3. Momentum, collision and center of mass

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Define **momentum** $\vec{p} = m\vec{v},$

SI unit: kg·m/s

Newton's second law in terms of momentum:

$$\sum \vec{F} = m\frac{d\vec{v}}{dt} = \frac{d\vec{p}}{dt}$$

Suppose net force $\sum \vec{F}$ is constant → ???

Define **impulse**

$$\vec{J} = \sum \vec{F}\,(t_2 - t_1) = \sum \vec{F}\,\Delta t,$$

SI unit: N·s

⚠ Most useful if the force is in effect for a short time, i.e., when Δt is small

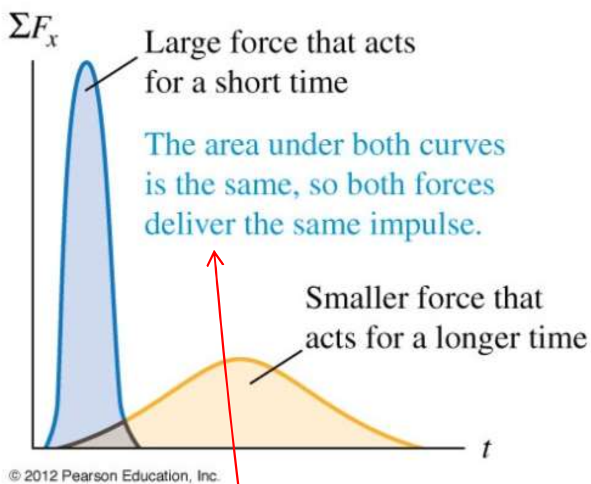
From Newton's second law

$$\sum \vec{\boldsymbol{F}}\ (t_2 - t_1) = \vec{\boldsymbol{p}}_2 - \vec{\boldsymbol{p}}_1$$

i.e., $$\vec{\boldsymbol{J}} = \vec{\boldsymbol{p}}_2 - \vec{\boldsymbol{p}}_1$$

Impulse-momentum theorem:
The change in momentum of a particle during a time interval equals the impulse of the net force acting on the particle during that interval

But in general, $\sum \vec{F}$ is not constant!



area under graph

Define impulse as $$\vec{J} = \int_{t_1}^{t_2} \sum \vec{F}\, dt = \int_{t_1}^{t_2} \frac{d\vec{p}}{dt}\, dt = \vec{p}_2 - \vec{p}_1$$

Impulse-momentum theorem again !!

⚠ Different forces can give the same impulse

Q8.2

You are testing a new car using crash test dummies. Consider two ways to slow the car from 90 km/h (56 mi/h) to a complete stop:

(i) You let the car slam into a wall, bringing it to a sudden stop.

(ii) You let the car plow into a giant tub of gelatin so that it comes to a gradual halt.

In which case is there a greater *impulse* of the net force on the car?

A. in case (i)

B. in case (ii)

C. The impulse is the same in both cases.

D. not enough information given to decide

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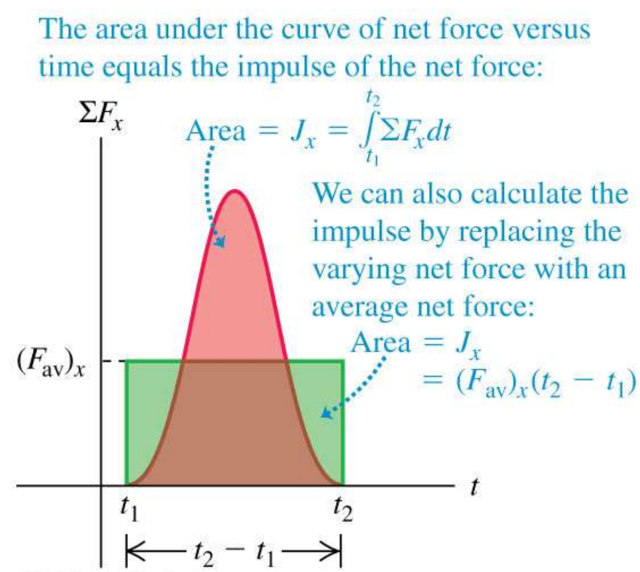
Define average net force $\vec{\boldsymbol{F}}_{\text{av}}$ as the constant force that gives the same impulse

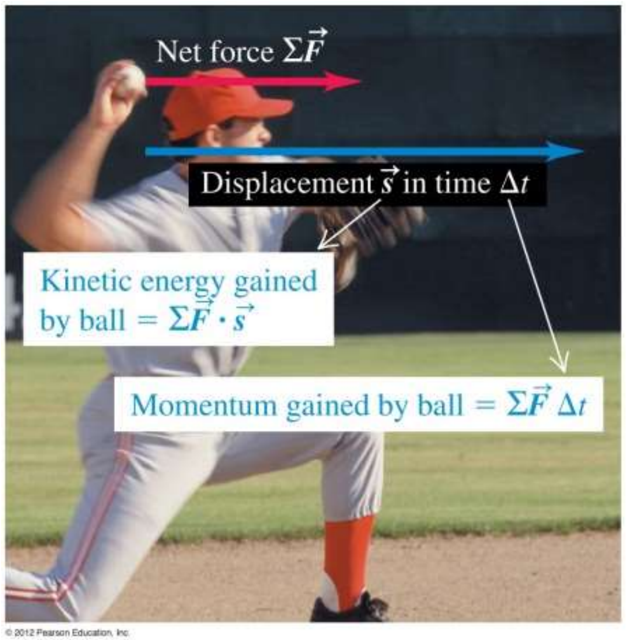
$$\vec{\boldsymbol{J}} = \int_{t_1}^{t_2} \sum \vec{\boldsymbol{F}}\, dt = \vec{\boldsymbol{F}}_{\text{av}}(t_2 - t_1)$$

$$\Longrightarrow \qquad \vec{\boldsymbol{F}}_{\text{av}} = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} \sum \vec{\boldsymbol{F}}\, dt$$

Geometric interpretation: $\vec{\boldsymbol{F}}_{\text{av}}$ is a constant force that has the same area under it as the variable force



Example: Catching a ball

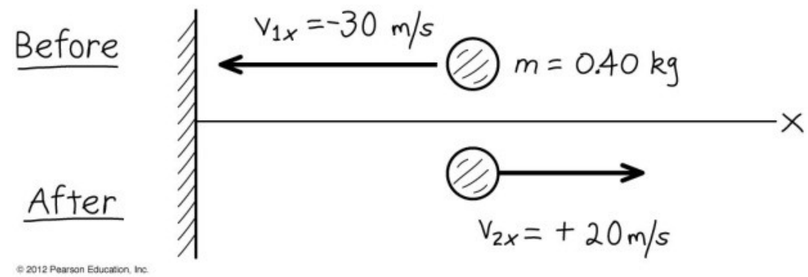


Case 1: 0.50 kg ball moving at 4.0 m/s,
p = 2.0 kg·m/s, K = 4.0 J
Case 2: 0.10 kg ball moving at 20 m/s,
p = 2.0 kg·m/s, K = 20 J
Which one is easier to catch?

Suppose your hand exerts the same force in both cases:
Both stop within the same time interval (∵ same impulse)
But case 2 stops at 5 times the distance (∵K is 5 times larger)

Example

A ball hits a wall and bounced back. Assume the ball is in contact with the wall for 0.010 s



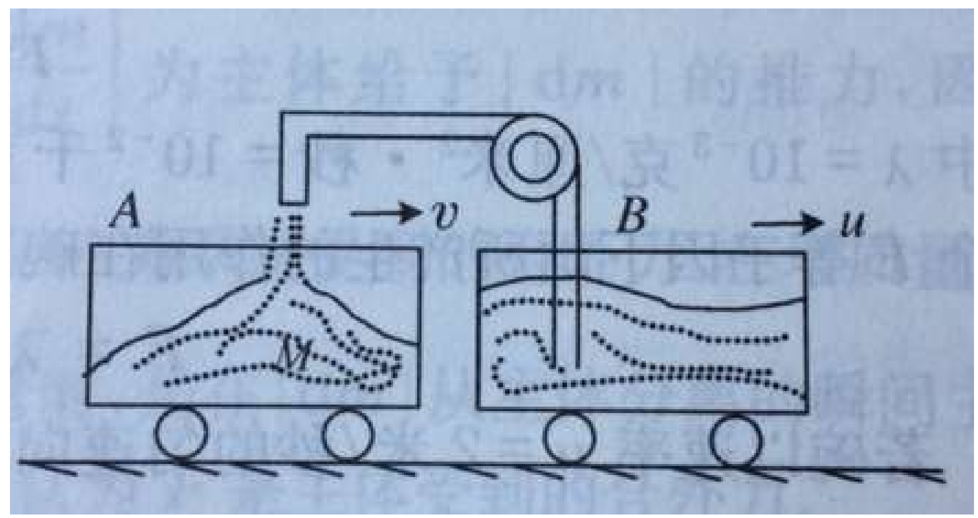
Impulse $J = m(v_{2x} - v_{1x})$

= (0.40 kg)(20 – (– 30) m/s) = 20 N·s

J is a vector, be careful about the direction

average force $F_{av} = \frac{J}{\Delta t} = \frac{20\ \text{N·s}}{0.010\ \text{s}} = 2000\ \text{N}$

Example There are some powder pumping continuously from car B to car A with the rate of b-kg per second. The powder is pumped vertically and thus has the same velocity u as car B. At the instance when the car A has mass M and velocity v. Find the acceleration a of car A.



Example

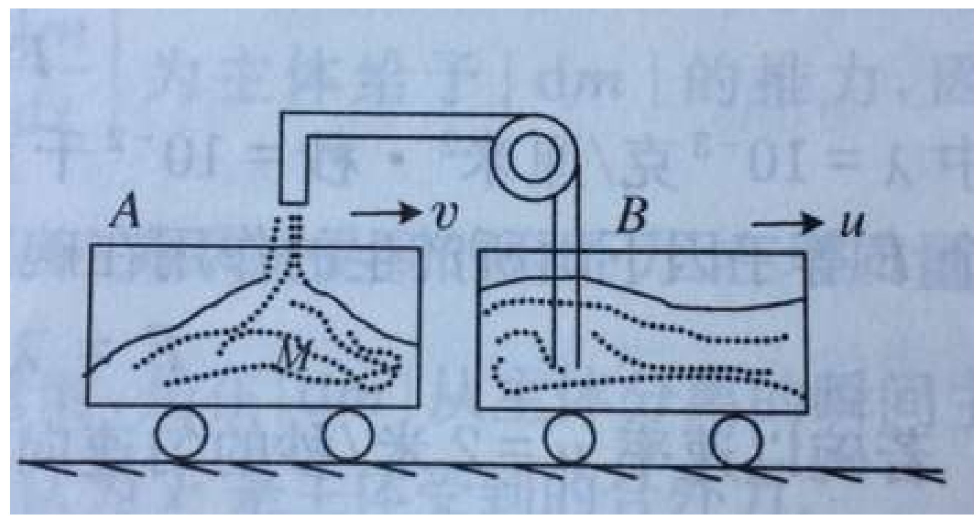
There are some powder pumping continuously from car B to car A with the rate of b-kg per second. The powder is pumped vertically and thus has the same velocity u as car B. At the instance when the car A has mass M and velocity v. Find the acceleration a of car A.

We consider what happen after a short time Δt.
powder with mass bΔt with velocity u entered into car A
By conservation of momentum, we have

$$b\Delta t u + Mv = (M + b\Delta t)(v + \Delta v)$$

$$bu\Delta t = M\Delta v + bv\Delta t$$

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{b}{M}(u - v)$$



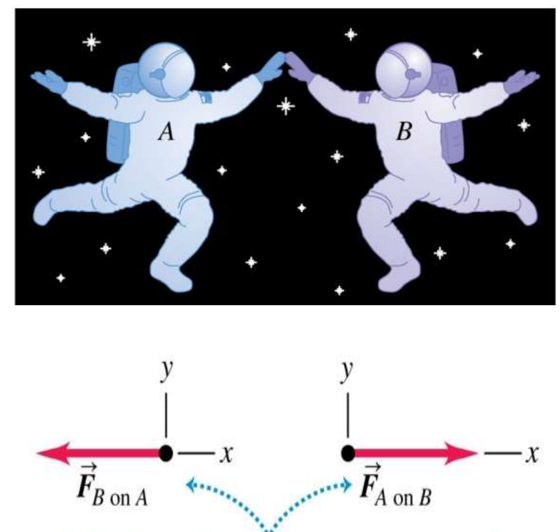
Some terminologies:

A **system** means a collection of bodies, e.g. the 2 astronauts form a system.

- **Internal forces** are forces which individual bodies in the same system exert on others, e.g., the push between the astronauts. **Internal forces always exist as action and reaction pairs.**

- **External forces** are forces exerted on one or more bodies of the system by another object outside it, e.g., gravitational (if any) pull on the astronauts.

A system with no external forces is called an **isolated system**.



The forces the astronauts exert on each other form an action–reaction pair.

Consider a 2 body system,

Net force on *A*, $\vec{\boldsymbol{F}}_A = \frac{d\vec{\boldsymbol{p}}_A}{dt}$,

net force on *B*, $\vec{\boldsymbol{F}}_B = \frac{d\vec{\boldsymbol{p}}_B}{dt}$

If it is an isolated system, $\vec{\boldsymbol{F}}_A$ and $\vec{\boldsymbol{F}}_B$ are action and reaction pair

$$\vec{\boldsymbol{F}}_A = -\vec{\boldsymbol{F}}_B \quad \Longrightarrow \quad \frac{d\vec{\boldsymbol{p}}_A}{dt} + \frac{d\vec{\boldsymbol{p}}_B}{dt} = 0$$

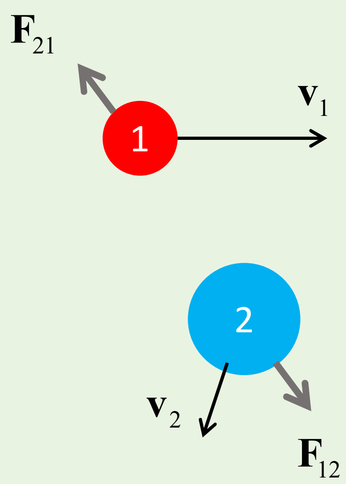
Define **total momentum** of the system $\vec{\boldsymbol{P}} = \vec{\boldsymbol{p}}_A + \vec{\boldsymbol{p}}_B$

$$\Rightarrow \quad \frac{d\vec{\boldsymbol{P}}}{dt} = 0,$$

$\vec{\boldsymbol{P}}$ is constant or conserved

Two Bodies

$$\frac{d}{dt}\mathbf{p}_{\text{total}} = \frac{d}{dt}\left(m_1\mathbf{v}_1 + m_2\mathbf{v}_2\right)$$
$$= m_1\frac{d\mathbf{v}_1}{dt} + m_2\frac{d\mathbf{v}_2}{dt}$$
$$= m_1\mathbf{a}_1 + m_2\mathbf{a}_2$$
$$= \mathbf{F}_{21} + \mathbf{F}_{12}$$
$$= \mathbf{0}$$



$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = \text{Constant}$$

Three Bodies

$$\frac{d}{dt}\mathbf{p}_{\text{total}}$$

$$= \frac{d}{dt}\left(m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + m_3\mathbf{v}_3\right)$$

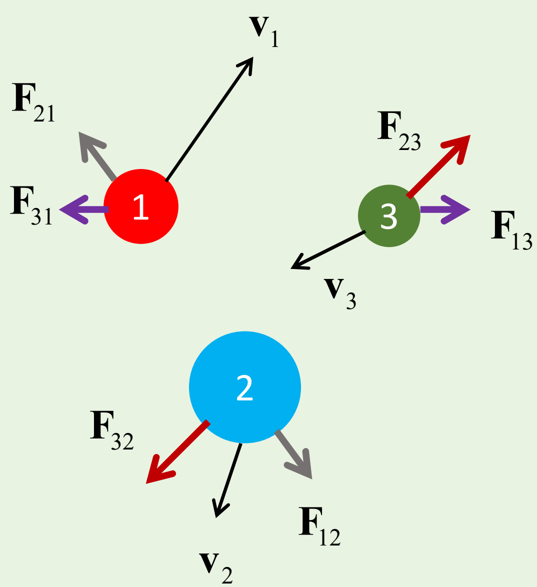
$$= m_1\frac{d\mathbf{v}_1}{dt} + m_2\frac{d\mathbf{v}_2}{dt} + m_3\frac{d\mathbf{v}_3}{dt}$$

$$= m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + m_3\mathbf{a}_3$$

$$= \left(\mathbf{F}_{21} + \mathbf{F}_{31}\right) + \left(\mathbf{F}_{12} + \mathbf{F}_{32}\right) + \left(\mathbf{F}_{13} + \mathbf{F}_{23}\right)$$

$$= \left(\mathbf{F}_{12} + \mathbf{F}_{21}\right) + \left(\mathbf{F}_{23} + \mathbf{F}_{32}\right) + \left(\mathbf{F}_{31} + \mathbf{F}_{13}\right)$$

$$= \mathbf{0}$$



$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + m_3\mathbf{v}_3 = \text{Constant}$$

N Bodies

$$
\begin{aligned}
\frac{d}{dt}\mathbf{p}_{\text{total}} &= \frac{d}{dt}\sum_{i=1}^{N} m_i \mathbf{v}_i = \sum_{i=1}^{N} m_i \frac{d\mathbf{v}_i}{dt} \\
&= \sum_{i=1}^{N} m_i \mathbf{a}_i = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \mathbf{F}_{ji} = \sum_{i}^{N} \sum_{j=i+1}^{N} \left(\mathbf{F}_{ij} + \mathbf{F}_{ji}\right) \\
&= \mathbf{0}
\end{aligned}
$$

$$
\mathbf{p}_{\text{total}} = \sum_{i=1}^{N} m_i \mathbf{v}_i = \text{ Constant}
$$

Question

- A spring-loaded toy sits at rest on a horizontal, frictionless surface. When the spring releases, the toy breaks into three equal mass pieces, A, B, and C, which slide along the surface. A moves off in the negative x direction, while B moves off in the negative y direction.
  a) What are the signs of the velocity components of C along the x and y directions?
  b) Which of the three pieces is moving the fastest?

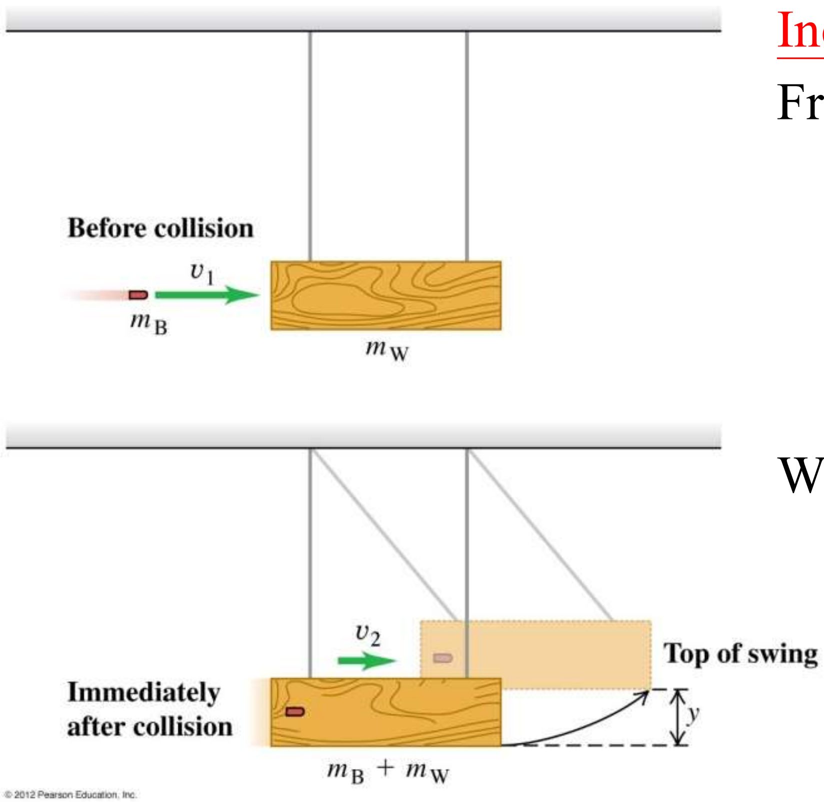
Under no <u>net</u> external force, momentum always conserved, but not mechanical energy.

In an **elastic collision**, the KE is the same before and after the collision. (No change in PE during the impact.)

In an **inelastic collision**, the KE before the collision is larger.

In a **completely inelastic collision**, the bodies stick together after collision.

Example The ballistic pendulum – one way to measure the speed of a bullet



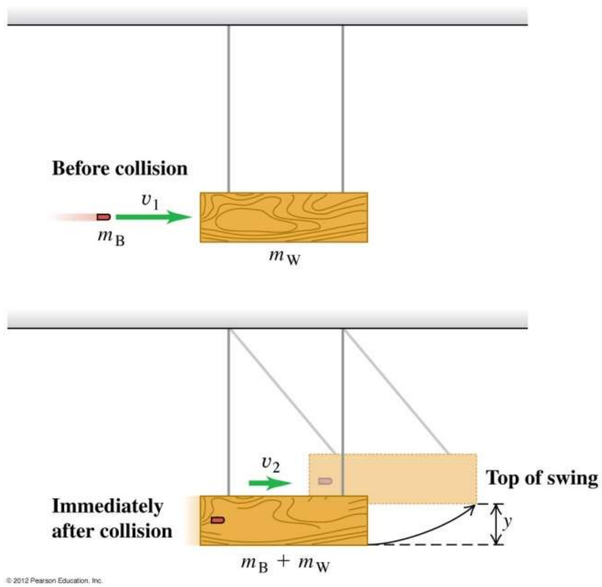
Incorrect solution:

From conservation of energy

$$\frac{1}{2} m_B v_1^2 = (m_B + m_W) g y$$

$$\Rightarrow v_1 = \sqrt{\frac{2(m_B + m_W) g y}{m_B}}$$

What is wrong?



Correct solution:
Conservation of momentum:

$$m_B v_1 = (m_B + m_W) v_2 \Rightarrow \quad v_2 = \frac{m_B v_1}{m_B + m_W}$$

Conservation of energy after collision:

$$\tfrac{1}{2}(m_B + m_W) v_2^2 = (m_B + m_W) g y$$

$$\Rightarrow \frac{1}{2}\left(\frac{m_B v_1}{m_B + m_W}\right)^2 = gy$$

$$\Rightarrow \quad v_1 = \frac{m_B + m_W}{m_B}\sqrt{2gy}$$

Put in realistic numbers,
m_B = 5.00 g, m_W = 2.00 kg, y = 3.00 cm, then
v_1 = 307 m/s

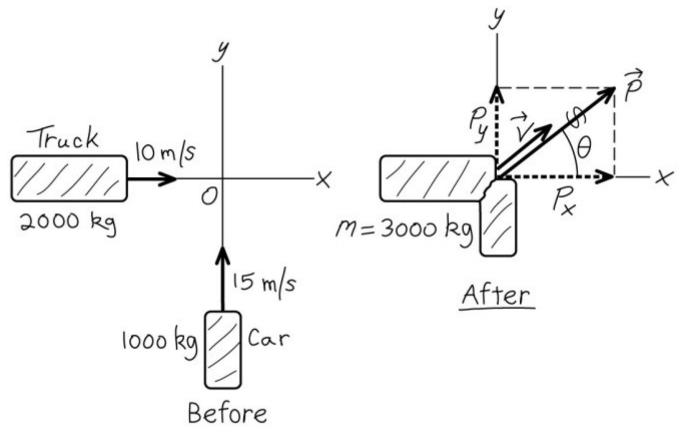
KE before impact is
½(0.00500 kg)(307 m/s)^2 = 236 J

KE after impact is
(0.00500 + 2.00 kg)(9.80 m/s^2)(0.0300 m) = 0.590 J

Most of the original KE is lost! What happens to this amount of energy?

Example An automobile collision

A 1000-kg car traveling north collides with a 2000-kg truck traveling east. Just before the collision, the speed of the car is 15 m/s and that of the truck is 10 m/s. The two vehicles move away from the impact point as one. Find the velocity just after the collision.



By conservation of momentum:

$$(m_C + m_T)V_x = m_T v_{Tx} + m_C v_{Cx}$$

$$\Rightarrow \quad V_x = \frac{m_T\, v_{Tx}}{(m_C + m_T)} = 6.7 \text{ m/s}$$

$$(m_C + m_T)V_y = m_T v_{Ty} + m_C v_{Cy}$$

$$\Rightarrow \quad V_y = \frac{m_C\, v_{Cy}}{(m_C + m_T)} = 5.0 \text{ m/s}$$

$$\therefore V = \sqrt{V_x^2 + V_y^2} = 8.3 \text{ m/s}$$

$$\tan\theta = \frac{V_y}{V_x} = 0.75 \Rightarrow \quad \theta = 37^{\circ}$$

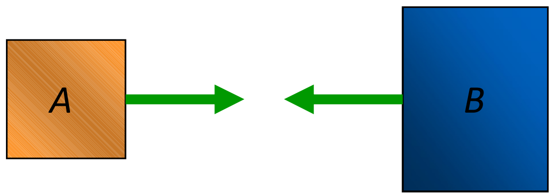
Are there **external** forces acting on the vehicles?
Yes!
Then how to justify using conservation of momentum?

Weight and normal reaction: cancel each other, does not contribute to the net external force.

Friction: contribute to the net external force, but can we neglect it?
The friction f between the vehicles and the road has finite magnitude.
Suppose the collision is ideal and takes time $\Delta t \to 0$, then the impulse is $f\Delta t \to 0$. Hence friction can be neglected.
In general, any external forces with bounded magnitude can be neglected in ideal collisions.
Of course no collision is ideal in the real word. From the given speeds, it is reasonable to assume that the collision takes a time $\Delta t{\sim}0.1\text{s}$. Suppose μ_k= 0.5. Then the frictions are of the order $\mu_k mg$~(0.5)(2000 kg)(10 m/s^2) = 10^4 N. The impulses are of the order $\sim 10^4\text{N} \times 0.1\text{s} = 10^3\text{Ns}$. The initial momenta of the vehicles are of the order of $2 \times 10^4\text{Ns}$. Therefore momentum is conserved approximately and we can simplify the question by neglecting friction.

Q8.4

Two objects with different masses collide and *stick* to each other. Compared to *before* the collision, the system of two objects *after* the collision has



A. the same total momentum and the same total kinetic energy.

B. the same total momentum but less total kinetic energy.

C. less total momentum but the same total kinetic energy.

D. less total momentum and less total kinetic energy.

E. not enough information given to decide

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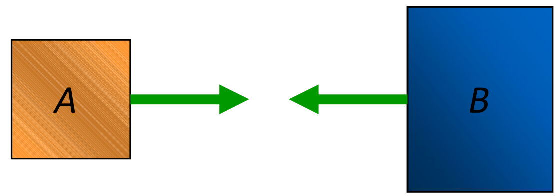

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Q8.6

Block *A* has mass 1.00 kg and block *B* has mass 3.00 kg. The blocks collide and stick together on a level, frictionless surface. After the collision, the kinetic energy (KE) of block *A* is

A. 1/9 the KE of block *B*.

B. 1/3 the KE of block *B*.

C. 3 times the KE of block *B*.

D. 9 times the KE of block *B*.

E. the same as the KE of block *B*.

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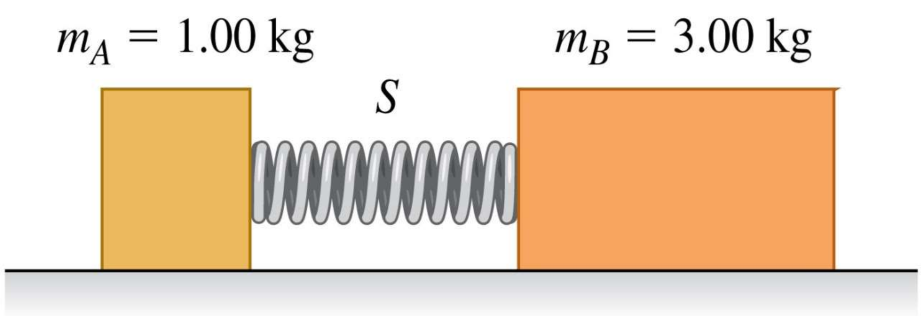
Q8.7

Block *A* on the left has mass 1.00 kg. Block *B* on the right has mass 3.00 kg. The blocks are forced together, compressing the spring. Then the system is released from rest on a level, frictionless surface. After the blocks are released, the kinetic energy (KE) of block *A* is



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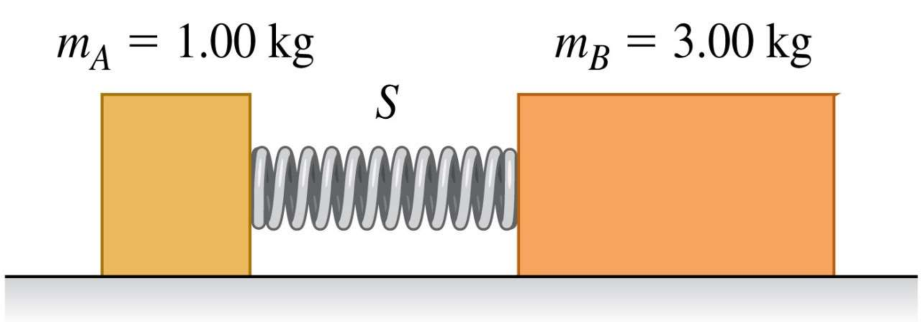

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B. 1/3 the KE of block *B.*

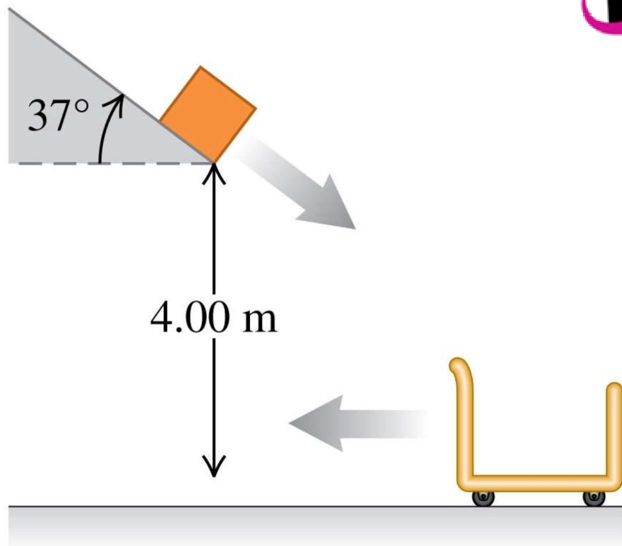
C. 3 times the KE of block *B.*

D. 9 times the KE of block *B.*

E. the same as the KE of block *B.*

Q8.8

An open cart is rolling to the left on a horizontal surface. A package slides down a chute and lands in the cart. Which quantities have the same value just *before* and just *after* the package lands in the cart?



A. the horizontal component of total momentum

B. the vertical component of total momentum

C. the total kinetic energy

D. two of A., B., and C.

E. all of A., B., and C.

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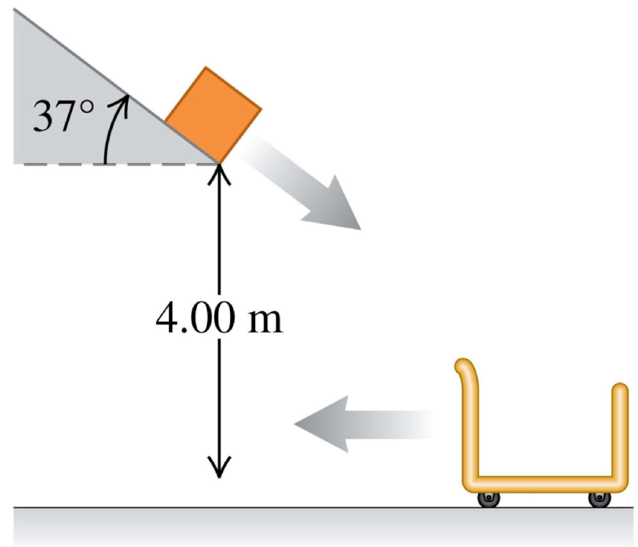


A. the horizontal component of total momentum

B. the vertical component of total momentum

C. the total kinetic energy

D. two of A., B., and C.

E. all of A., B., and C.

Elastic collision

Conservation of energy: $\frac{1}{2}m_A v_{A1x}^2 + \frac{1}{2}m_B v_{B1x}^2 = \frac{1}{2}m_A v_{A2x}^2 + \frac{1}{2}m_B v_{B2x}^2$

Conservation of momentum: $m_A v_{A1x} + m_B v_{B1x} = m_A v_{A2x} + m_B v_{B2x}$

Want to solve for v_{A2x} and v_{B2x}.

Trick:

$$m_A(v_{A1x}^2 - v_{A2x}^2) = m_B(v_{B2x}^2 - v_{B1x}^2)$$

$$m_A(v_{A1x} + v_{A2x})(v_{A1x} - v_{A2x}) = m_B(v_{B2x} + v_{B1x})(v_{B2x} - v_{B1x})$$

From momentum conservation we have

$$m_A(v_{A1x} - v_{A2x}) = m_B(v_{B2x} - v_{B1x})$$

$$\Rightarrow \quad v_{A1x} + v_{A2x} = v_{B1x} + v_{B2x}$$

Physical meaning: $$v_{B2x} - v_{A2x} = -(v_{B1x} - v_{A1x})$$

relative velocity *after* collision = – (relative velocity *before* collision)

⚠ In an elastic collision we can write down three equations:

1. conservation of energy
2. conservation of momentum.
3. relative velocity *after* collision = – (relative velocity *before* collision)

But only two of them are independent. Usually 2 and 3 are preferred because they are linear.

$$\begin{cases} m_A v_{A1x} + m_B v_{B1x} = m_A v_{A2x} + m_B v_{B2x} \\ v_{B2x} - v_{A2x} = -(v_{B1x} - v_{A1x}) \end{cases}$$

$$\Rightarrow v_{A2x} = \frac{m_A - m_B}{m_A + m_B} v_{A1x} + \frac{2m_B}{m_A + m_B} v_{B1x},$$

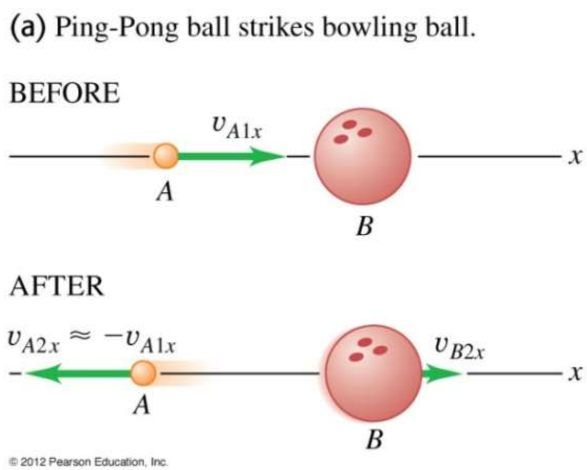
$$v_{B2x} = \frac{2m_A}{m_A + m_B} v_{A1x} + \frac{m_B - m_A}{m_A + m_B} v_{B1x}$$

Special case: Elastic collision with one body initially at rest

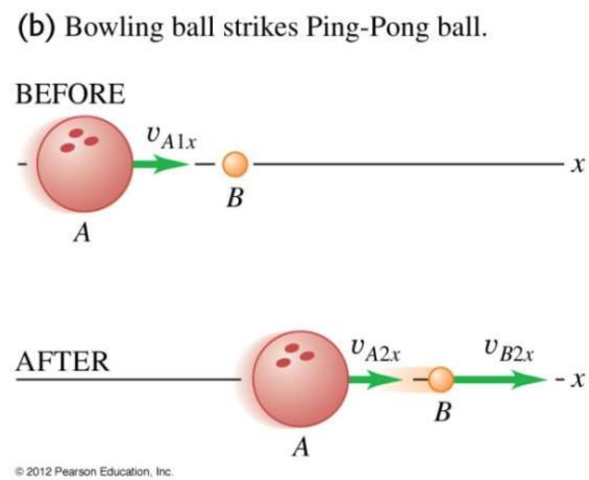
B initially at rest

$$v_{A2x} = \frac{m_A - m_B}{m_A + m_B} v_{A1x}, \qquad v_{B2x} = \frac{2m_A}{m_A + m_B} v_{A1x}$$

⚠ v_{B2x} same direction (same sign) as v_{A1x}, but direction of v_{A2x} depends on m_A-m_B

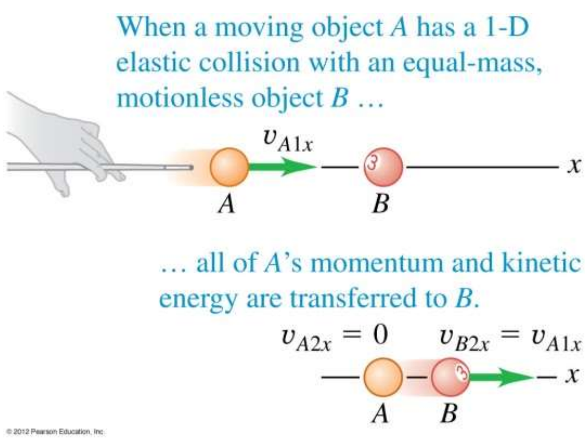


$m_B > m_A$, *A* reflected back



$m_B < m_A$, *A* continue forward and $v_{A2x} < v_{B2x}$

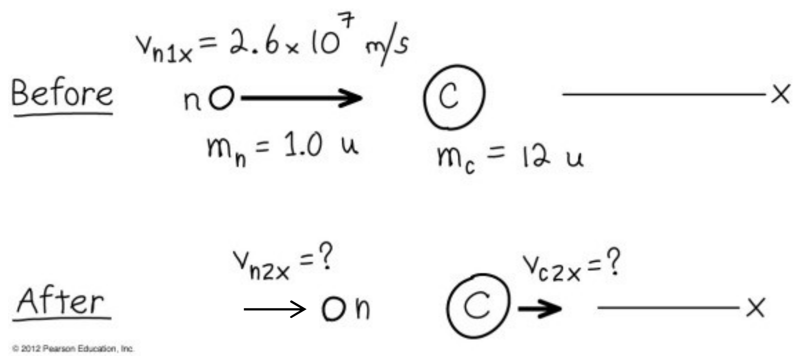
$$v_{A2x} = \frac{m_A - m_B}{m_A + m_B} v_{A1x}, \qquad v_{B2x} = \frac{2m_A}{m_A + m_B} v_{A1x}$$



$$m_A = m_B, v_{A2x} = 0, v_{B2x} = v_{A1x}$$

Example

Moderating fission neutrons in a nuclear reactor
Fission of uranium produces high speed neutrons which must be slowed down before it can initiate another fission process. Suppose graphite (carbon) is used as moderator to slow down neutrons.



assuming elastic collision

relative velocities: $v_{C2x} - v_{n2x} = -(0 - v_{n1x})$

conservation of momentum:

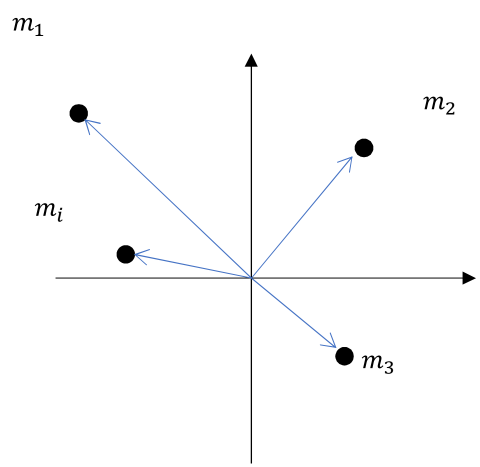
$$m_n v_{n1x} = m_C v_{C2x} + m_n v_{n2x}$$

get: $v_{n2x} = -2.2 \times 10^7$ m/s, $v_{C2x} = 0.4 \times 10^7$ m/s

⚠ Don't worry about the direction (forward or backward) of neutron after collision. Assume all v are +ve

Center of Mass

The center of mass (CM) of a system of point particles is defined as

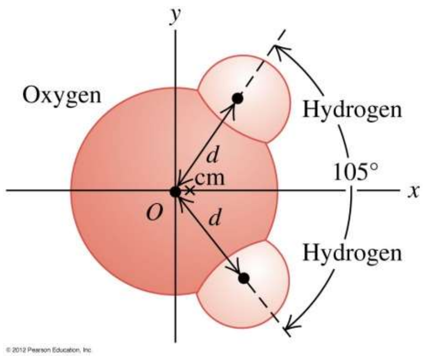


$$\vec{\boldsymbol{r}}_{\text{cm}} = \frac{m_1\vec{\boldsymbol{r}}_1 + m_2\vec{\boldsymbol{r}}_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum m_i\vec{\boldsymbol{r}}_i}{\sum m_i}$$

i.e.

$$x_{\text{cm}} = \frac{\sum m_i x_i}{\sum m_i}, \qquad y_{\text{cm}} = \frac{\sum m_i y_i}{\sum m_i}, \qquad z_{\text{cm}} = \frac{\sum m_i z_i}{\sum m_i}$$

Example



Need to worry about mass of nuclei only (why not e^-?)

⚠ choose symmetry axis of the molecule as the *x*, *y*, and *z* directions

meaning under rotation about that axis, the molecule looks the same

$$x_{\mathrm{cm}} = \frac{(1.0\ \mathrm{u})(d\cos 52.5°) + (1.0\ \mathrm{u})(d\cos 52.5°) + (16.0\ \mathrm{u})(0)}{1.0\ \mathrm{u} + 1.0\ \mathrm{u} + 16.0\ \mathrm{u}}$$
$$= 0.068d = 6.5 \times 10^{-12}\ \mathrm{m}$$

$$y_{\mathrm{cm}} = \frac{(1.0\ \mathrm{u})(d\sin 52.5°) + (1.0\ \mathrm{u})(-d\sin 52.5°) + (16.0\ \mathrm{u})(0)}{1.0\ \mathrm{u} + 1.0\ \mathrm{u} + 16.0\ \mathrm{u}}$$
$$= 0$$

Solid bodies

If not point particles, need integration....

$$x_{\mathrm{cm}} = \frac{1}{M}\int x\,dm,$$

$$y_{\mathrm{cm}} = \frac{1}{M}\int y\,dm,$$

$$z_{\mathrm{cm}} = \frac{1}{M}\int z\,dm.$$

If the object has uniform density,

$$\rho = \frac{dm}{dV} = \frac{M}{V}.$$

Rewriting $dm = \rho dV$ and $m = \rho V$, we obtain

$$x_{\mathrm{cm}} = \frac{1}{V}\int x\,dV,$$

$$y_{\mathrm{cm}} = \frac{1}{V}\int y\,dV,$$

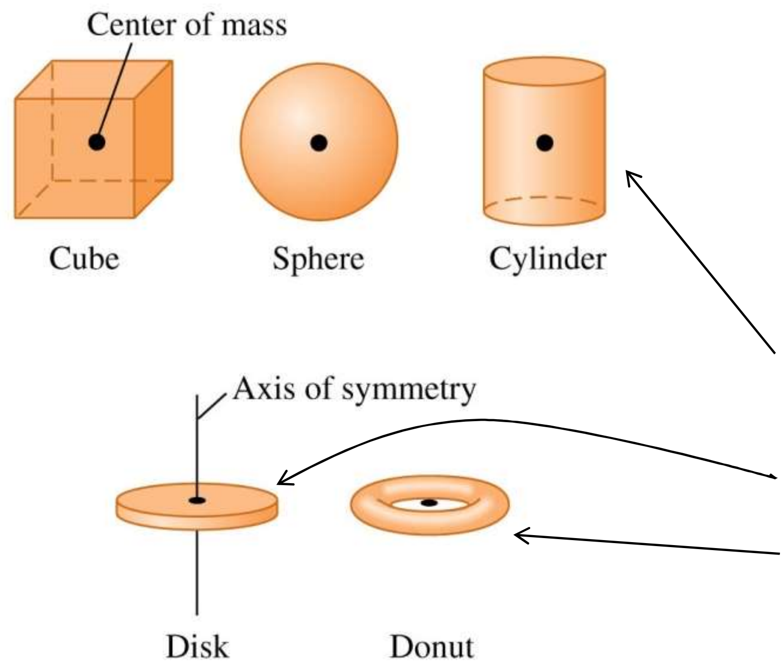
$$z_{\mathrm{cm}} = \frac{1}{V}\int z\,dV.$$

Ref. to tutorial 2

Note:

- If the object has a point of symmetry, then the center of mass lies at that point.
- If the object has a line of symmetry, then the center of mass lies on that line.
- If the object has a plane of symmetry, then the center of mass lies in that plane.

- The center of mass of an object need not lie within the object e.g. a doughnut.



Make use of *symmetry* rules to find CM of homogeneous (constant density) solids:

1) at the geometric center of regular solids
2) lies on symmetry axis
3) CM can lie outside a body

From definition of $\vec{\boldsymbol{r}}_{\text{cm}}$, (by differentiation)

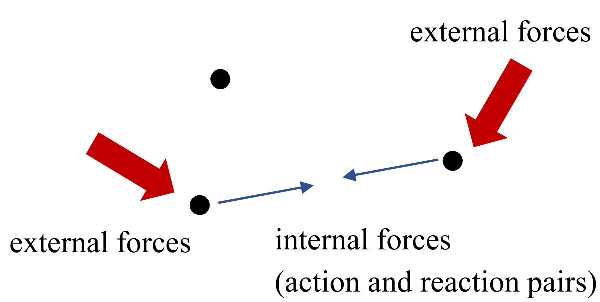
$$\vec{\boldsymbol{r}}_{\text{cm}} = \frac{m_1\vec{\boldsymbol{r}}_1 + m_2\vec{\boldsymbol{r}}_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum m_i\vec{\boldsymbol{r}}_i}{\sum m_i} = \frac{1}{M}\sum_{i=1}^{N} m_i\vec{r}_i$$

$$\vec{\boldsymbol{v}}_{\text{cm}} = \frac{1}{M}(m_1\vec{\boldsymbol{v}}_1 + m_2\vec{\boldsymbol{v}}_2 + \cdots) \Longrightarrow \quad M\vec{\boldsymbol{v}}_{\text{cm}} = m_1\vec{\boldsymbol{v}}_1 + \cdots = \vec{\boldsymbol{P}}$$

total linear momentum

$$M\vec{\boldsymbol{a}}_{\text{cm}} = m_1\vec{\boldsymbol{a}}_1 + m_2\vec{\boldsymbol{a}}_2 + \cdots = \sum \vec{\boldsymbol{F}}_{\text{ext}} + \sum \vec{\boldsymbol{F}}_{\text{int}} = \sum \vec{\boldsymbol{F}}_{\text{ext}}$$

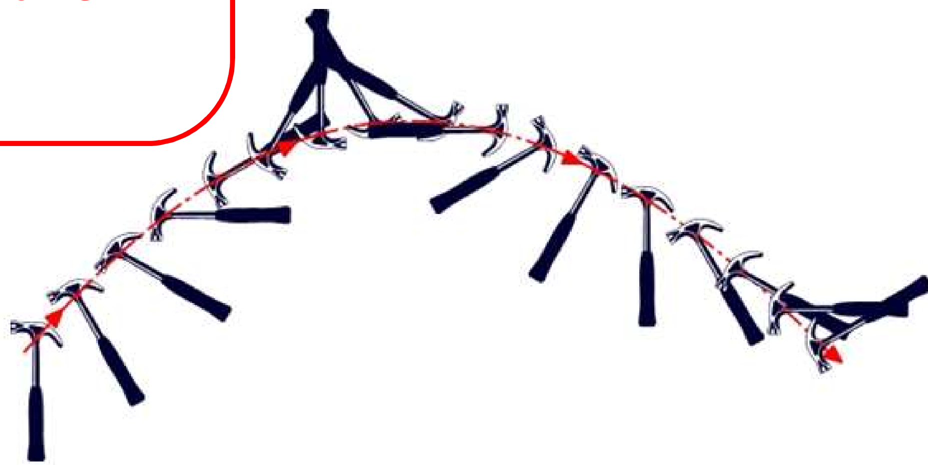
in equal and opposite pairs,
add up to zero



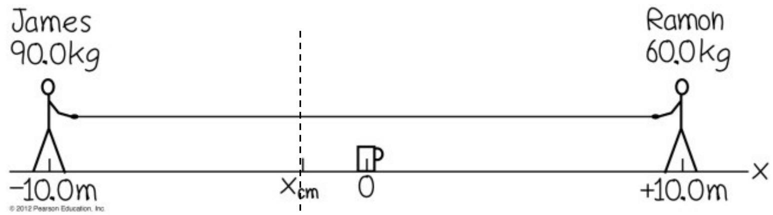
Conclusion:

$$M\vec{\boldsymbol{a}}_{\text{cm}} = \sum \vec{\boldsymbol{F}}_{\text{ext}} = \frac{d\vec{\boldsymbol{P}}}{dt}$$

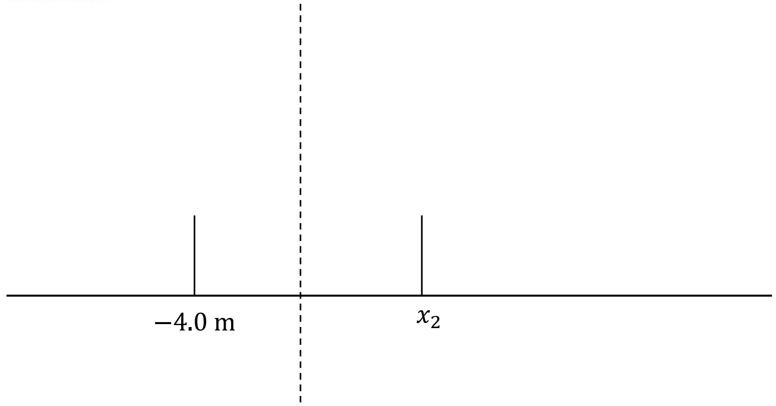
When a body or a collection of particles is acted on by external forces, the CM moves just as though all the mass were concentrated at that point and it were acted on by a net force equal to the sum of the external forces on the system.



An example with no external force – tug of war on ice



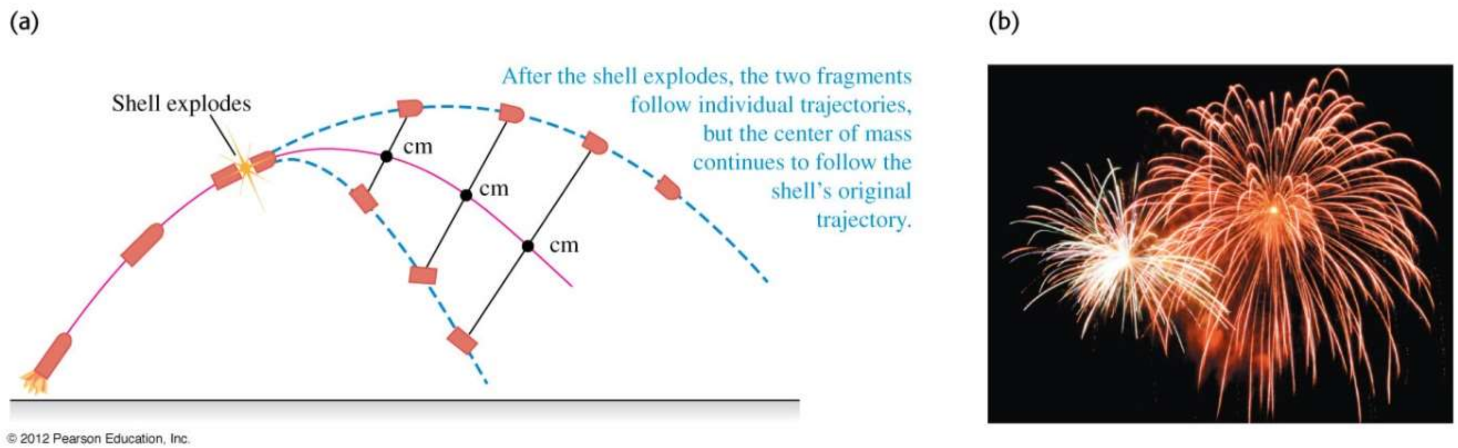
$$x_{cm} = \frac{(90.0\text{ kg})(-10.0\text{ m}) + (60.0\text{ kg})(10.0\text{ m})}{90.0\text{ kg} + 60.0\text{ kg}}$$



$$x_{cm} = \frac{(90.0\text{ kg})(-4.0\text{ m}) + (60.0\text{ kg})x_2}{90.0\text{ kg} + 60.0\text{ kg}}$$

$$\Rightarrow \quad x_2 = 1.0\text{ m}$$

An example with external force – A shell explodes into two fragments in flight.



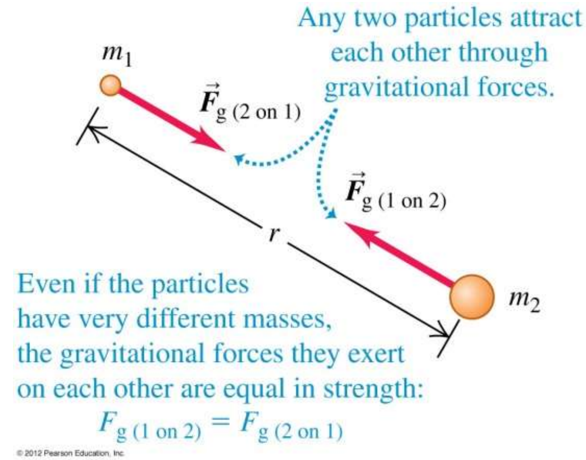
Question: Will the CM in the above problem continue on the same parabolic trajectory even after one of the fragments hits the ground?

GRAVITATION

Newton's Law of Gravitation

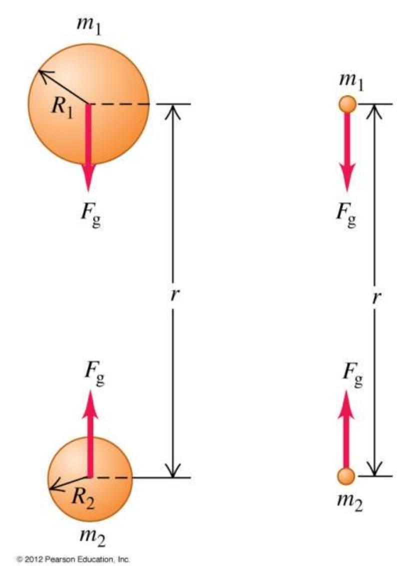
$$F_g = \frac{Gm_1 m_2}{r^2}$$

inverse square law



G: **gravitational constant** 6.67×10^{-11} Nm^2/kg^2

Gravitational attraction between two masses is always along the line joining them (called **central force**), and forms an action-reaction pair



When outside a spherically symmetric body (i.e., density $\rho(r)$ depends on radial distance r only, not on direction), the gravitational effect is the same as if all of the mass were concentrated at its center

Q14.1

The mass of the Moon is 1/81 of the mass of the Earth.

Compared to the gravitational force that the Earth exerts on the Moon, the gravitational force that the Moon exerts on the Earth is

A. 81^2 = 6561 times greater.

B. 81 times greater.

C. equally strong.

D. 1/81 as great.

E. $(1/81)^2$ = 1/6561 as great.

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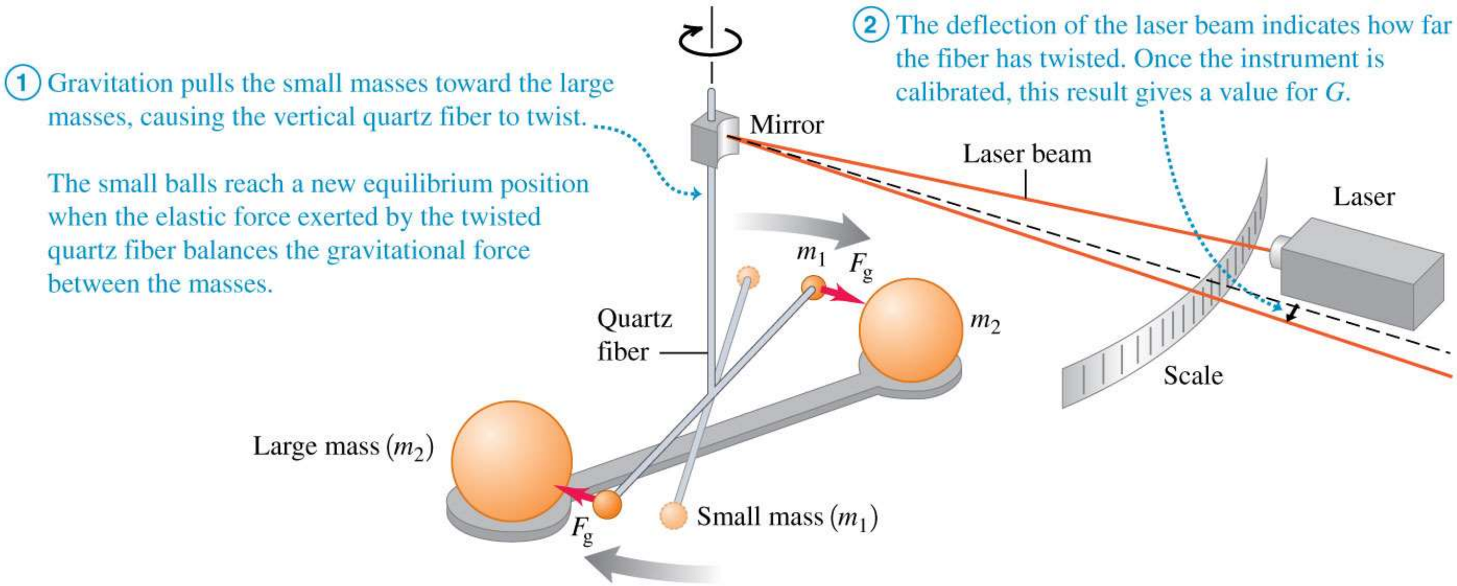
Cavendish Experiment – to measure *G* (or to "weight the earth")



Question

Saturn is about 100 times the mass of the earth and about 10 times farther from the sun than the earth. Compared to the acceleration of the earth caused by the sun's gravitational pull, the acceleration of Saturn due to the sun's gravitation is (100 times greater / 10 times greater / the same / $\frac{1}{10}$ as great / $\frac{1}{100}$ as great).

Four fundamental forces of nature:

Force	Example	Range
Gravitation force	Hold planets together	∞
Electromagnetic force	Hold molecules together	∞
Strong force	Hold nucleons (protons and neutrons in an atomic nucleus) together	10^{-15} m
Weak force	Beta decay of nuclei	10^{-18} m

Weight – defined as the total gravitational forces exert on the body by all other bodies in the universe

On earth's surface, gravitational attraction by the earth dominates over others

$$W = F_g = \frac{Gm_E m}{R_E^2} = mg \quad \Longrightarrow \quad g = \frac{Gm_E}{R_E^2}$$

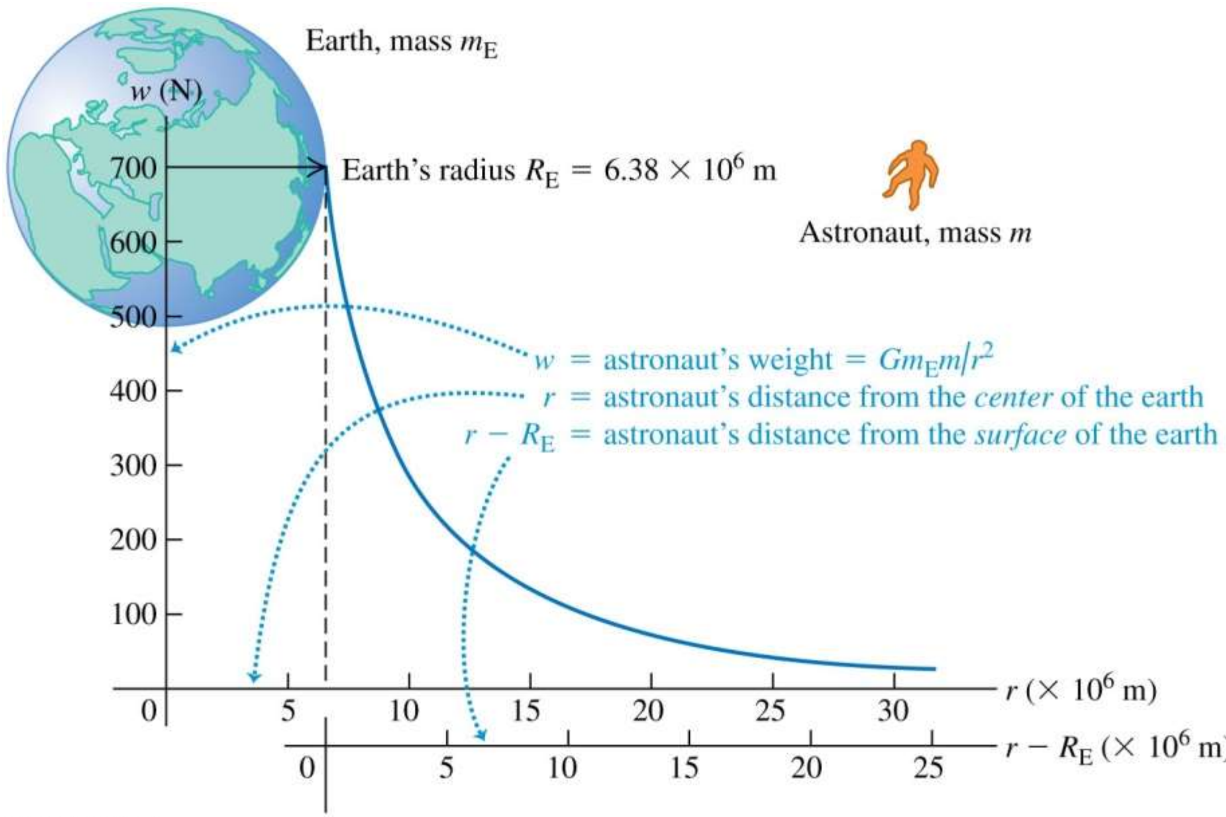
↗

assuming earth is a sphere
with radius R_E and mass m_E

measure $g = 9.8\ \mathrm{m}/s^2$

$R_E = 6.38 \times 10^6$ m

$\Longrightarrow \quad m_E = 5.974 \times 10^{24}$ kg



when $r > R_E$, weight decreases as $1/r^2$,

$$W = Gm_E m/r^2$$

Gravitational Potential Energy – beyond $U = mgy$ (near earth surface only)
Recall: gravitation is a conservative force
Reminder: revisit the properties of conservative forces in Lecture 5

work done by gravitational attraction from 1 to 2

path independent, just choose radial path (straight line)

$$-\Delta U = W_{\text{grav}} = \int_1^2 \vec{\boldsymbol{F}}_g \cdot d\vec{\boldsymbol{r}}$$

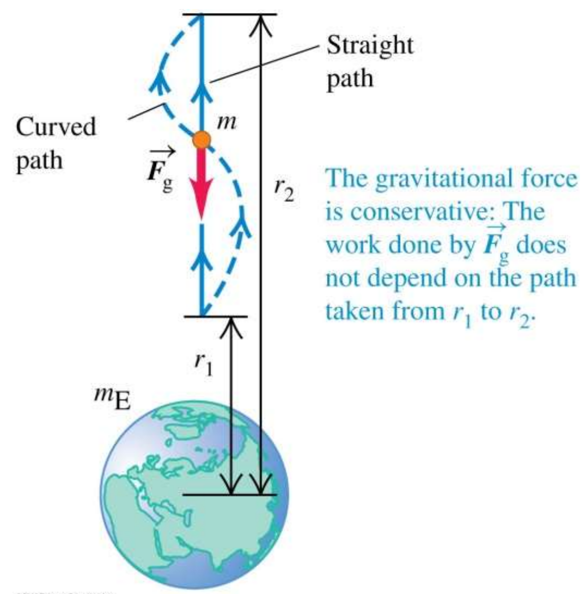
$$= \int_{r_1}^{r_2} (-F_g dr)$$

$$\vec{\boldsymbol{F}}_g \cdot d\vec{\boldsymbol{r}} = -F_g dr$$

inwards outwards

$$= -\left[\left(-\frac{Gm_E m}{r_2}\right) - \left(-\frac{Gm_E m}{r_1}\right)\right]$$

$$= -[U(2) - U(1)]$$



Define

$$\boxed{U(r) = -\frac{Gm_E m}{r}}$$

$U(\infty) = 0$, i.e. zero level of PE at ∞
$U(r) < 0$, decreases (more negative) as r decreases

When close to earth surface, $r_1, r_2 \approx R_E$

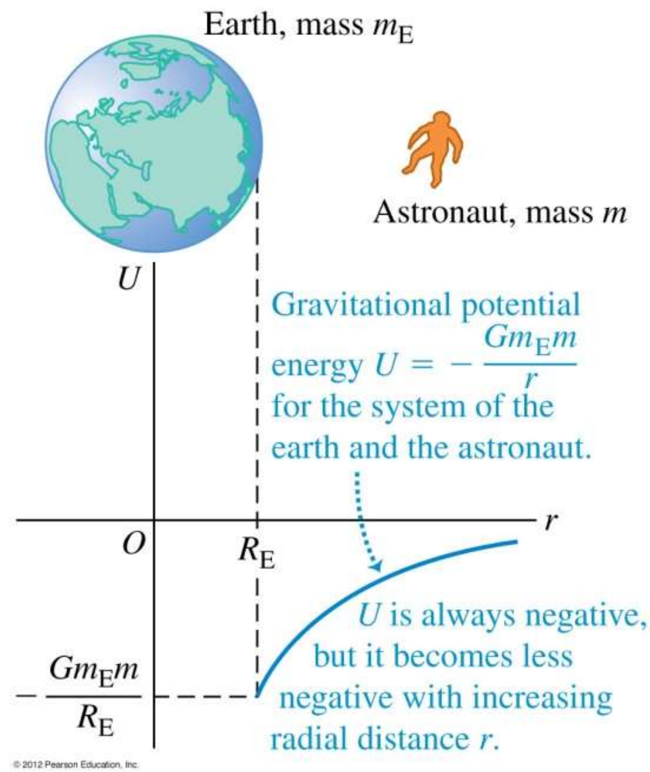
$$W_{\text{grav}} = Gm_E m\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

$$= Gm_E m\frac{r_1 - r_2}{r_1 r_2} \approx m\frac{Gm_E}{R_E^2}(r_1 - r_2)$$

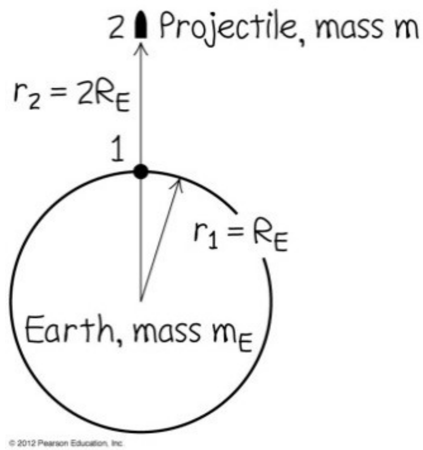
$$= -mg(r_2 - r_1)$$

↑ same as defining $U = mgy$ with zero level of PE arbitrary

g (← $\frac{Gm_E}{R_E^2}$)



Example Escape speed



Shoot a projectile vertically with speed v_1, can it escape from earth's gravitational attraction?

$$\frac{1}{2}mv_1^2 + \left(-\frac{Gm_E m}{R_E}\right) = 0 + 0$$

zero KE, i.e., correspond to minimum v_1

PE at ∞

$$\Rightarrow \boxed{v_1 = \sqrt{\frac{2Gm_E}{R_E}}}$$

escape speed,
⚠ independent of m

On substitution

$$v_1 = \sqrt{\frac{2(6.67\times10^{-11}\ \mathrm{Nm^2/kg^2})(5.97\times10^{24}\ \mathrm{kg})}{6.38\times10^6\ \mathrm{m}}} = 1.12\times10^4\ \mathrm{m/s}$$

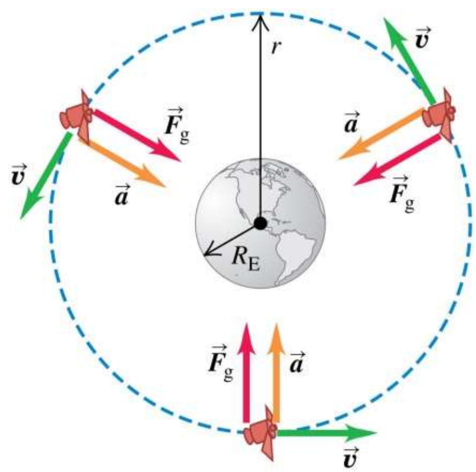
⚠ better to launch a spacecraft towards to east ∵ before launching, its already moving to the east at 410 m/s due to earth's rotation

⚠ air molecules at room temperature $\sim 500\ \mathrm{m/s}$, → atmosphere exists

Question

Is it possible for a planet to have the same surface gravity as the earth (i.e., same g) and yet have a greater escape speed?

Satellites – Assuming circular orbit



⚠ No tangential force, v must be constant

$$\frac{Gm_E m}{r^2} = m\left(\frac{v^2}{r}\right)$$ centripetal acceleration

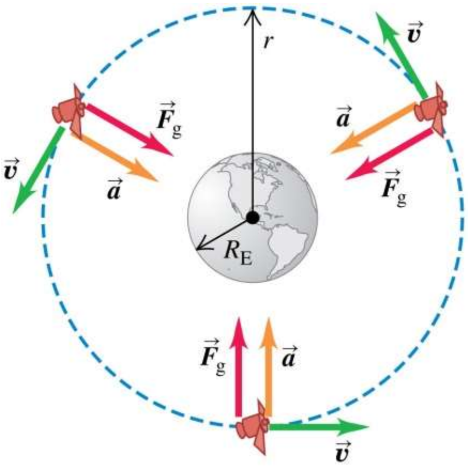
$$\Rightarrow \quad v = \sqrt{\frac{Gm_E}{r}}$$

c.f. escape speed $v_1 = \sqrt{2}v$

⚠ v independent of mass, astronauts orbit about the earth together with spacecraft – apparent weightlessness

True weightlessness only if object is infinitely far from other masses

Satellites – Assuming circular orbit



$$v = \sqrt{\frac{Gm_E}{r}}$$

Period of orbit

$$T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{Gm_E}} = \frac{2\pi}{\sqrt{Gm_E}}r^{3/2}$$

⚠ larger orbit ⟶ longer period

Total energy in an orbit

$$E = K + U = \tfrac{1}{2}mv^2 + \left(-\frac{Gm_E m}{r}\right) = \frac{1}{2}\left(-\frac{Gm_E m}{r}\right) = \frac{U}{2}$$

⚠ larger orbit, larger (less –ve) *E*. If spacecraft loses energy (due to air resistance when it is too close to the earth's atmosphere), *r* decreases and eventually falls to the earth

Example

In order to launch a 1000-kg satellite into a circular orbit 300 km above the earth

$$\underbrace{\frac{1}{2}\left(-\frac{Gm_E m}{R_E + 300\text{ km}}\right)}_{\text{total energy in orbit}} = W_{\text{required}} + \underset{\uparrow \text{ assume no initial KE}}{0} + \left(-\frac{Gm_E m}{R_E}\right)$$

$$\Longrightarrow \quad W_{\text{required}} = 3.26 \times 10^{10}\text{ J}$$

⚠ Ignore rotation of the earth so that no KE before launching.
Its contribution is about $\frac{1}{2}(1000\text{ kg})(410\text{ m/s})^2 = 8.41 \times 10^7$ J, insignificant compared to W_{required}.

Question

A spacecraft is in a low-altitude circular orbit around the earth. Air resistance from the outer regions of the atmosphere does negative work on the spacecraft, causing the orbital radius to decrease slightly. The speed of the spacecraft (remains the same / increases / decreases).

Q14.8

Star X has twice the mass of the Sun. One of Star X's planets has the same mass as the Earth and orbits Star X at the same distance at which the Earth orbits the Sun.

The orbital *speed* of this planet of Star X is

A. faster than the Earth's orbital speed.

B. the same as the Earth's orbital speed.

C. slower than the Earth's orbital speed.

D. not enough information given to decide

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Q14.9

Suppose the Sun were to shrink to half of its present radius while maintaining the same mass. What effect would this have on the Earth's orbit?

A. The size of the orbit would decrease and the orbital period would decrease.

B. The size of the orbit would increase and the orbital period would increase.

C. The size of the orbit and the orbital period would remain unchanged.

D. none of these

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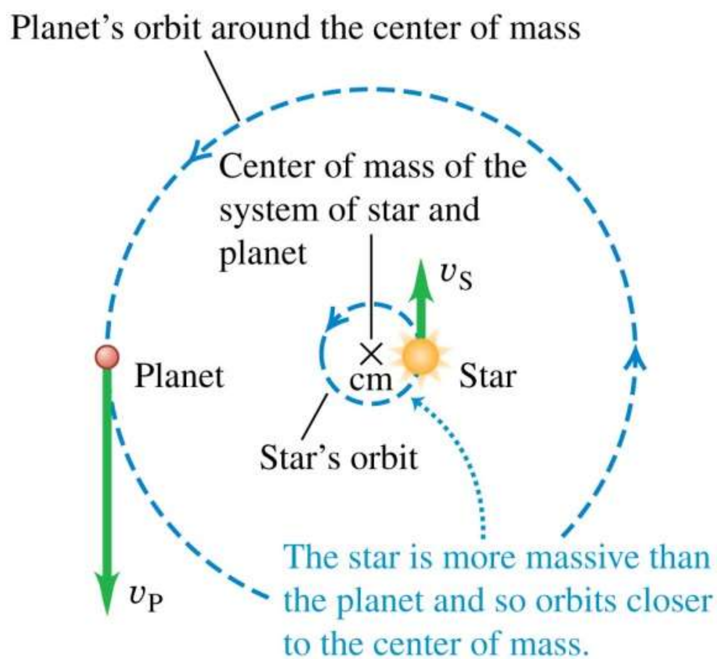
Center of mass



Both sun and planet orbit around their center of mass

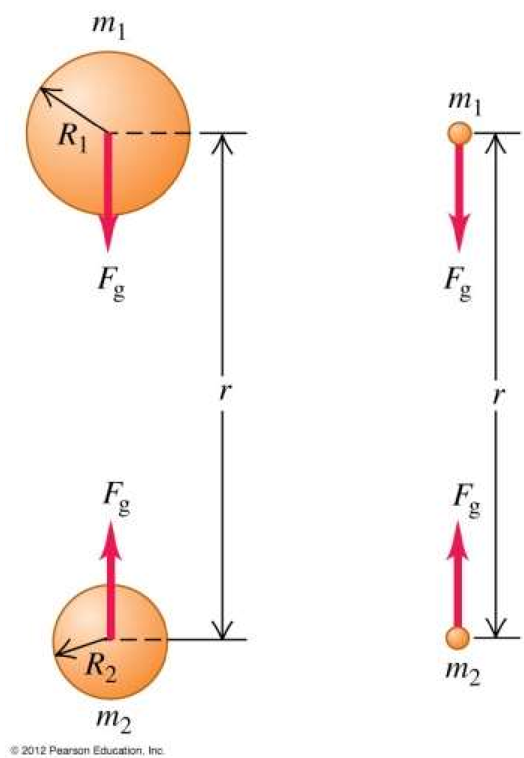
Mass of sun ~ 750 times the total mass of planets → sun effectively at rest

A **binary star** consists of two stars orbiting about their common CM, one called primary (the brighter one) and the other secondary. Can detect the secondary based on wobbling of the primary around their CM

Spherical Mass Distribution

Means density $\rho(r)$ depends on distance from the center only, not on the direction. Can be a shell or solid.

1. The gravitational effect *outside* a spherical mass distribution is the same as if all the mass is concentrated at the center of the sphere.



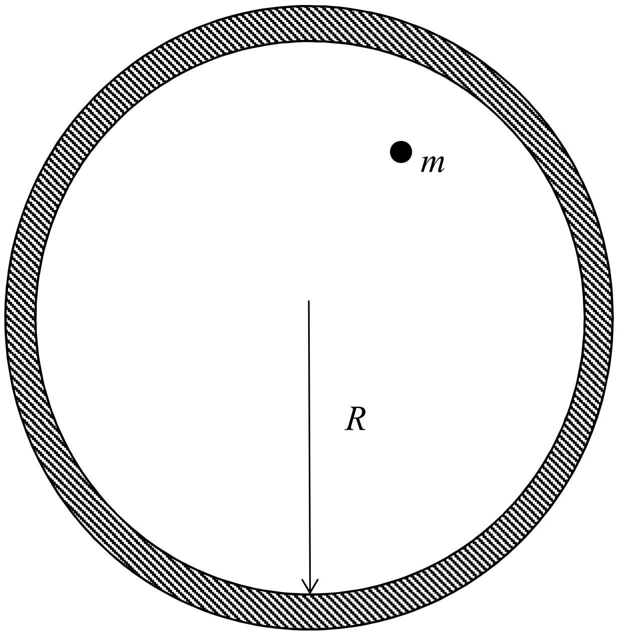
2. The gravitational effect *inside* a hollow sphere is zero.

The gravitational PE of a mass m *anywhere inside* a hollow sphere is a constant

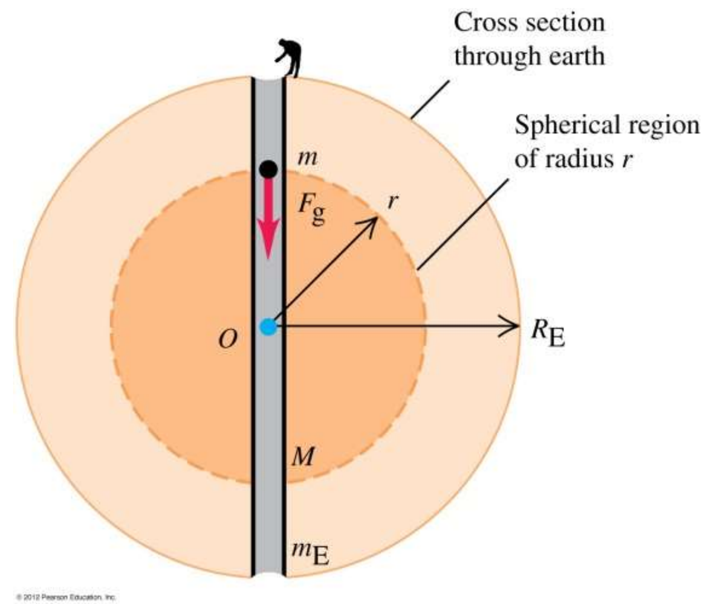
$$U = -\frac{GMm}{R}$$

Gravitational force

$$\vec{\boldsymbol{F}}_g = -\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right) = 0$$



3. The gravitational effect *inside* a spherical mass distribution is the same as if all the mass *interior* to that point is concentrated at the center of the sphere.



Example: when passing through a tunnel through the earth (assume constant density ρ), only the spherical region of radius r contributes to the gravitational force at r

$$F_g = \frac{GMm}{r^2} = \frac{Gm}{r^2}\left[\left(\frac{4}{3}\pi r^3\right)\rho\right] = \frac{Gm_E m}{R_E^3} r$$

$$\rho = \frac{m_E}{\frac{4}{3}\pi R_E^3}$$

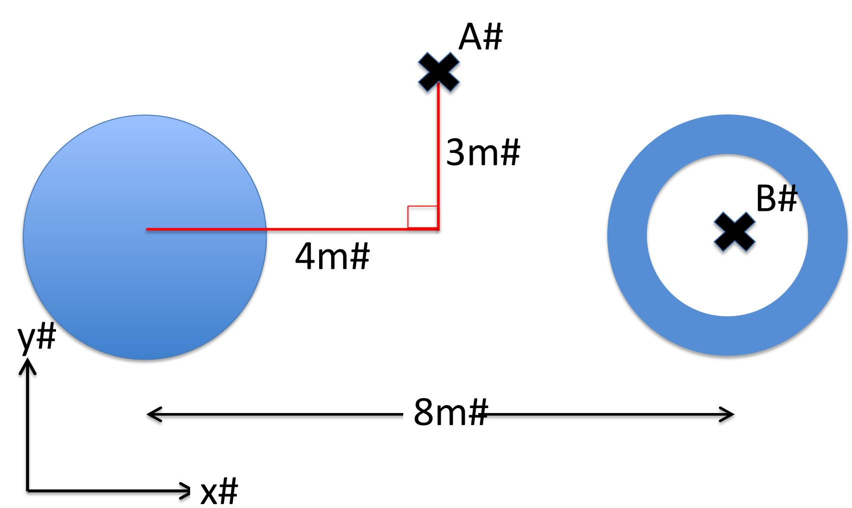
$\propto r$, not $1/r^2$

Example

There is a solid sphere and a spherical shell which are separated by 8m. Both of them have the same mass M=100kg.

(a) What are the magnitude and the direction of the resultant gravitational force acting on this 1-kg mass at point A?

(b) Now, the 1-kg mass is held at the center of the spherical shell (point B), what are the magnitude and direction of the gravitational force acting on it.



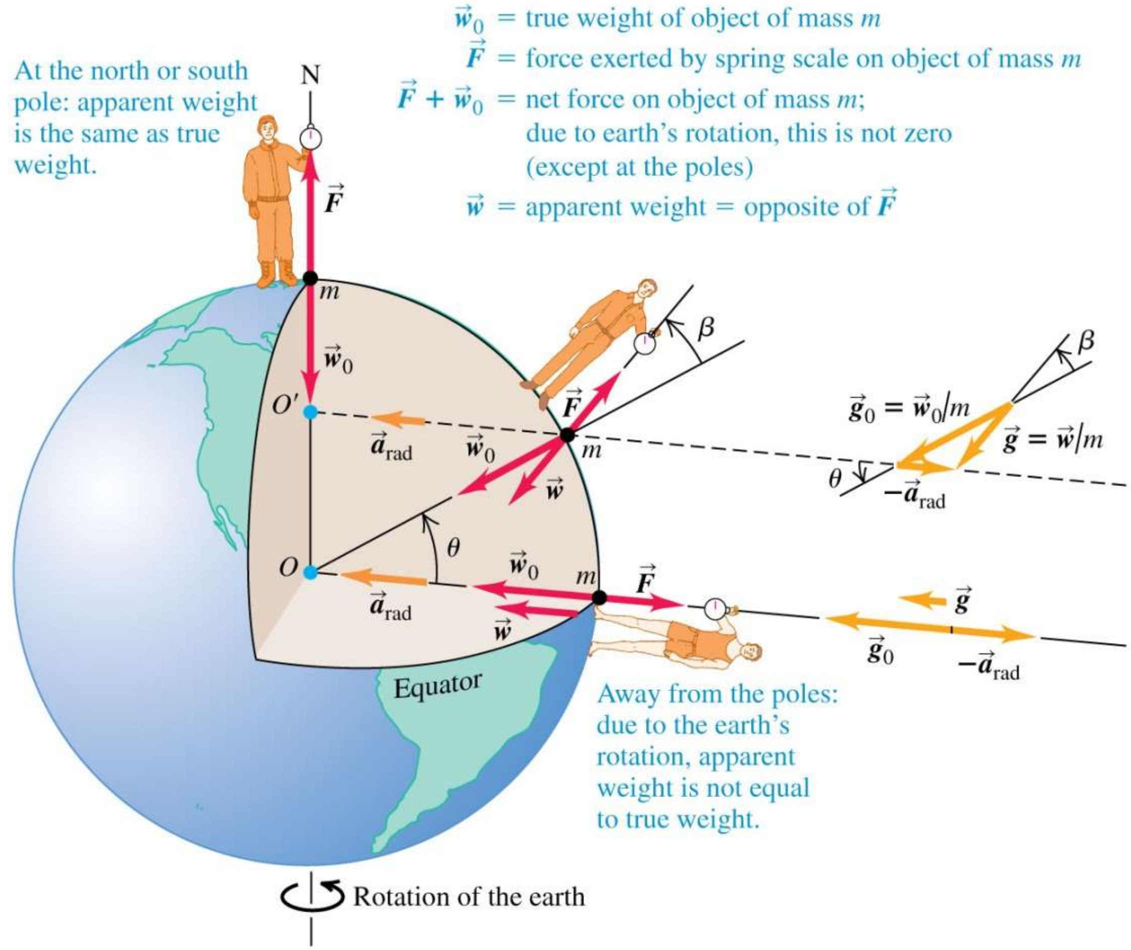
Apparent Weight and the Earth's Rotation

$\vec{w}_0$ true weight (due to earth's gravitational attraction)

If earth not rotating, body in equilibrium,

$$\vec{F} = -\vec{w}_0$$

(this is true at the north/south poles of the rotating earth)



At equator

$$w_0 - F = \frac{mv^2}{R_E}$$

Apparent weight

$$w = F$$

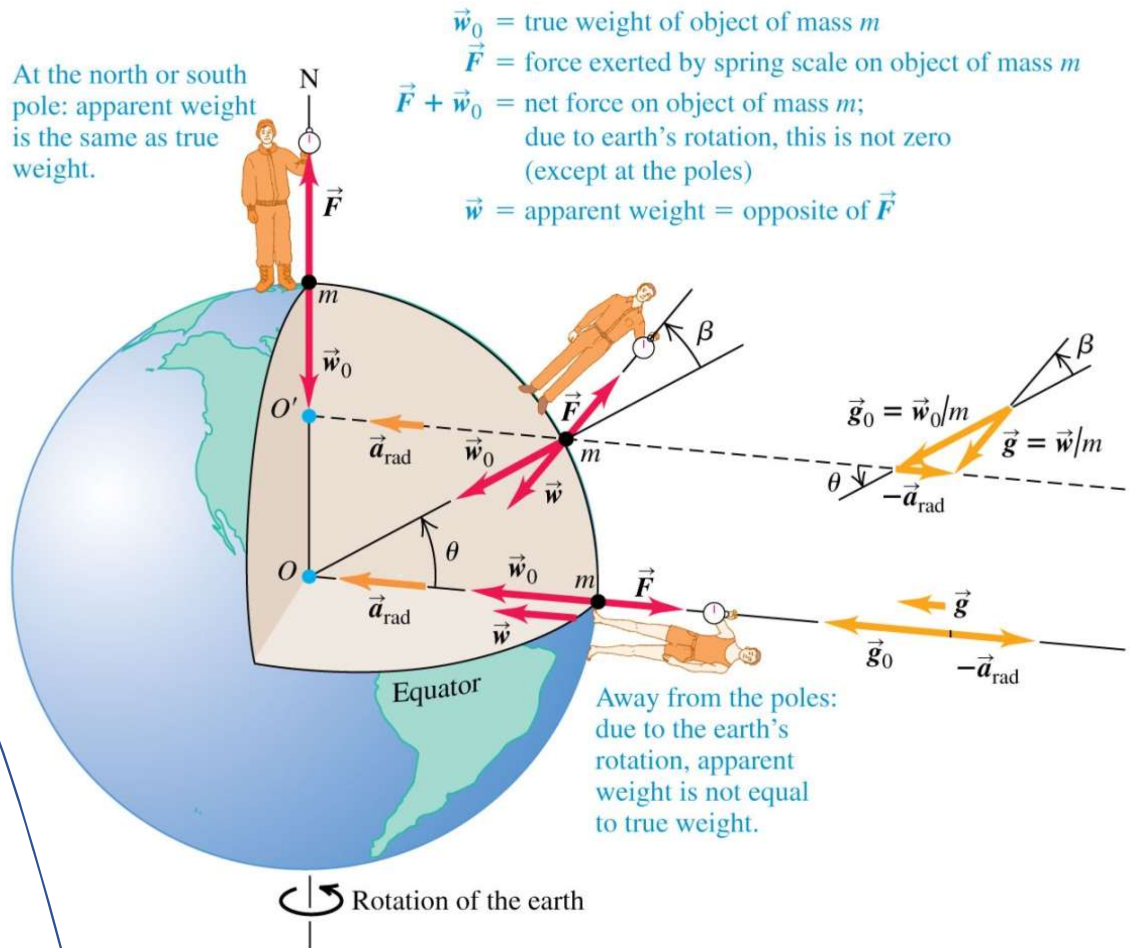
$$= \frac{Gm_E}{R_E^2} m - \frac{mv^2}{R_E}$$

$$= mg$$

acceleration due to gravity of a rotating earth

$$g = g_0 - \frac{v^2}{R_E}$$

$\sim 0.0339\ \mathrm{m/s^2}$



g_0, acceleration due to gravity of a non-rotating earth

Black Holes

Recall *escape speed* from a star of mass *M* and radius *R*

$$v = \sqrt{\frac{2GM}{R}}$$

What if a star collapses, keeping the same *M* but *R* decreases? v increases.

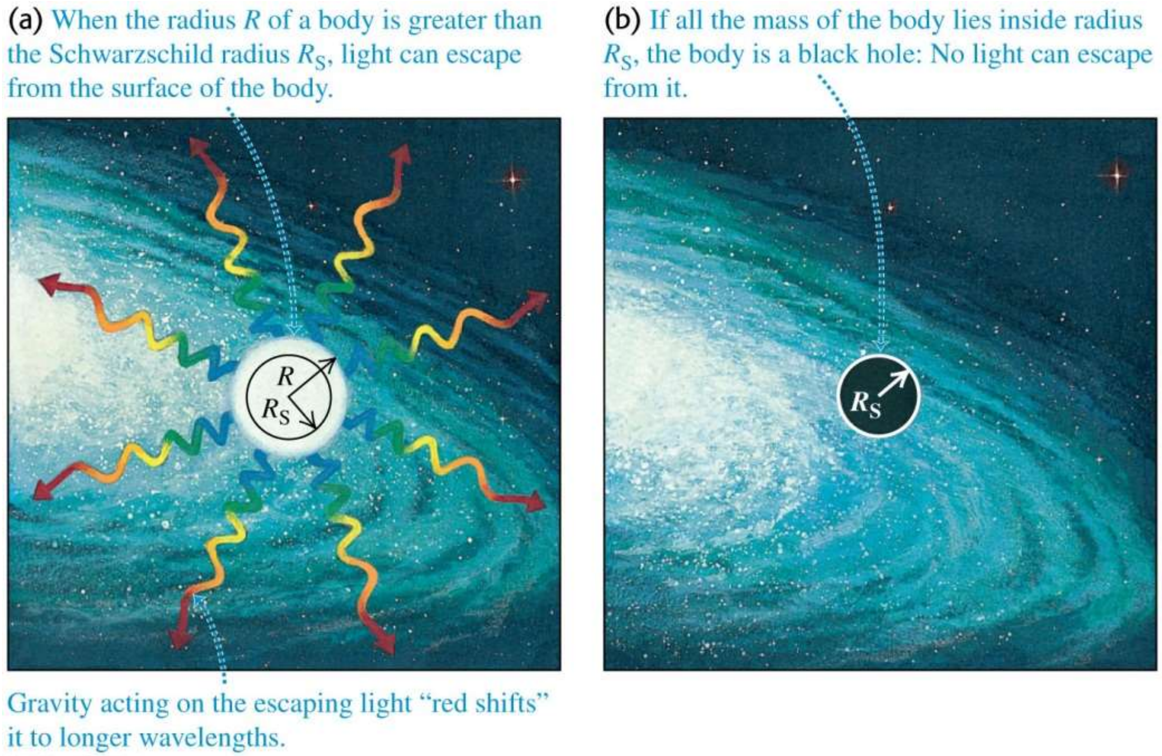
When *R* small enough (reaches a critical value R_S), $v \rightarrow c$, no light can escape?

$$c = \sqrt{\frac{2GM}{R_S}} \quad \rightarrow \quad \boxed{R_S = \frac{2GM}{c^2}} \quad \textbf{Schwarzschild radius}$$

Problems: 1) KE of light (photon) is not $\frac{1}{2}mc^2$

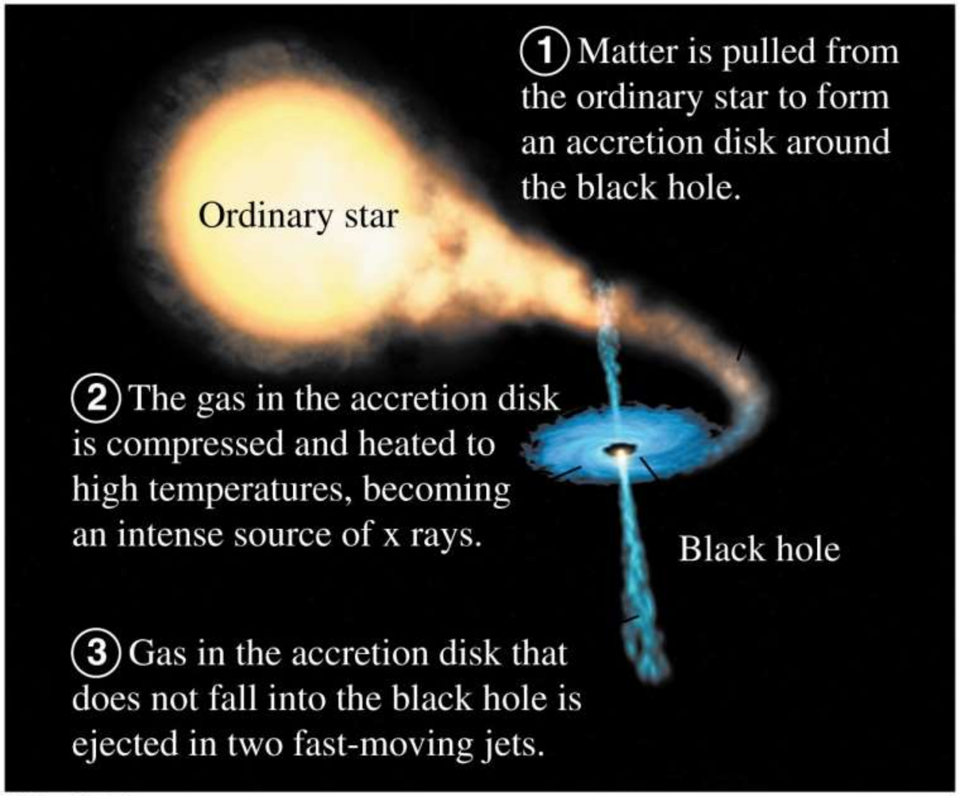
2) gravitational PE near black hole is not $-GMm/r$

Schwarzschild (1916) derived exactly the same critical radius using *General Relativity* (a relativistic theory of gravitation)



Surface of the sphere with radius R_S surrounding a black hole is called the **event horizon**:
light inside cannot escape ∴ cannot know what happens inside a black hole
We know a black hole's
•mass – through its gravitational force on others
•electric charge – through its electric force on others
•angular momentum – through its effect on surrounding space

Can’t see light from a black hole, how to detect it?



In a binary star system (ordinary star + black hole), look for **x-ray source**

Or study orbits of surrounding stars in other cases