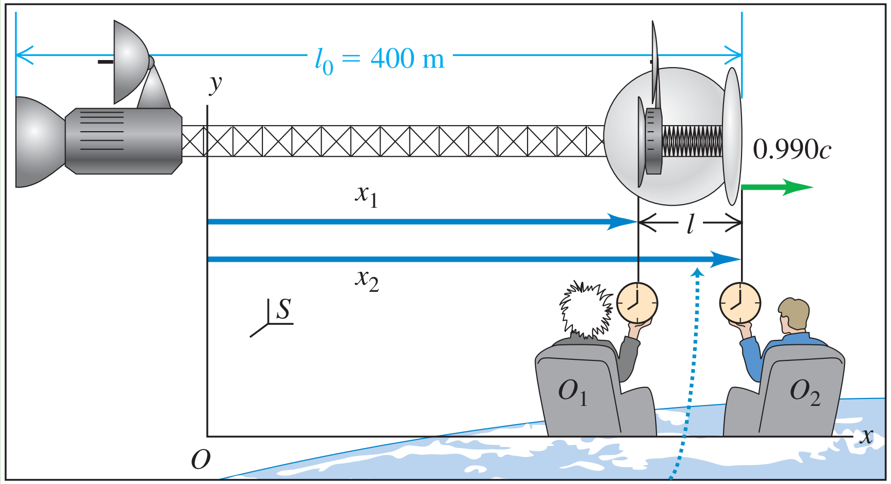
IDENTIFY and SET UP: This problem is about the nose-to-tail length of the spaceship as measured on the spaceship and on earth. This length is along the direction of relative motion (Fig. 37.13), so there will be length contraction. The spaceship's 400-m length is the *proper* length l_0 because it is measured in the frame in which the spaceship is at rest. Our target variable is the length l measured in the earth frame, relative to which the spaceship is moving at $u = 0.990c$.

EXECUTE: From Eq. (37.16), the length in the earth frame is

$$l = l_0\sqrt{1 - \frac{u^2}{c^2}} = (400\text{ m})\sqrt{1 - (0.990)^2} = 56.4\text{ m}$$

EVALUATE: The spaceship is shorter in a frame in which it is in motion than in a frame in which it is at rest. To measure the length l, two earth observers with synchronized clocks could measure the positions of the two ends of the spaceship simultaneously in the earth's reference frame, as shown in Fig. 37.13. (These two measurements will *not* appear simultaneous to an observer in the spaceship.)

37.13 Measuring the length of a moving spaceship.



The two observers on earth (S) must measure x_2 and x_1 simultaneously to obtain the correct length $l = x_2 - x_1$ in their frame of reference.

Example 37.5 How far apart are the observers?

Observers O_1 and O_2 in Fig. 37.13 are 56.4 m apart on the earth. How far apart does the spaceship crew measure them to be?

SOLUTION

IDENTIFY and SET UP: In this example the 56.4-m distance is the *proper* length l_0. It represents the length of a ruler that extends from O_1 to O_2 and is at rest in the earth frame in which the observers are at rest. Our target variable is the length l of this ruler measured in the spaceship frame, in which the earth and ruler are moving at $u = 0.990c$.

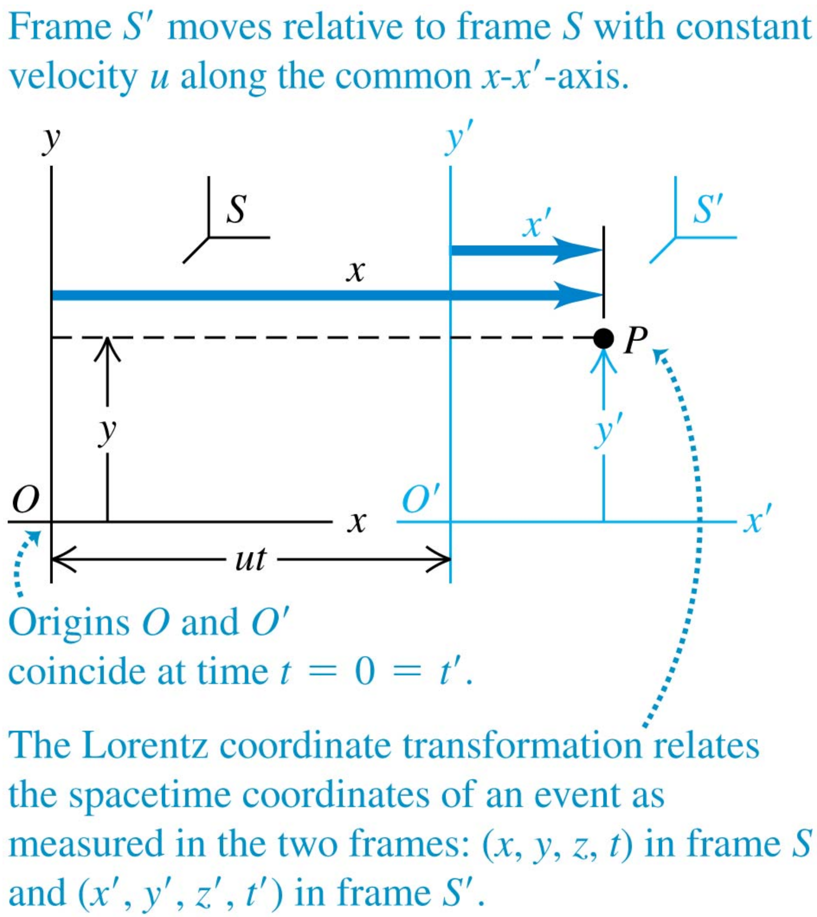
EXECUTE: As in Example 37.4, but with $l_0 = 56.4$ m,

$$l = l_0\sqrt{1 - \frac{u^2}{c^2}} = (56.4\text{ m})\sqrt{1 - (0.990)^2} = 7.96\text{ m}$$

EVALUATE: This answer does *not* say that the crew measures their spaceship to be both 400 m long and 7.96 m long. As measured on earth, the tail of the spacecraft is at the position of O_1 at the same instant that the nose of the spacecraft is at the position of O_2. Hence the length of the spaceship measured on earth equals the 56.4-m distance between O_1 and O_2. But in the spaceship frame O_1 and O_2 are only 7.96 m apart, and the nose (which is 400 m in front of the tail) passes O_2 before the tail passes O_1.

The Lorentz transformations

- This Galilean transformation, as we have seen, is valid only in the limit when u approaches zero.
- The more general relationships are called the **Lorentz transformations**.



The Lorentz transformations for coordinates

- The **Lorentz transformations** relate the coordinates and velocities in two inertial reference frames.
- They are more general than the Galilean transformations and are consistent with the principle of relativity.

Lorentz coordinate transformation: Spacetime coordinates of an event are x, y, z, t in frame S and x', y', z', t' in frame S'.

Velocity of S' relative to S in positive direction along x-x'-axis

$$x' = \frac{x - ut}{\sqrt{1 - u^2/c^2}} = \gamma(x - ut)$$

Lorentz factor relating the two frames

$$y' = y$$

Speed of light in vacuum

$$z' = z$$

$$t' = \frac{t - ux/c^2}{\sqrt{1 - u^2/c^2}} = \gamma(t - ux/c^2)$$

The Lorentz transformations for velocities

Lorentz velocity transformation (velocity in S' in terms of velocity in S):

x-velocity of object in frame S'

x-velocity of object in frame S

$$v'_x = \frac{v_x - u}{1 - uv_x/c^2} \equiv \frac{dx'}{dt'}$$

Velocity of S' relative to S in positive direction along x-x'-axis

Speed of light in vacuum

Lorentz velocity transformation (velocity in S in terms of velocity in S'):

x-velocity of object in frame S

x-velocity of object in frame S'

$$v_x = \frac{v'_x + u}{1 + uv'_x/c^2} \equiv \frac{dx}{dt}$$

Velocity of S' relative to S in positive direction along x-x'-axis

Speed of light in vacuum

- The Lorentz velocity transformations show us that a body moving with a speed less than *c* in one frame of reference always has a speed less than *c* in *every other* frame of reference.

Example 37.6 Was it received before it was sent?

Winning an interstellar race, Mavis pilots her spaceship across a finish line in space at a speed of $0.600c$ relative to that line. A "hooray" message is sent from the back of her ship (event 2) at the instant (in her frame of reference) that the front of her ship crosses the line (event 1). She measures the length of her ship to be 300 m. Stanley is at the finish line and is at rest relative to it. When and where does he measure events 1 and 2 to occur?

SOLUTION

IDENTIFY and SET UP: This example involves the Lorentz coordinate transformation. Our derivation of this transformation assumes that the origins of frames S and S' coincide at $t = 0 = t'$. Thus for simplicity we fix the origin of S at the finish line and the origin of S' at the front of the spaceship so that Stanley and Mavis measure event 1 to be at $x = 0 = x'$ and $t = 0 = t'$.

Mavis in S' measures her spaceship to be 300 m long, so she has the "hooray" sent from 300 m behind her spaceship's front at the instant she measures the front to cross the finish line. That is, she measures event 2 at $x' = -300$ m and $t' = 0$.

Our target variables are the coordinate x and time t of event 2 that Stanley measures in S.

EXECUTE: To solve for the target variables, we modify the first and last of Eqs. (37.21) to give x and t as functions of x' and t'. We do so in the same way that we obtained Eq. (37.23) from Eq. (37.22). We remove the primes from x' and t', add primes to x and t, and replace each u with $-u$. The results are

$$x = \gamma(x' + ut') \quad \text{and} \quad t = \gamma(t' + ux'/c^2)$$

From Eq. (37.7), $\gamma = 1.25$ for $u = 0.600c = 1.80 \times 10^8$ m/s. We also substitute $x' = -300$ m, $t' = 0$, $c = 3.00 \times 10^8$ m/s, and $u = 1.80 \times 10^8$ m/s in the equations for x and t to find $x = -375$ m at $t = -7.50 \times 10^{-7}$ s $= -0.750\ \mu$s for event 2.

EVALUATE: Mavis says that the events are simultaneous, but Stanley says that the "hooray" was sent *before* Mavis crossed the finish line. This does not mean that the effect preceded the cause. The fastest that Mavis can send a signal the length of her ship is $300\text{ m}/(3.00 \times 10^8\text{ m/s}) = 1.00\ \mu$s. She cannot send a signal from the front at the instant it crosses the finish line that would cause a "hooray" to be broadcast from the back at the same instant. She would have to send that signal from the front at least $1.00\ \mu$s before then, so she had to slightly anticipate her success.

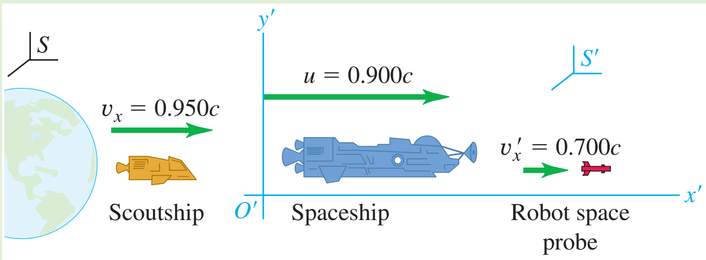
Example 37.7 Relative velocities

(a) A spaceship moving away from the earth at $0.900c$ fires a robot space probe in the same direction as its motion at $0.700c$ relative to the spaceship. What is the probe's velocity relative to the earth? (b) A scoutship is sent to catch up with the spaceship by traveling at $0.950c$ relative to the earth. What is the velocity of the scoutship relative to the spaceship?

SOLUTION

IDENTIFY and SET UP: This example uses the Lorentz velocity transformation. Let the earth and spaceship reference frames be S and S', respectively (Fig. 37.16); their relative velocity is $u = 0.900c$. In part (a) we are given the probe velocity $v'_x = 0.700c$ with respect to S', and the target variable is the velocity v_x of the probe relative to S. In part (b) we are given the velocity $v_x = 0.950c$ of the scoutship relative to S, and the target variable is its velocity v'_x relative to S'.

37.16 The spaceship, robot space probe, and scoutship.



EXECUTE: (a) We use Eq. (37.23) to find the probe velocity relative to the earth:

$$v_x = \frac{v'_x + u}{1 + uv'_x/c^2} = \frac{0.700c + 0.900c}{1 + (0.900c)(0.700c)/c^2} = 0.982c$$

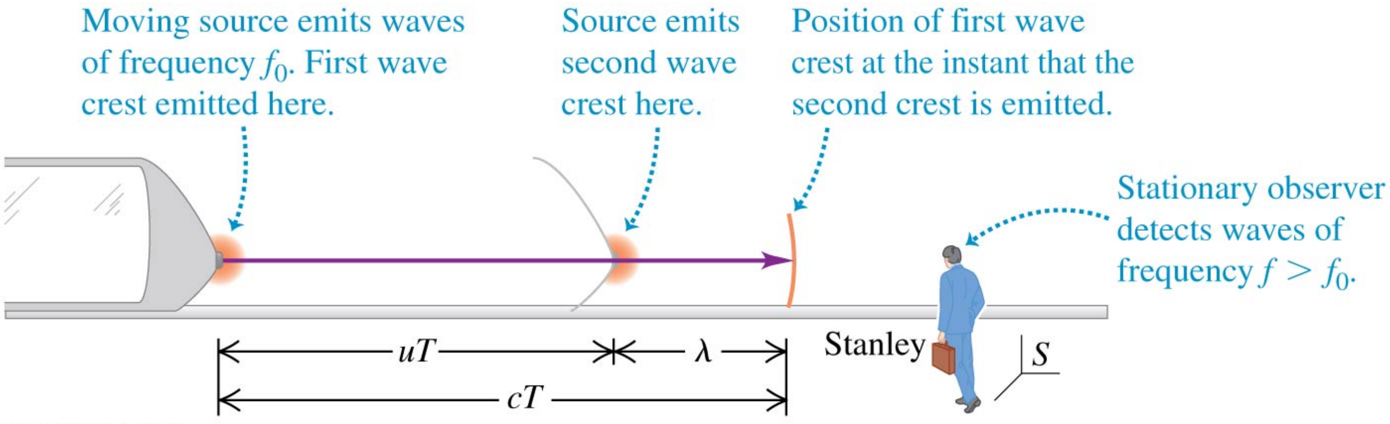
(b) We use Eq. (37.22) to find the scoutship velocity relative to the spaceship:

$$v'_x = \frac{v_x - u}{1 - uv_x/c^2} = \frac{0.950c - 0.900c}{1 - (0.900c)(0.950c)/c^2} = 0.345c$$

EVALUATE: What would the Galilean velocity transformation formula, Eq. (37.2), say? In part (a) we would have found the probe's velocity relative to the earth to be $v_x = v'_x + u = 0.700c + 0.900c = 1.600c$, which is greater than c and hence impossible. In part (b), we would have found the scoutship's velocity relative to the spaceship to be $v'_x = v_x - u = 0.950c - 0.900c = 0.050c$; the relativistically correct value, $v'_x = 0.345c$, is almost seven times greater than the incorrect Galilean value.

Doppler effect for electromagnetic waves

- When a source moves toward the observer, the observed frequency f is greater than the emitted frequency f_0.



Doppler effect, electromagnetic waves, source approaching observer:

Frequency measured by observer

$$f = \sqrt{\frac{c+u}{c-u}}\, f_0$$

Frequency measured in rest frame of source

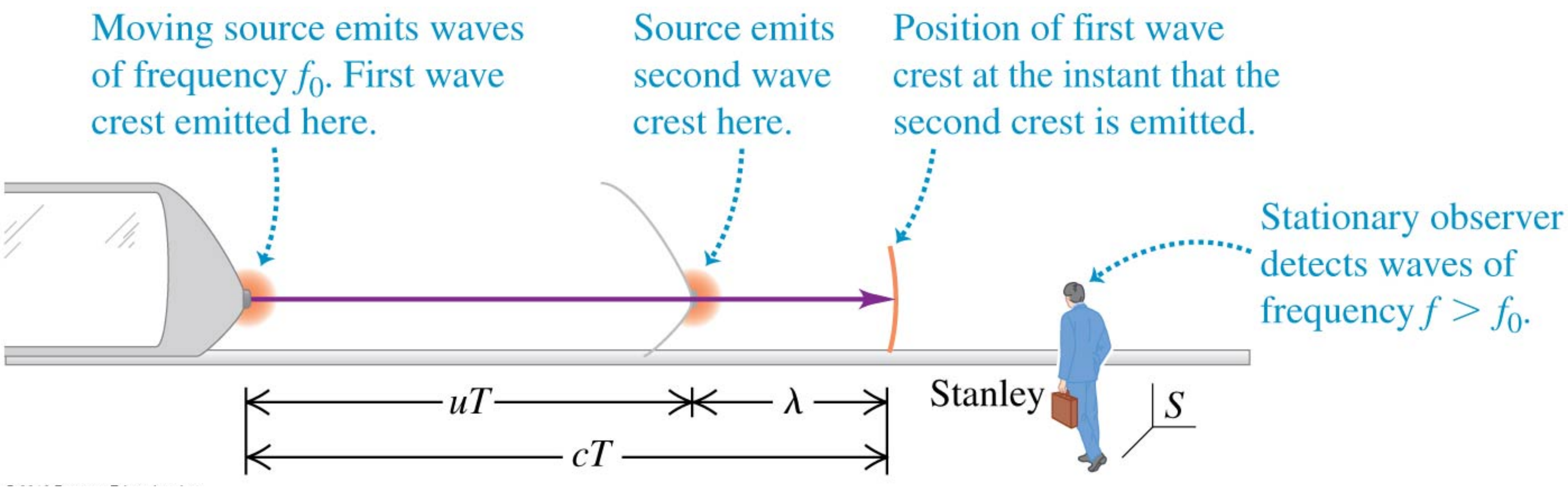
Speed of light in vacuum

Speed of source relative to observer

Doppler effect for electromagnetic waves

- $\lambda = (c - u)T$ and $\mathrm{f} = \frac{c}{\lambda}$
- The period measured in the rest frame of the source (T_0) and the observer S (T) are related by:

$$T = \frac{T_0}{\sqrt{1 - u^2/c^2}} = \frac{cT_0}{\sqrt{c^2 - u^2}}$$



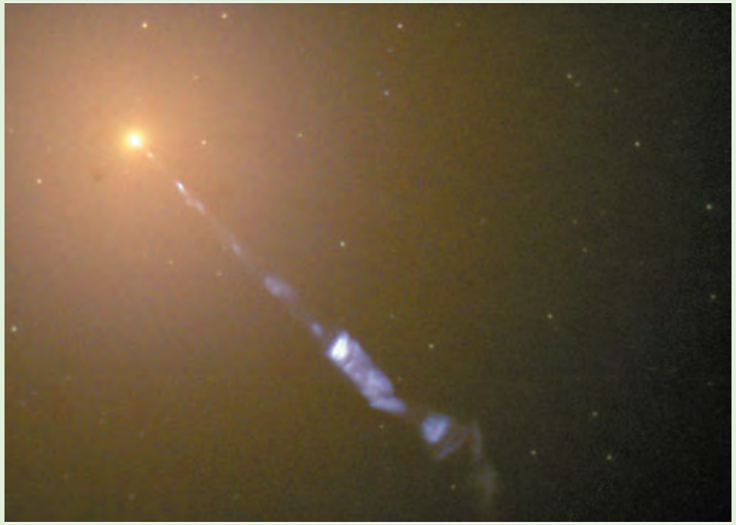
Doppler effect for electromagnetic waves

- This handheld radar gun emits a radio beam of frequency f_0, which in the frame of reference of an approaching car has a higher frequency f.
- The reflected beam also has frequency f in the car's frame, but has an even higher frequency f' in the police officer's frame.
- The radar gun calculates the car's speed by comparing the frequencies of the emitted beam and the doubly Doppler-shifted reflected beam.

Example 37.8 **A jet from a black hole**

Many galaxies have supermassive black holes at their centers (see Section 13.8). As material swirls around such a black hole, it is heated, becomes ionized, and generates strong magnetic fields. The resulting magnetic forces steer some of the material into high-speed jets that blast out of the galaxy and into intergalactic space (Fig. 37.19). The light we observe from the jet in Fig. 37.19 has a frequency of 6.66×10^{14} Hz (in the far ultraviolet region of the electromagnetic spectrum; see Fig. 32.4), but in the reference frame of the jet material the light has a frequency of 5.55×10^{13} Hz (in the infrared). What is the speed of the jet material with respect to us?

37.19 This image shows a fast-moving jet 5000 light-years in length emanating from the center of the galaxy M87. The light from the jet is emitted by fast-moving electrons spiraling around magnetic field lines (see Fig. 27.18).



SOLUTION

IDENTIFY and SET UP: This problem involves the Doppler effect for electromagnetic waves. The frequency we observe is $f = 6.66 \times 10^{14}$ Hz, and the frequency in the frame of the source is $f_0 = 5.55 \times 10^{13}$ Hz. Since $f > f_0$, the jet is approaching us and we use Eq. (37.25) to find the target variable u.

EXECUTE: We need to solve Eq. (37.25) for u. We'll leave it as an exercise for you to show that the result is

$$u = \frac{(f/f_0)^2 - 1}{(f/f_0)^2 + 1}c$$

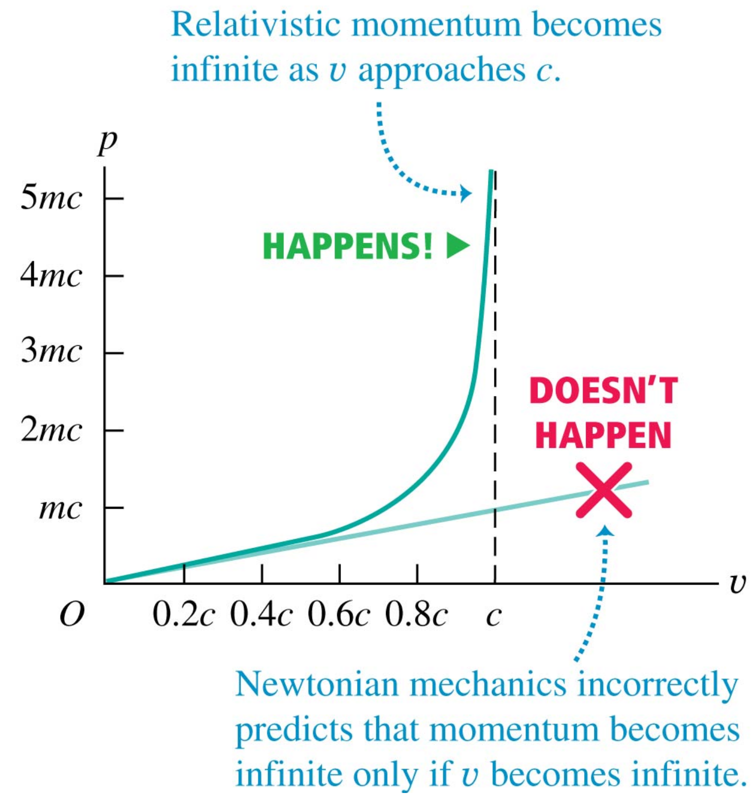
We have $f/f_0 = (6.66 \times 10^{14}\ \text{Hz})/(5.55 \times 10^{13}\ \text{Hz}) = 12.0$, so

$$u = \frac{(12.0)^2 - 1}{(12.0)^2 + 1}c = 0.986c$$

EVALUATE: Because the frequency shift is quite substantial, it would have been erroneous to use the approximate expression $\Delta f/f = u/c$. Had you done so, you would have found $u = c(\Delta f/f_0) = c(6.66 \times 10^{14}\ \text{Hz} - 5.55 \times 10^{13}\ \text{Hz})/(5.55 \times 10^{13}\ \text{Hz}) = 11.0c$. This result cannot be correct because the jet material cannot travel faster than light.

Relativistic momentum

- Principle of conservation of momentum should valid in **ALL inertia frames**
- From the Lorentz transformation of relative velocities, we can show that the Newtonian prediction, $p = mv$, only gives correct results at speeds much less than c.
- Shown is a graph of the magnitude of the momentum of a particle of rest mass m as a function of speed v.



Relativistic momentum

- Suppose we measure the mass of a particle to be *m* when it is at rest relative to us: We call *m* the **rest mass**.
- When such a particle has a velocity ***v***, its relativistic momentum is:

Rest mass of particle

Velocity of particle

Relativistic momentum $$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}}$$

Speed of particle

Speed of light in vacuum

- We can rewrite this in terms of the Lorentz factor of the particle's rest frame with respect to the rest frame of the system:

Rest mass of particle

Velocity of particle

Relativistic momentum $$\vec{p} = \gamma m\vec{v}$$

Lorentz factor relating rest frame of particle and frame of observer

Example 37.9 Relativistic dynamics of an electron

An electron (rest mass 9.11×10^{-31} kg, charge -1.60×10^{-19} C) is moving opposite to an electric field of magnitude $E = 5.00 \times 10^5$ N/C. All other forces are negligible in comparison to the electric-field force. (a) Find the magnitudes of momentum and of acceleration at the instants when $v = 0.010c$, $0.90c$, and $0.99c$. (b) Find the corresponding accelerations if a net force of the same magnitude is perpendicular to the velocity.

SOLUTION

IDENTIFY and SET UP: In addition to the expressions from this section for relativistic momentum and acceleration, we need the relationship between electric force and electric field from Chapter 21. In part (a) we use Eq. (37.31) to determine the magnitude of momentum; the force acts along the same line as the velocity, so we use Eq. (37.32) to determine the magnitude of acceleration. In part (b) the force is perpendicular to the velocity, so we use Eq. (37.33) rather than Eq. (37.32).

EXECUTE: (a) For $v = 0.010c$, $0.90c$, and $0.99c$ we have $\gamma = \sqrt{1 - v^2/c^2} = 1.00$, 2.29, and 7.09, respectively. The values of the momentum magnitude $p = \gamma m v$ are

$$\begin{aligned} p_1 &= (1.00)(9.11 \times 10^{-31}\ \text{kg})(0.010)(3.00 \times 10^8 \text{m/s}) \\ &= 2.7 \times 10^{-24}\ \text{kg}\cdot\text{m/s at } v_1 = 0.010c \end{aligned}$$

$$\begin{aligned} p_2 &= (2.29)(9.11 \times 10^{-31}\ \text{kg})(0.90)(3.00 \times 10^8\ \text{m/s}) \\ &= 5.6 \times 10^{-22}\ \text{kg}\cdot\text{m/s at } v_2 = 0.90c \end{aligned}$$

$$\begin{aligned} p_3 &= (7.09)(9.11 \times 10^{-31}\ \text{kg})(0.99)(3.00 \times 10^8\ \text{m/s}) \\ &= 1.9 \times 10^{-21}\ \text{kg}\cdot\text{m/s at } v_3 = 0.99c \end{aligned}$$

From Eq. (21.4), the magnitude of the force on the electron is

$$\begin{aligned} F &= |q|E = (1.60 \times 10^{-19}\ \text{C})(5.00 \times 10^5\ \text{N/C}) \\ &= 8.00 \times 10^{-14}\ \text{N} \end{aligned}$$

From Eq. (37.32), $a = F/\gamma^3 m$. For $v = 0.010c$ and $\gamma = 1.00$,

$$a_1 = \frac{8.00 \times 10^{-14}\ \text{N}}{(1.00)^3(9.11 \times 10^{-31}\ \text{kg})} = 8.8 \times 10^{16}\ \text{m/s}^2$$

The accelerations at the two higher speeds are smaller than the nonrelativistic value by factors of $\gamma^3 = 12.0$ and 356, respectively:

$$a_2 = 7.3 \times 10^{15}\ \text{m/s}^2 \qquad a_3 = 2.5 \times 10^{14}\ \text{m/s}^2$$

(b) From Eq. (37.33), $a = F/\gamma m$ if $\vec{F}$ and $\vec{v}$ are perpendicular. When $v = 0.010c$ and $\gamma = 1.00$,

$$a_1 = \frac{8.00 \times 10^{-14}\ \text{N}}{(1.00)(9.11 \times 10^{-31}\ \text{kg})} = 8.8 \times 10^{16}\ \text{m/s}^2$$

Now the accelerations at the two higher speeds are smaller by factors of $\gamma = 2.29$ and 7.09, respectively:

$$a_2 = 3.8 \times 10^{16}\ \text{m/s}^2 \qquad a_3 = 1.2 \times 10^{16}\ \text{m/s}^2$$

These accelerations are larger than the corresponding ones in part (a) by factors of γ^2.

EVALUATE: Our results in part (a) show that at higher speeds, the relativistic values of momentum differ more and more from the nonrelativistic values calculated from $p = mv$. The momentum at $0.99c$ is more than three times as great as at $0.90c$ because of the increase in the factor γ. Our results also show that the acceleration drops off very quickly as v approaches c.

Relativistic work and energy

- Experiments show that the net force,

$$\boxed{\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\frac{m\vec{v}}{\sqrt{1 - v^2/c^2}}}$$

- And we can re-derive the work-energy theorem:

$$W = \int_{x_1}^{x_2} F\,dx = \int_{x_1}^{x_2} \frac{ma\,dx}{(1 - v^2/c^2)^{3/2}}$$ = change of kinetic energy

Relativistic energy and rest energy

- The relativistic kinetic energy is:

Rest mass of particle

Speed of light in vacuum

Relativistic kinetic energy

$$K = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 = (\gamma - 1)mc^2$$

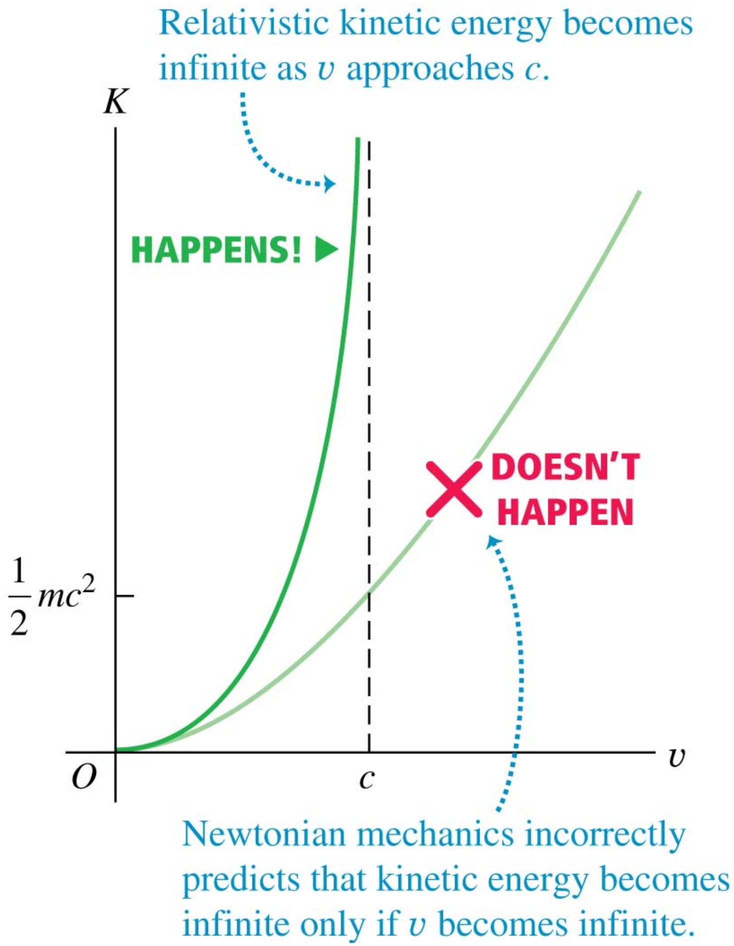
Speed of particle

Lorentz factor relating rest frame of particle and frame of observer

- Note that the kinetic energy approaches infinity as the speed approaches the speed of light.
- The **rest energy** is mc^2.

Relativistic work and energy

- Graph of the kinetic energy of a particle of rest mass *m* as a function of speed *v*.
- Also shown is the Newtonian prediction, which gives correct results only at speeds much less than *c*



Relativistic energy and momentum

- The **total energy** of a particle is:

Total energy of a particle

$$E = \overset{\text{Kinetic energy}}{K} + \overbrace{mc^2}^{\text{Rest energy}} = \frac{\overset{\text{Rest mass of particle}}{m}\,\overset{\text{Speed of light in vacuum}}{c^2}}{\sqrt{1 - \underset{\text{Speed of particle}}{v}^2/c^2}} = \underset{\substack{\text{Lorentz factor relating rest frame}\\ \text{of particle and frame of observer}}}{\gamma} mc^2$$

- The total energy, rest energy, and momentum are related by:

Total energy, rest energy, and momentum:

$$\overset{\text{Total energy}}{E^2} = (\overbrace{\underset{\text{Rest mass}}{m}\,\underset{\text{Speed of light in vacuum}}{c^2}}^{\text{Rest energy}})^2 + (\overset{\text{Magnitude of momentum}}{p}\,\underset{\text{Speed of light in vacuum}}{c})^2$$

Example 37.10 Energetic electrons

(a) Find the rest energy of an electron ($m = 9.109 \times 10^{-31}$ kg, $q = -e = -1.602 \times 10^{-19}$ C) in joules and in electron volts. (b) Find the speed of an electron that has been accelerated by an electric field, from rest, through a potential increase of 20.0 kV or of 5.00 MV (typical of a high-voltage x-ray machine).

SOLUTION

IDENTIFY and SET UP: This problem uses the ideas of rest energy, relativistic kinetic energy, and (from Chapter 23) electric potential energy. We use $E = mc^2$ to find the rest energy and Eqs. (37.7) and (37.38) to find the speed that gives the stated total energy.

EXECUTE: (a) The rest energy is

$$mc^2 = (9.109 \times 10^{-31}\ \text{kg})(2.998 \times 10^8\ \text{m/s})^2 = 8.187 \times 10^{-14}\ \text{J}$$

From the definition of the electron volt in Section 23.2, 1 eV = 1.602×10^{-19} J. Using this, we find

$$mc^2 = (8.187 \times 10^{-14}\ \text{J})\frac{1\ \text{eV}}{1.602 \times 10^{-19}\ \text{J}} = 5.11 \times 10^5\ \text{eV} = 0.511\ \text{MeV}$$

(b) In calculations such as this, it is often convenient to work with the quantity $\gamma = 1/\sqrt{1 - v^2/c^2}$ from Eq. (37.38). Solving this for v, we find

$$v = c\sqrt{1 - (1/\gamma)^2}$$

The total energy E of the accelerated electron is the sum of its rest energy mc^2 and the kinetic energy eV_{ba} that it gains from the work done on it by the electric field in moving from point a to point b:

$$E = \gamma mc^2 = mc^2 + eV_{ba} \quad \text{or}$$

$$\gamma = 1 + \frac{eV_{ba}}{mc^2}$$

An electron accelerated through a potential increase of V_{ba} = 20.0 kV gains 20.0 keV of energy, so for this electron

$$\gamma = 1 + \frac{20.0 \times 10^3\ \text{eV}}{0.511 \times 10^6\ \text{eV}} = 1.039$$

and

$$v = c\sqrt{1 - (1/1.039)^2} = 0.272c = 8.15 \times 10^7\ \text{m/s}$$

Repeating the calculation for $V_{ba} = 5.00$ MV, we find $eV_{ba}/mc^2 = 9.78$, $\gamma = 10.78$, and $v = 0.996c$.

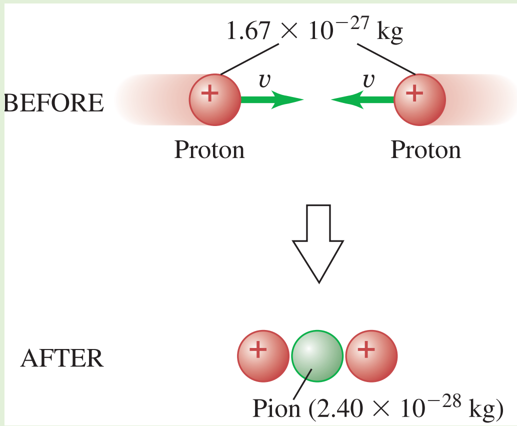
EVALUATE: With $V_{ba} = 20.0$ kV, the added kinetic energy of 20.0 keV is less than 4% of the rest energy of 0.511 MeV, and the final speed is about one-fourth the speed of light. With V_{ba} = 5.00 MV, the added kinetic energy of 5.00 MeV is much greater than the rest energy and the speed is close to c.

CAUTION **Three electron energies** All electrons have *rest* energy 0.511 MeV. An electron accelerated from rest through a 5.00-MeV potential increase has *kinetic* energy 5.00 MeV (we call it a "5.00-MeV electron") and *total* energy 5.51 MeV. Be careful to distinguish these energies from one another.

Example 37.11 A relativistic collision

Two protons (each with mass $M_p = 1.67 \times 10^{-27}$ kg) are initially moving with equal speeds in opposite directions. They continue to exist after a head-on collision that also produces a neutral pion of mass $M_\pi = 2.40 \times 10^{-28}$ kg (Fig. 37.22). If all three particles are at rest after the collision, find the initial speed of the protons. Energy is conserved in the collision.

37.22 In this collision the kinetic energy of two protons is transformed into the rest energy of a new particle, a pion.



SOLUTION

IDENTIFY and SET UP: Relativistic total energy is conserved in the collision, so we can equate the (unknown) total energy of the two protons before the collision to the combined rest energies of the two protons and the pion after the collision. We then use Eq. (37.38) to find the speed of each proton.

EXECUTE: The total energy of each proton before the collision is γMc^2. By conservation of energy,

$$2(\gamma M_p c^2) = 2(M_p c^2) + M_\pi c^2$$

$$\gamma = 1 + \frac{M_\pi}{2M_p} = 1 + \frac{2.40 \times 10^{-28}\ \text{kg}}{2(1.67 \times 10^{-27}\ \text{kg})} = 1.072$$

From Eq. (37.38), the initial proton speed is

$$v = c\sqrt{1 - (1/\gamma)^2} = 0.360c$$

EVALUATE: The proton rest energy is 938 MeV, so the initial kinetic energy of each proton is $(\gamma - 1)Mc^2 = 0.072Mc^2 = (0.072)(938\ \text{MeV}) = 67.5$ MeV. You can verify that the rest energy $M_\pi c^2$ of the pion is twice this, or 135 MeV. All the kinetic energy "lost" in this completely inelastic collision is transformed into the rest energy of the pion.