1. Frictional force

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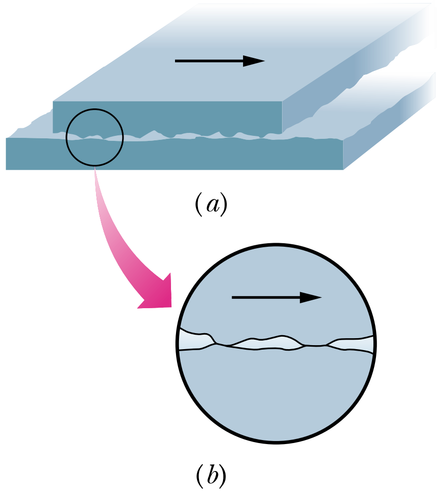
Some particular Forces – Friction

If we either slide or attempt to slide a body over a surface, the motion is resisted by a bonding between the body and the surface. The resistance is considered to be a single force $\mathbf{F_f}$, called either the frictional force or simply friction. This force is directed along the surface, opposite the direction of the intended motion



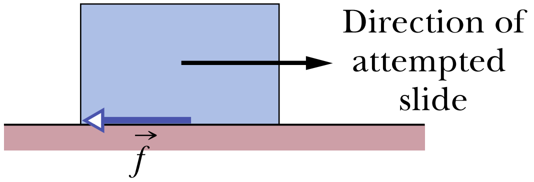
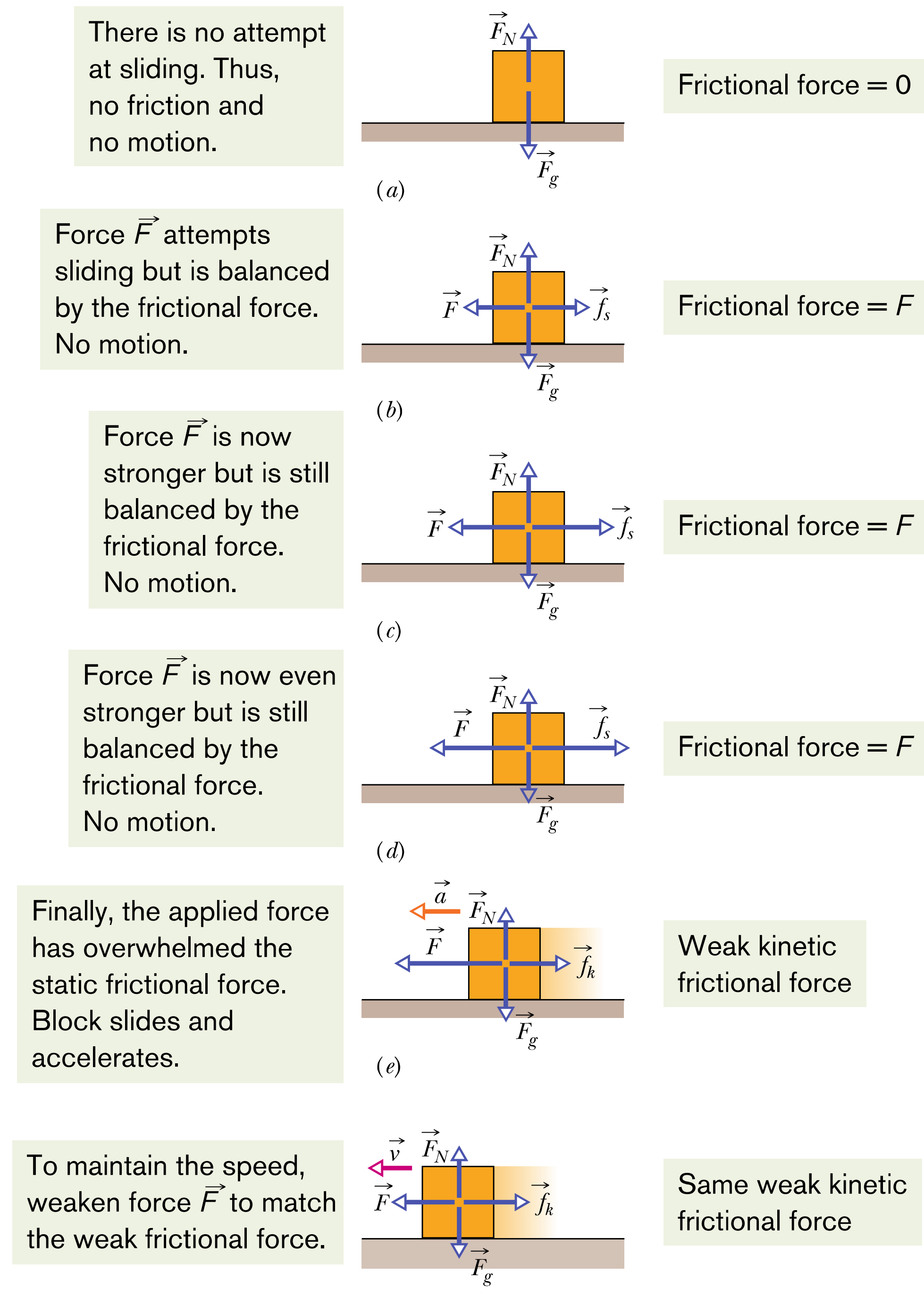
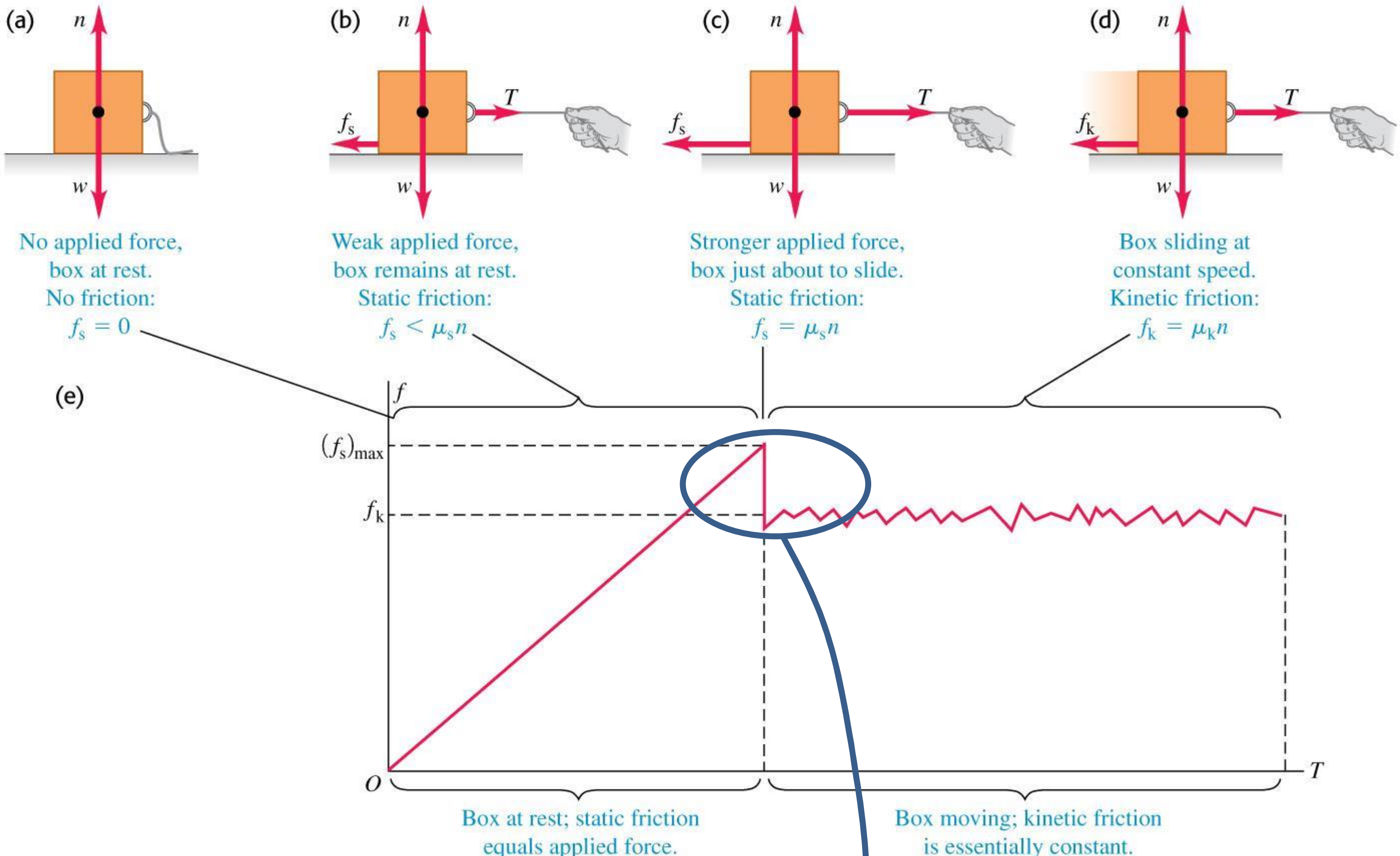


Fig. 5-8 A frictional force $\vec{f}$ opposes the attempted slide of a body over a surface.

Properties of Friction

1. If the body does not move, then the static frictional force $\mathbf{F_s}$ and the component of $\mathbf{F}$ that is parallel to the surface balance each other. They are equal in magnitude, and $\mathbf{F_s}$ is directed opposite that component of $\mathbf{F}$.
2. The maximum value of the static friction is given by, $\mathbf{F_{s,max}=\mu_s F_N}$ where $\boldsymbol{\mu}_s$ is the coefficient of static friction.
3. If the body begins to slide along the surface, the magnitude of the frictional force rapidly decreases to a value $\mathbf{F_k}$ given by, $\mathbf{F_k=\mu_k F_N}$ where $\boldsymbol{\mu}_k$ is the coefficient of kinetic friction.





Interpretation: easier to keep the
block moving than to start it moving

Static & Kinetic Friction Coefficients

Material	Coefficient of Static Friction μ_S	Coefficient of Kinetic Friction μ_S
Rubber on Glass	2.0+	2.0
Rubber on Concrete	1.0	0.8
Steel on Steel	0.74	0.57
Wood on Wood	0.25 – 0.5	0.2
Metal on Metal	0.15	0.06
Paper on paper	0.28	
Synovial Joints in Humans	0.01	0.003

Example 2

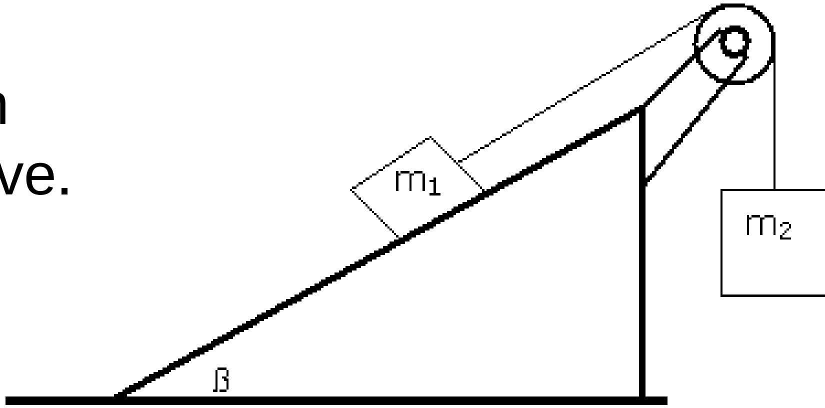
The pulley is frictionless and weightless. The block of mass m_1 is on the plane, inclined at an angle β with the horizontal. The block of mass m_2 is connected to m_1 by a string.

1. Assuming there is no friction, show a formula for the acceleration of the system in terms of m_1, m_2, β and g.

2. What condition is required for m_1 to go up the incline?

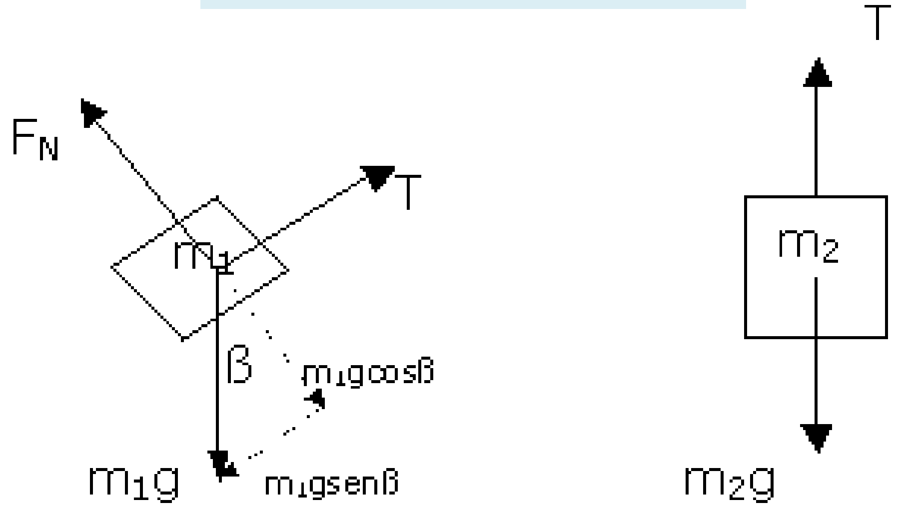
3. Assume that the coefficient of kinetic friction between m_1 and the plane is 0.2, m_1=2kg, m_2=2.5kg and the angle $\beta=30^o$. Calculate the acceleration of m_1 and m_2.

4. What is the maximum value of friction coefficient so the system can still move.



Free Body Diagram - In every problem where the Second Newton's Law applies it is fundamental to draw what is called the Free Body Diagram. This diagram must show all the external forces acting on a body. We isolate the body and the forces due to that strings and surfaces are replaced by arrows; of course, the friction forces and the force of gravity must be included. If there are several bodies, a separate diagram should be drawn for each one.

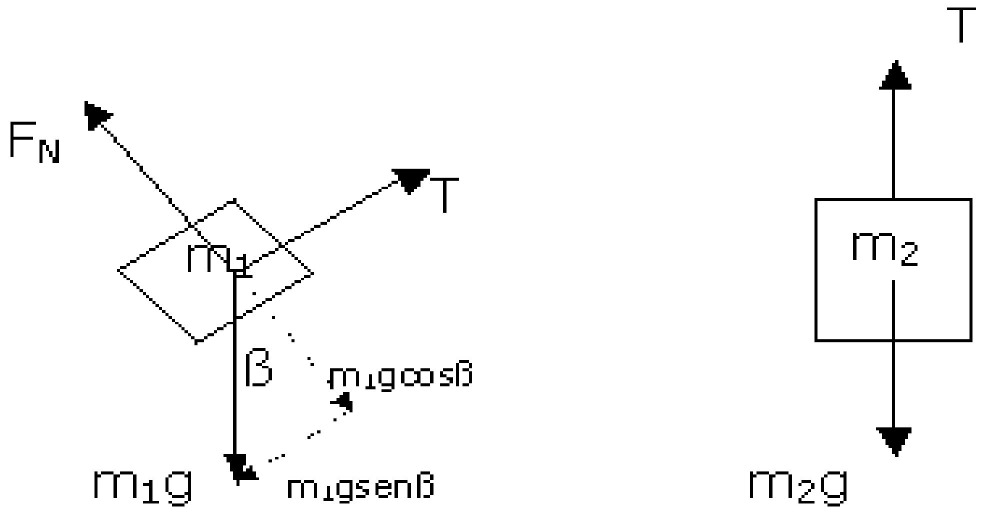
Free Body diagram



Key Observations:

- The (tension) force that m_1 exerts on m_2 through the rope has the same magnitude T. This is so because a rope only changes the direction of a force, not its magnitude assuming a weightless rope.
- The magnitude of the acceleration is the same at both ends of the rope assuming an inextensible rope.

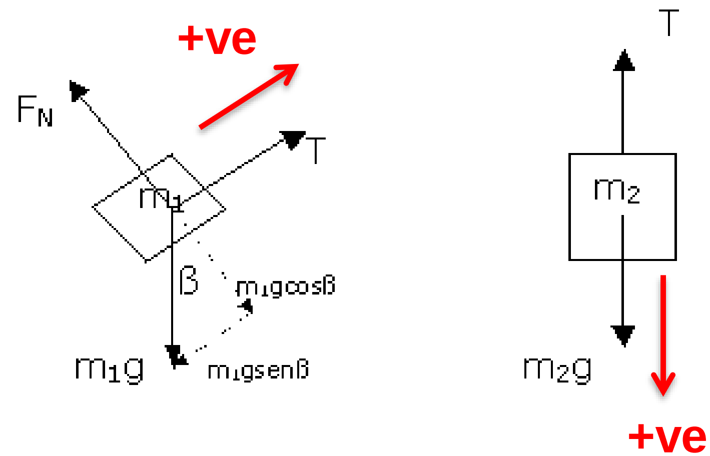
Free Body diagram



Components of forces

Free Body diagram

Notice from the diagram the weight of m_1 has been split into the components $m_1 g\sin\beta$ parallel to the incline, and $m_1 g\cos\beta$ perpendicular to it.



Without friction

1) Let's assume the direction of the acceleration makes m_1 to go upward.

Sum of forces on m_1 in the dirction of the incline plane: $T - m_1 g \sin\beta = m_1 a$

Sum of verticeal forces on m_2 : $m_2 g - T = m_2 a$

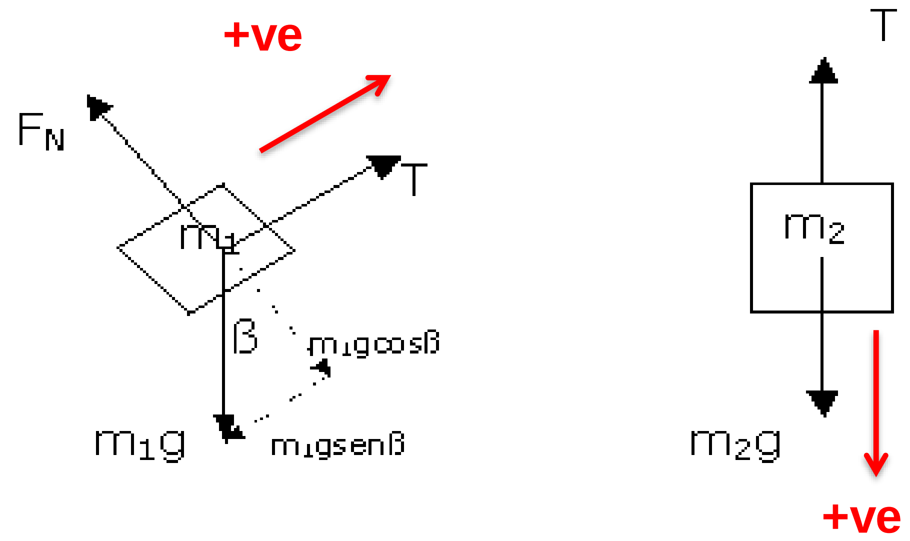
Adding both equations we get $m_2 g - m_1 g \sin\beta = a(m_1 + m_2)$

$$a = g\frac{m_2 - m_1 \sin\beta}{m_1 + m_2}$$

The acceleration of the masses is:

$$a = g\frac{m_2 - m_1 \sin b}{m_1 + m_2}$$

Free Body diagram

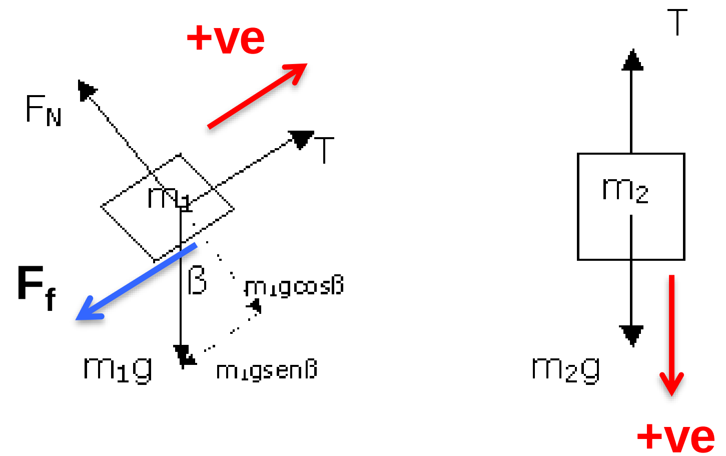


2) For a to be positive (i.e. m_1 going up): $m_2 > m_1 \sin b$

For a to be negative (i.e. m_1 going down): $m_2 < m_1 \sin b$

3) Now appears a friction force, always in an opposite direction to the movement. The magnitude of this friction force is $F_f = \mu F_N$. Where μ is the coefficient of kinetic friction.

Free Body diagram



$F_N - m_1 g\cos\beta = 0$ OR $F_N = m_1 g\cos\beta$

The friction force is then $F_f = \mu m_1 g\cos\beta$.

Hence the sum of forces on m_1 on the incline plane is now:

$T - m_1 g\sin\beta - \mu m_1 g\cos\beta = m_1 a$

The sume of vertical forces on m_2 is:

$m_2 g - T = m_2 a$

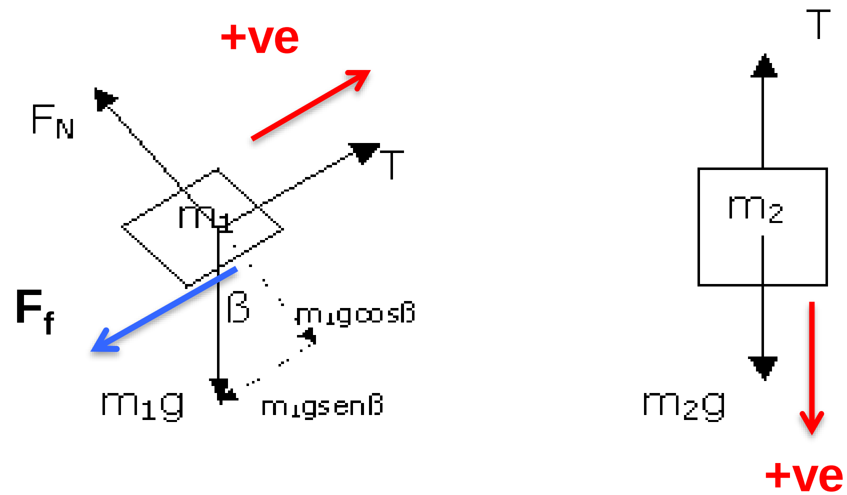
$$\therefore \quad a = \frac{m_2 g - m_1 g(\sin\beta + \mu\cos\beta)}{m_1 + m_2}$$

Replacing values, we have $a = 2.51\,m/s^2$

The acceleration of the masses is:

$$\boxed{a=\frac{m_2 g - m_1 g(\sin\beta+\mu\cos\beta)}{m_1+m_2}}$$

Free Body diagram



As the coefficient of friction μ increases, the acceleration decreases until the acceleration becomes zero. The condition is obtained when:

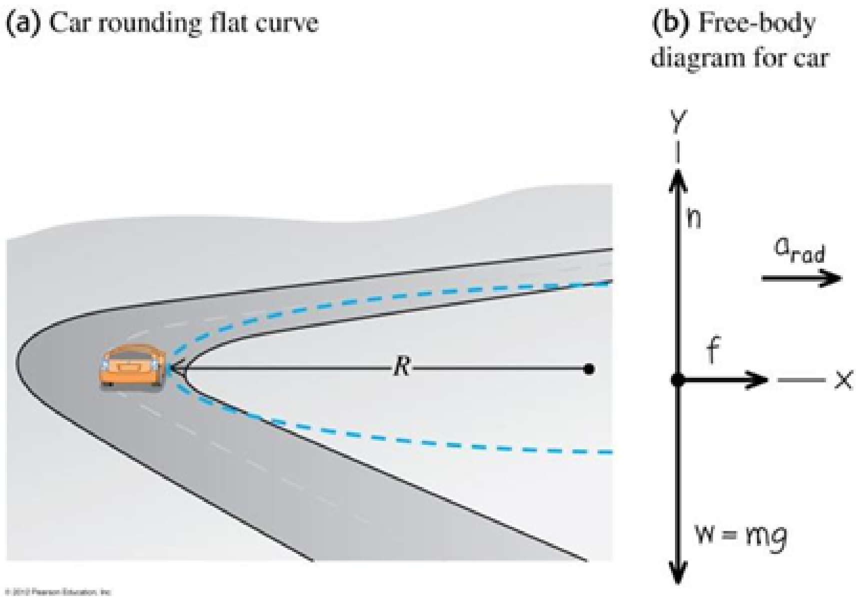
$$m_2 g - m_1 g(\sin\beta+\mu\cos\beta)=0$$

$$\Rightarrow \quad \mu=\frac{m_2 g - m_1 g\sin\beta}{m_1\cos\beta}$$

Replacing values we get μ=0.87.

Example 3: Why banked curves in a racing track help?

On a flat curve



What supplies the centripetal force?

(Static / Kinetic) friction!

Max. speed without skidding:

$$f = f_{max} = m\frac{v_{max}^2}{R} \Rightarrow v_{max} = \sqrt{\mu_s g R}$$

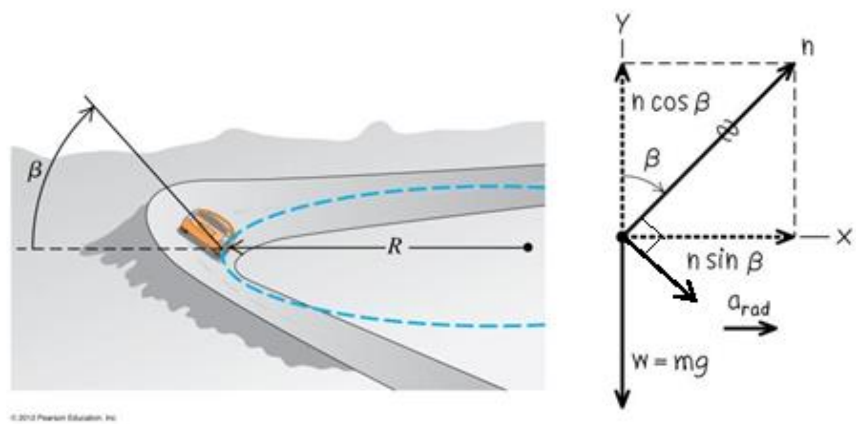
$$\mu_s n = \mu_s m g$$

Example 3: Why banked curves in a racing track help?

If banked at angle

(a) Car rounding banked curve

(b) Free-body diagram for car



What supplies the centripetal force? n and f!

$$\sum F_x = n \sin\beta + f \cos\beta = ma_{\text{rad}}$$

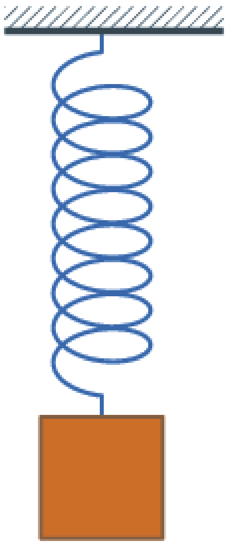
$$\sum F_y = n \cos\beta - f \sin\beta - mg = 0$$

$$f = m\left(\frac{v^2}{R}\cos\beta - g\sin\beta\right) = m\left(\frac{v^2}{R}\sin\beta + g\cos\beta\right)$$

$$f \leq \mu_s n \Rightarrow v \leq v_{\max} = \sqrt{\frac{\tan\beta + \mu_s}{1 - \mu_s \tan\beta} gR} \geq \sqrt{\mu_s g R}$$

Challenging Question:

What happen to the friction f if $v < \sqrt{gR\tan\beta}$? How would you interpret this situation?



1. Hooke’s Law and Simple harmonic motion (SHM)

The Force Law of Springs

Hooke's Law for Springs

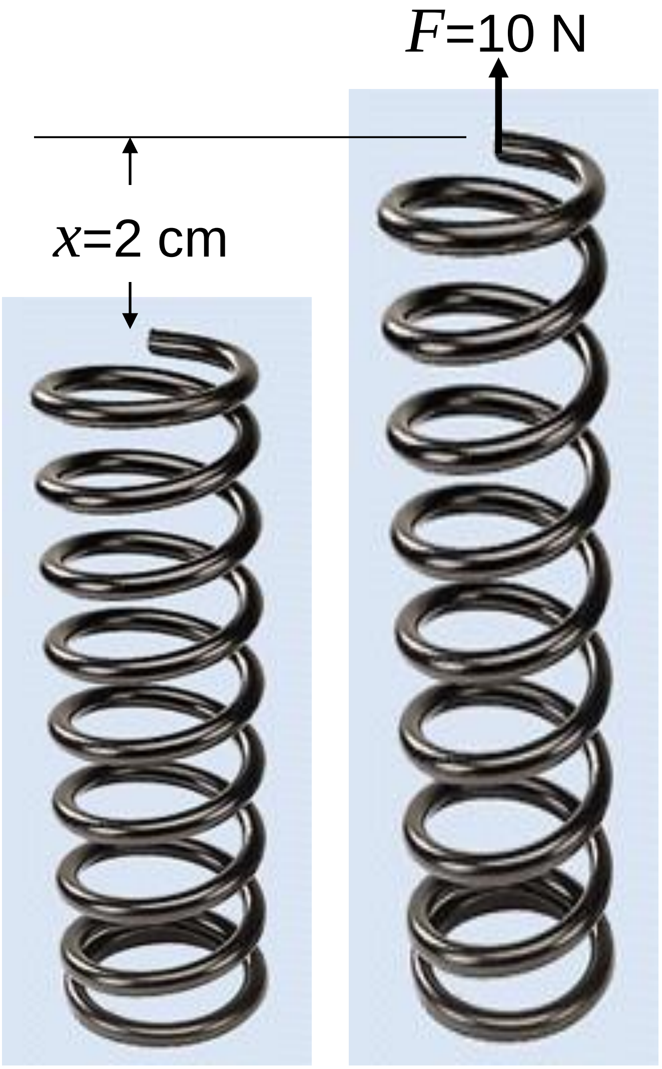
$$F = kx$$

k = spring constant

x = distance that the spring is extended beyond its natural length

For real springs, this is usually a good approximation when x is not too large

$$F = kx$$



Example:

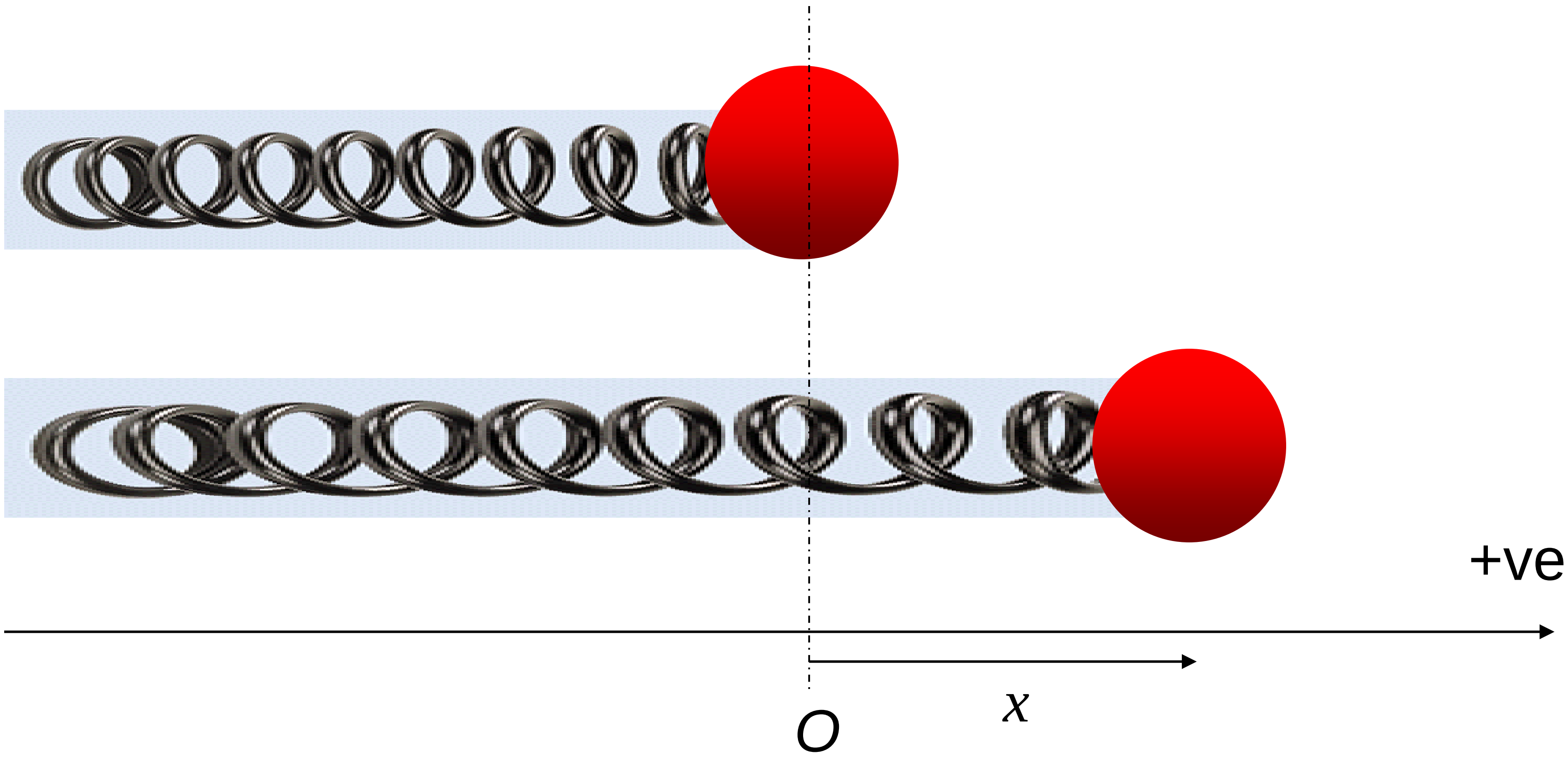
It takes 10 newtons to stretch a spring 2 cm beyond its natural length.

$$10\text{N} = k \cdot 0.02\text{m}$$

$$k = 500 \text{ N/m} \quad \longrightarrow \quad F = 500x$$

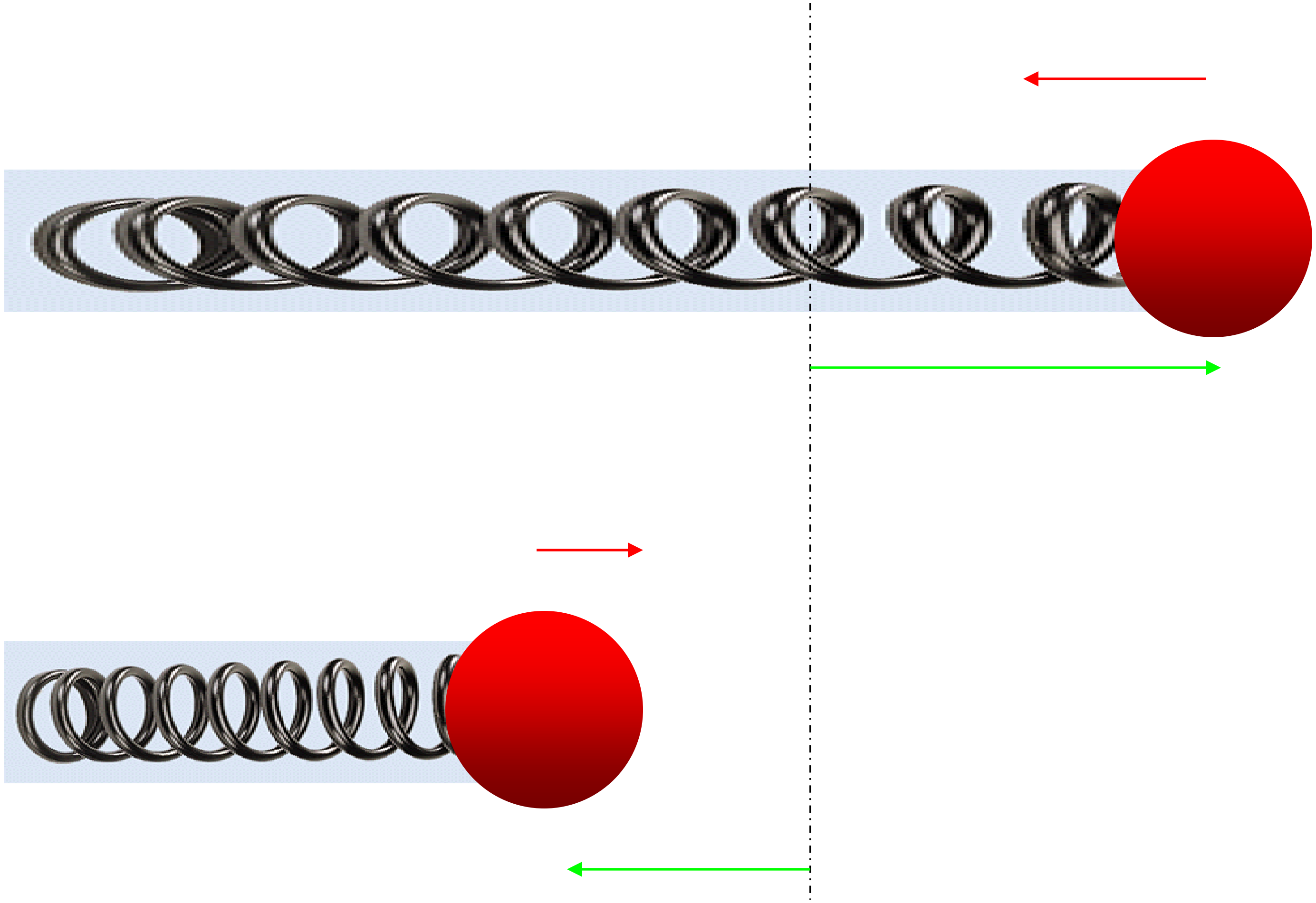
Now consider an object attached to a spring to move along the x-axis

For simplicity, let us take the equilibrium position as the origin, and take the direction at which the spring is stretched as positive



Notice that x and F always have opposite directions
Hence we should write

$$\boxed{F = -kx}$$



If the mass of the object is m, then the equation of motion which leads to the simple harmonic motion (SHM) is

$$a = \frac{d^2 x}{dt^2} = -\frac{k}{m} x = -\omega_0^2 x$$

Hence

$$\boxed{\omega_0 = \sqrt{\frac{k}{m}}}$$

k: the spring constant is the property of the string
m: mass of the object the spring attached to

What happen as the acceleration is not a constant?

Simple Harmonic Motions

- When the acceleration of an object
  - is in the opposite direction to its displacement from a certain position *O*
  - has magnitude directly proportional to its distance from *O*

  its motion is called Simple Harmonic Motion (SHM)
- At *O*, the acceleration is zero, and *O* is called the equilibrium position
- The equation of motion of SHM is

$$\boxed{\frac{d^2x}{dt^2} = -\omega_0^2 x}$$

where $\omega_0 > 0$ is called natural frequency

To study the motion, we need to solve the differential equation:

$$\frac{d^2x}{dt^2} = -\omega_0^2 x$$

What function(s), when differentiated twice, equals $-\omega_0^2$ times itself?

$\cos \omega_0 t$ $\sin \omega_0 t$

$$x(t) = B\cos \omega_0 t + C\sin \omega_0 t$$

where B, C are arbitrary constants

Check:

$$
\begin{aligned}
\frac{d^2}{dt^2}x(t) &= \frac{d^2}{dt^2}\left(B\cos\omega_0 t + C\sin\omega_0 t\right) \\
&= \frac{d}{dt}\left(-B\omega_0\sin\omega_0 t + \omega_0 C\cos\omega_0 t\right) \\
&= -\omega_0^2 B\cos\omega_0 t - \omega_0^2 C\sin\omega_0 t \\
&= -\omega_0^2\left(B\cos\omega_0 t + C\sin\omega_0 t\right) \\
&= -\omega_0^2 x(t)
\end{aligned}
$$

Natural Frequency

The general solution is:

$$x(t) = B\cos\omega_0 t + C\sin\omega_0 t$$

ω_0 has unit s^{-1} = Hz and is called the ***natural frequency*** of the SHM

Any values of B and C satisfy the differential equation.

How do we determine the values of B and C uniquely for a specific motion?

Initial Conditions

$$x(t) = B\cos\omega_0 t + C\sin\omega_0 t$$

$$v(t) = \frac{d}{dt}x(t) = \omega_0\left(-B\sin\omega_0 t + C\cos\omega_0 t\right)$$

B, C can be determined by initial conditions:
$v(t_0)$, $x(t_0)$

For simplicity, take $t_0 = 0$

Given $v(0) = v_0$, $x(0) = x_0$

$$x(0) = x_0 \Rightarrow B = x_0 \qquad v(0) = v_0 \Rightarrow C = v_0/\omega_0$$

Summary

General Solution of SHM

$$x(t) = x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t$$

$$v(t) = -\omega_0 x_0 \sin \omega_0 t + v_0 \cos \omega_0 t$$

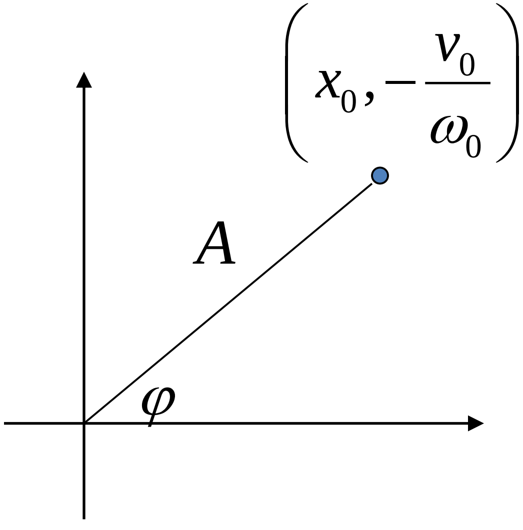
$$a(t) = -\omega_0^2 x(t) = -\omega_0^2 x_0 \cos \omega_0 t - \omega_0 v_0 \sin \omega_0 t$$

The motion is sinusoidal oscillations
We can rewrite it in another form

$$x(t) = A\cos(\omega_0 t + \varphi)$$
$$= A\cos\varphi\cos\omega_0 t - A\sin\varphi\sin\omega_0 t$$

Comparing this with

$$x(t) = x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t)$$



we can find A and φ by locating the point $(x_0, -v_0/\omega_0)$

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_0}\right)^2} \qquad \tan\varphi = -\frac{v_0}{\omega_0 x_0}$$

General Solution of SHM

$$x(t) = A\cos(\omega_0 t + \varphi)$$

$$v(t) = -\omega_0 A\sin(\omega_0 t + \varphi)$$

$$a(t) = -\omega_0^2 A\cos(\omega_0 t + \varphi)$$

where A and φ are obtained by solving

$$\begin{cases} A\cos\varphi = x_0 \\ A\sin\varphi = -v_0/\omega_0 \end{cases}$$

A is called the *amplitude* of the SHM

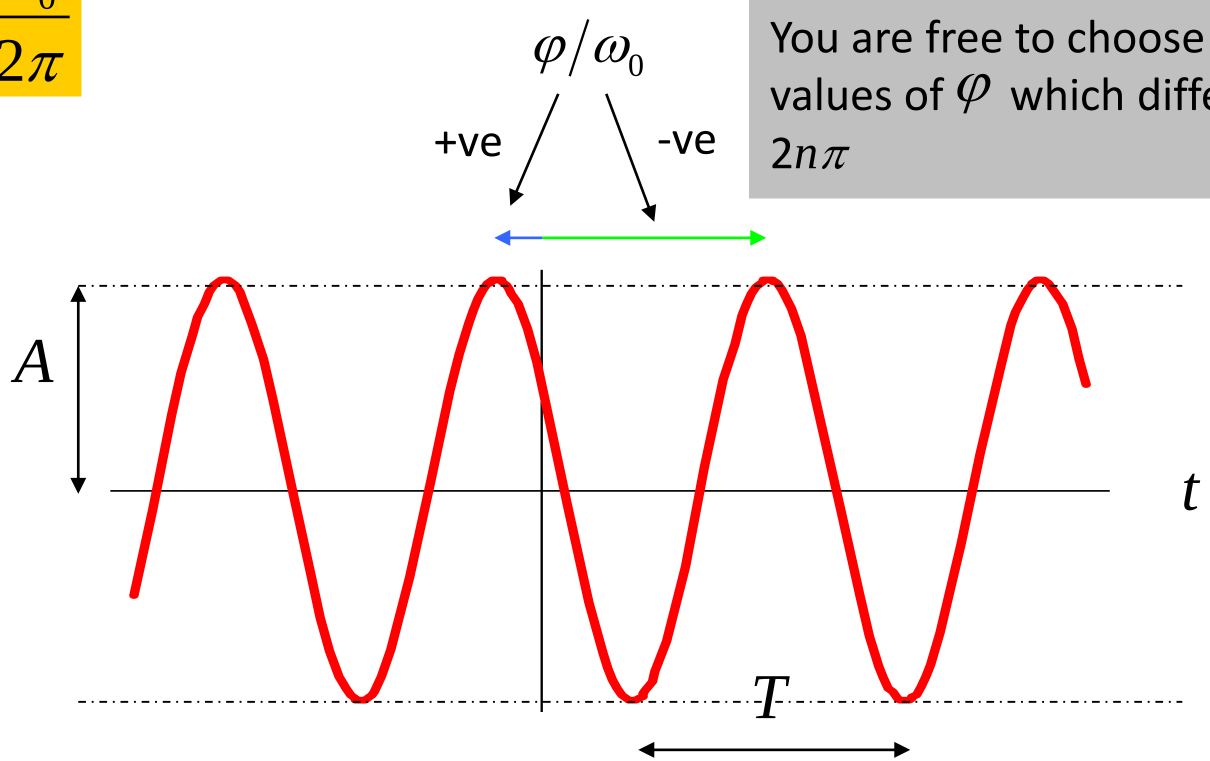
The *period* of the SHM is given by $$T = \frac{2\pi}{\omega_0}$$

The *natural frequency* of the oscillation is given by

$$f_0 = \frac{1}{T} = \frac{\omega_0}{2\pi}$$

You are free to choose any values of φ which differ by $2n\pi$

Note: f_0 and ω_0 are both called the natural frequency



Example: An object is attached to a spring so that it performs SHM with ω_0 = 2 s^{-1} on a smooth table. The spring is initially compressed by 10 cm, and the object has initial speed of 0.5 ms^{-1} (towards the equilibrium position).
Find the period and frequency of the oscillation.

Solution:

$$T = \frac{2\pi}{\omega_0} = \pi \text{ s}$$

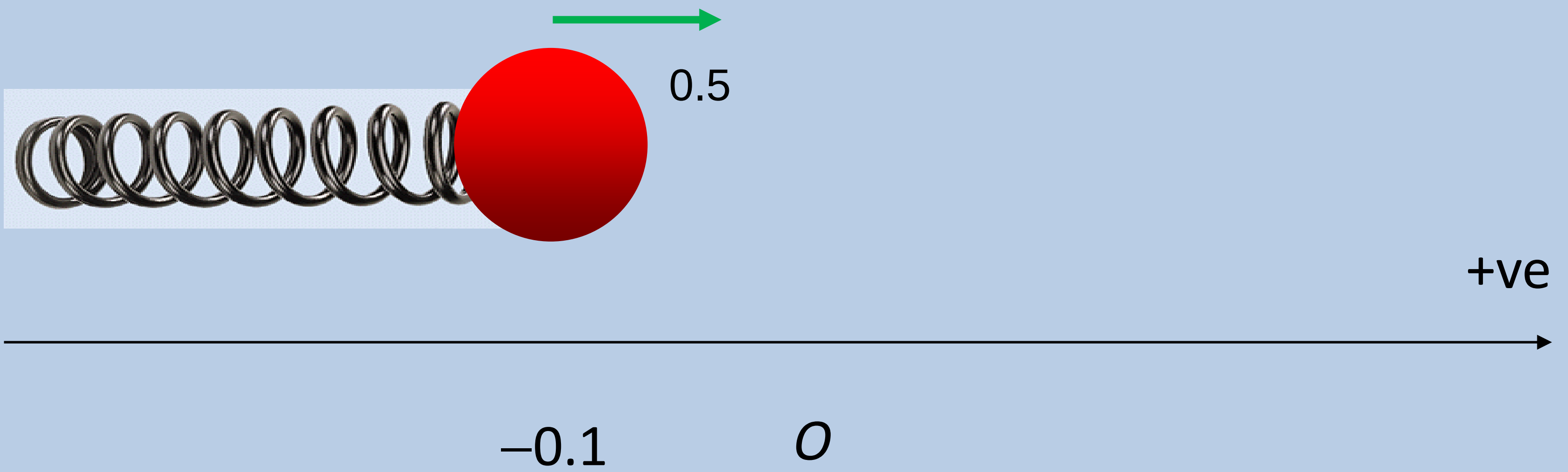
$$f_0 = \frac{1}{T} = \pi^{-1} \text{ Hz}$$

Example: Following the last example, find the amplitude of the oscillation and the phase angle

Solution:

Let the equilibrium position be the origin and the direction at which the spring is stretched be positive, so that

$$x_0 = -0.1 \text{ m},\ v_0 = 0.5 \text{ ms}^{-1}$$

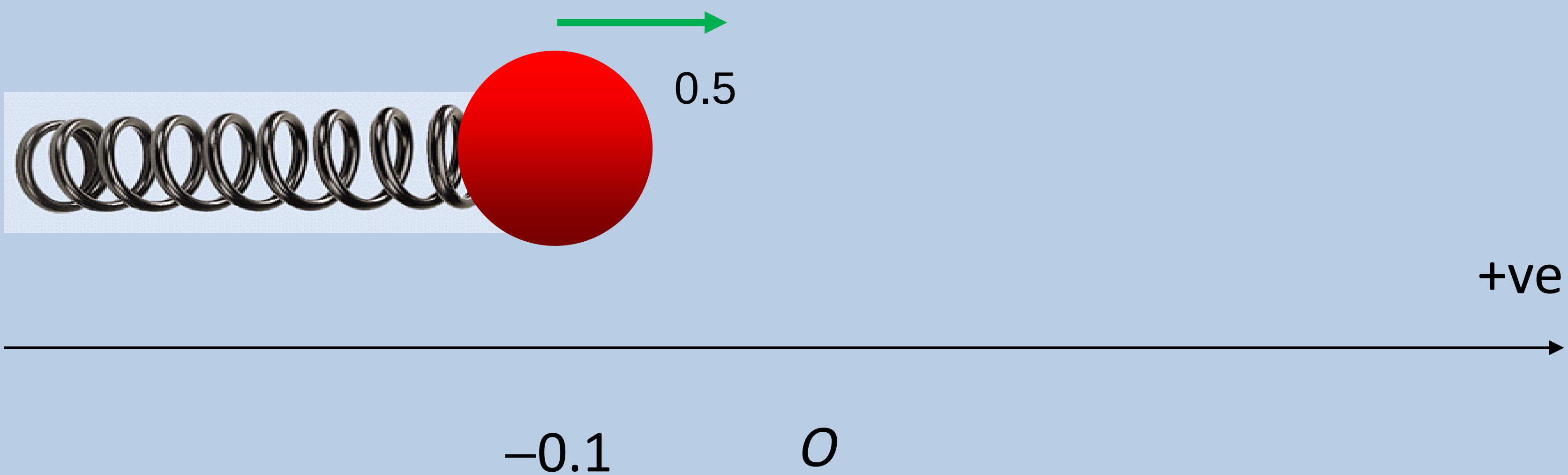


Solution:

To find the amplitude, solve

$$\begin{cases} A\cos\varphi = x_0 = -0.1 \\ A\sin\varphi = -v_0/\omega_0 = -0.5/2 = -0.25 \end{cases}$$

$$A = \sqrt{x_0^2 + \left(v_0/\omega_0\right)^2} = \sqrt{0.1^2 + 0.25^2} \approx 0.27 \text{ m}$$



Solution:

To find the phase angle, solve

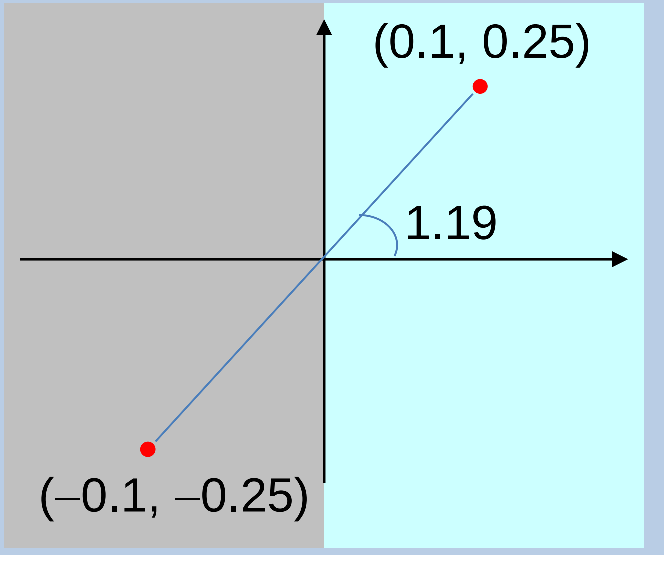
$$\begin{cases} A\cos\varphi = x_0 = -0.1 \\ A\sin\varphi = -v_0/\omega_0 = -0.5/2 = -0.25 \end{cases}$$

$$\tan\varphi = -\frac{v_0}{\omega_0 x_0} = -\frac{0.5}{2\times(-0.1)} = 2.5$$

$$\Rightarrow \varphi = \tan^{-1}(2.5) + n\pi \approx 1.19 + n\pi = 1.19 \text{ or } 1.19 - \pi$$

Obviously 1.19 should be rejected

Hence $\varphi = 1.19 - \pi$



Example: Following the last example, find the position, velocity, and acceleration of the object after 0.7 s

Solution:

$$x(0.7) = A\cos(0.7\omega_0 + \varphi)$$
$$\approx 0.23 \text{ m}$$

$$v(0.7) = -\omega_0 A\sin(0.7\omega_0 + \varphi)$$
$$= 0.28 \text{ ms}^{-1}$$

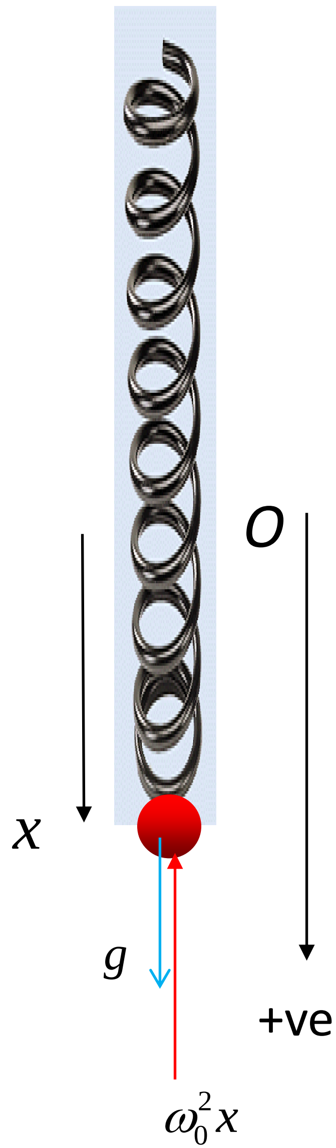
$$a(0.7) = -\omega_0^2 x(0.7) = -0.92 \text{ ms}^{-2}$$

Example: A spring drives an object attached to it to perform SHM with frequency ω_0 in the horizontal direction. If now the spring is vertical and with the same mass attached to it, what will be the motion of the mass?

Solution:

Let *O* be the origin equilibrium position without gravity. Choose *O* as the origin of the vertical axis and take downward as positive. When the object is at x, its acceleration due to the spring is $-\omega_0^2 x$ Now there is another downward acceleration g due to gravity. Hence the acceleration of the mass is

$$a = \frac{d^2x}{dt^2} = -\omega_0^2 x + g$$

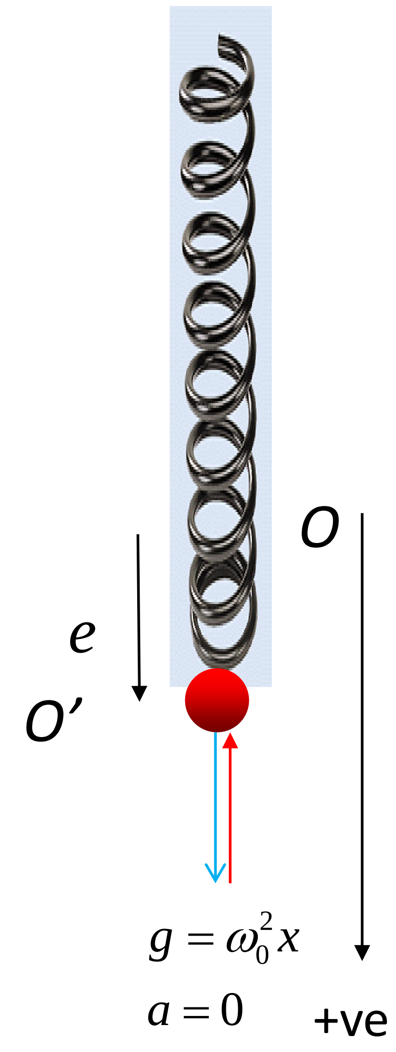


Solution:

Consider the position at which these two accelerations cancel each other, leading to zero total acceleration. This happens at

$$\omega_0^2 e = g \quad \Rightarrow \quad e = g/\omega_0^2$$

Let's call this new equilibrium position *O'*.



Solution:

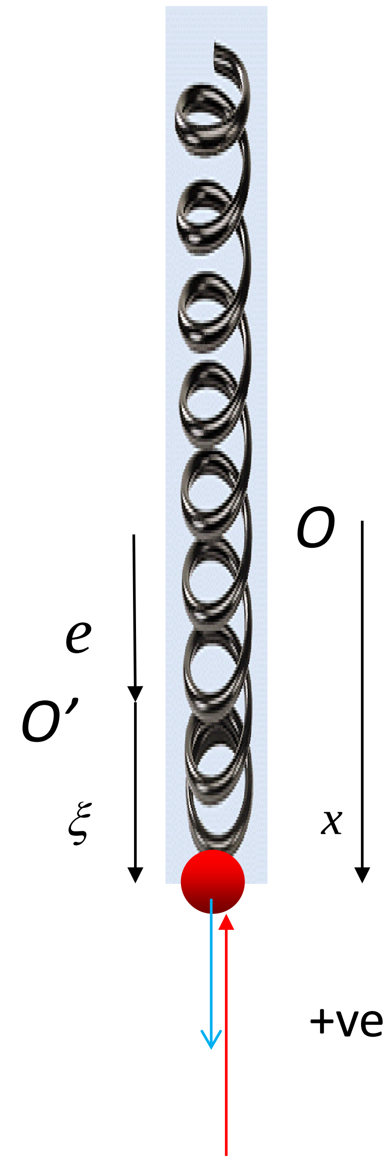
Now if we shift the origin to *O′*, the new coordinate of the mass becomes

$$\xi = x - e = x - g / \omega_0^2$$

The equation of motion becomes

$$\frac{d^2\xi}{dt^2} = \frac{d^2}{dt^2}(x-e) = \frac{d^2x}{dt^2} = -\omega_0^2 x + g$$

$$= -\omega_0^2\left(\xi + g/\omega_0^2\right) + g = -\omega_0^2\xi$$



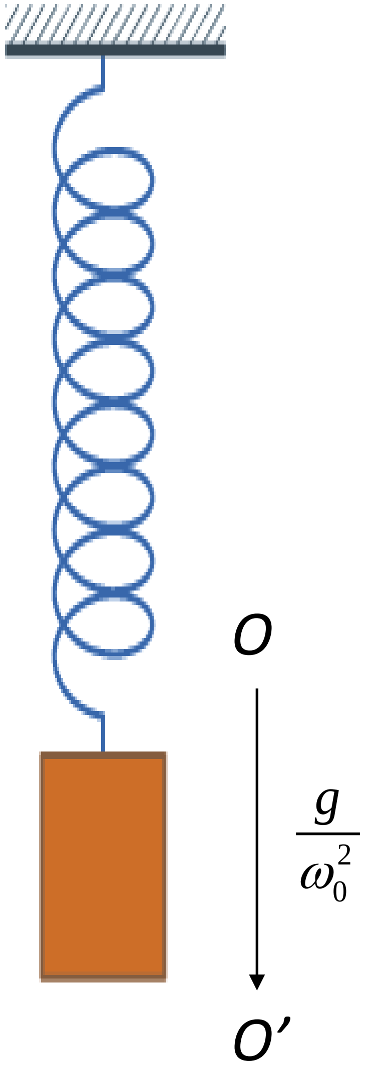
Solution:

$$\frac{d^2\xi}{dt^2} = -\omega_0^2 \xi$$

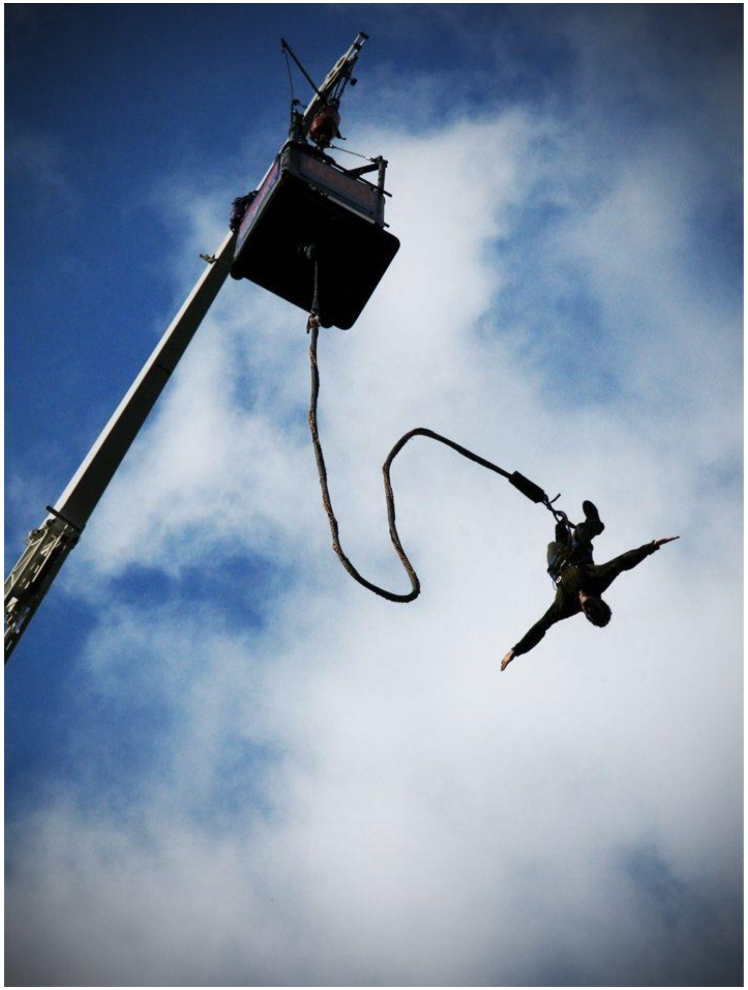
The motion is still SHM but with new equilibrium position at

$$\xi = 0 \;\Rightarrow\; x = g/\omega_0^2$$

The equilibrium position shifts downwards to *O'*



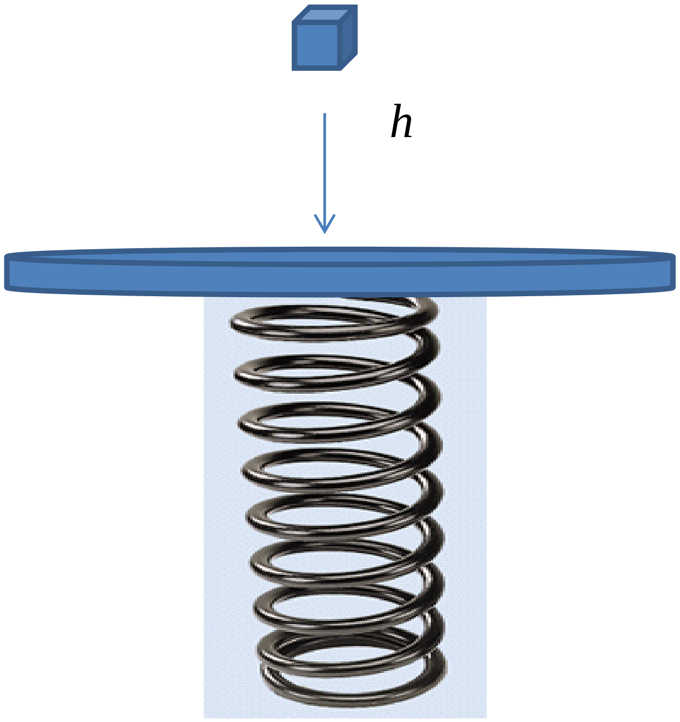
Example: Bungee Jump



https://www.youtube.com/watch?v=zG22qQydPVQ

(Challenge) Example: A spring drives an object M attached to it to perform SHM with frequency ω_0 in the horizontal direction. Now the spring is glued to a plate so that it becomes a balance, which is put on a table.

The object M is now released at a height h above the balance. It is assumed that air resistance and the plate have no effect on the motions. Find the lowest position of the object.



Solution:

First let us find the velocity of the object when it hits the plate.

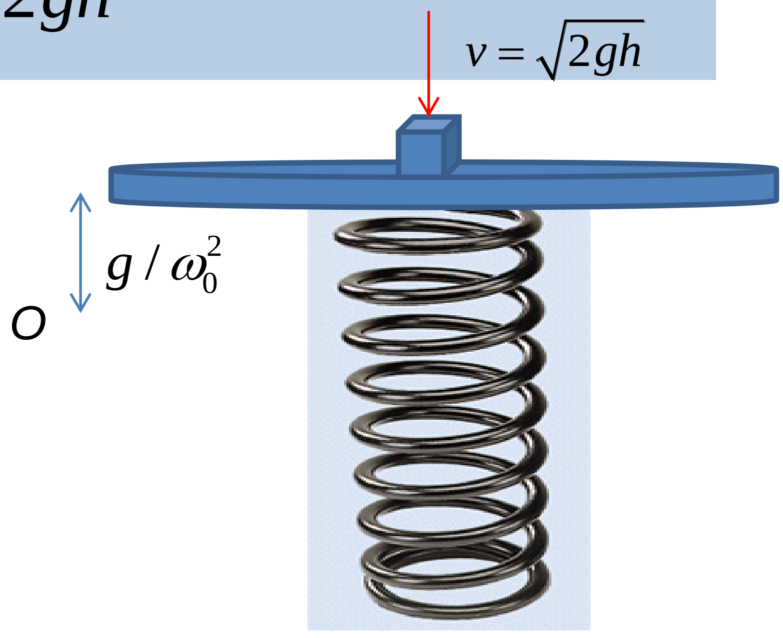
Take downward as positive

$$v^2 - 0^2 = 2gh \Rightarrow v = \sqrt{2gh}$$

Afterwards, the motion will be SHM. The equilibrium position is g/ω_0^2 below the initial height of the plate.

Hence, the initial conditions of the SHM is

$$x_0 = -g/\omega_0^2,\ v_0 = \sqrt{2gh}$$



Solution:

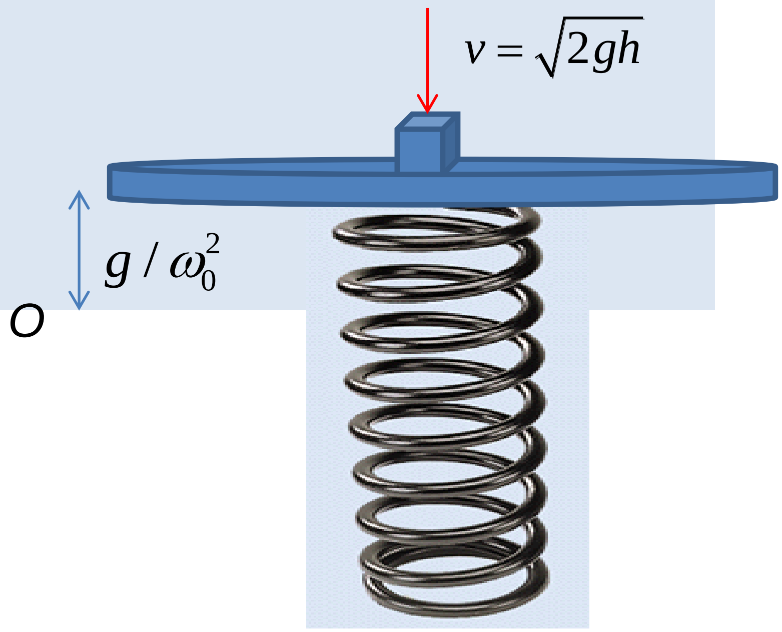
The amplitude of the SHM motion is

$$A=\sqrt{x_0^2+\left(\frac{v_0}{\omega_0}\right)^2}=\sqrt{\left(-\frac{g}{\omega_0^2}\right)^2+\left(\frac{\sqrt{2gh}}{\omega_0}\right)^2}=\frac{\sqrt{g^2+2gh\omega_0^2}}{\omega_0^2}$$

The lowest position is at a distance

$$\frac{\sqrt{g^2+2gh\omega_0^2}}{\omega_0^2}+\frac{g}{\omega_0^2}=\frac{g+\sqrt{g^2+2gh\omega_0^2}}{\omega_0^2}$$

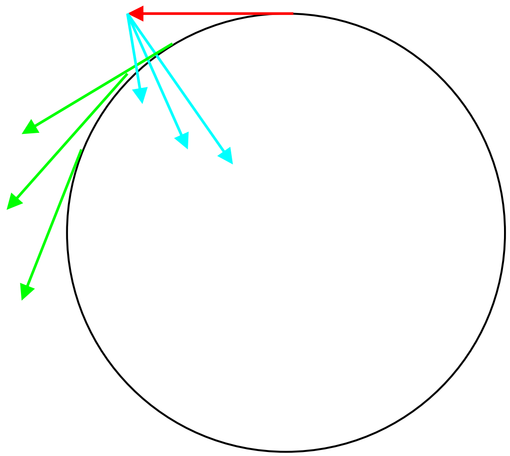
below the original position of the plate

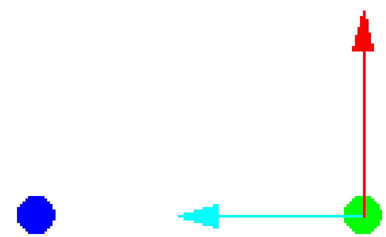


2. Uniform Circular Motion

Centripetal Acceleration

Centripetal acceleration: The acceleration towards the center at which objects under circular motion are falling.





When $\Delta t \rightarrow 0$,
acceleration perpendicular to velocity

To show this rigorously, and to obtain the formula of the acceleration, we need to use calculus

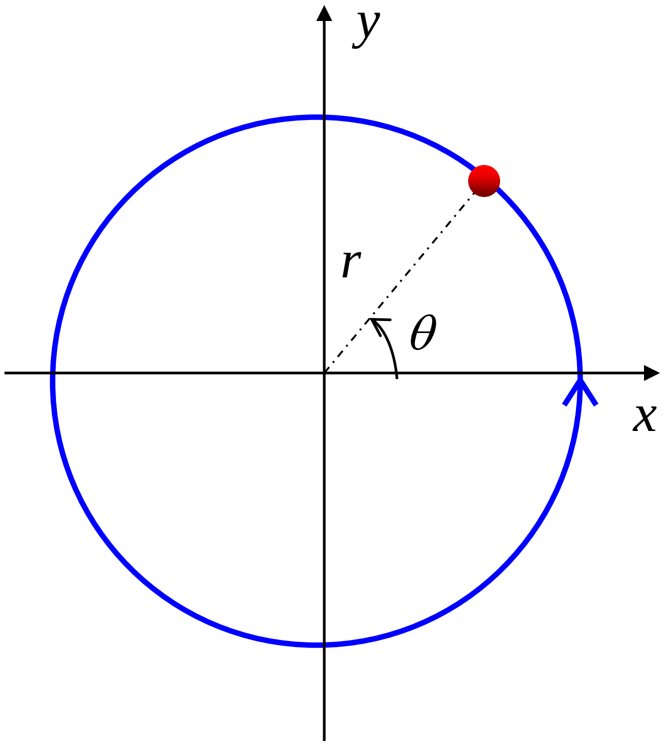
Uniform Circular Motion

Consider an object moving along a circular path of radius r

The position is completely determined by the angle between the positive x-axis and the line joining it to the center, θ

Counterclockwise angle: Positive
Clockwise angle: Negative

θ is a function of time



Angular velocity, ω: rate of change of θ w.r.t. t

$$\omega = \frac{d\theta}{dt}$$

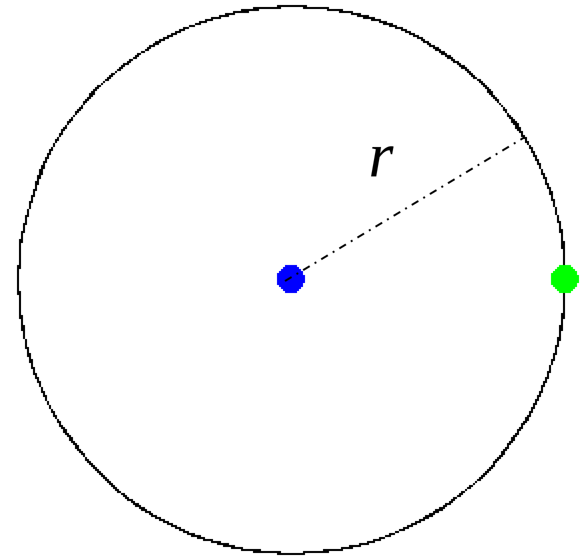
When ω is a constant, it is called
uniform circular motion

If at t = t_0, θ = θ_0, then

$$\boxed{\theta(t) = \theta_0 + \omega(t - t_0)}$$

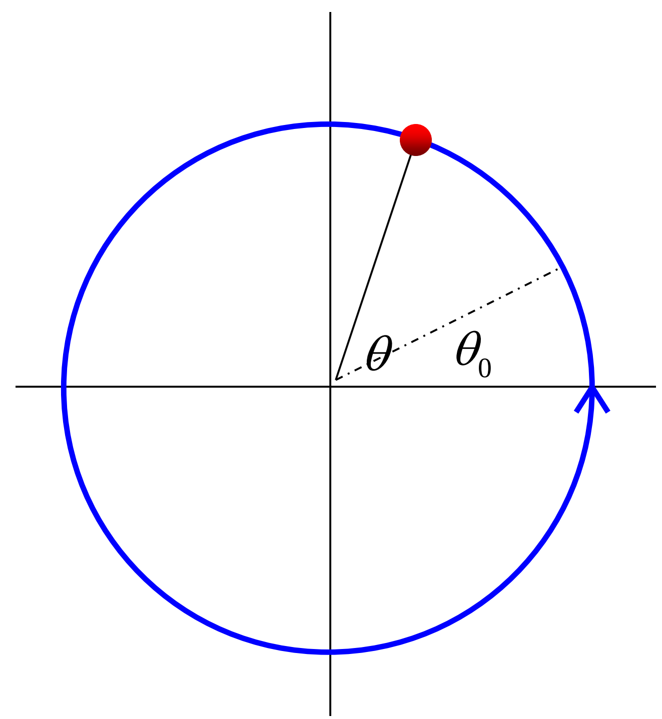
cf.

$$x(t) = x_0 + v_0(t - t_0)$$



For simplicity, hereafter, we shall take
t_0 = 0:

$$\boxed{\theta(t) = \theta_0 + \omega t}$$



Period of circular motion, T:
Time taken to complete one cycle

$$\omega T = 2\pi \quad \Rightarrow \quad \boxed{\omega = \frac{2\pi}{T}}$$

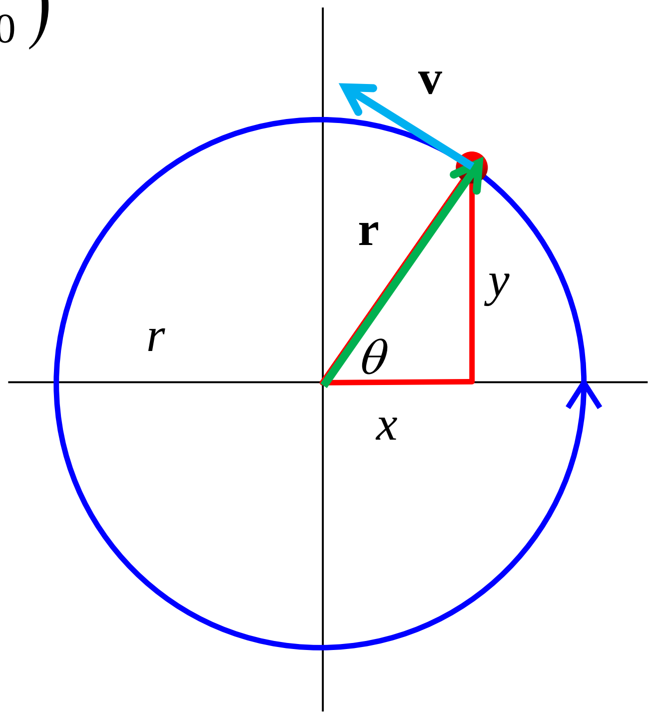
Velocity

The position of the object is $\mathbf{r} = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}}$

where

$$x(t) = r\cos\theta = r\cos(\omega t + \theta_0)$$

$$y(t) = r\sin\theta = r\sin(\omega t + \theta_0)$$



Then the velocity

$$\mathbf{v} = v_x(t)\hat{\mathbf{x}} + v_y(t)\hat{\mathbf{y}}$$

is given by

$$v_x(t) = \frac{dx}{dt} = -\omega r\sin(\omega t + \theta_0)$$

$$v_y(t) = \frac{dy}{dt} = \omega r\cos(\omega t + \theta_0)$$

Speed

The object moves at a constant speed

$$v = \sqrt{v_x^2 + v_y^2} = \omega r$$

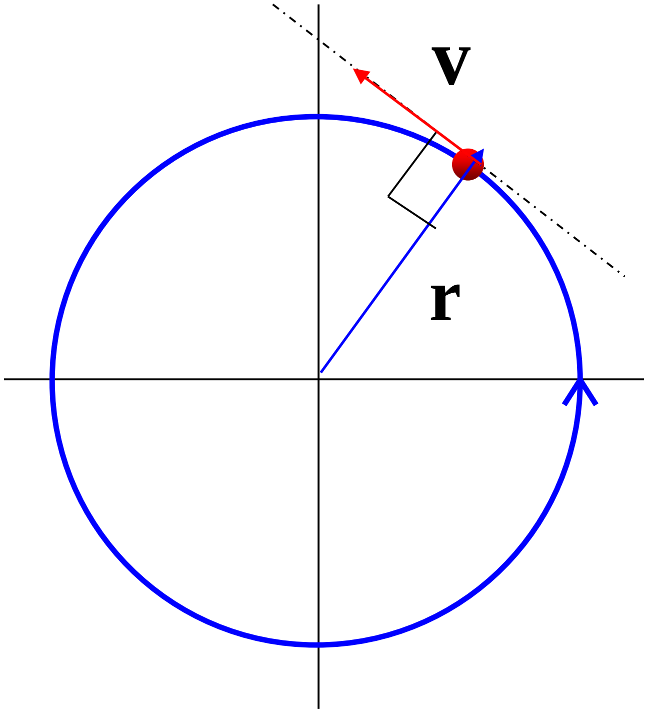
$$v_x(t) = \frac{dx}{dt} = -\omega r \sin(\omega t + \theta_0)$$
$$v_y(t) = \frac{dy}{dt} = \omega r \cos(\omega t + \theta_0)$$

Note: Not constant velocity

The direction of velocity is constantly changing

Direction of Velocity

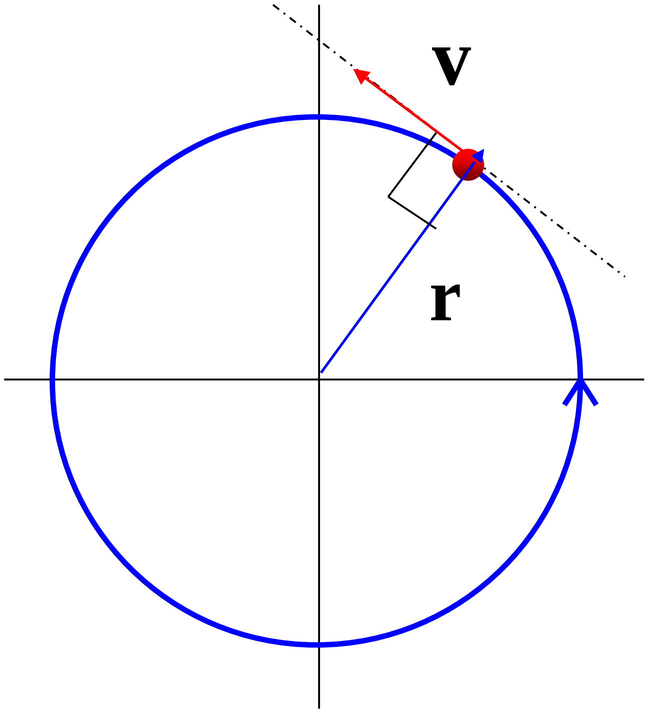
The velocity is always tangential



Direction of Velocity

To prove this, notice that the dot product

$$
\begin{aligned}
\mathbf{r}\cdot\mathbf{v} &= \left(x\hat{\mathbf{x}} + y\hat{\mathbf{y}}\right)\cdot\left(v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}}\right) \\
&= xv_x + yv_y \\
&= \left[r\cos\left(\omega t+\theta_0\right)\right]\times\left[-\omega r\sin\left(\omega t+\theta_0\right)\right] \\
&\quad +\left[r\sin\left(\omega t+\theta_0\right)\right]\times\left[\omega r\cos\left(\omega t+\theta_0\right)\right] \\
&= -\omega r^2\sin\left(\omega t+\theta_0\right)\cos\left(\omega t+\theta_0\right) \\
&\quad +\omega r^2\sin\left(\omega t+\theta_0\right)\cos\left(\omega t+\theta_0\right) \\
&= 0
\end{aligned}
$$



Centripetal Acceleration

The acceleration $\mathbf{a} = a_x(t)\hat{\mathbf{x}} + a_y(t)\hat{\mathbf{y}}$

is given by

$$a_x(t) = \frac{dv_x}{dt} = -\omega^2 r\cos(\omega t + \theta_0)$$

$$a_y(t) = \frac{dv_y}{dt} = -\omega^2 r\sin(\omega t + \theta_0)$$

It is readily observed that

$$\mathbf{a} = -\omega^2\mathbf{r}$$

The magnitude of acceleration

$$a = \omega^2 r = v^2 / r = \omega v$$

Remark

Notice that the speed is constant, although the velocity is constantly changing under acceleration

This is because the acceleration is always perpendicular to velocity

Instantaneous acceleration perpendicular to velocity will not change speed, but only the direction of motion

Example: The orbital period of the moon around the Earth is about 27 days 8 hours. The orbit is approximately circular with a radius of 384000 km. Find (a) its orbital speed and (b) the magnitude of the centripetal acceleration.

Solution:

The period is $T = \left(27 \times 24 + 8\right) \times 3600 = 2361600 \text{ s}$

Hence the angular velocity is $\omega = 2\pi / T \approx 2.66 \times 10^{-6} \text{ s}^{-1}$

The radius is $r = 3.84 \times 10^{8} \text{ m}$

Hence the orbital speed is $v = \omega r \approx 1.02 \text{ km/s}$

The centripetal acceleration is $a = \omega^2 r \approx 0.0027 \text{ ms}^{-2}$

SHM and Circular Motion

Consider an object in constant speed circular motion with angular velocity ω_0 and radius A

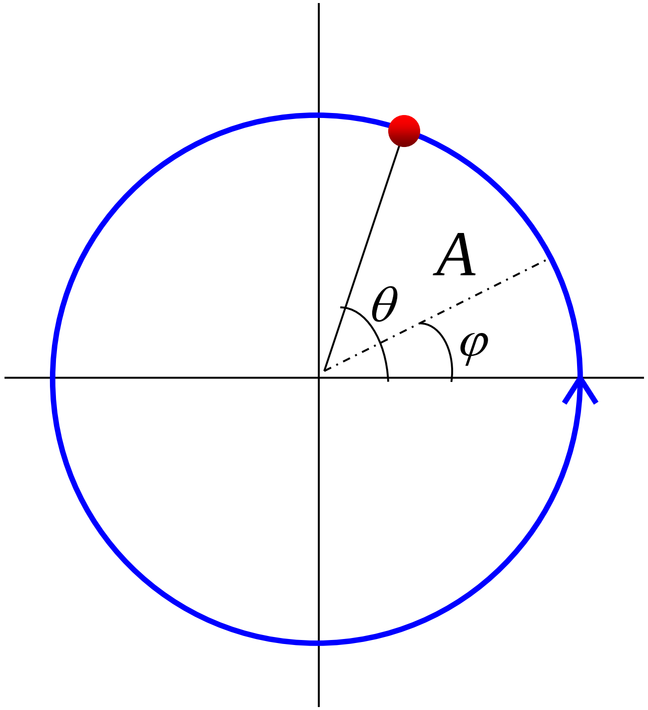
If at t = 0, the object starts at an angle φ , then

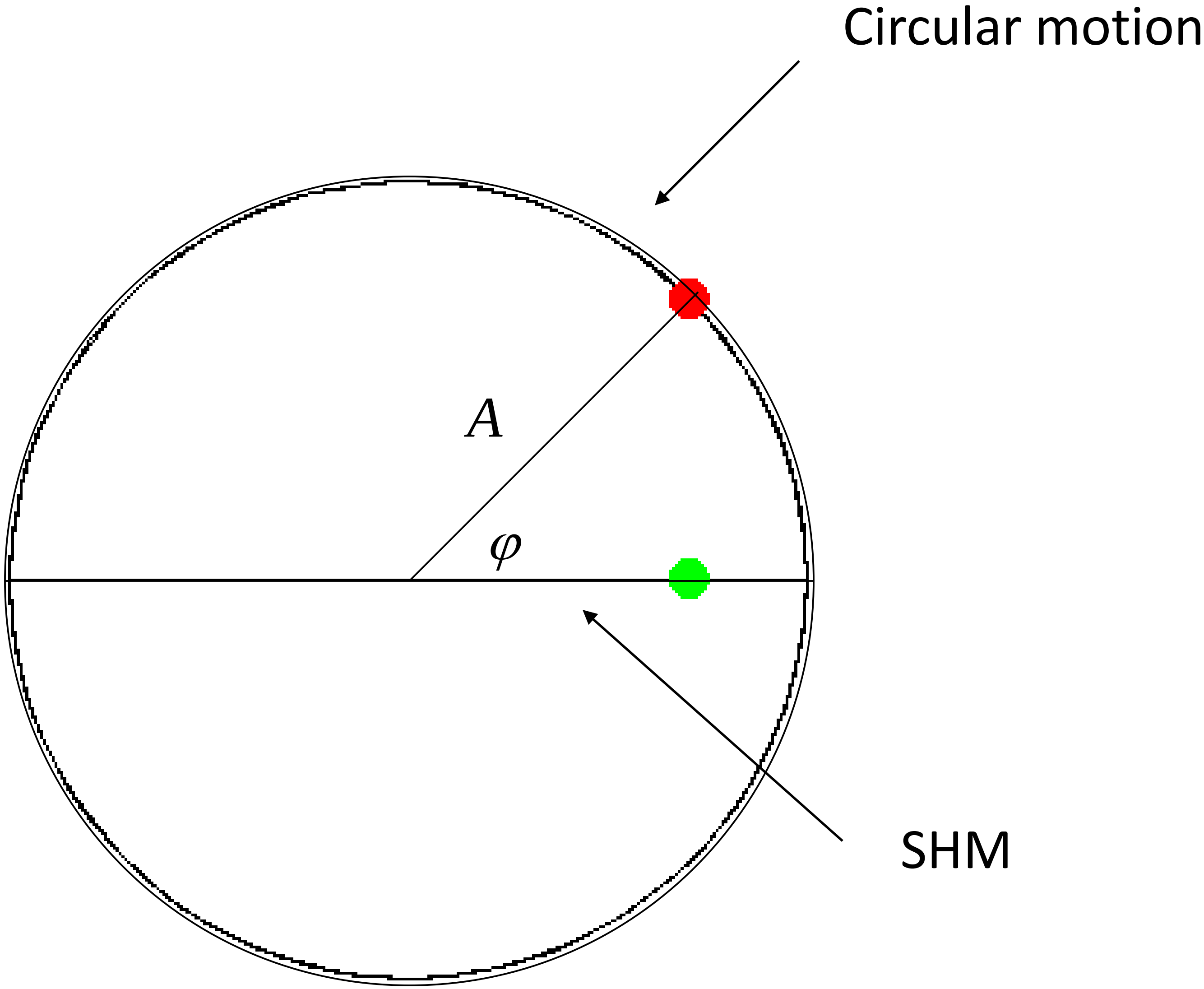
$$\theta(t) = \omega_0 t + \varphi$$

And its x-coordinate is

$$A\cos(\omega_0 t + \varphi)$$

SHM can be visualized as
the projection of circular motion



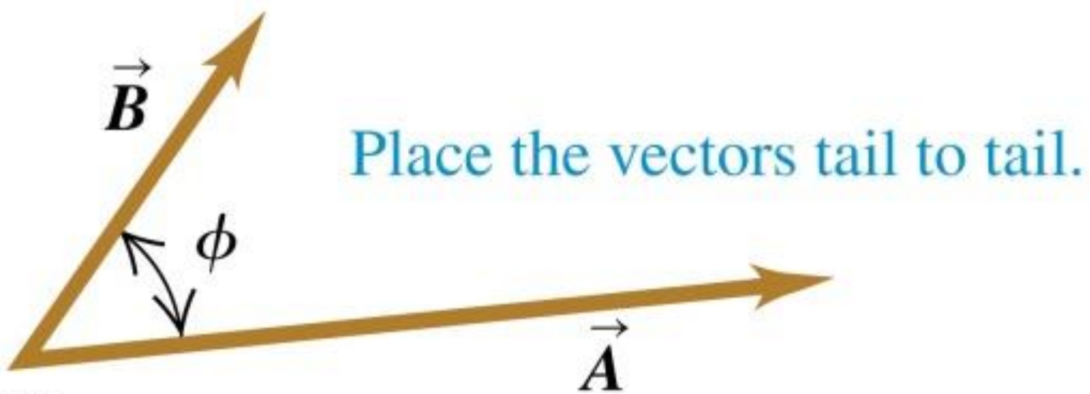

Circular motion
A
φ
SHM

3. Work and Kinetic Energy

Recall

Scalar Product

$$\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = AB\cos\phi$$



Recall

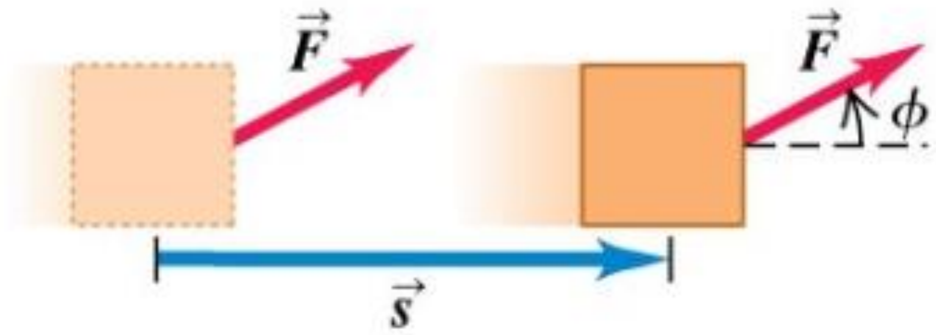
Special cases:

(i) if $\vec{\boldsymbol{A}} \parallel \vec{\boldsymbol{B}}$, $\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = AB$, in particular, $\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = 1$

(ii) if $\vec{\boldsymbol{A}} \perp \vec{\boldsymbol{B}}$, $\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = 0$, in particular, $\hat{\imath} \cdot \hat{\jmath} = \hat{\jmath} \cdot \hat{k} = \hat{k} \cdot \hat{\imath} = 0$

In analytical form, $\boxed{\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = A_x B_x + A_y B_y + A_z B_z}$

From high school,



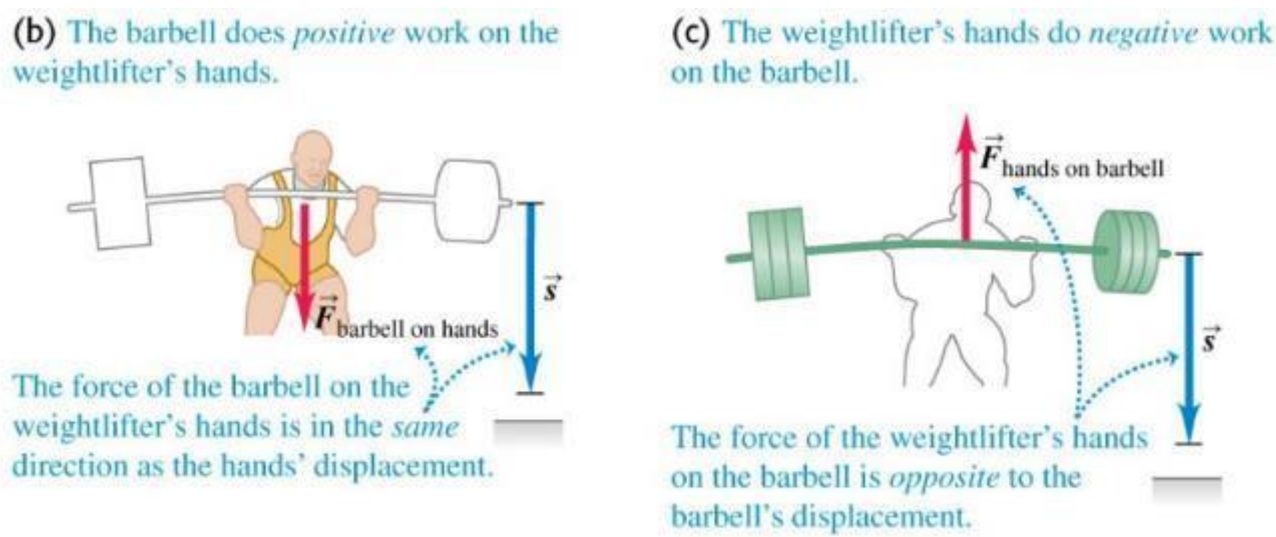
work done $W = Fs\cos\phi$ SI unit: joule 1 J = 1 Nm

$\vec{F} \cdot \vec{s}$, see how useful vector notation is!!

In general, $W = \vec{F} \cdot \vec{s} = F_x s_x + F_y s_y + F_z s_z$

W can be +ve (work done on a body), -ve (work done by a body), or zero

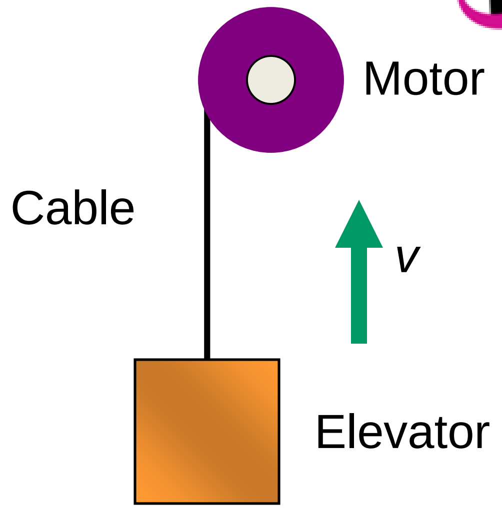
what does this mean?



In this example, a body does –ve work on a second body, the second body does an equal amount of +ve work on the first body

Q6.1

An elevator is being *lifted* at a constant speed by a steel cable attached to an electric motor. Which statement is correct?



A. The cable does positive work on the elevator, and the elevator does positive work on the cable.

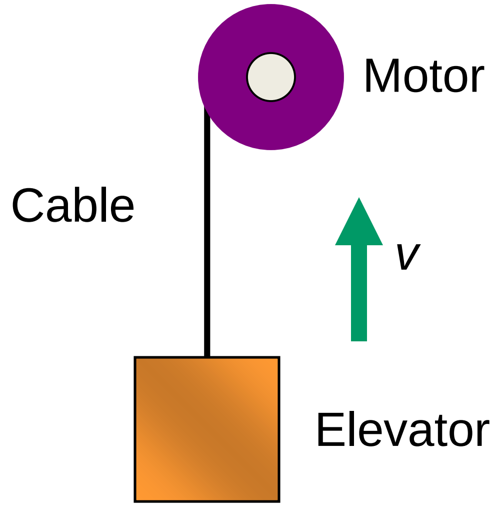
B. The cable does positive work on the elevator, and the elevator does negative work on the cable.

C. The cable does negative work on the elevator, and the elevator does positive work on the cable.

D. The cable does negative work on the elevator, and the elevator does negative work on the cable.

A6.1

An elevator is being *lifted* at a constant speed by a steel cable attached to an electric motor. Which statement is correct?



A. The cable does positive work on the elevator, and the elevator does positive work on the cable.

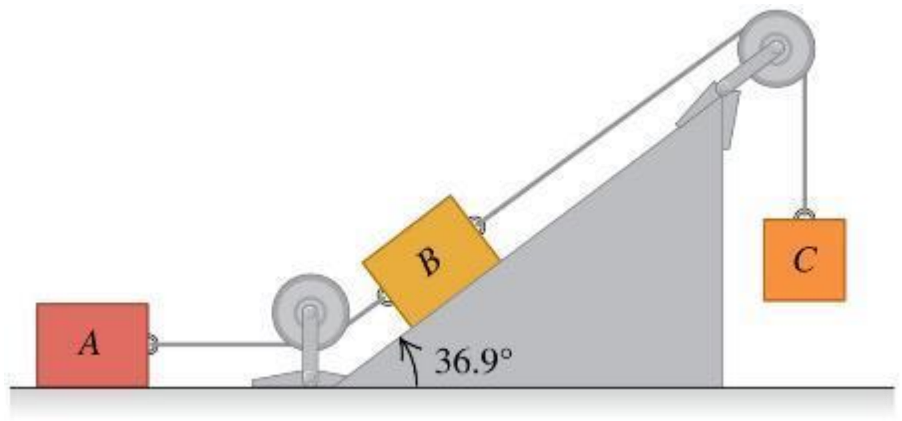
B. The cable does positive work on the elevator, and the elevator does negative work on the cable.

C. The cable does negative work on the elevator, and the elevator does positive work on the cable.

D. The cable does negative work on the elevator, and the elevator does negative work on the cable.

Q6.8

Three blocks are connected as shown. The ropes and pulleys are of negligible mass. When released, block *C* moves downward, block *B* moves up the ramp, and block *A* moves to the right.



After each block has moved a distance *d*, the force of gravity has done

A. positive work on *A*, *B*, and *C.*

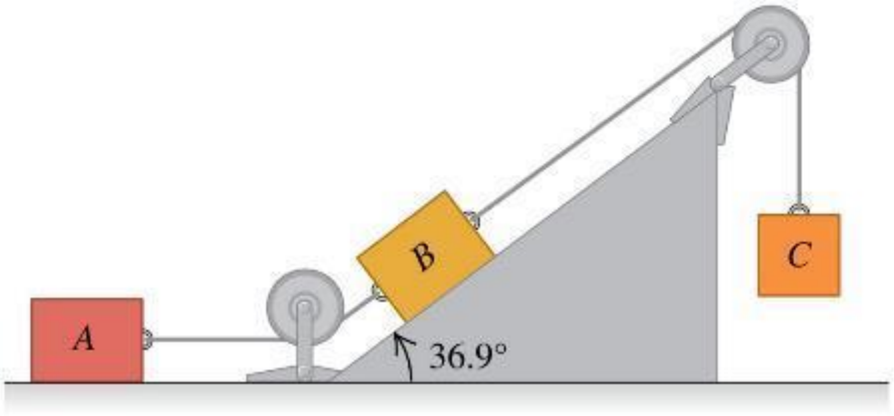
B. zero work on *A,* positive work on *B*, and negative work on *C.*

C. zero work on *A,* negative work on *B*, and positive work on *C.*

D. none of these

A6.8

Three blocks are connected as shown. The ropes and pulleys are of negligible mass. When released, block *C* moves downward, block *B* moves up the ramp, and block *A* moves to the right.



After each block has moved a distance *d*, the force of gravity has done

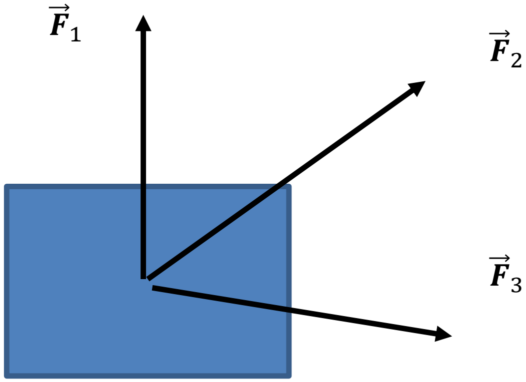
A. positive work on *A*, *B*, and *C*.

B. zero work on *A*, positive work on *B*, and negative work on *C*.

C. zero work on *A*, negative work on *B*, and positive work on *C*.

D. none of these

Workdone by multiple forces:



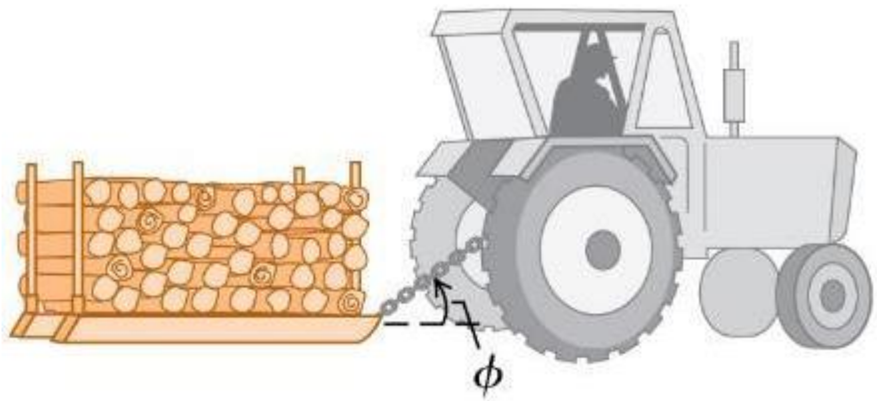
$$W = \left(\sum \vec{\boldsymbol{F}}\right) \cdot \vec{s} = \sum\left(\vec{\boldsymbol{F}} \cdot \vec{s}\right)$$

work done by resultant force

sum of work done by individual forces

Q6.4

A tractor driving at a constant speed pulls a sled loaded with firewood. There is friction between the sled and the road.



The total work done on the sled after it has moved a distance *d* is

A. positive.

B. negative.

C. zero.

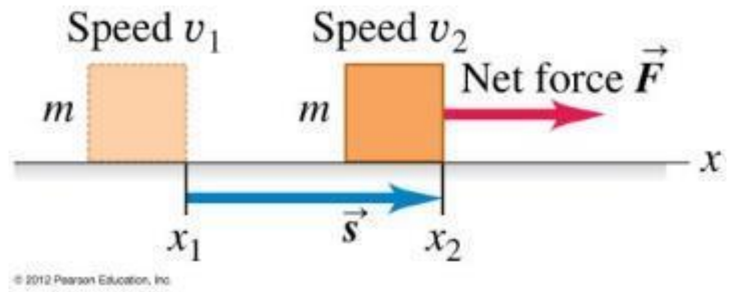
D. not enough information given to decide

Also from high school:

•Definition of **kinetic energy**, $K = \frac{1}{2}mv^2$

•**Work-energy theorem**

Work done by the net external force = change in KE of the particle



⚠ When accelerating a particle, work done by an external force $W = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 > 0$,

i.e, work is done on the particle.

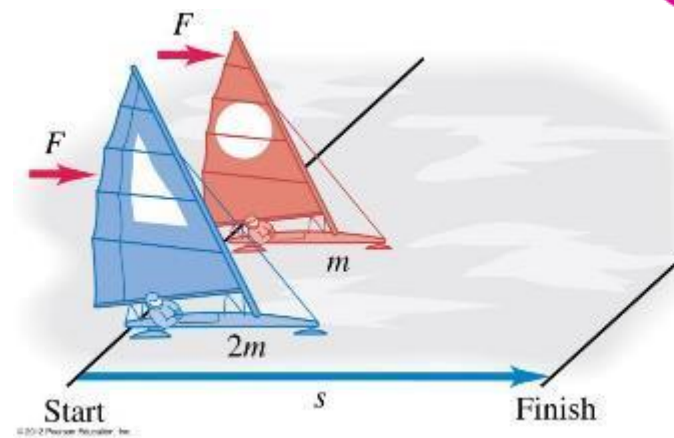
⚠ When decelerating a particle, $W < 0$,

i.e, work is done by the particle.

The above results are easy to prove if you consider 1D motion under a constant external force (as you have done in high school).

Q6.3

Two iceboats (one of mass *m,* one of mass 2*m*) hold a race on a frictionless, horizontal, frozen lake. Both iceboats start at rest, and the wind exerts the same constant force on both iceboats.



Which iceboat crosses the finish line with more kinetic energy (KE)?

A. The iceboat of mass *m*: it has twice as much KE as the other.

B. The iceboat of mass *m*: it has 4 times as much KE as the other.

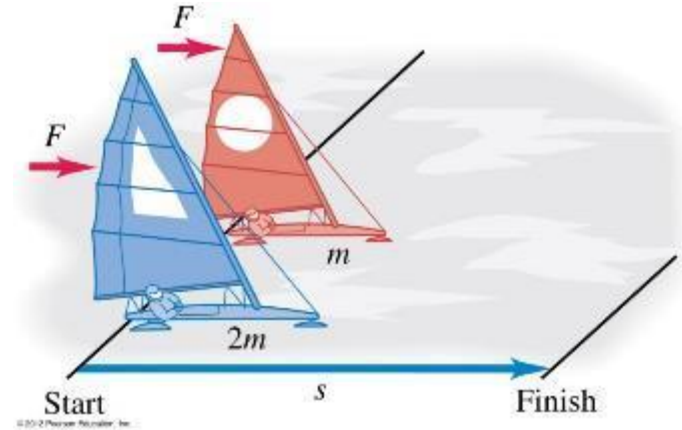
C. The iceboat of mass 2*m*: it has twice as much KE as the other.

D. The iceboat of mass 2*m*: it has 4 times as much KE as the other.

E. They both cross the finish line with the same kinetic energy.

A6.3

Two iceboats (one of mass *m,* one of mass 2*m*) hold a race on a frictionless, horizontal, frozen lake. Both iceboats start at rest, and the wind exerts the same constant force on both iceboats.



Which iceboat crosses the finish line with more kinetic energy (KE)?

A. The iceboat of mass *m*: it has twice as much KE as the other.

B. The iceboat of mass *m*: it has 4 times as much KE as the other.

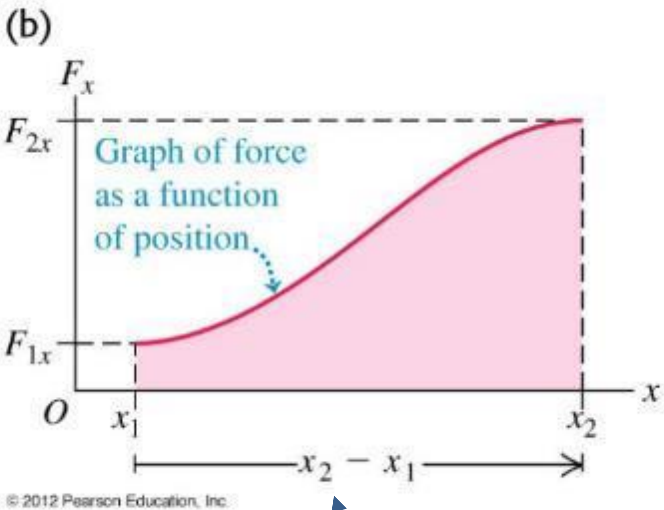
C. The iceboat of mass 2*m*: it has twice as much KE as the other.

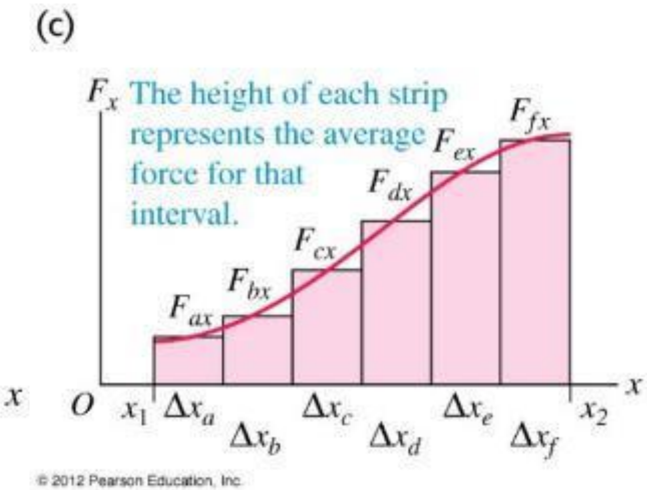
D. The iceboat of mass 2*m*: it has 4 times as much KE as the other.

E. They both cross the finish line with the same kinetic energy.

Question:

What if the force is not constant (but still in 1D)?





approximate each sub-interval by a constant force

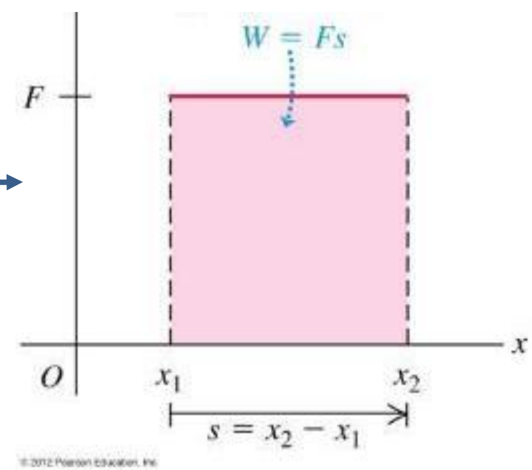
$W = F_{ax}\Delta x_a + F_{bx}\Delta x_b + \cdots$

as $\Delta x \to 0$

W = area under curve

$= \int_{x_1}^{x_2} F_x \, dx$

c.f. constant force



Example: An ideal spring

Hooke's law – restoring force (i.e., tension in the spring) $= -kx$

Hooke's law (Robert Hooke, 1678)

– restoring force (i.e., tension in the spring) $= -kx$

displacement from natural length

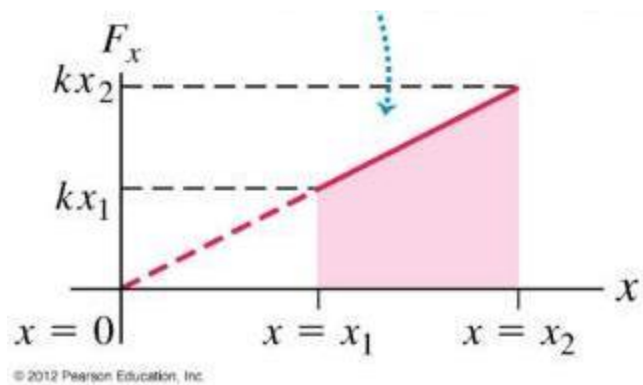
direction opposite to displacement

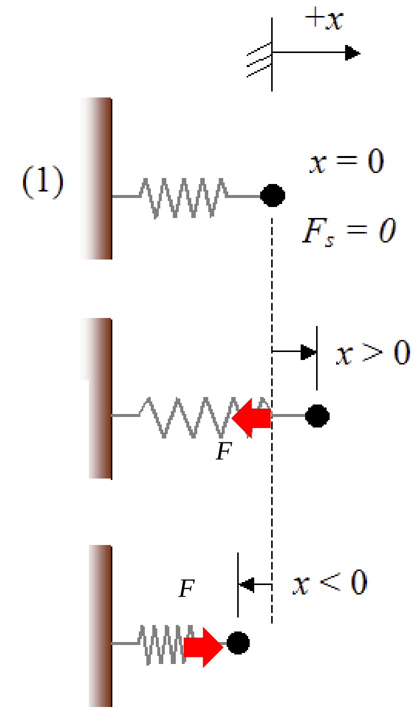
force constant
unit: N/m

Work done by an external force (⚠ not tension in the spring) in stretching a spring from x_1 to x_2

$$\boxed{W = \int_{x_1}^{x_2} F\,dx = k\int_{x_1}^{x_2} x\,dx = \underbrace{\frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2}_{\text{area under the line}}}$$

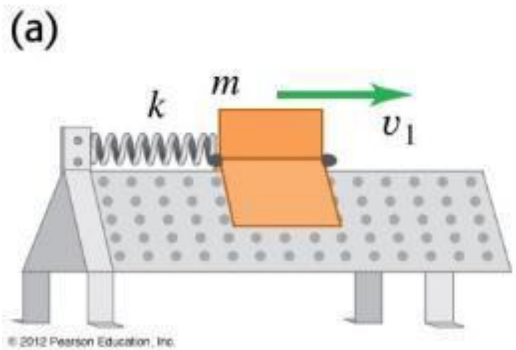
- on stretching, $x_2 > x_1$, $W > 0$, work is done (on / by) the spring
- on releasing, $x_2 < x_1$, $W < 0$, work is done (on / by) the spring





Example 6.7

A glider of mass *m*, and a spring with force constant *k*. Initially the spring is unstretched and the glider is moving with speed v_1. What is the maximum displacement *d* to the right if the frictional coefficient is μ_k?



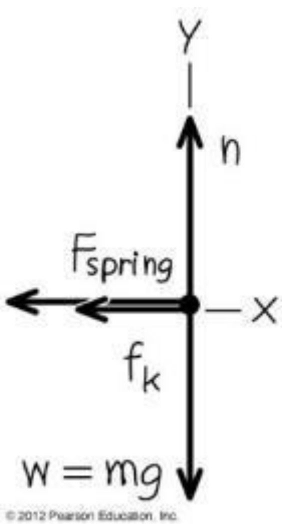
By the work-energy theorem

$$-\mu_k mgd - \int_0^d kx\,dx = \underbrace{0 - \frac{1}{2}mv_1^2}_{\text{change in KE}}$$

work done by f_k

work done by

$F_{\text{spring}} = -\frac{1}{2}kd^2$



$$\frac{1}{2}kd^2 + \mu_k mgd - \frac{1}{2}mv_1^2 = 0$$

$$\Rightarrow \quad d = -\frac{\mu_k mg}{k} \pm \sqrt{\left(\frac{\mu_k mg}{k}\right)^2 + \frac{mv_1^2}{k}}$$

1D motion with variable force,

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx},$$

i.e.,

$$F = ma = mv\frac{dv}{dx}$$

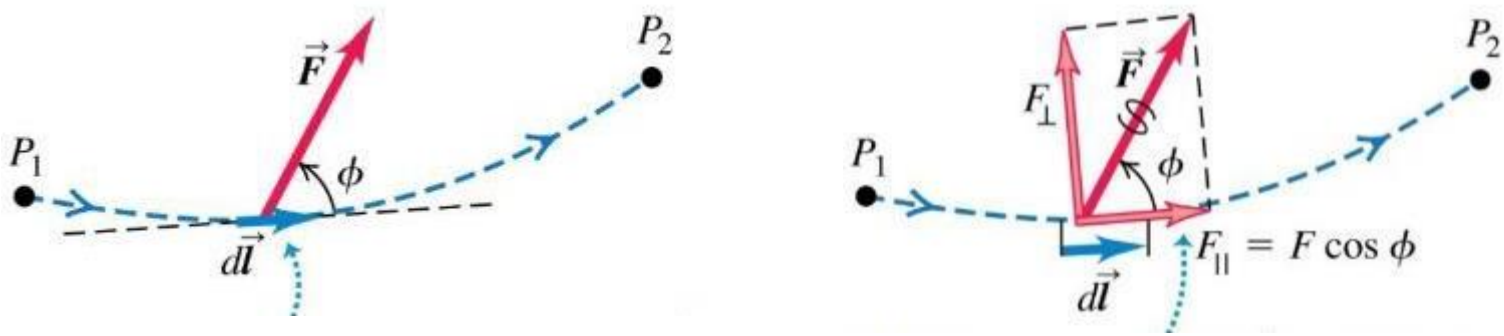
∴ work done by an external force

$$\boxed{W = \int_{x_1}^{x_2} F\,dx = m\int_{x_1}^{x_2} v\frac{dv}{dx}\,dx = m\int_{v_1}^{v_2} v\,dv = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2}$$

Work-energy theorem works for variable force!

3D motion with variable force

Idea: break up the path into very short segments so that in each segment, $\vec{\boldsymbol{F}}$ is approximately constant



v along tangential direction (why?)

work done in this small segment $dW = \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{l}} = F_\parallel dl = mv\frac{dv}{dl}dl = mvdv$

total work done = sum over all segments

$$W_{tot} = \sum \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{l}} \rightarrow \int_{P_1}^{P_2} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{l}} = \int_{P_1}^{P_2} mvdv$$

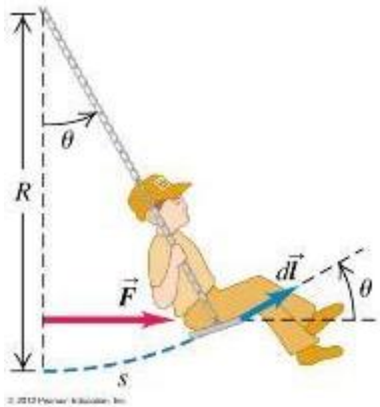
$$= \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

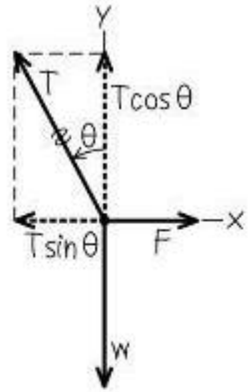
Conclusion: work-energy theorem holds for motion along a curve under variable force.

Example 6.8

Apply a horizontal force $\vec{\boldsymbol{F}}$ to push the swing up from $\theta = 0$ to θ_0

Assumption: $\vec{\boldsymbol{F}}$ is just enough to push it up so that the swing is in equilibrium any time





$$\sum F_x = F - T\sin\theta = 0$$

$$\sum F_y = T\cos\theta - w = 0$$

$$\Rightarrow\ T = w\sec\theta$$

$$F = w\tan\theta$$

Work done by net force, $W_{\text{net}} =$ _____

Work done by $\vec{\boldsymbol{T}}$, $W_T =$ _______ $(\because \vec{\boldsymbol{T}} \perp d\vec{\boldsymbol{l}})$

Work done by $\vec{\boldsymbol{F}}$,

$$W_F = \int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{l}} = \int_0^{\theta_0} F\cos\theta\, dl = \int_0^{\theta_0} w\tan\theta\cos\theta\, Rd\theta = wR(1-\cos\theta_0)$$

Work done by $\vec{\boldsymbol{w}}$ $W_w = \int \vec{\boldsymbol{w}} \cdot d\vec{\boldsymbol{l}} = \int_0^{\theta_0} w\cos\left(\frac{\pi}{2}+\theta\right) dl =$

$$-\int_0^{\theta_0} w\sin\theta\, Rd\theta = -wR(1-\cos\theta_0)$$

Check that $W_{\text{net}} = W_T + W_F + W_w$

Power

Average over a period Δt, $P_{av} = \frac{\Delta W}{\Delta t}$

Instantaneous power ($\Delta t \to 0$), $P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}$

SI unit: watt 1 W = 1 J/s

Another unit of *energy* besides J – kilowatt hour, common in electric bills

1 KWh = (10^3 J/s)(3600 s) = 3.6×10^6 J

$$P = \frac{dW}{dt} = \vec{\boldsymbol{F}} \cdot \frac{d\vec{\boldsymbol{s}}}{dt} = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}$$

Q6.10

An object is initially at rest. A net force (which always points in the same direction) is applied to the object so that the *power* of the net force is constant. As the object gains speed,

A. the magnitude of the net force remains constant.

B. the magnitude of the net force increases.

C. the magnitude of the net force decreases.

D. not enough information given to decide

A6.10

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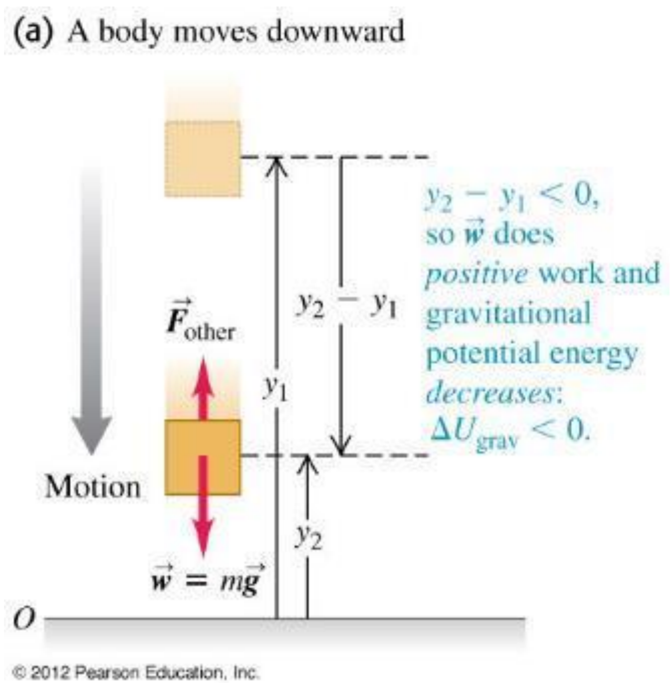
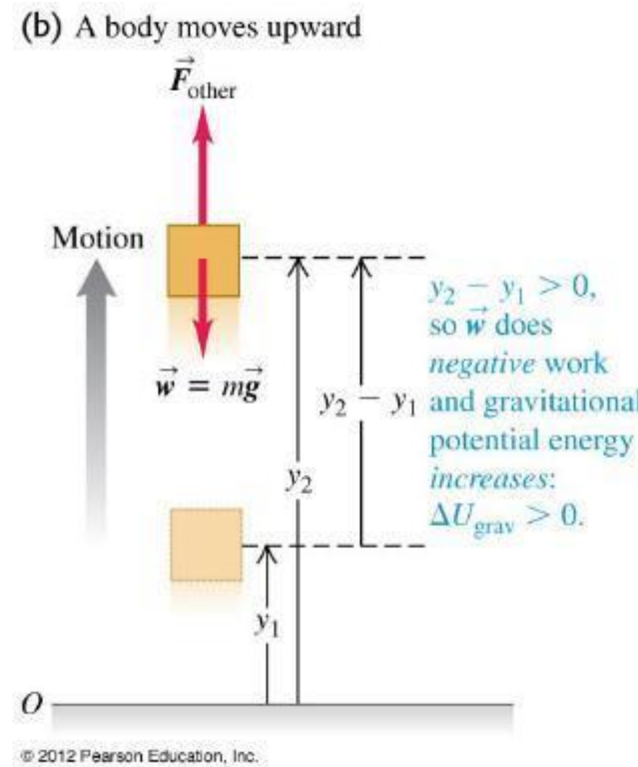
Potential Energy & Energy Conservation

Potential energy –
energy associated with the position of bodies in a system

Gravitational PE

Defined by $U_{grav} = mgy$



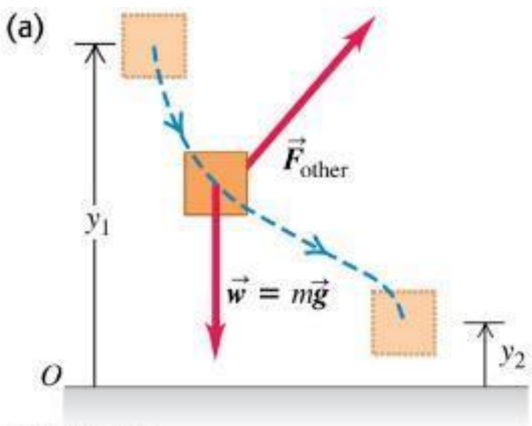


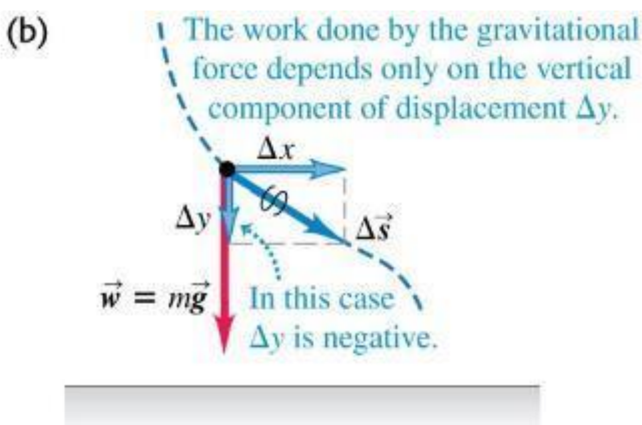
Work done by the weight of the body

$W_{\text{grav}} = mg(y_1 - y_2) > 0$,
$\vec{\boldsymbol{w}}$ does +ve work
$\Delta U_{\text{grav}} = mg(y_2 - y_1) = -W_{\text{grav}} < 0$
gravitational PE decreases

$W_{\text{grav}} = -mg(y_2 - y_1) < 0$,
$\vec{\boldsymbol{w}}$ does –ve work
$\Delta U_{\text{grav}} = mg(y_2 - y_1) = -W_{\text{grav}} > 0$
gravitational PE increases

Along a curved path





work done by the weight

$$W_{\text{grav}} = \vec{\boldsymbol{w}} \cdot \overrightarrow{\Delta \boldsymbol{s}} = -mg\Delta y$$

$$= -\Delta U_{\text{grav}}$$

same as vertical motion!

Conclusion: $W_{grav} = -\Delta U_{grav}$

c.f. drawing money from the bank and spending it

Gravitational PE does not belong to the body only, it belongs to both the body and the earth

A piece of fruit falls straight down. As it falls,

A. the gravitational force does positive work on it and the gravitational potential energy increases.

B. the gravitational force does positive work on it and the gravitational potential energy decreases.

C. the gravitational force does negative work on it and the gravitational potential energy increases.

D. the gravitational force does negative work on it and the gravitational potential energy decreases.

A7.1

A piece of fruit falls straight down. As it falls,

A. the gravitational force does positive work on it and the gravitational potential energy increases.

B. the gravitational force does positive work on it and the gravitational potential energy decreases.

C. the gravitational force does negative work on it and the gravitational potential energy increases.

D. the gravitational force does negative work on it and the gravitational potential energy decreases.

By **work-energy theorem**

$$\Delta K = -\Delta U_{\text{grav}} \quad \Rightarrow \; \Delta K + \Delta U_{\text{grav}} = 0,$$

or $$K_{\text{initial}} + U_{\text{grav,initial}} = K_{\text{final}} + U_{\text{grav,final}}$$

Conservation of mechanical energy

What if other forces also do work?

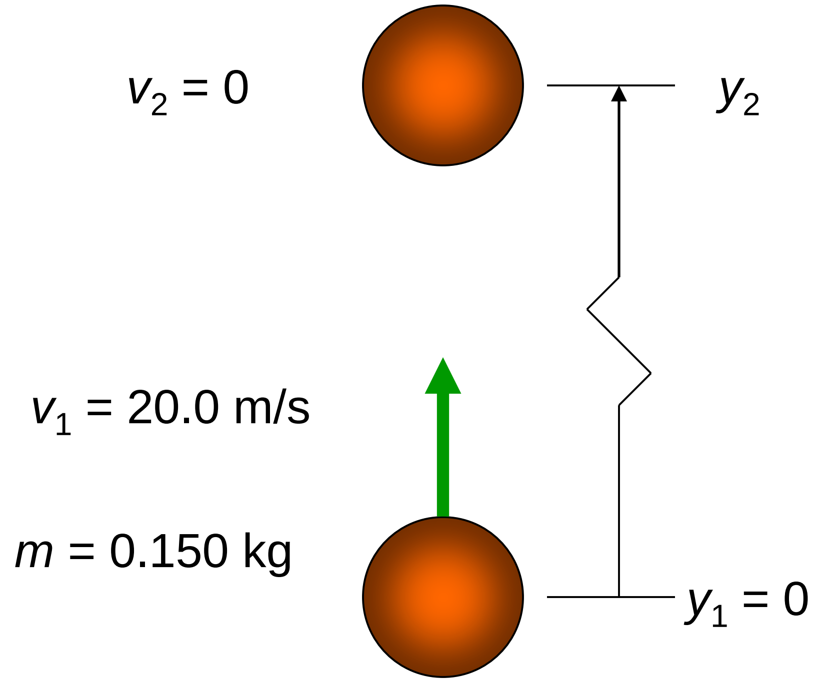
Work-energy theorem $\Rightarrow W_{other} + W_{grav} = \Delta K$

$$\Rightarrow W_{other} = \Delta K + \Delta U_{grav}$$

Q7.2

You toss a 0.150-kg baseball straight upward so that it leaves your hand moving at 20.0 m/s. The ball reaches a maximum height y_2.

What is the speed of the ball when it is at a height of $y_2/2$? Ignore air resistance.



A. 10.0 m/s

B. less than 10.0 m/s but greater than zero

C. greater than 10.0 m/s

D. not enough information given to decide

A7.2

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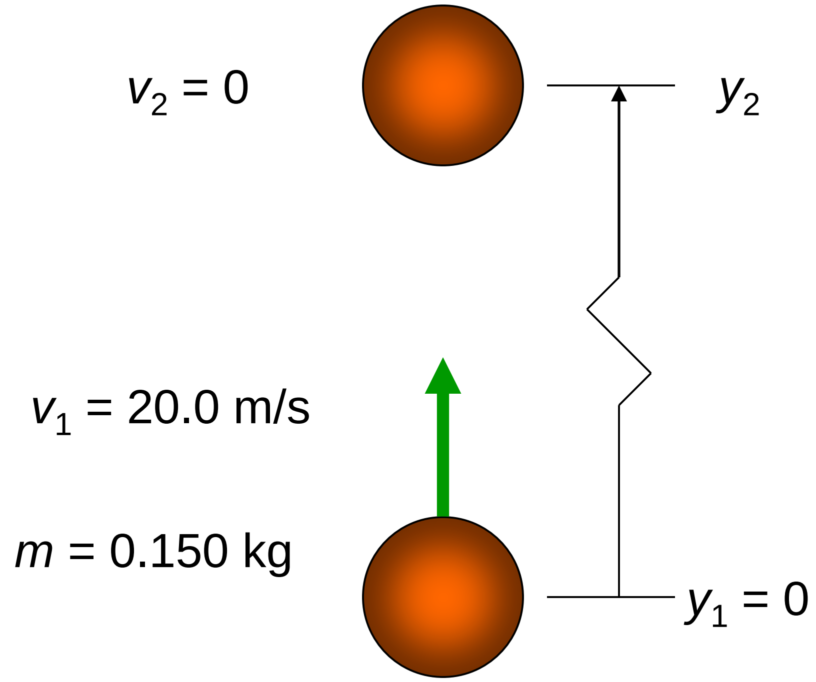


A. 10.0 m/s

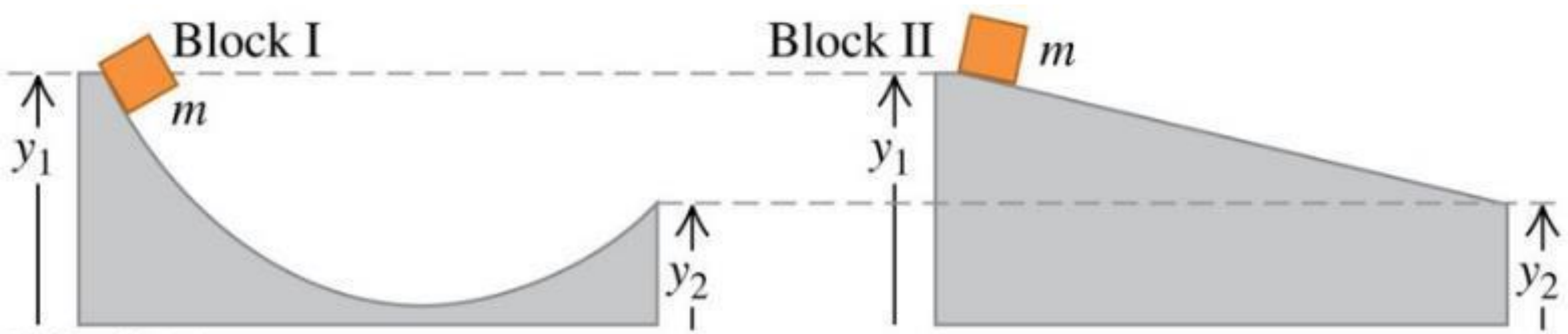
B. less than 10.0 m/s but greater than zero

C. greater than 10.0 m/s

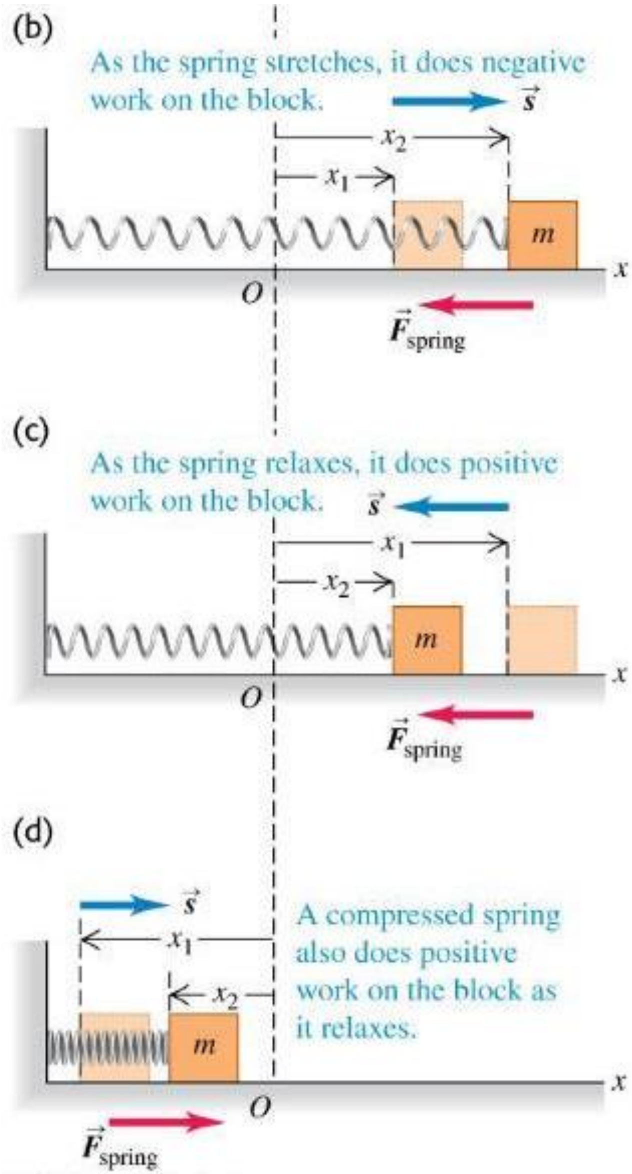
D. not enough information given to decide

Question

- The figure shows two different frictionless ramps. The heights y_1 and y_2 are the same for both ramps. If a block of mass m is released from rest at the left-hand end of each ramp, which block arrives at the right-hand end with the greater speed?
  1) block I;
  2) block II;
  3) the speed is the same for both blocks.



Elastic PE – spring



Work done by restoring force in spring

$$W_{\text{el}} = \int_{x_1}^{x_2} (-kx)dx = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2$$

Define elastic PE of spring $U_{\text{el}} = \frac{1}{2}kx^2$

$$W_{\text{el}} = -\Delta U_{\text{el}}$$

c.f. gravitational PE

⚠U_{grav} free to choose zero level position, but for U_{el}, zero level position must correspond to unstretched position.

In the presence of gravitational, elastic, and other forces

Work-energy theorem $\Rightarrow \quad W_{\text{grav}} + W_{\text{el}} + W_{\text{other}} = \Delta K$

$$\Rightarrow \quad W_{\text{other}} = \Delta K + \Delta\left(U_{\text{grav}} + U_{\text{el}}\right)$$
$$= \Delta K + \Delta PE$$

If $\quad W_{\text{other}} = 0, \quad \Delta K + \Delta PE = 0\,,$

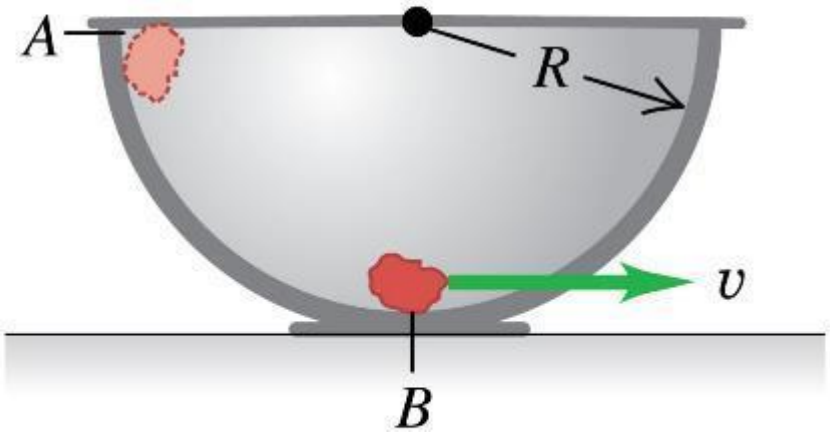
or $\quad K_{\text{initial}} + PE_{\text{initial}} = K_{\text{final}} + PE_{\text{final}}$

Conservation of mechanical energy

Q7.3

As a rock slides from *A* to *B* along the inside of this frictionless hemispherical bowl, mechanical energy is conserved. Why?

(Ignore air resistance.)



A. The bowl is hemispherical.

B. The normal force is balanced by centrifugal force.

C. The normal force is balanced by centripetal force.

D. The normal force acts perpendicular to the bowl’s surface.

E. The rock’s acceleration is perpendicular to the bowl’s surface.

A7.3

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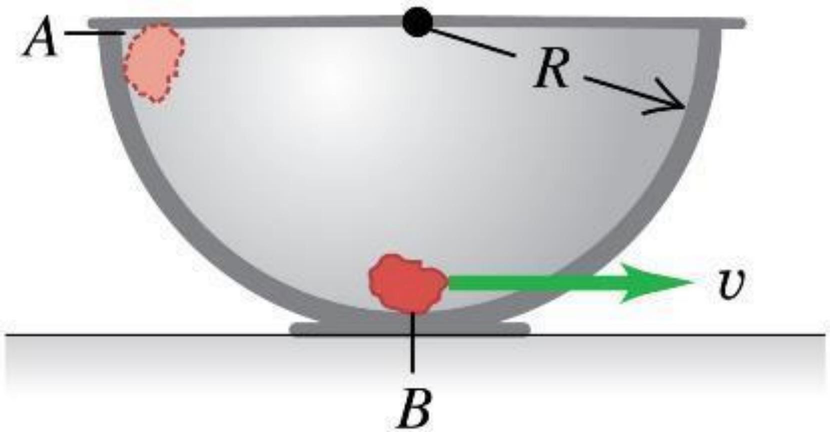


A. The bowl is hemispherical.

B. The normal force is balanced by centrifugal force.

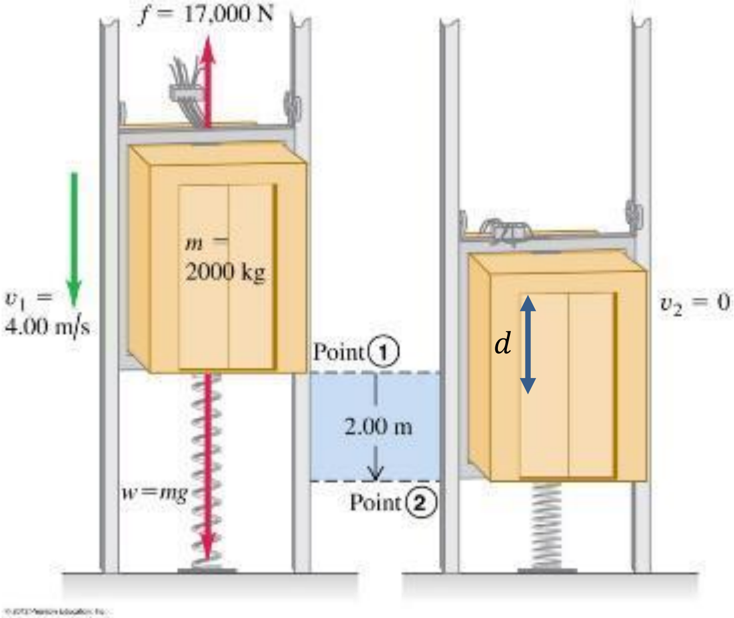
C. The normal force is balanced by centripetal force.

D. The normal force acts perpendicular to the bowl's surface.

E. The rock's acceleration is perpendicular to the bowl's surface.

Example

An elevator with a broken cable. Friction between the rail and the elevator is f. What is the spring constant k if the elevator has initial speed v_1 when it just touches the spring, and comes to rest at a distance $d = 2.00$ m?



work done by friction $W_{\text{other}} = -fd$

$$\Delta K = 0 - \frac{1}{2}mv_1^2$$

$$\Delta PE = -mgd + \frac{1}{2}kd^2$$

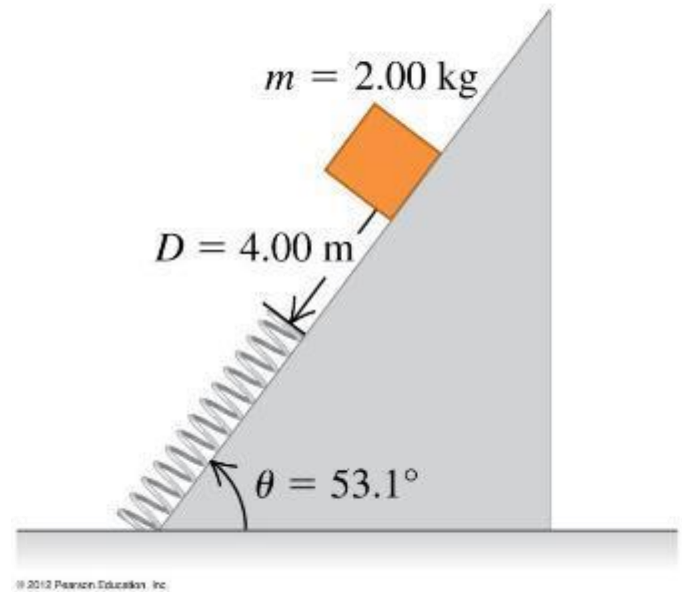
$$W_{\text{other}} = \Delta K + \Delta PE$$

$$\Rightarrow \quad -fd = -\tfrac{1}{2}mv_1^2 - mgd + \tfrac{1}{2}kd^2$$

$$\Rightarrow \qquad k = \frac{2\left(mgd + \frac{1}{2}mv_1^2 - fd\right)}{d^2}$$

Q7.5

A block is released from rest on a frictionless incline as shown. When the moving block is in contact with the spring and compressing it, what is happening to the gravitational potential energy U_{grav} and the elastic potential energy U_{el}?



A. U_{grav} and U_{el} are both increasing.

B. U_{grav} and U_{el} are both decreasing.

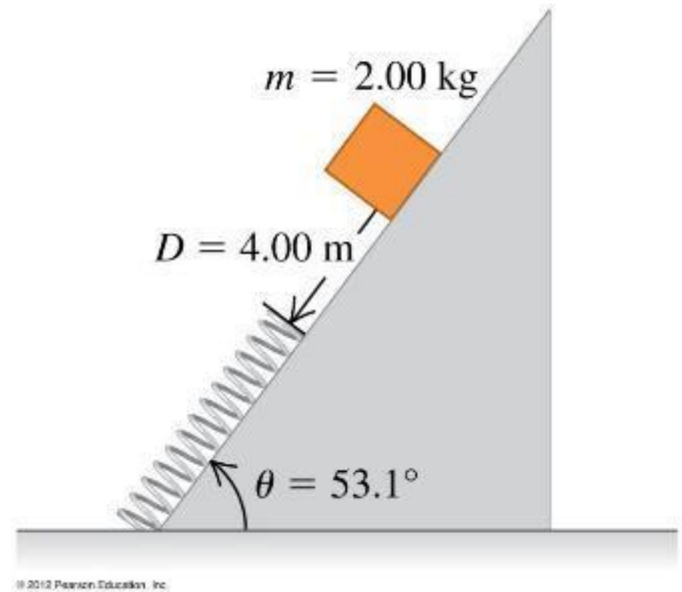
C. U_{grav} is increasing; U_{el} is decreasing.

D. U_{grav} is decreasing; U_{el} is increasing.

E. The answer depends on how the block's speed is changing.

A7.5

A block is released from rest on a frictionless incline as shown. When the moving block is in contact with the spring and compressing it, what is happening to the gravitational potential energy U_{grav} and the elastic potential energy U_{el}?



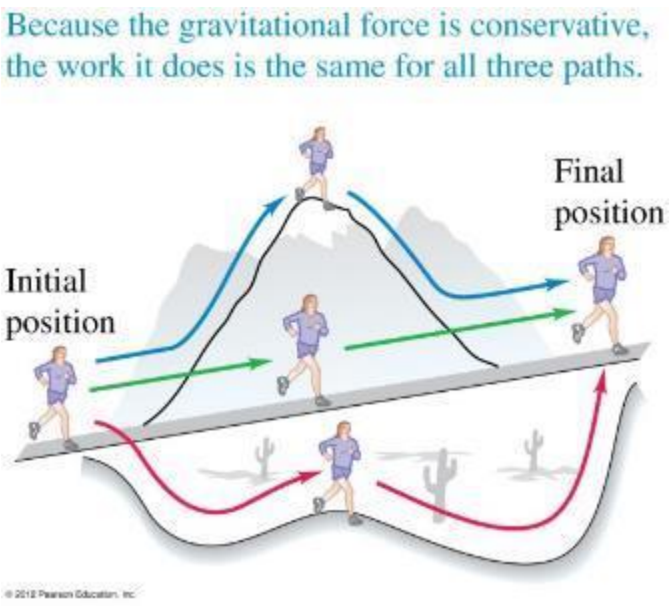
A. U_{grav} and U_{el} are both increasing.

B. U_{grav} and U_{el} are both decreasing.

C. U_{grav} is increasing; U_{el} is decreasing.

D. U_{grav} is decreasing; U_{el} is increasing.

E. The answer depends on how the block's speed is changing.



<u>Properties of the work done by conservative forces</u>:

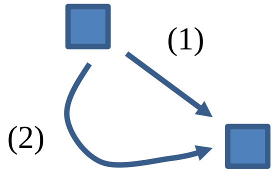
1. It can be expressed as the difference between the initial and final values of a potential energy function.
2. It is reversible.

Consequences:

1. It is independent of the path of the body.
2. When the starting and ending points are the same (path forms a close loop), the total work is zero.

c.f. **–**ve work done by friction cannot be "reclaimed", called **non-conservative forces**.

⚠ Work done by non-conservative force is <u>path dependent</u>



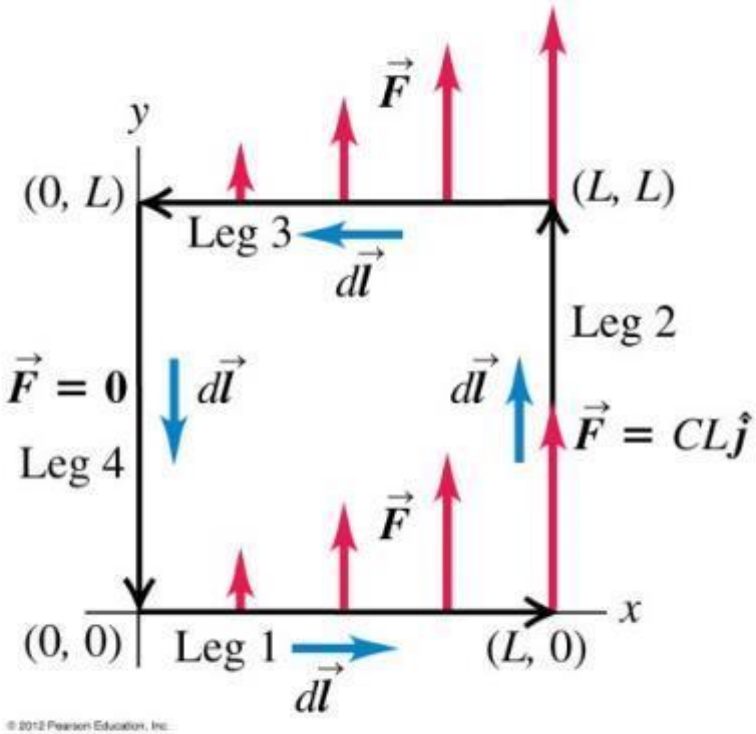
work done by friction in path (2) is more negative than in path (1).

⚠ The term PE is reserved for conservative forces only

To test whether a force is conservative **–** check if the work done is zero around a close loop.

Example

An electron goes counter clockwise around a square loop under a force $\vec{\boldsymbol{F}} = Cx\hat{j}$, C constant



Leg 1, $W_1 = \int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{l}} = 0$

Leg 2, $W_2 = CL^2$

Leg 3, $W_3 = 0$

Leg 4, $W_4 = 0$

$\vec{\boldsymbol{F}}$ is (conservative / non-conservative)

To derive a conservative force $\vec{\boldsymbol{F}}$ from its potential energy function U:

Work done by a conservative force $W = -\Delta U(x)$ in 1D

$F\Delta x$

$$\Rightarrow \quad F = -\frac{\Delta U}{\Delta x} \xrightarrow{\Delta x \to 0} F = -\frac{dU}{dx}$$

Free to add a constant to $U(x)$ without changing the force

Check: $U_{\text{grav}} = mgh$, $F = -mg$

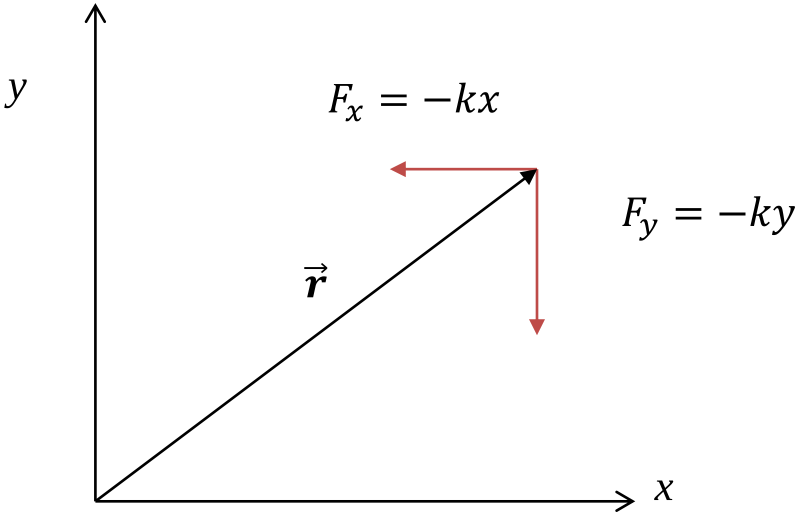
$U_{\text{el}} = \frac{1}{2}kx^2$, $F = -kx$

In 3D, $F_x = -\frac{\partial U}{\partial x}$, $F_y = -\frac{\partial U}{\partial y}$, $F_z = -\frac{\partial U}{\partial z}$

Example

$$U(x, y) = \frac{1}{2}k(x^2 + y^2)$$

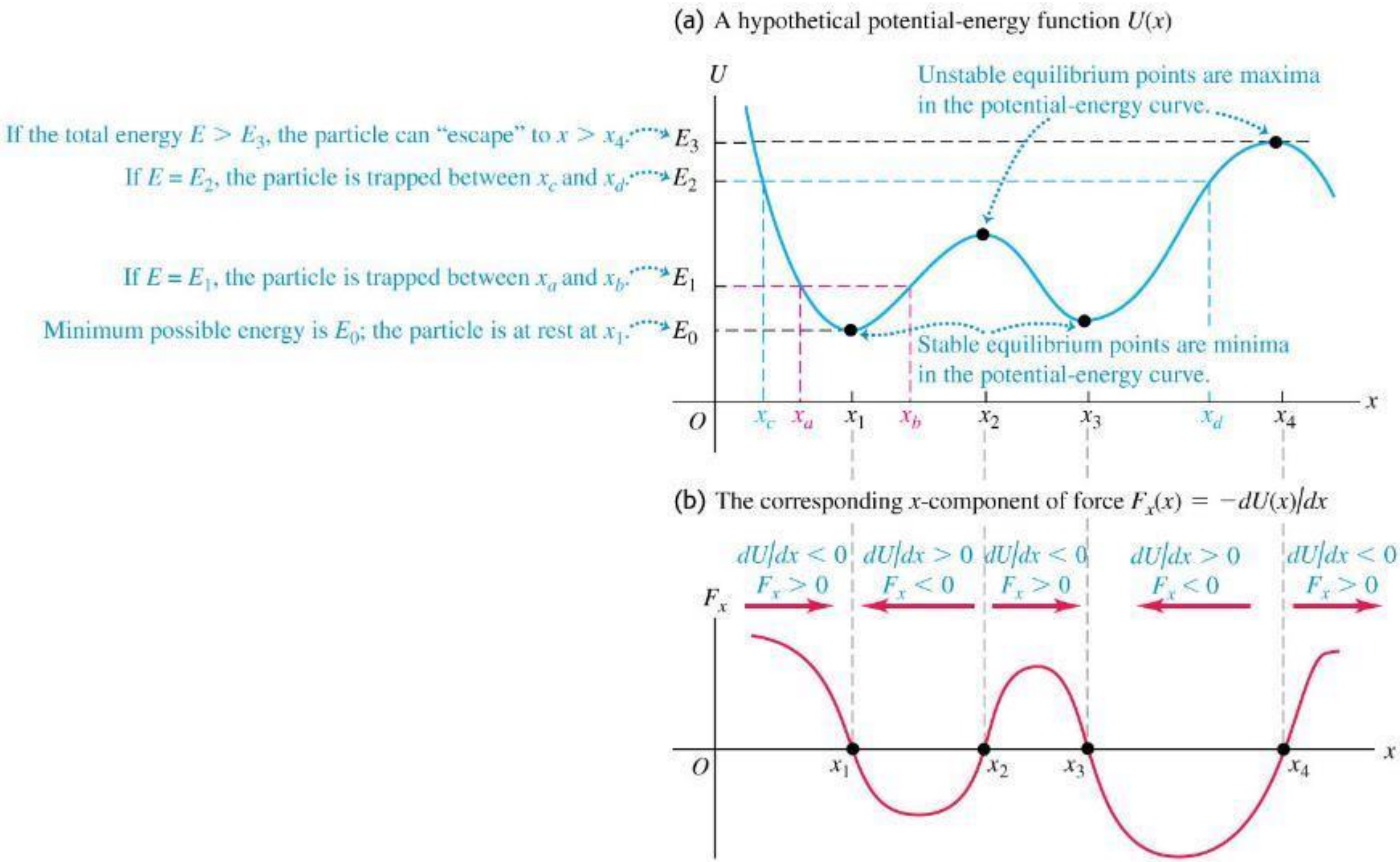
$$F_x = -\frac{\partial U}{\partial x} = -kx, F_y = -\frac{\partial U}{\partial y} = -ky$$



Question

- A particle moving along the x-axis is acted on by a conservative force F_x.
- At a certain point, the force is zero.
- At that point the value of the potential energy function $U(x)$ is
  1) $= 0$
  2) > 0
  3) < 0
  4) not enough information to decide
- dU/dx is
  1) $= 0$
  2) > 0
  3) < 0
  4) not enough information to decide

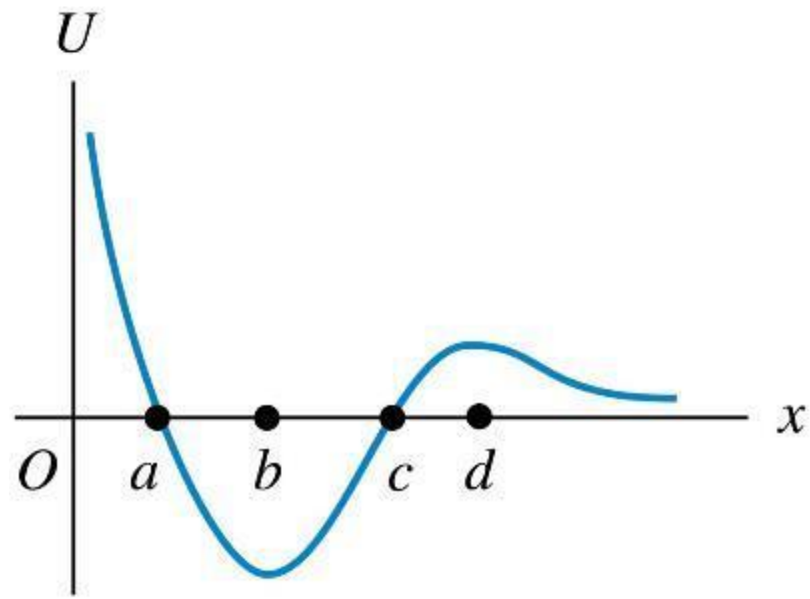
Interpretation of an **energy diagram**:
Note the meanings of stable and unstable equilibrium.



Q7.6

The graph shows the potential energy U for a particle that moves along the x-axis.

The particle is initially at $x = d$ and moves in the negative x-direction. At which of the labeled x-coordinates does the particle have the greatest *speed*?



A. at $x = a$

B. at $x = b$

C. at $x = c$

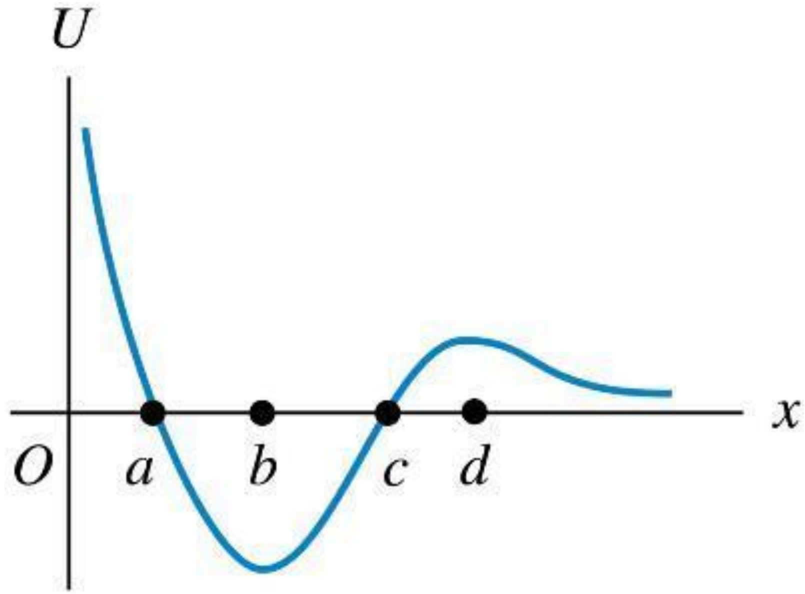
D. at $x = d$

E. more than one of the above

A7.6

The graph shows the potential energy *U* for a particle that moves along the *x*-axis.

The particle is initially at $x = d$ and moves in the negative *x*-direction. At which of the labeled *x*-coordinates does the particle have the greatest *speed*?



A. at $x = a$

B. at $x = b$

C. at $x = c$

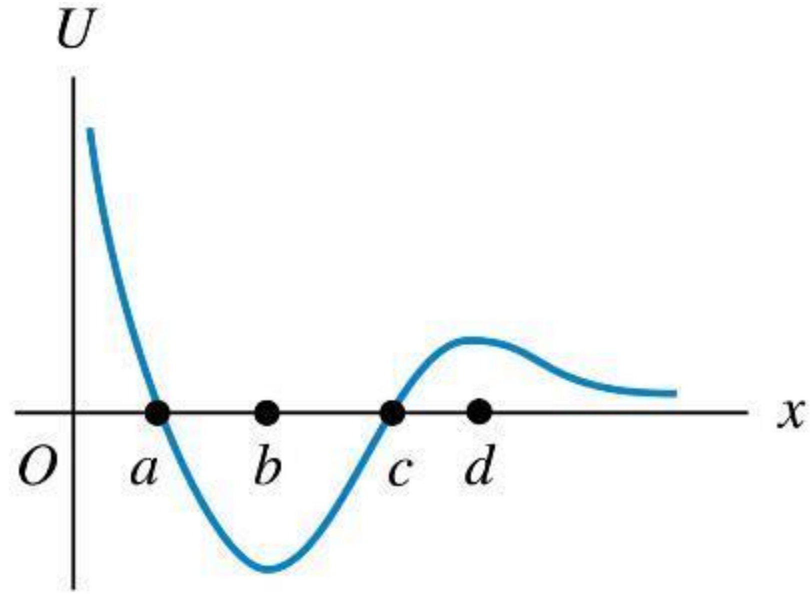
D. at $x = d$

E. more than one of the above

Q7.7

The graph shows the potential energy U for a particle that moves along the x-axis.

The particle is initially at $x = d$ and moves in the negative x-direction. At which of the labeled x-coordinates is the particle *slowing down*?



A. at $x = a$

B. at $x = b$

C. at $x = c$

D. at $x = d$

E. more than one of the above

A7.7

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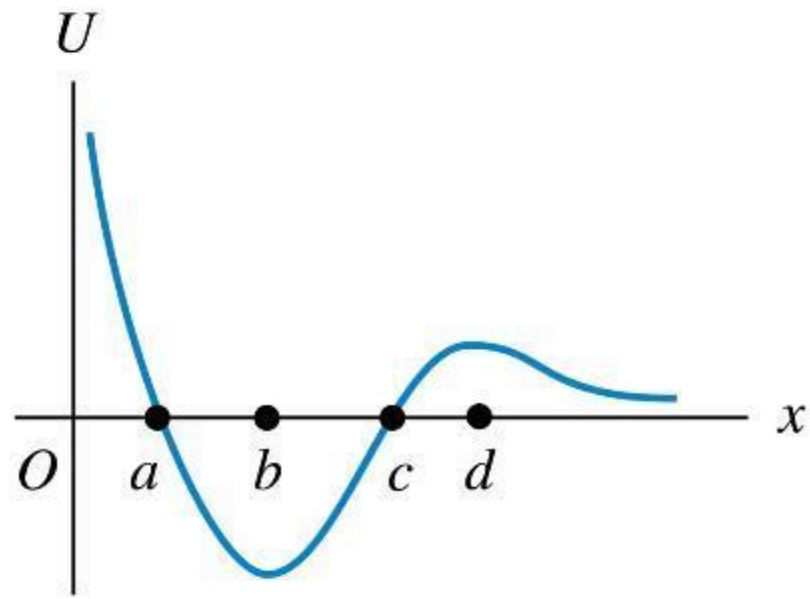


A. at $x = a$

B. at $x = b$

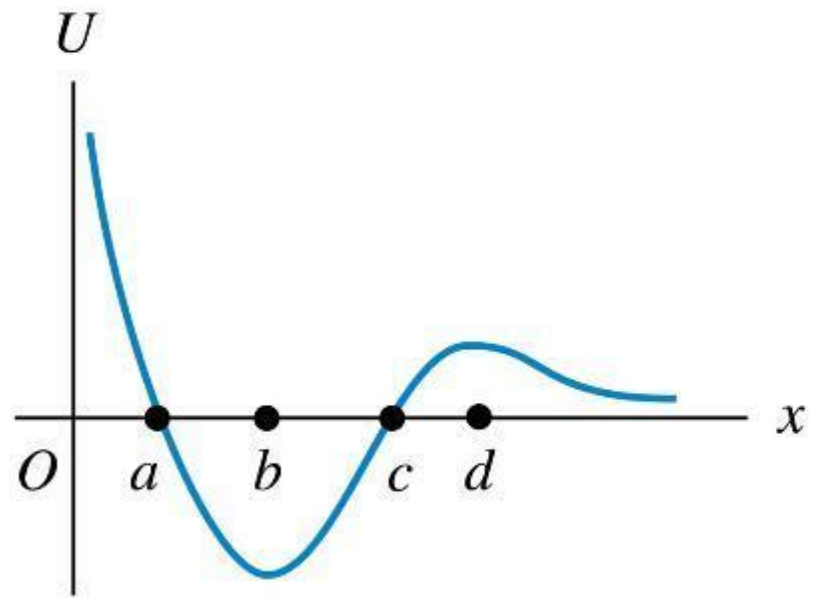
C. at $x = c$

D. at $x = d$

E. more than one of the above

Q7.8

The graph shows the potential energy *U* for a particle that moves along the *x*-axis. At which of the labeled *x*-coordinates is there *zero* force on the particle?



A. at $x = a$ and $x = c$

B. at $x = b$ only

C. at $x = d$ only

D. at $x = b$ and d

E. misleading question—there is a force at all values of x

A7.8

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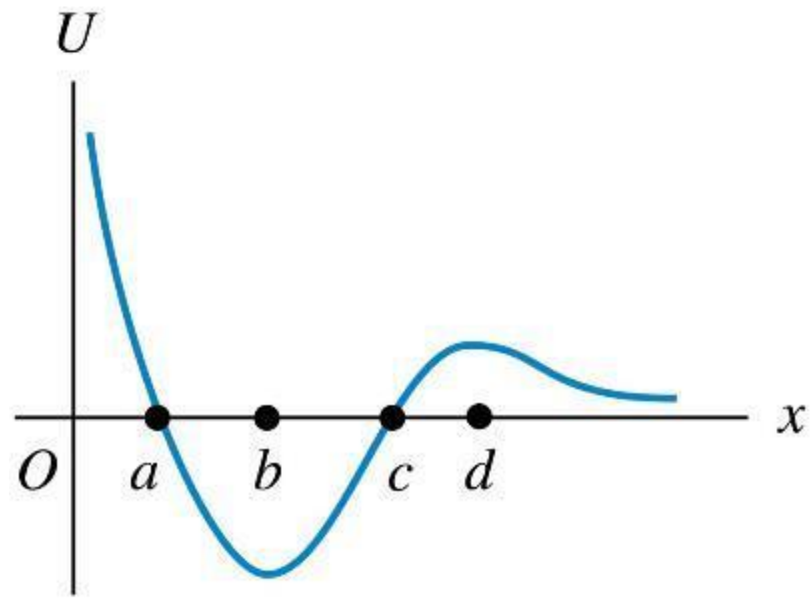


A. at $x = a$ and $x = c$

B. at $x = b$ only

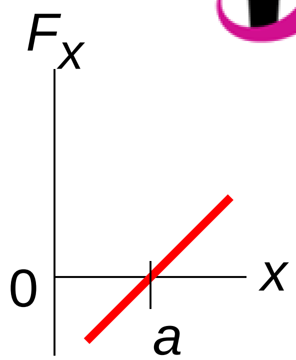
C. at $x = d$ only

D. at $x = b$ and d

E. misleading question—there is a force at all values of x

Q7.9

The graph shows a conservative force F_x as a function of x in the vicinity of $x = a$. As the graph shows, $F_x = 0$ at $x = a$. Which statement about the associated *potential energy* function U at $x = a$ is correct?



A. $U = 0$ at $x = a$

B. U is a maximum at $x = a$.

C. U is a minimum at $x = a$.

D. U is neither a minimum or a maximum at $x = a$, and its value at $x = a$ need not be zero.

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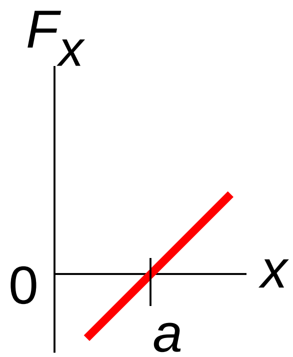

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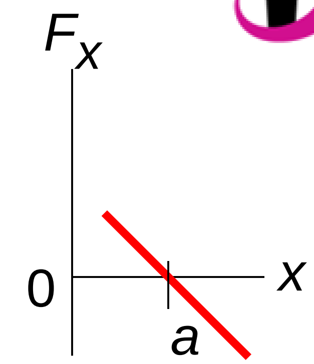

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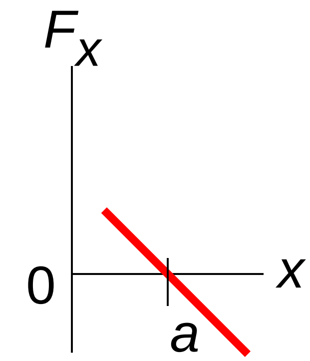

A. $U = 0$ at $x = a$

B. U is a maximum at $x = a$.

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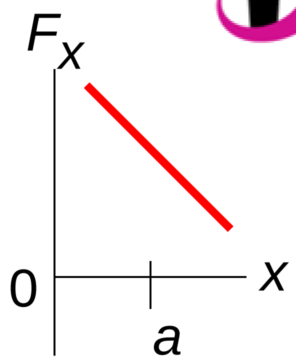
Q7.11



The graph shows a conservative force F_x as a function of x in the vicinity of $x = a$. As the graph shows, $F_x > 0$ and $dF_x/dx < 0$ at $x = a$. Which statement about the associated *potential energy* function U at $x = a$ is correct?

A. $dU/dx > 0$ at $x = a$

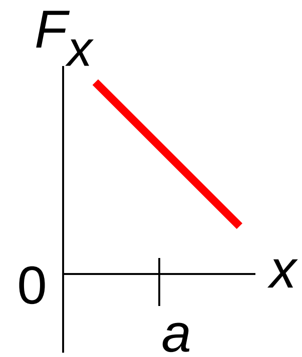
B. $dU/dx < 0$ at $x = a$

C. $dU/dx = 0$ at $x = a$

D. Any of the above could be correct.

A7.11

The graph shows a conservative force F_x as a function of x in the vicinity of $x = a$. As the graph shows, $F_x > 0$ and $dF_x/dx < 0$ at $x = a$. Which statement about the associated *potential energy* function U at $x = a$ is correct?



A. $dU/dx > 0$ at $x = a$

B. $dU/dx < 0$ at $x = a$

C. $dU/dx = 0$ at $x = a$

D. Any of the above could be correct.