DYNAMICS OF RIGID BODIES

香港資優教育學苑
The Hong Kong Academy for Gifted Education

香港科技大學
THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY

Measuring angles in radian

Define the value of an angle θ in **radian**

as $\theta = \frac{s}{r}$,

or arc length $s = r\theta$

⚠ a pure number, without dimension

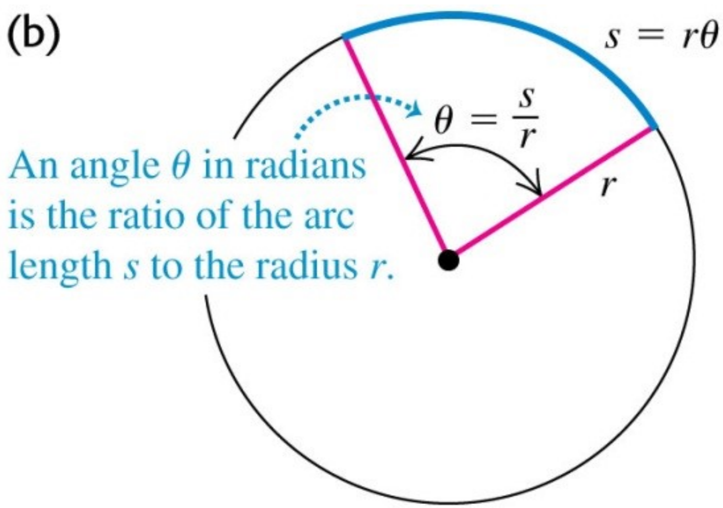
⚠ independent of radius r of the circle

⚠ one complete circle

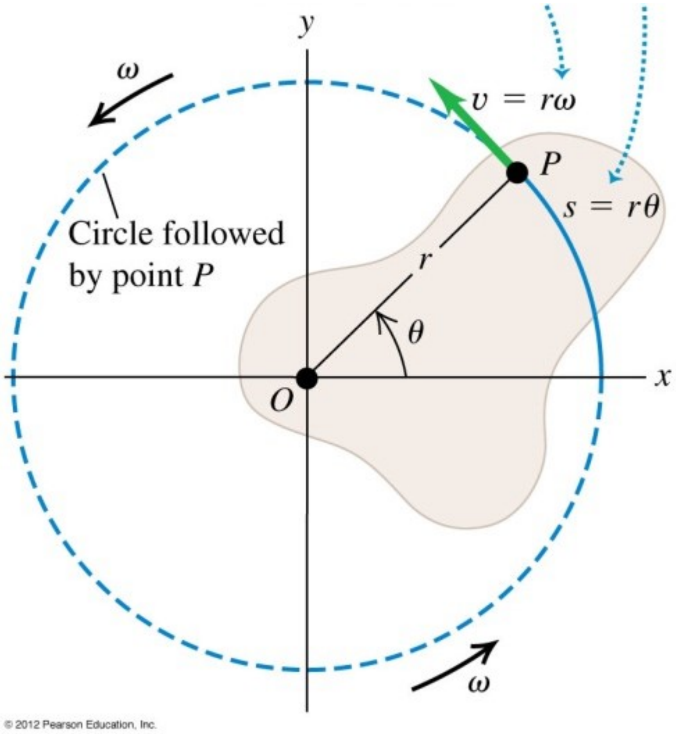
$$\theta = \frac{2\pi r}{r} = 2\pi \text{ (in radian)} \leftrightarrow 360°$$

$$\pi \text{ (in radian)} \leftrightarrow 180°$$

$$\pi/2 \text{ (in radian)} \leftrightarrow 90°$$



Consider a rigid body rotating about a fixed axis



© 2012 Pearson Education, Inc.

angular displacement: $\Delta\theta = \theta_2 - \theta_1$

angular velocity:

$$\omega = \frac{\Delta\theta}{\Delta t} \xrightarrow{\Delta t \to 0} \frac{d\theta}{dt}$$

(average) (instantaneous)

angular acceleration:

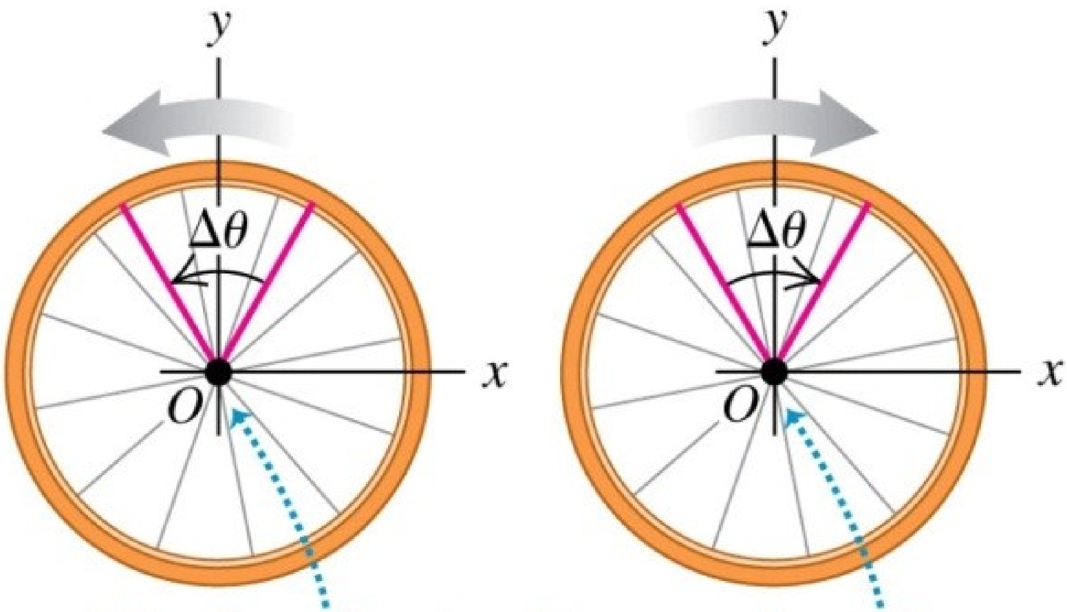
$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

Convention: θ measured from x axis in counterclockwise direction

Convention: θ measured from x axis in counterclockwise direction

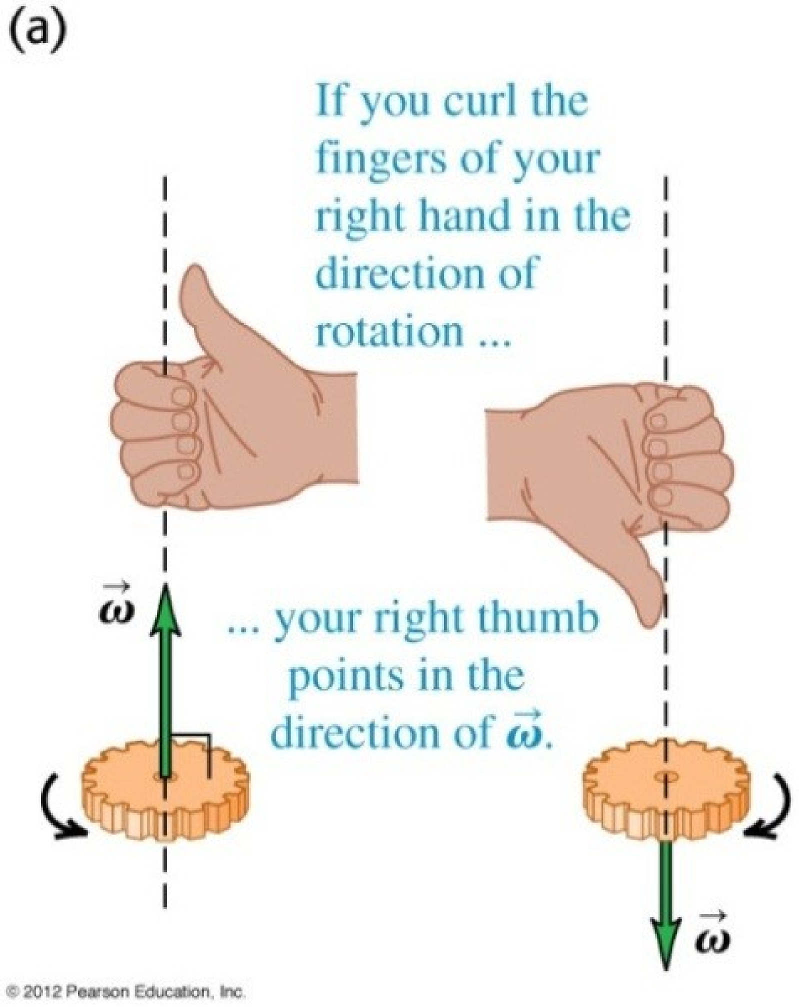
Counterclockwise rotation positive:
$\Delta\theta > 0$, so
$\omega_{\text{av-}z} = \Delta\theta/\Delta t > 0$

Clockwise rotation negative:
$\Delta\theta < 0$, so
$\omega_{\text{av-}z} = \Delta\theta/\Delta t < 0$



Axis of rotation (z-axis) passes through origin and points out of page.

Angular velocity is a vector, direction defined by the **right hand rule**

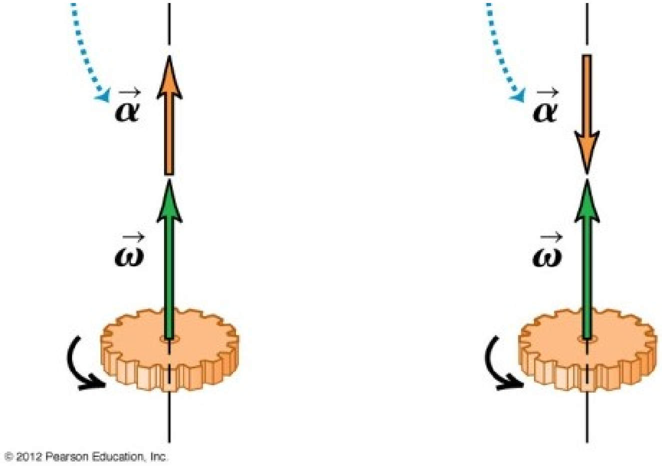


direction of $\vec{\boldsymbol{\omega}}$ represents sense of rotation

Angular acceleration is defined as $\vec{\boldsymbol{\alpha}} = d\vec{\boldsymbol{\omega}}/dt$

⚠ if rotation axis is fixed, $\vec{\boldsymbol{\alpha}}$ along the direction of $\vec{\boldsymbol{\omega}}$

Rotation speeding up, $\vec{\boldsymbol{\alpha}}$ and $\vec{\boldsymbol{\omega}}$ in the same direction



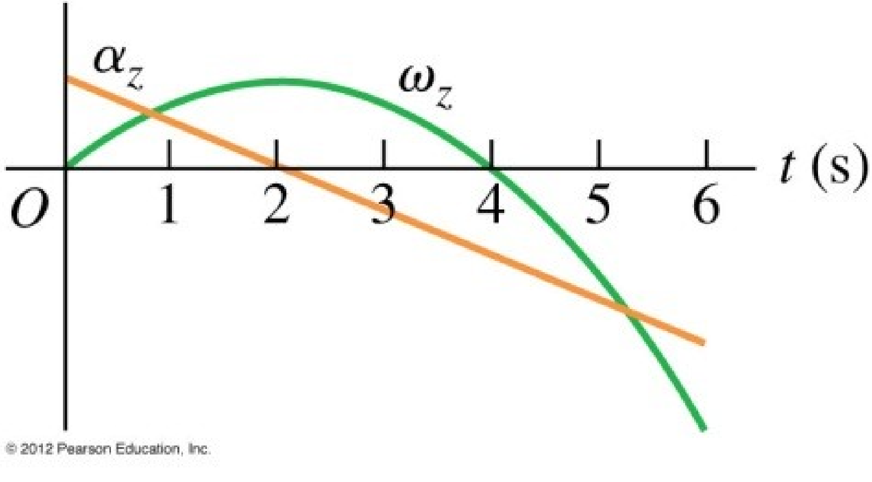
© 2012 Pearson Education, Inc.

Rotation slowing down, $\vec{\boldsymbol{\alpha}}$ and $\vec{\boldsymbol{\omega}}$ in the opposite direction

Question

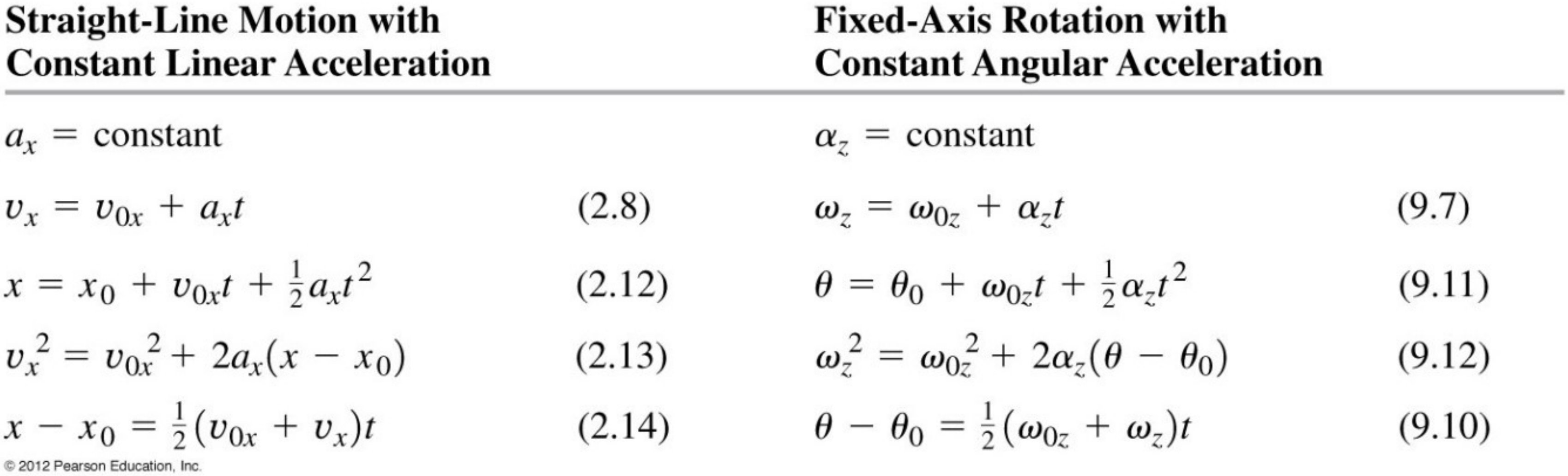
- The figure shows a graph of ω and α versus time. During which time intervals is the rotation speeding up?

(i) 0 < t < 2 s; (ii) 2 s < t < 4 s; (iii) 4 s < t < 6 s.



© 2012 Pearson Education, Inc.

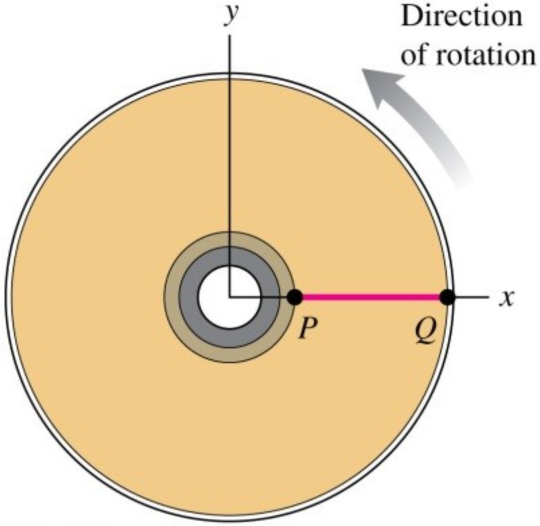
Rotation with constant angular acceleration



Straight-Line Motion with Constant Linear Acceleration		Fixed-Axis Rotation with Constant Angular Acceleration	
$a_x = \text{constant}$		$\alpha_z = \text{constant}$	
$v_x = v_{0x} + a_x t$	(2.8)	$\omega_z = \omega_{0z} + \alpha_z t$	(9.7)
$x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2$	(2.12)	$\theta = \theta_0 + \omega_{0z} t + \frac{1}{2} \alpha_z t^2$	(9.11)
$v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$	(2.13)	$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$	(9.12)
$x - x_0 = \frac{1}{2}(v_{0x} + v_x)t$	(2.14)	$\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t$	(9.10)

Example

A Blu-ray disc is slowing down to a stop with constant angular acceleration $\alpha = -10.0$ rad/s^2. At $t = 0$, $\omega_0 = 27.5$ rad/s, and a line *PQ* marked on the disc surface is along the *x* axis.



© 2012 Pearson Education, Inc.

angular velocity at $t = 0.300$ s:

$$\omega = \omega_0 + \alpha t$$

$$= 27.5 \text{ rad/s} + \left(-10.0 \text{ rad/s}^2\right)(0.300 \text{ s})$$

$$= 24.5 \text{ rad/s}$$

Suppose θ is the angular position of *PQ* at $t = 0.300$ s

$$\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2 = 7.80 \text{ rad}$$

$$= (7.8 \text{ rad})\left(\frac{360°}{2\pi \text{ rad}}\right) = 447° = 87°$$

What are the directions of $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{\alpha}}$?

Question

- In the above example, suppose the initial angular velocity is doubled to $2\omega_0$, and the angular acceleration (deceleration) is also doubled to 2α, it will take (more / less / the same amount of) time for the disc to come to a stop compared to the original problem.

Q9.2

A DVD is initially at rest so that the line *PQ* on the disc's surface is along the +*x*-axis. The disc begins to turn with a constant α_z = 5.0 rad/s^2.

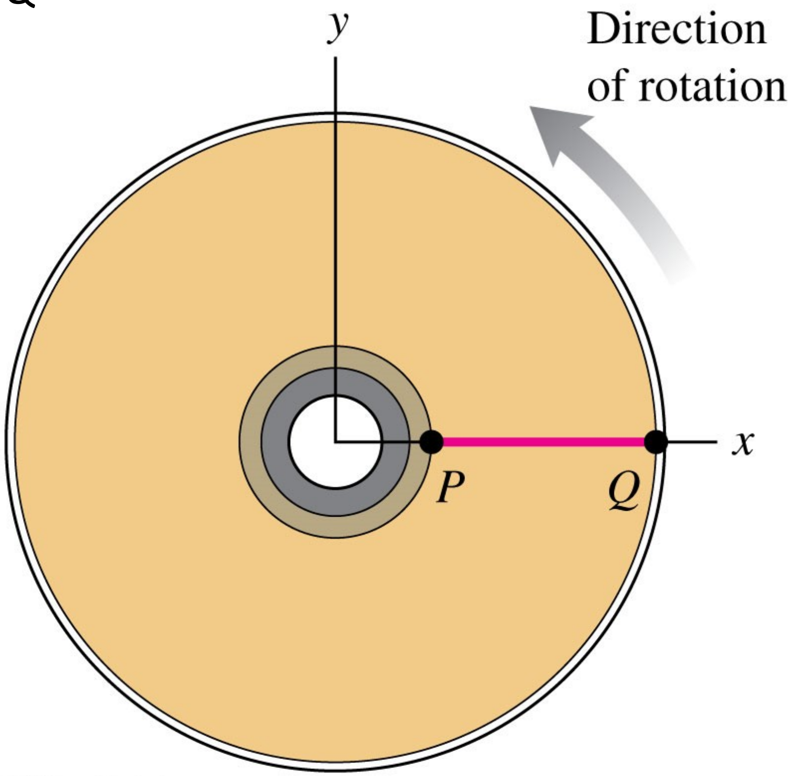
At *t* = 0.40 s, what is the angle between the line *PQ* and the +*x*-axis?

A. 0.40 rad

B. 0.80 rad

C. 1.0 rad

D. 2.0 rad



A9.2

A DVD is initially at rest so that the line *PQ* on the disc’s surface is along the +*x*-axis. The disc begins to turn with a constant α_z = 5.0 rad/s^2.

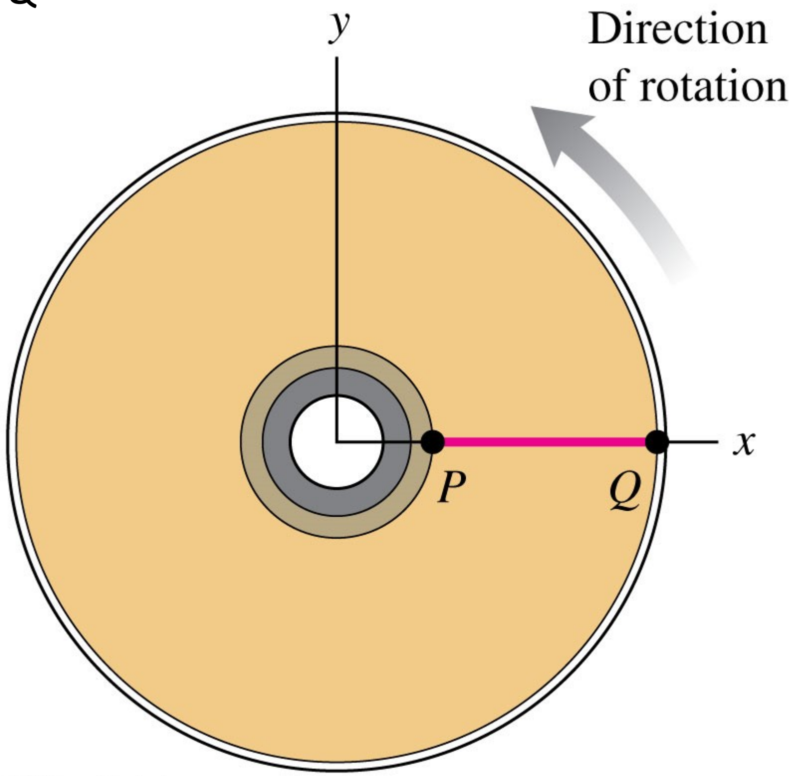
At *t* = 0.40 s, what is the angle between the line *PQ* and the +*x*-axis?

[✓).40 rad]

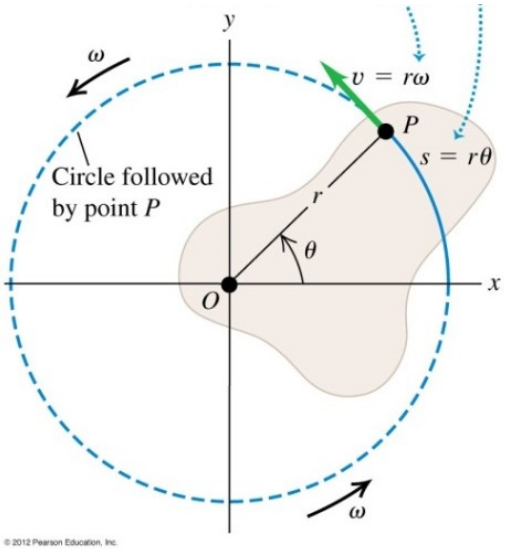
B. 0.80 rad

C. 1.0 rad

D. 2.0 rad



Rigid body rotation



© 2012 Pearson Education, Inc.

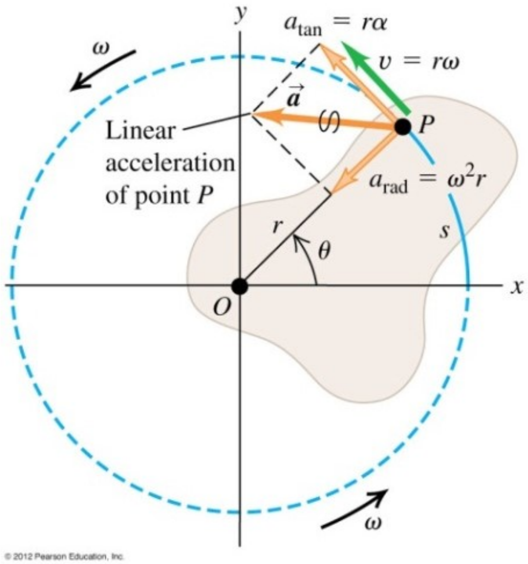
In time Δt, angular displacement is $\Delta\theta$, tangential displacement (arc length) is

$$\Delta s = r\Delta\theta$$

∴ tangential speed

$$v = \frac{\Delta s}{\Delta t} = r\frac{\Delta\theta}{\Delta t} \rightarrow r\frac{d\theta}{dt} = r\omega$$

Velocity of point P, $\vec{v}$, is tangential and has magnitude $v = r\omega$

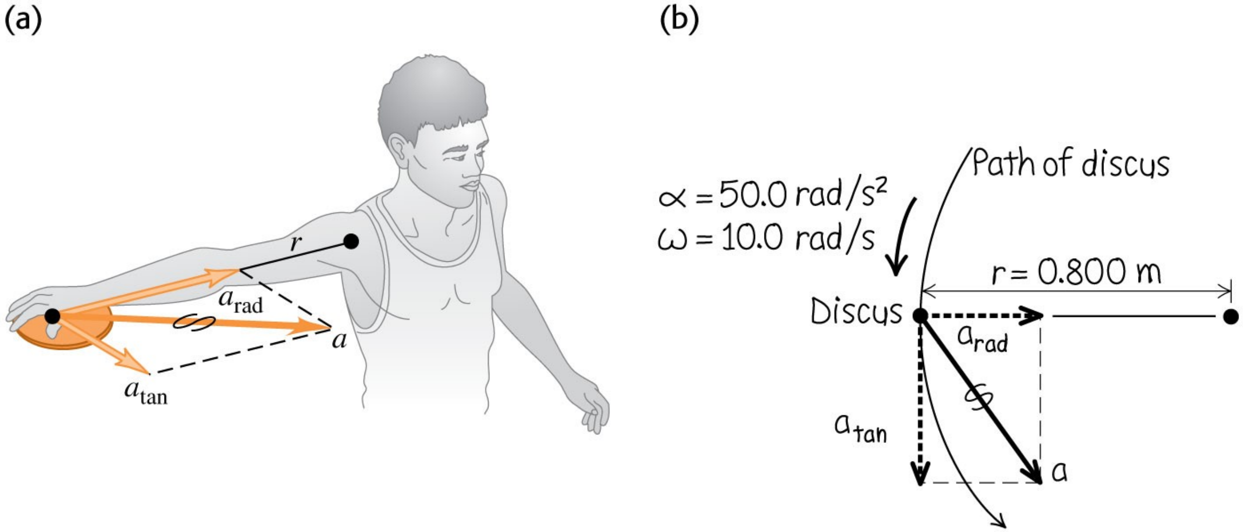


© 2012 Pearson Education, Inc.

tangential acceleration $$a_{\mathrm{tan}} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha$$

⚠ radial acceleration (from circular motion of P) $$a_{\mathrm{rad}} = \frac{v^2}{r} = \omega^2 r$$

Example



An athlete whirls a discus in a circle of radius 80.0 cm. At some instant $\omega = 10.0$ rad/s, and $\alpha = 50.0$ rad/s^2. Then

$$a_{\text{tan}} = r\alpha = (0.800\ \text{m})(50.0\ \ \text{rad/s}^2) = 40.0\ \text{m/s}^2$$

$$a_{\text{rad}} = \omega^2 r = (10.0\ \ \text{rad/s})^2(0.800\ \text{m}) = 80.0\ \text{m/s}^2$$

Magnitude of the linear acceleration is

$$a = \sqrt{a_{\text{tan}}^2 + a_{\text{rad}}^2} = 89.4\ \text{m/s}^2$$

Rotational kinetic energy of a rigid body

Consider a rigid body as a collection of particles, the kinetic energy due to rotation is

$$K = \sum \tfrac{1}{2} m_i v_i^2 = \sum \tfrac{1}{2} m_i r_i^2 \omega^2 = \frac{1}{2} \underbrace{\left(\sum m_i r_i^2 \right)}_{} \omega^2$$

moment of inertia *I*, analogous to *mass* in rectilinear motion

$$\boxed{\begin{array}{l} K = \tfrac{1}{2} I \omega^2 \\ I = \sum m_i r_i^2 \end{array}}$$

c.f. in rectilinear motion,
$K = \tfrac{1}{2} m v^2$

⚠

I depends on distribution of mass, and therefore on the location of the rotation axis.

Q9.5

You want to double the radius of a rotating solid sphere while keeping its kinetic energy constant. (The mass does not change.) To do this, the final angular velocity of the sphere must be

A. 4 times its initial value.

B. twice its initial value.

C. the same as its initial value.

D. 1/2 of its initial value.

E. 1/4 of its initial value.

A9.5

You want to double the radius of a rotating solid sphere while keeping its kinetic energy constant. (The mass does not change.) To do this, the final angular velocity of the sphere must be

A. 4 times its initial value.

B. twice its initial value.

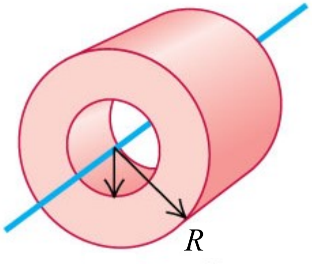
C. the same as its initial value.

✓ 1/2 of its initial value.

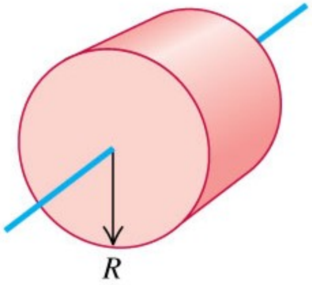
E. 1/4 of its initial value.

Q9.6

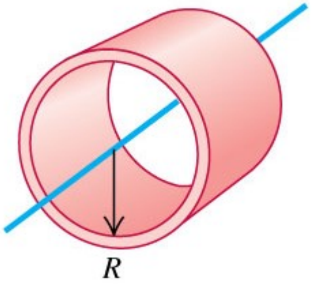
The three objects shown here all have the same mass *M*. Each object is rotating about its axis of symmetry (shown in blue). All three objects have the *same* rotational kinetic energy. Which one is rotating with *fastest* angular speed?



A



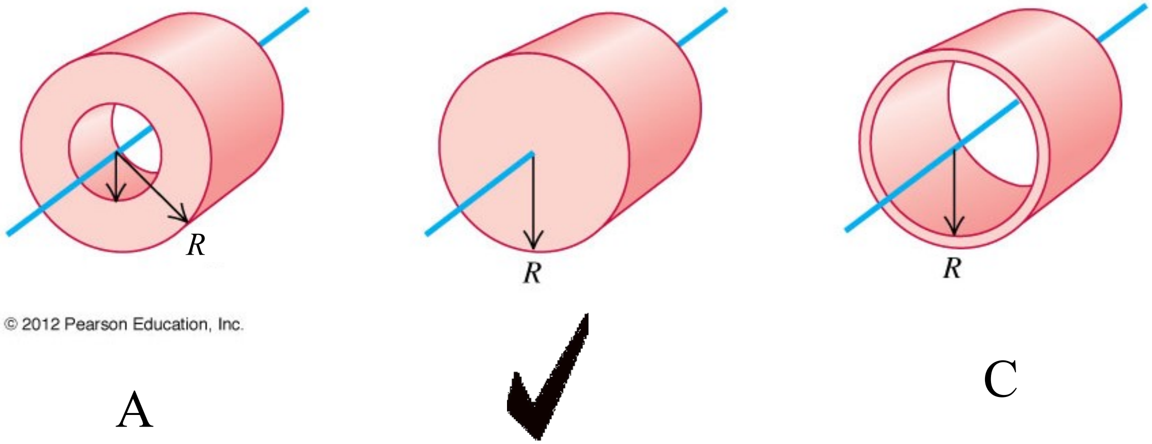
B



C

Q9.6

The three objects shown here all have the same mass *M*. Each object is rotating about its axis of symmetry (shown in blue). All three objects have the *same* rotational kinetic energy. Which one is rotating with *fastest* angular speed?



Gravitational potential energy of a rigid body

$$\begin{aligned} U &= m_1 g y_1 + m_2 g y_2 + \cdots \\ &= (m_1 y_1 + m_2 y_2 + \cdots)g = Mgy_{\text{cm}} \end{aligned}$$

Gravitational PE is as if all the mass is concentrated at the CM.

Example

Assumption: rotation of cylinder is frictionless

no slipping between cylinder and cable

At the moment the block hits the ground, speed of block is v, angular speed of cylinder is ω

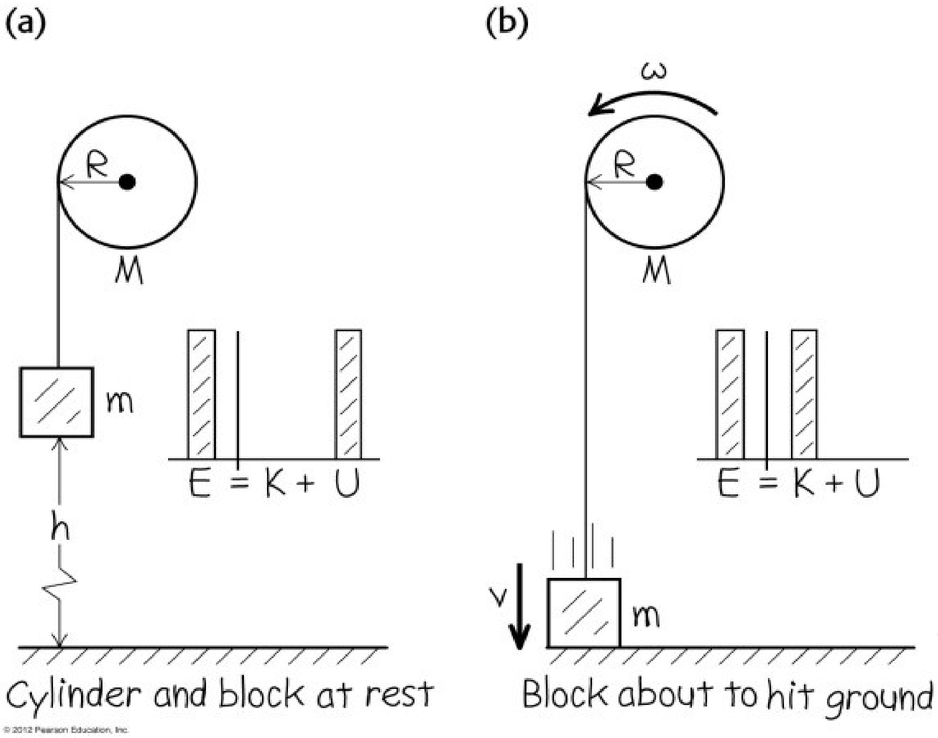
$$v = R\omega$$

$$0 + \underbrace{mgh} = \frac{1}{2}mv^2 + \underbrace{\frac{1}{2}I\omega^2}$$

initial PE of block

rotational KE, $I = \frac{1}{2}MR^2$

(we will tell you why later)



Cylinder and block at rest

Block about to hit ground

$$\Rightarrow \quad v = \sqrt{\frac{2gh}{1 + M/2m}}$$

if $M = 0$, $v = \sqrt{2gh}$, same as free falling

Question: Is there friction between the string and pulley? Does it dissipate energy?

Question

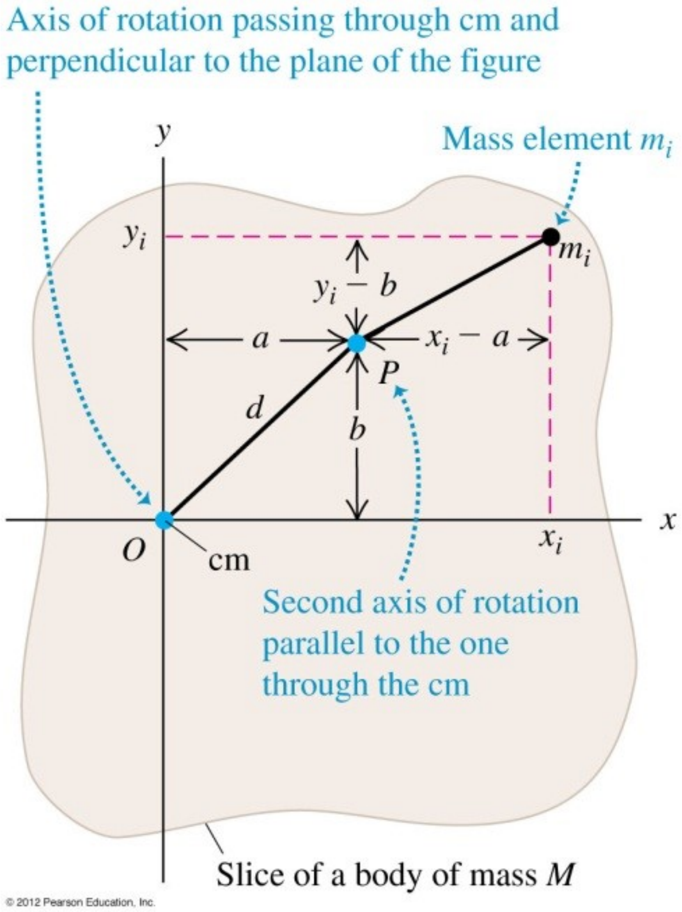
- Suppose the cylinder and block have the same mass, $m = M$. Just before the block hits the floor, it’s KE is (larger than / less than / the same as) the KE of the cylinder.

Parallel axis theorem

I_{cm}: moment of inertia about an axis through its CM

I_p: moment of inertia about another axis ∥ to the original one and at ⊥ distance d

$$I_p = I_{\text{cm}} + Md^2$$



© 2012 Pearson Education, Inc.

Proof:

$$I_{\text{cm}} = \sum m_i \overbrace{\left(x_i^2 + y_i^2\right)}^{\text{square of } \perp \text{ distance of } m_i \text{ to rotation axis}}$$

$$I_p = \sum m_i\left[(x_i - a)^2 + (y_i - b)^2\right]$$

$$\underbrace{= \sum m_i\left(x_i^2 + y_i^2\right)}_{I_{\text{cm}}} - \underbrace{2a\sum m_i x_i}_{Mx_{\text{cm}} = 0} - \underbrace{2b\sum m_i y_i}_{My_{\text{cm}} = 0}$$

$$+\left(a^2 + b^2\right)\underbrace{\sum m_i}_{M}$$

Question

- A pool cue is a wooden rod with a uniform composition and tapered with a larger diameter at one end than at the other end. Does it have a larger moment of inertia

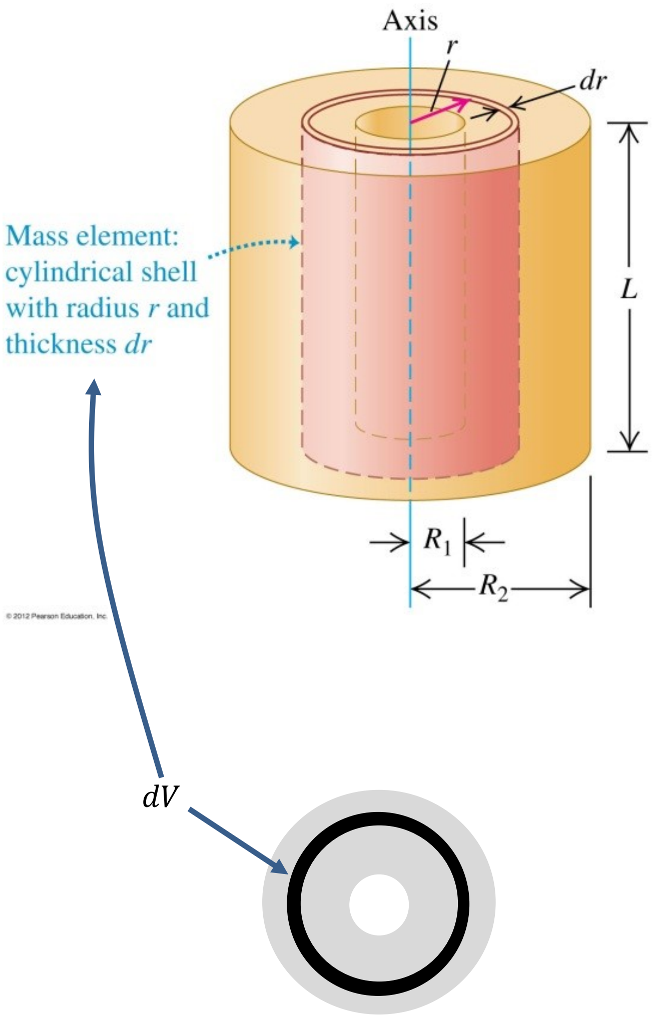
① for an axis through the thicker end of the rod and perpendicular to the length of the rod, or

② for an axis through the thinner end of the rod and perpendicular to the length of the rod?

Significance of the parallel axis theorem:

need formula for I_{cm} only

Example A cylinder with uniform density



⚠ Before calculating moment of inertia, must specify rotation axis

CM along axis of symmetry

$$I = \sum m_i \underset{\substack{\uparrow \\ \perp \text{ distance of } m_i \text{ to rotation axis}}}{r_i^2} \longrightarrow \int r^2\,dm = \int r^2 \underbrace{\rho dV}_{\text{uniform density}}$$

Key: choose dV (the volume element) wisely, as symmetric as possible

$$dV = (2\pi r)(dr)L$$

$$I = 2\pi\rho L \int_{R_1}^{R_2} r^3\,dr = \frac{\pi\rho L}{2}(R_2^4 - R_1^4)$$

$$= \frac{\pi\rho L}{2}(R_2^2 - R_1^2)(R_2^2 + R_1^2)$$

But $M = \rho(\pi R_2^2 L - \pi R_1^2 L) = \pi\rho L(R_2^2 - R_1^2)$

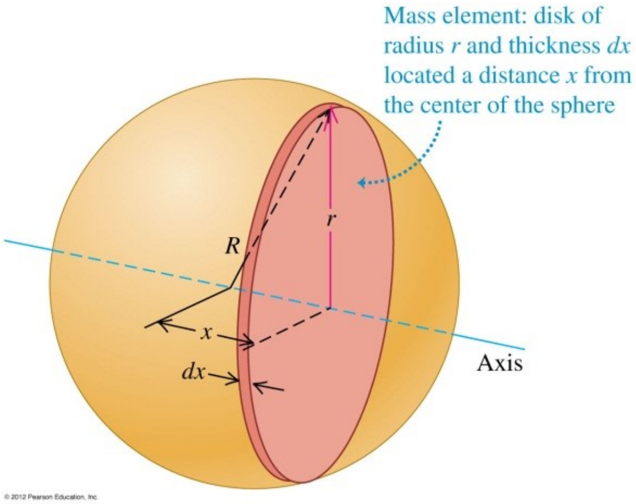
$$\boxed{I = \tfrac{1}{2}M(R_2^2 + R_1^2)}$$

⚠ independent of length

Question

- Two hollow cylinders have the same inner and outer radii and the same mass, but they have different lengths. One is made of wood and the other of lead. The wooden cylinder has (larger / smaller / the same) moment of inertia about the symmetry axis than the lead one.

Example A uniform sphere



Choose dV to be a disk of radius $r = \sqrt{R^2 - x^2}$ and thickness dx

From Example 9.10, moment of inertia of this disk is

$$\frac{1}{2}(dm)r^2 = \frac{1}{2}(\rho\pi r^2 dx)r^2 = \frac{1}{2}\rho\pi(R^2 - x^2)^2 dx$$

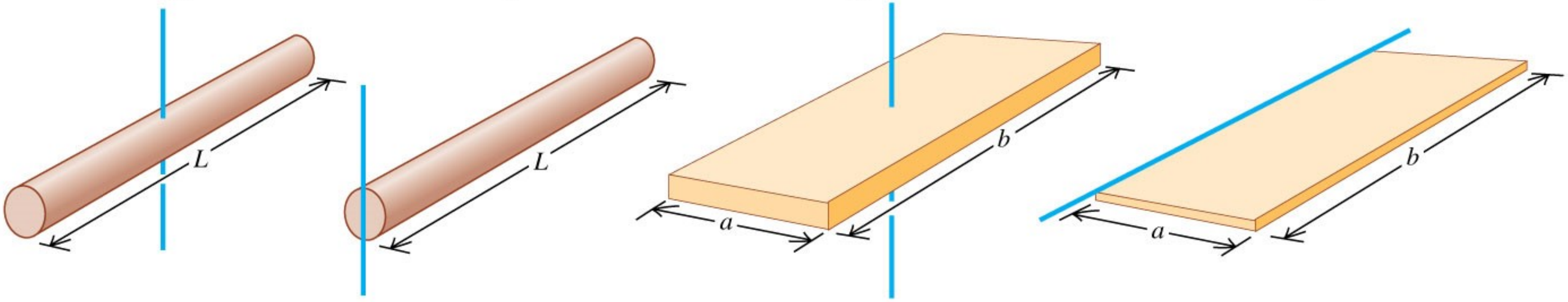
Therefore

$$I = \int \tfrac{1}{2}(dm)r^2 = \frac{\rho\pi}{2}\int_{-R}^{R}(R^2 - x^2)^2 dx = \frac{8\pi\rho R^5}{15}$$

Since $\rho = \frac{M}{V} = \frac{3M}{4\pi R^3}$

$$\boxed{I = \tfrac{2}{5}MR^2}$$

Table 9.2 Moments of Inertia of Various Bodies



(a) Slender rod, axis through center

$$I = \frac{1}{12}ML^2$$

(b) Slender rod, axis through one end

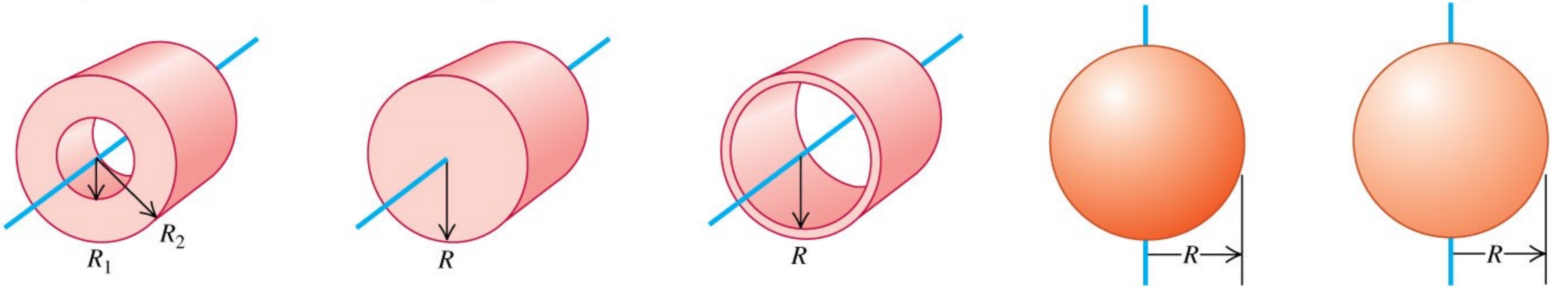
$$I = \frac{1}{3}ML^2$$

(c) Rectangular plate, axis through center

$$I = \frac{1}{12}M(a^2 + b^2)$$

(d) Thin rectangular plate, axis along edge

$$I = \frac{1}{3}Ma^2$$



(e) Hollow cylinder

$$I = \frac{1}{2}M(R_1{}^2 + R_2{}^2)$$

(f) Solid cylinder

$$I = \frac{1}{2}MR^2$$

(g) Thin-walled hollow cylinder

$$I = MR^2$$

(h) Solid sphere

$$I = \frac{2}{5}MR^2$$

(i) Thin-walled hollow sphere

$$I = \frac{2}{3}MR^2$$

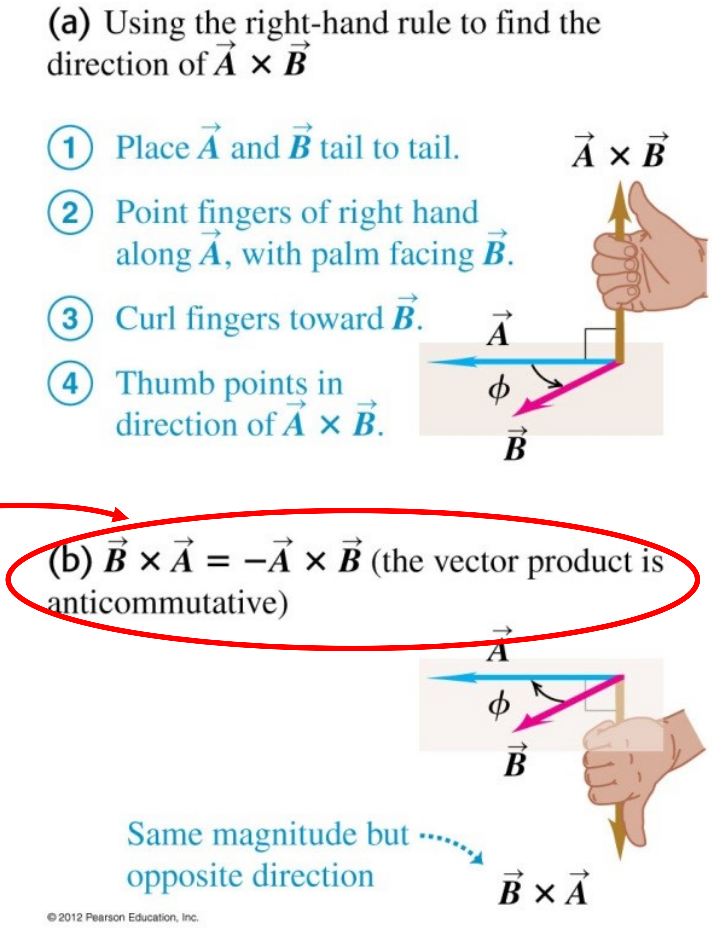
Vector (Cross) Product

$\vec{C} = \vec{A} \times \vec{B}$

Magnitude: $C = AB \sin \phi$

direction determined by *Right Hand Rule*

Important!



(a) Using the right-hand rule to find the direction of $\vec{A} \times \vec{B}$

① Place $\vec{A}$ and $\vec{B}$ tail to tail.

② Point fingers of right hand along $\vec{A}$, with palm facing $\vec{B}$.

③ Curl fingers toward $\vec{B}$.

④ Thumb points in direction of $\vec{A} \times \vec{B}$.

(b) $\vec{B} \times \vec{A} = -\vec{A} \times \vec{B}$ (the vector product is anticommutative)

Same magnitude but opposite direction

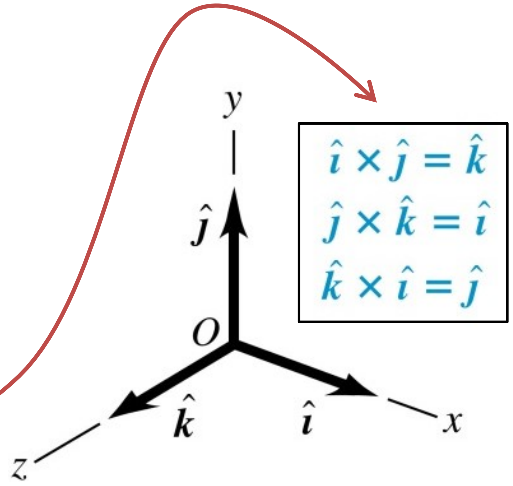
Special cases:

(i) if $\vec{\boldsymbol{A}} \parallel \vec{\boldsymbol{B}}$, $|\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}| = 0$,

in particular, $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$

(ii) if $\vec{\boldsymbol{A}} \perp \vec{\boldsymbol{B}}$, $|\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}| = AB$

in particular,



$\hat{\imath} \times \hat{\jmath} = \hat{k}$

$\hat{\jmath} \times \hat{k} = \hat{\imath}$

$\hat{k} \times \hat{\imath} = \hat{\jmath}$

In analytical form (no need to memorize)

$$\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}$$

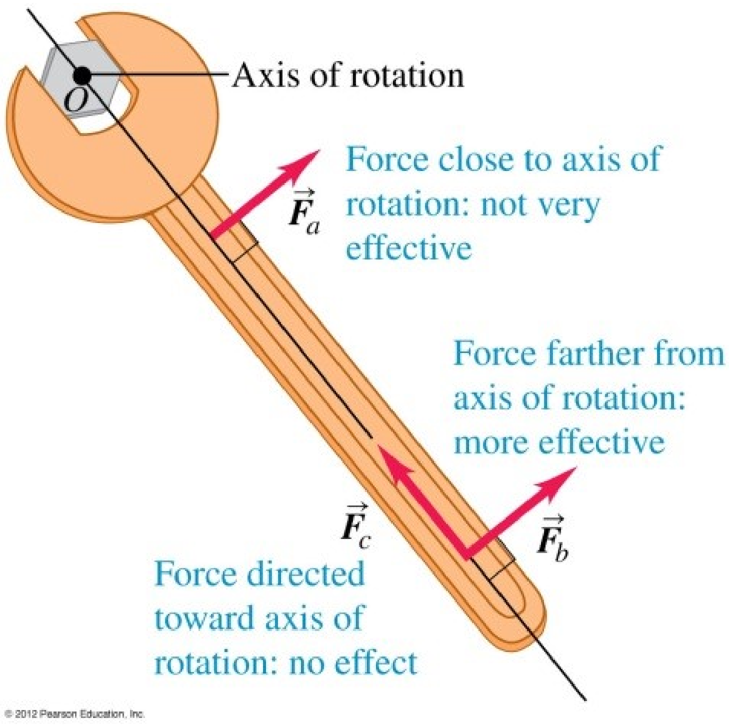
$$= (A_y B_z - A_z B_y)\hat{\imath} + (A_z B_x - A_x B_z)\hat{\jmath} + (A_x B_y - A_y B_x)\hat{k}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

don't worry if you have not learnt determinants in high school

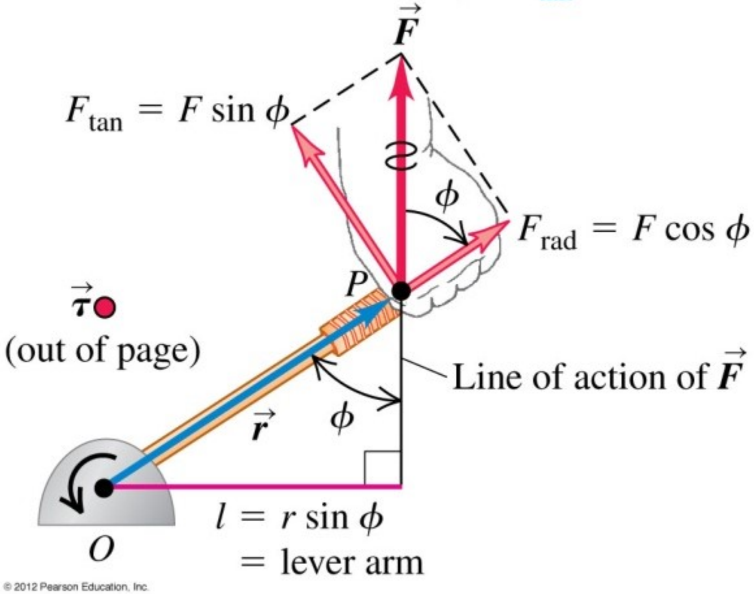
Torque

Besides magnitude and direction, the **line of action** of a force is important because it produces rotation effect.



© 2012 Pearson Education, Inc.

$\vec{\boldsymbol{F}}_a$ and $\vec{\boldsymbol{F}}_b$ have the same magnitudes and directions, but different line of action: they produce different physical effects – which force would you apply if you were to tighten/loosen the screw?



Define **torque** about a point O as a vector

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

⚠ $\vec{\tau}$ is ⊥ to both $\vec{r}$ and $\vec{F}$

Magnitude:

$$\tau = r\underbrace{(F\sin\phi\,)}_{\text{component of }\vec{F}\ \perp\text{ to }\vec{r}} = \underbrace{(r\sin\phi)}_{\perp\text{ distance from }O\text{ to line of actions of }\vec{F}}F$$

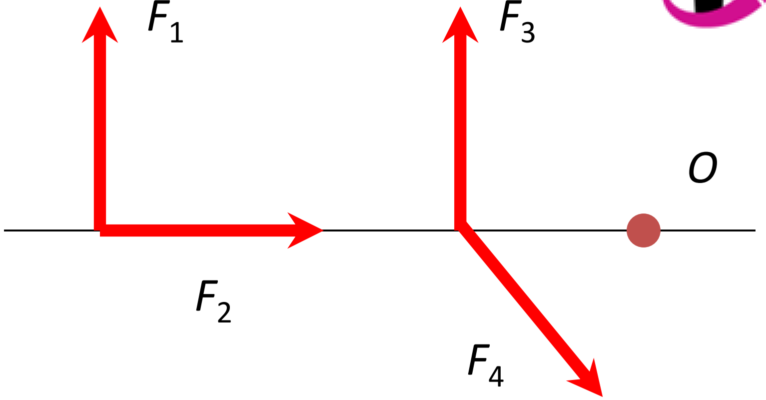
Direction gives the sense of rotation about O through the right-hand-rule.

Notation: $\odot$ out of the plane
$\otimes$ into the plane

SI unit for torque: Nm (just like work done)

Q10.2

Which of the four forces shown here produces a torque about *O* that is directed *out of* the plane of the drawing?



A. F_1

B. F_2

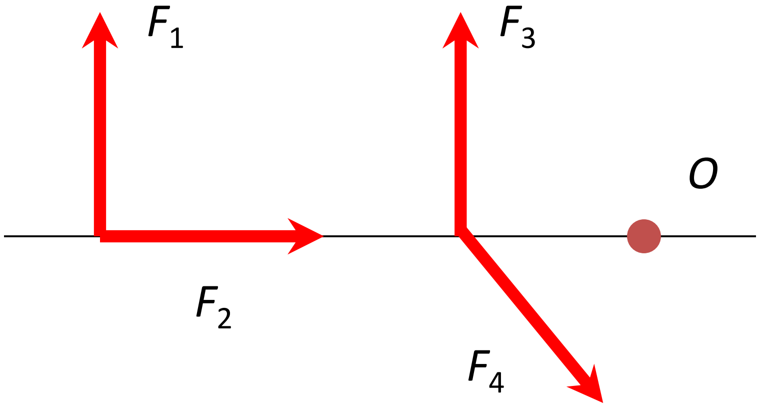
C. F_3

D. F_4

E. more than one of these

A10.2

Which of the four forces shown here produces a torque about *O* that is directed *out of* the plane of the drawing?



A. F_1

B. F_2

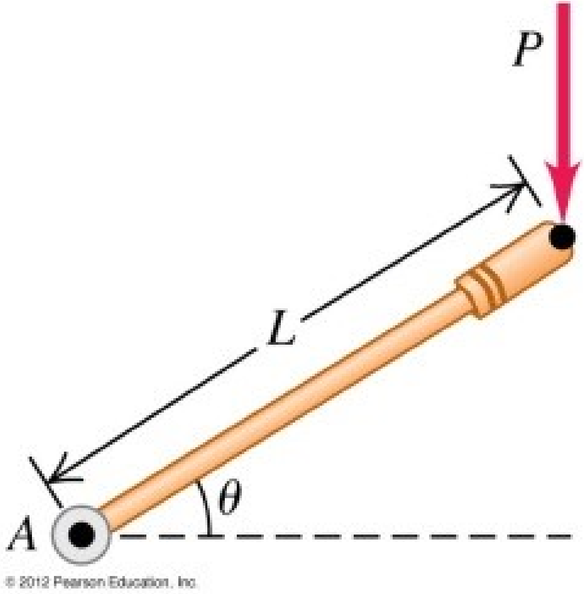
C. F_3

[✓ D. F_4]

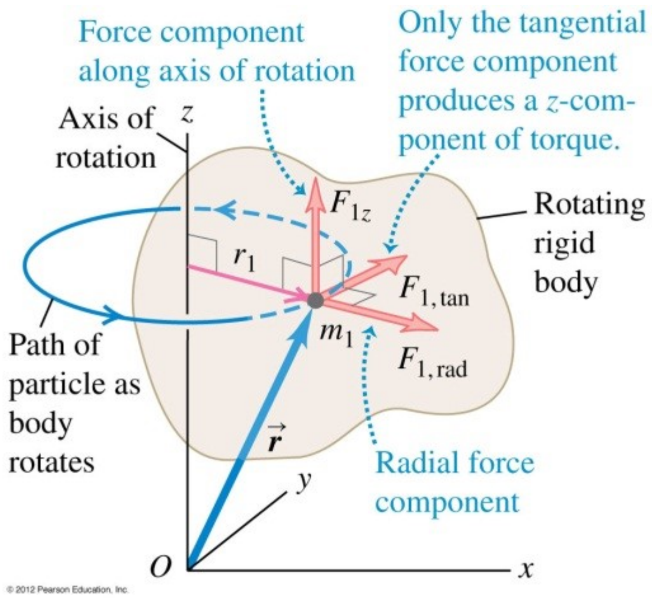
E. more than one of these

Question

A force P is applied to one end of a lever of length L. The magnitude of the torque of this force about point A is ($PL\sin\theta$ / $PL\cos\theta$ / $PL\tan\theta$)



Suppose a rigid body is rotating about a fixed axis which we arbitrarily call the *z* axis. m_1 is a small part of the total mass.



$F_{1,\mathrm{rad}}$, $F_{1,\tan}$, and $F_{1,\mathrm{z}}$ are the 3 components of the total force acting on m_1

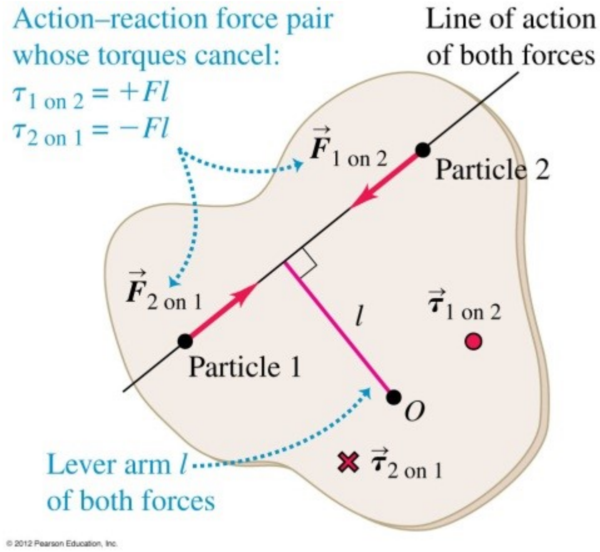
Only $F_{1,\tan}$ produces the desired rotation, $F_{1,\mathrm{rad}}$ and $F_{1,\mathrm{z}}$ produce some other effects which are irrelevant to the rotation about the *z* axis.

$$F_{1,\tan} = m_1 a_{1,\tan} = m_1(r_1\alpha_z)$$

$$\underbrace{F_{1,\tan} r_1}_{\text{torque on } m_1 \text{ about } z,\ \tau_{1z}} = m_1 r_1^2 \alpha_z$$

Sum over all mass in the body, since they all have the same α_z

$$\sum \tau_{iz} = \left(\sum m_r r_i^2\right)\alpha_z = I\alpha_z$$



Need to consider torque due to external forces only. Internal forces (action and reaction pairs) produce equal and opposite torques which have no net rotational effect.

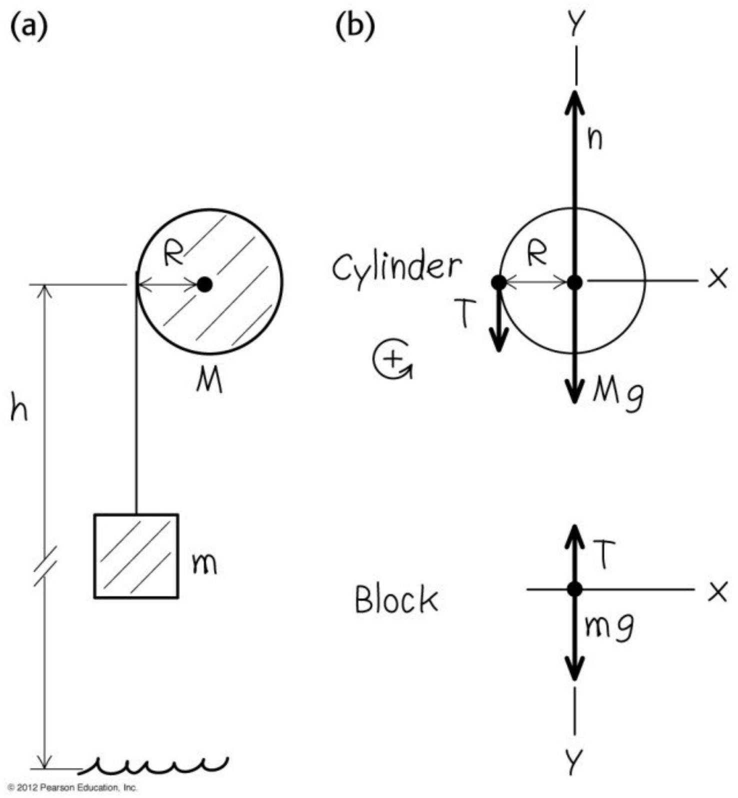
Conclusion: for rigid body rotation about a fixed axis,

$$\boxed{\sum \tau_{\text{ext}} = I\alpha}$$

c.f. Newton's second law $\sum \vec{\boldsymbol{F}}_{\text{ext}} = M\vec{\boldsymbol{a}}$

Example

Pulley rotates about a fixed axis. What is the acceleration *a* of the block?



For the cylinder

$$\underbrace{TR}_{\text{torque due to } T} = \underbrace{\left(\tfrac{1}{2}MR^2\right)}_{\text{moment of inertia of cylinder}} \underbrace{\left(\frac{a}{R}\right)}_{\text{angular acceleration}}$$

i.e. $$T = \tfrac{1}{2}Ma$$

For the block

$$mg - T = ma$$

Therefore

$$a = \frac{g}{1 + M/2m}$$

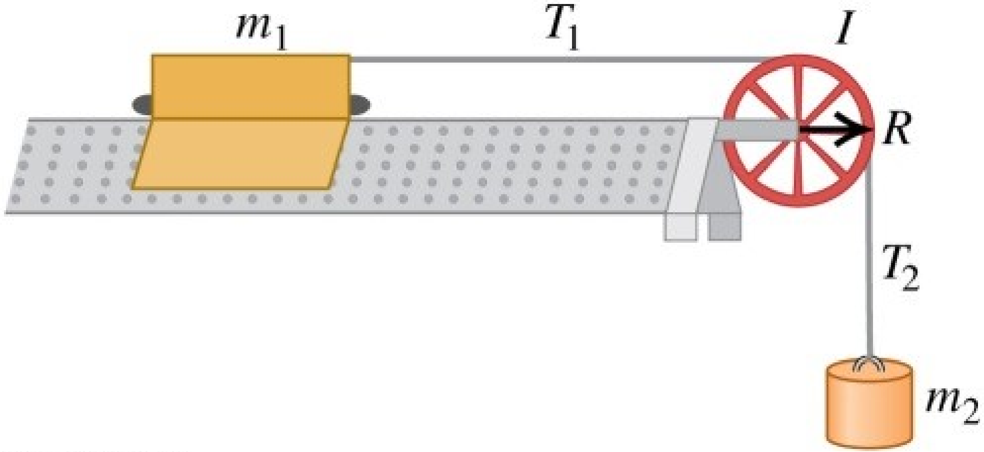
Suppose the block is initially at rest at height *h*. At the moment it hits the floor:

$$v^2 = 0 + 2\left(\frac{g}{1 + M/2m}\right)h \quad \Longrightarrow \quad v = \sqrt{\frac{2gh}{1 + M/2m}}$$

c.f. Previously we get the same result using energy conservation.

Question

Mass m_1 slides on a frictionless track. The pulley has moment of inertia *I* about its rotation axis, and the string does not slip nor stretch. When the hanging mass m_2 is released, arrange the forces T_1, T_2, and $m_2 g$ in increasing order of magnitude.



We know how to deal with:

translation of a point particle (or CM of a rigid body):

$$\sum \vec{\boldsymbol{F}}_{\text{ext}} = m\vec{\boldsymbol{a}}$$

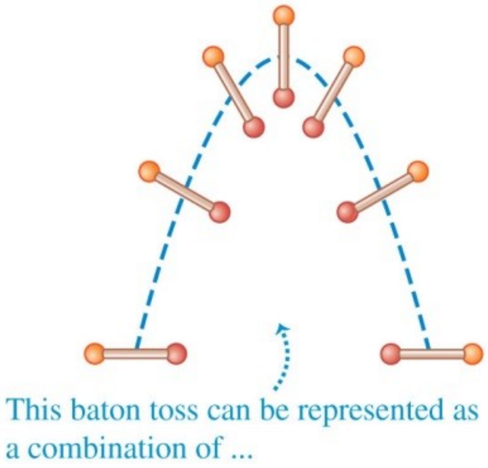
rotation of a rigid body about a *fixed* axis:

$$\sum \tau_{\text{ext}} = I\alpha$$

In general, a rigid body is rotating about a *moving* axis, i.e., has both types of motion simultaneously.

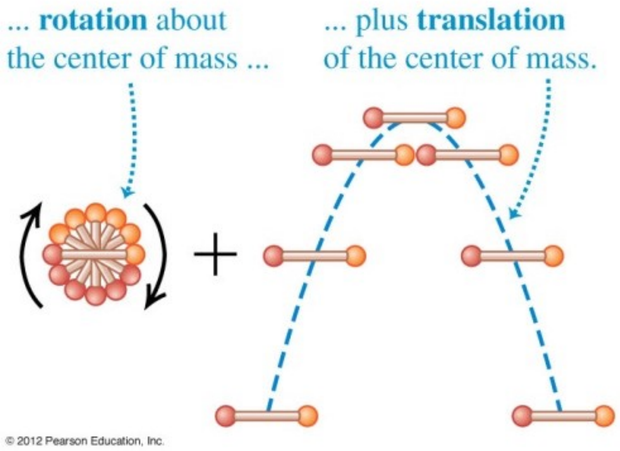
Every possible motion of a rigid body can be represented as a combination of translational motion of the CM and rotation about an axis through its CM.

e.g. tossing a baton



This baton toss can be represented as a combination of ...

translation + rotation



... **rotation** about the center of mass ...

... plus **translation** of the center of mass.

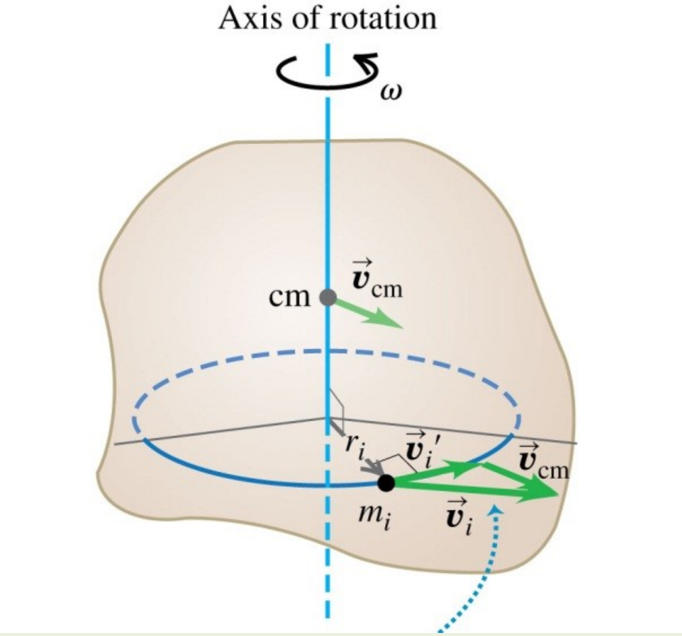
rotation about a fixed axis through CM

translation of CM (considered as a particle)

Energy consideration

m_i is a small mass of the rigid body

$\vec{\boldsymbol{v}}_i'$ its velocity relative to the CM, its velocity relative to the ground is $\vec{\boldsymbol{v}}_i = \vec{\boldsymbol{v}}_{\mathrm{cm}} + \vec{\boldsymbol{v}}_i'$



$$\begin{aligned} K_i &= \tfrac{1}{2} m_i |\vec{\boldsymbol{v}}_i|^2 = \tfrac{1}{2} m_i (\vec{\boldsymbol{v}}_{\mathrm{cm}} + \vec{\boldsymbol{v}}_i') \cdot (\vec{\boldsymbol{v}}_{\mathrm{cm}} + \vec{\boldsymbol{v}}_i') \\ &= \tfrac{1}{2} m_i (\vec{\boldsymbol{v}}_{\mathrm{cm}} \cdot \vec{\boldsymbol{v}}_{\mathrm{cm}} + 2\vec{\boldsymbol{v}}_{\mathrm{cm}} \cdot \vec{\boldsymbol{v}}_i' + \vec{\boldsymbol{v}}_i' \cdot \vec{\boldsymbol{v}}_i') \\ &= \tfrac{1}{2} m_i \left(v_{\mathrm{cm}}^2 + 2\vec{\boldsymbol{v}}_{\mathrm{cm}} \cdot \vec{\boldsymbol{v}}_i' + v_i'^2 \right) \end{aligned}$$

Total KE of the rigid body

$$K = \sum K_i$$

$$= \tfrac{1}{2} \underbrace{\left(\sum m_i \right)}_{M} v_{\mathrm{cm}}^2 + \vec{\boldsymbol{v}}_{\mathrm{cm}} \cdot \underbrace{\left(\sum m_i \vec{\boldsymbol{v}}_i' \right)}_{\text{center of mass velocity relative to CM – zero}} + \underbrace{\sum \left(\tfrac{1}{2} m_i v_i'^2 \right)}_{\tfrac{1}{2} m_i r_i^2 \omega^2}$$

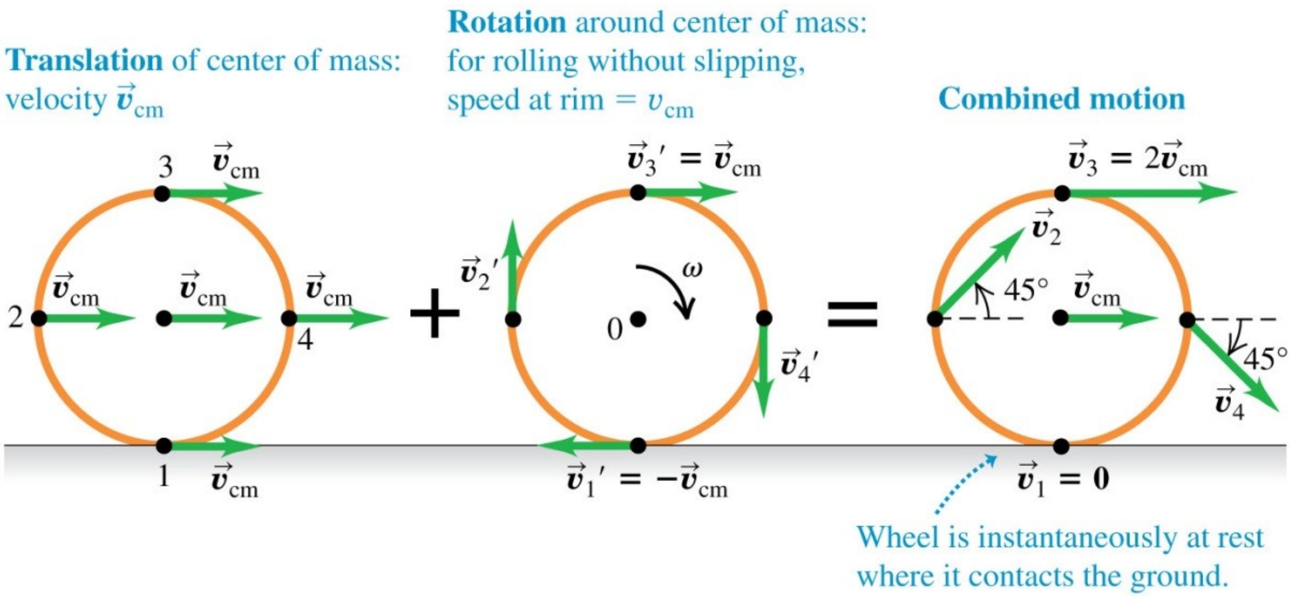
Therefore

$$\boxed{K = \tfrac{1}{2} M v_{\mathrm{cm}}^2 + \tfrac{1}{2} I \omega^2}$$

Rolling without slipping

No slipping at the point of contact ⟹ point of contact must be at rest (instantaneously), i.e.,

$$-R\omega + v_{\text{cm}} = 0 \quad \Longrightarrow \quad v_{\text{cm}} = R\omega$$



translation + rotation

$$K = \tfrac{1}{2}Mv_{\text{cm}}^2 + \tfrac{1}{2}I\omega^2$$

rotation about instantaneous axis of rotation (a moving axis)

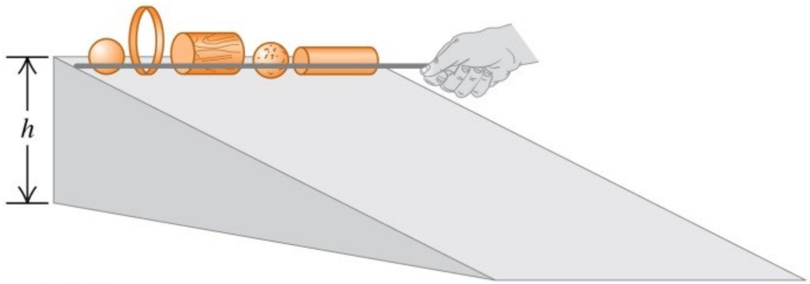
$$K = \tfrac{1}{2}(I + MR^2)\omega^2$$

$$= \tfrac{1}{2}Mv_{\text{cm}}^2 + \tfrac{1}{2}I\omega^2$$

parallel axis theorem

$v_{\text{cm}} = R\omega$

Example



⚠ Rolling without slipping, friction does no work

What determines which body rolls down the incline fastest?

Suppose a rigid body's moment of inertia about its symmetry axis is $I = cMR^2$

$$0 + Mgh = \underbrace{\tfrac{1}{2}Mv_{\text{cm}}^2}_{\text{translation KE of CM}} + \underbrace{\tfrac{1}{2}cMR^2\left(\frac{v_{\text{cm}}}{R}\right)^2}_{\text{rotation KE about a fixed axis}} + 0$$

initial KE

initial PE

final PE

$$\Rightarrow \quad v_{cm} = \sqrt{\frac{2gh}{1+c}}$$

⚠ depends on c only, independent of M and R

Rigid body with smaller c rolls faster :

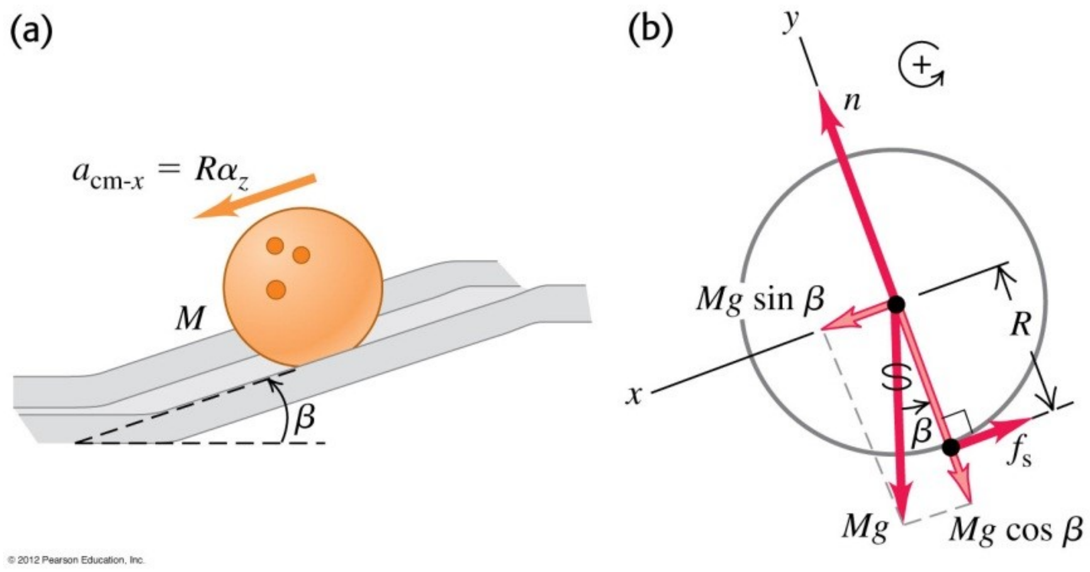
solid sphere ($c = \frac{2}{5}$)

> solid cylinder ($c = \frac{1}{2}$)

> thin walled hollow sphere ($c = \frac{2}{3}$)

> thin walled hollow cylinder ($c = 1$)

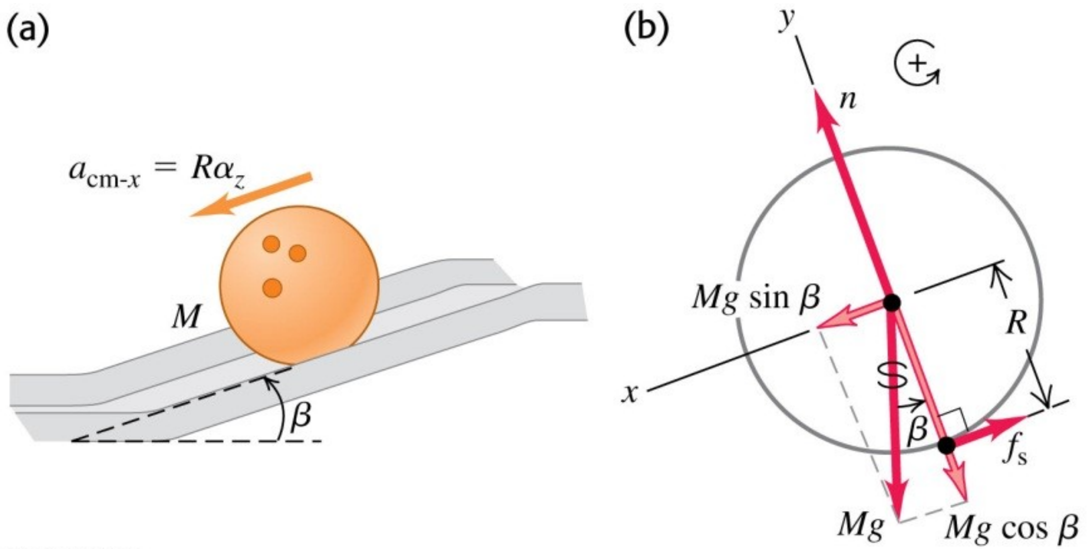
Role of friction: Example



Rolling without slipping is not possible without friction.
Consider a rigid sphere going freely down an inclined plane. If no friction, no torque about the center and the sphere slides down the plane.

Assume rolling without slipping, friction must be (static / dynamics) and must point (upward / downward) along the plane.

$$v_{\text{cm}} = R\omega \quad \Rightarrow \quad a_{\text{cm}} = R\alpha$$



$$v_{\text{cm}} = R\omega \quad \Rightarrow \quad a_{\text{cm}} = R\alpha$$

Translation of CM: $Mg\sin\beta - f = Ma_{\text{cm}}$

Rotation of sphere about its center: $fR = I_{\text{cm}}\alpha = \left(\frac{2}{5}MR^2\right)(a_{\text{cm}}/R)$

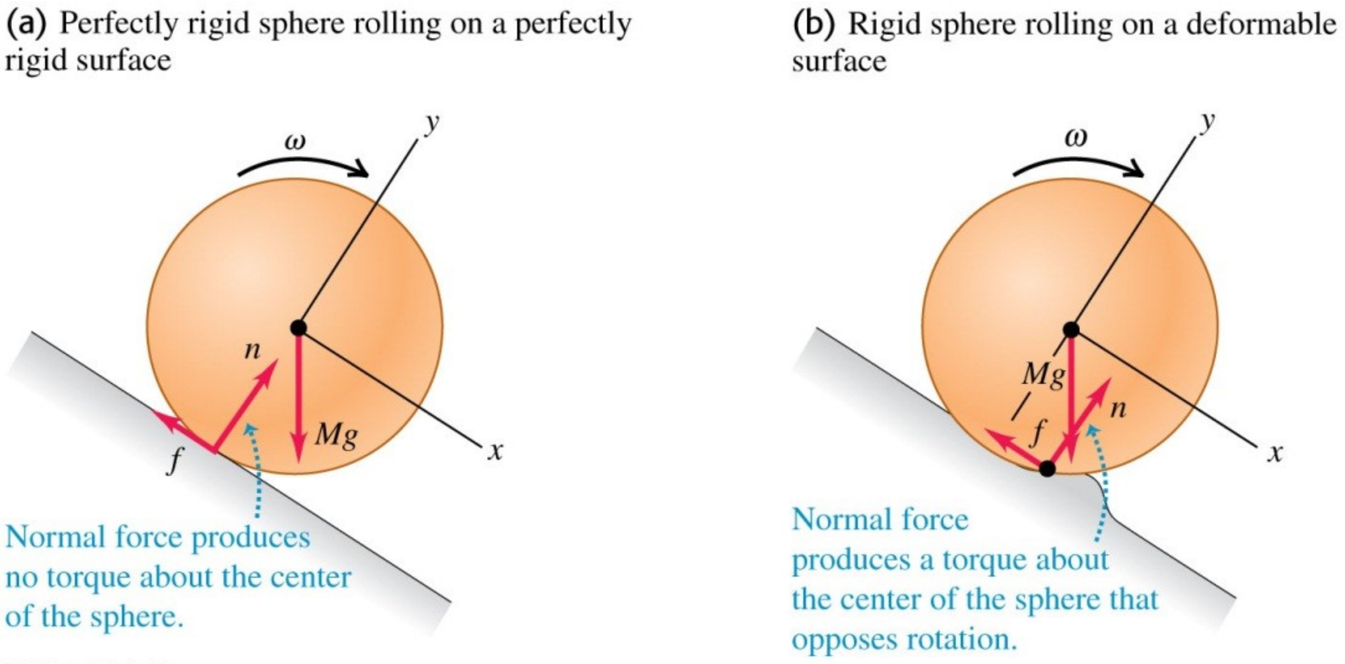
Get $a_{cm} = \frac{5}{7}g\sin\beta$ and $f = \frac{2}{7}Mg\sin\beta$

⚠ Rolling is slower than sliding because part of the PE is converted into rotation KE

⚠ If the sphere is rolling uphill with no slipping, the friction will point (upward / downward) along the plane because its effect is to *decelerate* the rotation.

Puzzle: For rolling without slipping,
friction does NO work.
Therefore a vehicle will go on forever
if there is no air resistance,
just like a magnetic levitated train.
Too good to be true! ☹

In reality energy is lost because the floor and/or the rolling body are deformed, e.g. vehicle tyre.



(a) Perfectly rigid sphere rolling on a perfectly rigid surface

(b) Rigid sphere rolling on a deformable surface

Normal force produces no torque about the center of the sphere.

Normal force produces a torque about the center of the sphere that opposes rotation.

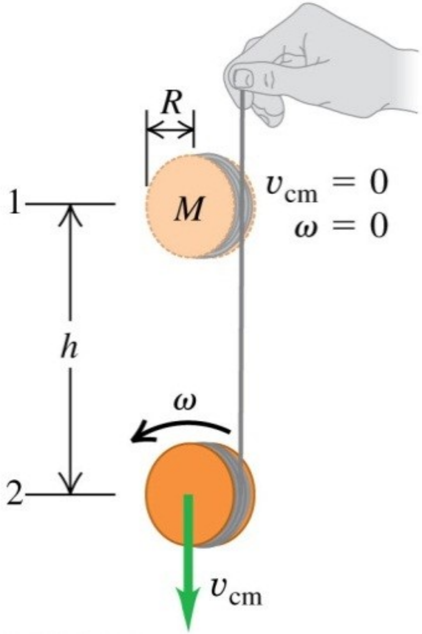
Energy is lost because:

- due to deformation, normal reaction produces a torque opposing the rotation.
- sliding of the deformed surfaces causes energy lost.

These two effects give rise to **rolling friction**.

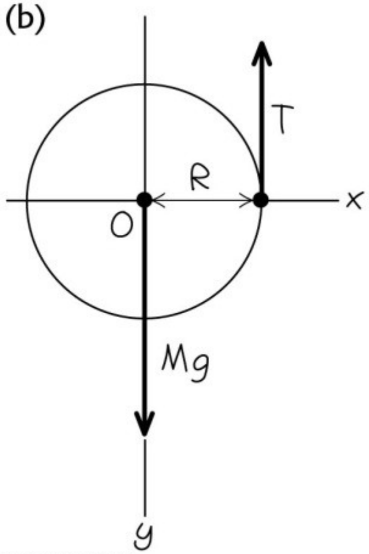
Consequence: trains, with metal wheels on metal tracks, are more fuel efficient than vehicles with rubber tires.

A Yo-yo



© 2012 Pearson Education, Inc.

(b)



© 2012 Pearson Education, Inc.

To find v_{cm} at point 2, need energy conservation

$$\underset{\substack{\uparrow\\ \text{initial}\\ \text{KE}}}{0} + \underset{\substack{\uparrow\\ \text{initial}\\ \text{PE}}}{Mgh} = \underbrace{\tfrac{1}{2}Mv_{\text{cm}}^2}_{\substack{\text{translation}\\ \text{KE of CM}}} + \underbrace{\tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)\left(\frac{v_{\text{cm}}}{R}\right)^2}_{\substack{\text{rotation KE about a}\\ \text{fixed axis}}} + \underset{\substack{\uparrow\\ \text{final}\\ \text{PE}}}{0}$$

$$\Rightarrow \quad v_{\text{cm}} = \sqrt{\tfrac{4}{3}gh}$$

c.f. for free falling $v_{\text{cm}} = \sqrt{2gh}$

To find the downward acceleration of the yo-yo, need dynamic equations

Translation of CM: $Mg - T = Ma_{\text{cm}}$

Rotation of cylinder about its axis:

$$TR = I_{\text{cm}}\alpha = \left(\tfrac{1}{2}MR^2\right)(a_{\text{cm}}/R)$$

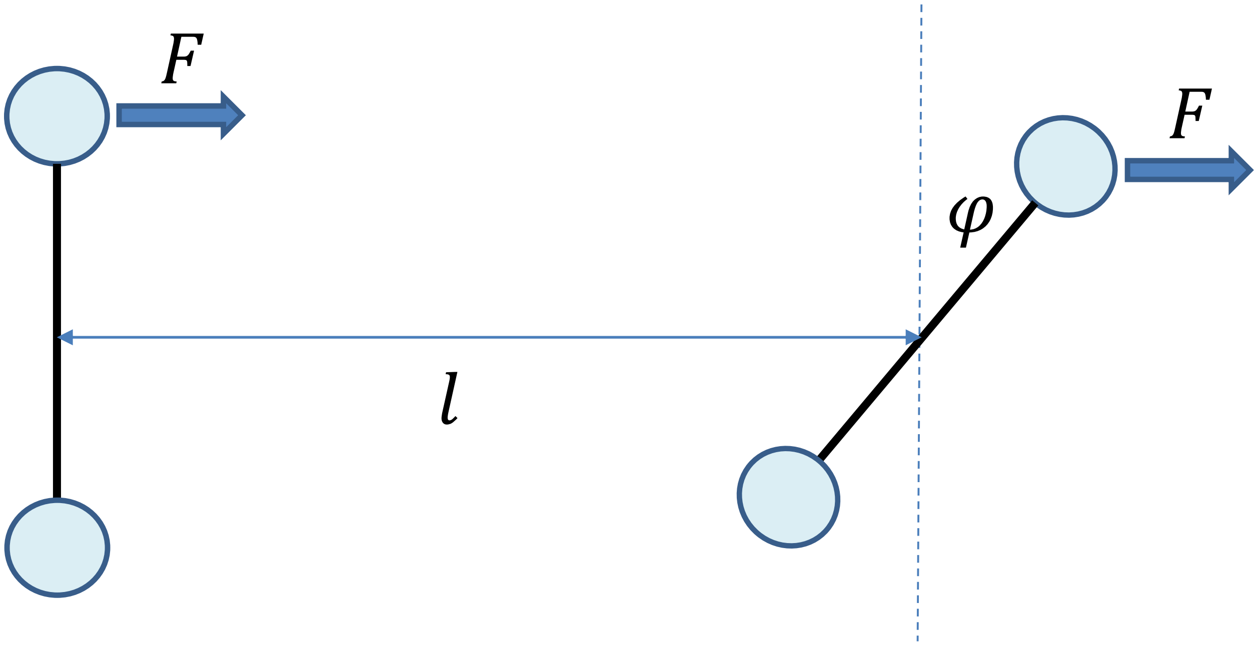
Get

$$a_{\text{cm}} = \tfrac{2}{3}g$$

$$T = \tfrac{1}{3}Mg$$

Exercise: rotation of a dumbbell

A dumbbell consists of a weightless rod of length L and two masses (each with mass M) on its two ends. Initially, the dumbbell sits on a frictionless table and points north. A constant force F (towards east) is applied on one of the ball. The dumbbell will accelerate and rotate due to the applied force.
Find the tension in the rod when the dumbbell rotation 90^o



Due to the constant external force F, the CM of the dumbbell accelerates with constant acceleration $a = F/2M$.

At the instance when the CM moves to the distance l, the CM velocity becomes $v = \sqrt{2al}$. And the work-energy theorem gives

$$F(l + \frac{1}{2}L\sin\varphi) = K_t + K_r$$

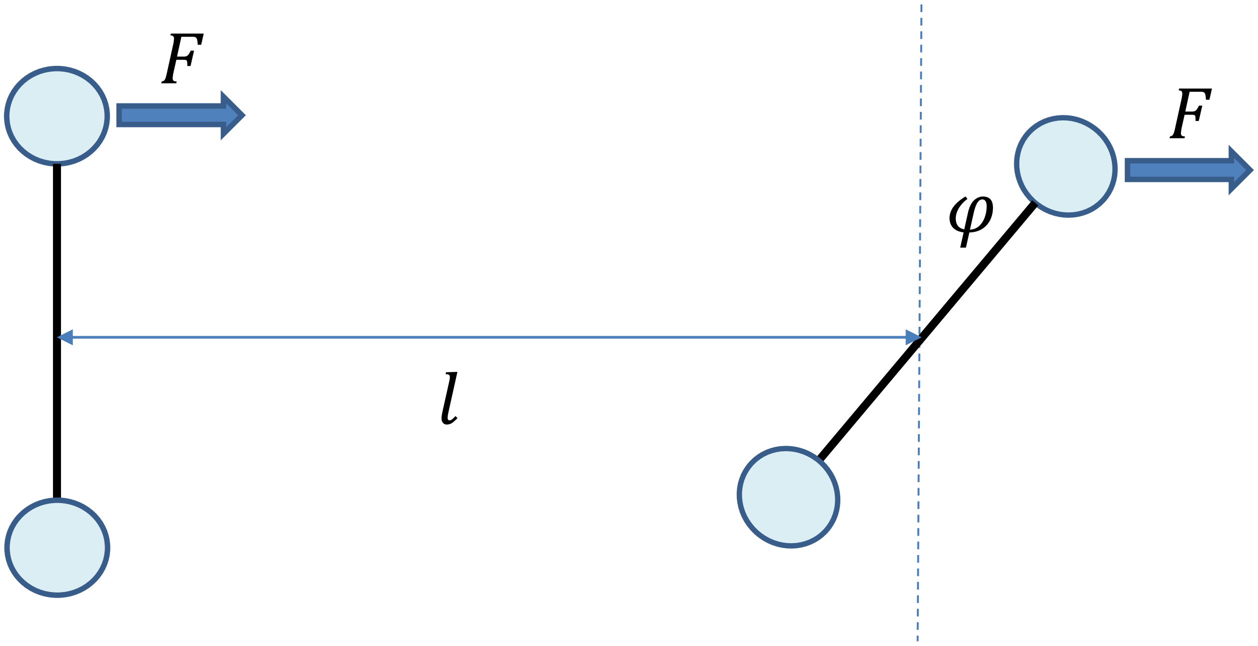
where

$$K_t = 2 \times \frac{1}{2}Mv^2 = Fl$$

$$K_r = 2 \times \frac{1}{2}M\left(\frac{L}{2}\right)^2\omega^2 = \frac{1}{4}ML^2\omega^2$$

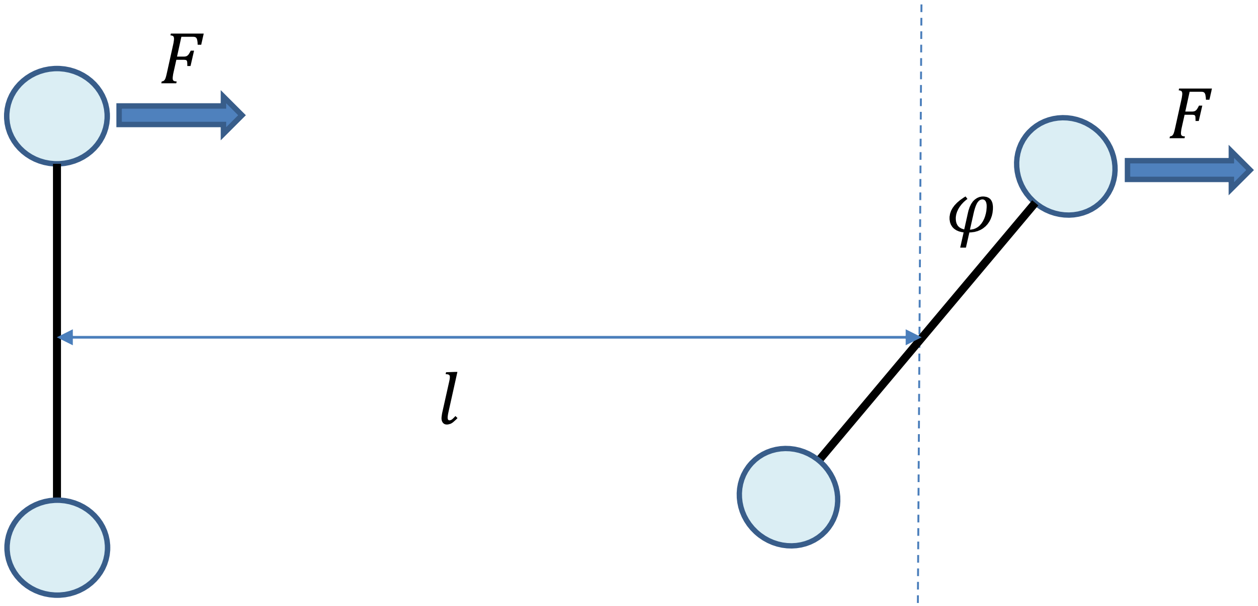
are the translational and rotational kinetic energies of the dumbbell respectively. Hence we have

$$\omega = \sqrt{\frac{2F\sin\varphi}{ML}}$$



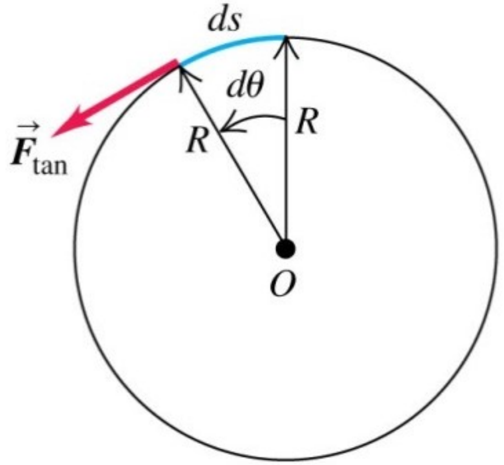
Finally, focusing on the centripetal force acting on the mass 1.

$$\begin{aligned} T - F\sin\varphi &= Ma_{\text{cir}} - Ma_{\text{CM}}\sin\varphi \\ &= M\omega^2\frac{L}{2} - M\frac{F}{2M}\sin\varphi \\ T = \frac{F}{2}\sin\varphi + \frac{ML}{2}\omega^2 &= \frac{3}{2}F\sin\varphi \end{aligned}$$



Work and power in rotational motion

A particle or rigid body, being pushed by an external force, is undergoing circular motion about a fixed axis (such as a merry-go-round).



⚠ only the tangential component $F_{\tan}$ does work – no displacement along the radial and *z* directions.

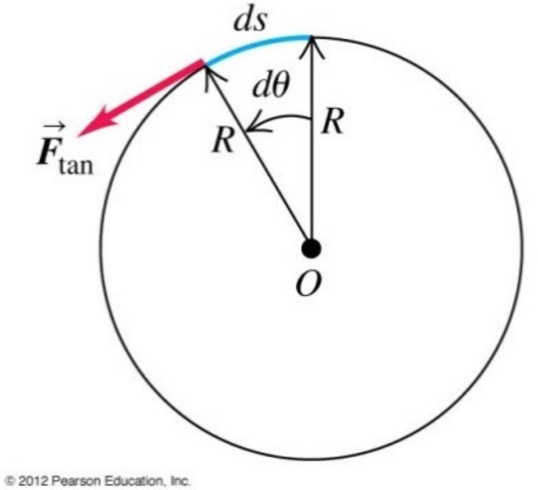
Work done after going through angle $d\theta$

$$dW = F_{\tan}(R d\theta) = \tau d\theta$$

$$\Rightarrow \quad \boxed{W = \int \tau \, d\theta}$$

c.f. in translation, $W = \int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$

$$\boxed{W = \int \tau \, d\theta}$$



By changing variable

$$\tau d\theta = (I\alpha)d\theta = I\frac{d\omega}{dt}d\theta = I(d\omega)\omega$$

$$\boxed{W_{\text{tot}} = \int_{\omega_1}^{\omega_2} I\omega d\omega = \tfrac{1}{2}I\omega_2^2 - \tfrac{1}{2}I\omega_1^2}$$

This is the **work-energy theorem** for rotational motion.

How about power?

$$P = \frac{dW}{dt} = \tau\frac{d\theta}{dt} = \tau\omega$$

c.f. $P = \vec{F} \cdot \vec{v}$ for translational motion.

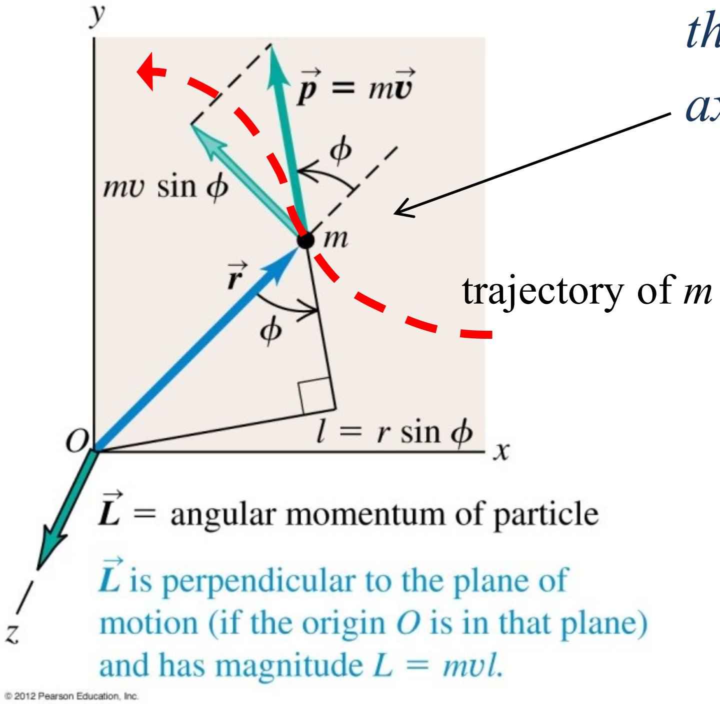
Question

- You apply equal torques to two different cylinders, one of which has a moment of inertial twice as large as the other. Each cylinder is initially at rest. After one complete rotation, the cylinder with larger moment of inertia will have (larger / smaller / the same) kinetic energy as the other one.

Angular momentum

For a point particle, define its **angular momentum** about the origin *O* by

$$\vec{\boldsymbol{L}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}$$



$\vec{L}$ is perpendicular to the plane of motion (if the origin O is in that plane) and has magnitude $L = mvl$.

© 2012 Pearson Education, Inc.

the particle need not be rotating about any axis, can be travelling in a straight line

$$L = mvr\sin\phi = (mv\sin\phi)r$$
$$= mv(r\sin\phi)$$

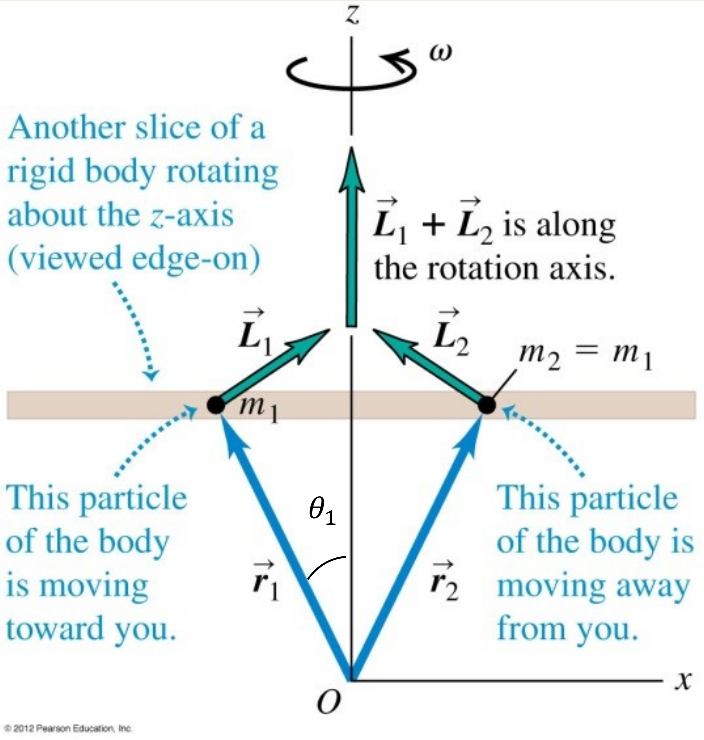
$$\frac{d\vec{\boldsymbol{L}}}{dt} = \left(\underset{m\frac{d\vec{\boldsymbol{r}}}{dt}}{\frac{d\vec{\boldsymbol{r}}}{dt}} \times \vec{\boldsymbol{p}}\right) + \left(\vec{\boldsymbol{r}} \times \underset{\vec{\boldsymbol{F}}}{\frac{d\vec{\boldsymbol{p}}}{dt}}\right) = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}$$
$$= \vec{\boldsymbol{\tau}}$$

i.e. $$\boxed{\frac{d\vec{\boldsymbol{L}}}{dt} = \vec{\boldsymbol{\tau}}}$$

c.f.

$$\frac{d\vec{\boldsymbol{P}}}{dt} = \vec{\boldsymbol{F}}$$

For a rigid body



© 2012 Pearson Education, Inc.

Take the rotation axis as the *z* axis,
m_1 is a small mass of the rigid body

$$L_1 = mv_1r_1 = m(\omega r_1 \sin\theta_1)r_1$$

If rotation axis is a symmetry axis, then there exist m_2 on the opposite side whose *x*-*y* components of angular momentum cancel those of m_1.
Therefore only *z* component of any $\vec{\boldsymbol{L}}_i$ is important.

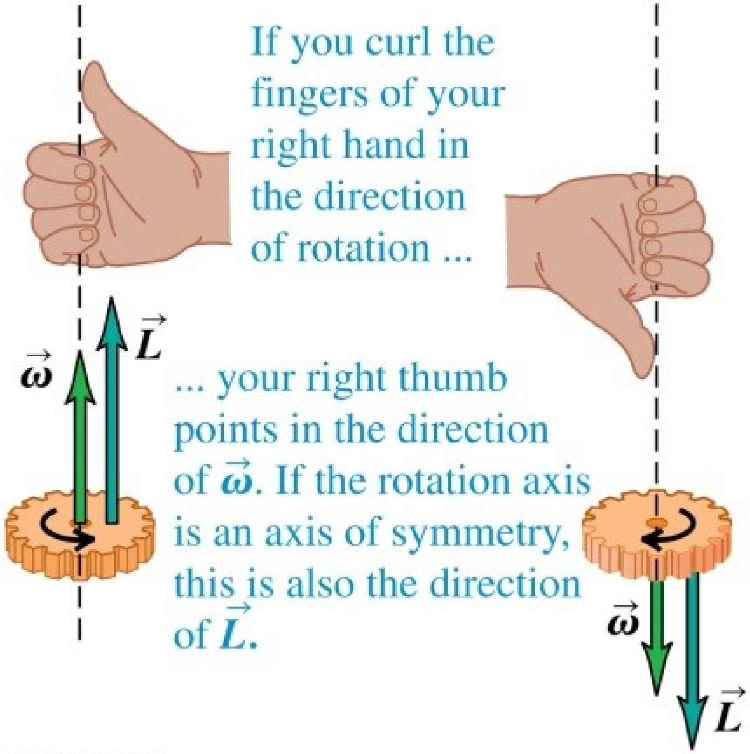
Total angular momentum $\vec{\boldsymbol{L}} = \sum \vec{\boldsymbol{L}}_i = \sum L_i \sin\theta_i \,\hat{\boldsymbol{k}}$, points along rotation axis with magnitude

$$L = \sum [m_i(\omega r_i \sin\theta_i)r_i] \sin\theta_i = \left(\sum m_i(\underbrace{r_i \sin\theta_i}_{\perp \text{ distance of } m_i \text{ to rotation axis}})^2\right)\omega$$

Conclusion: if rotation axis is a symmetry axis, then

$$\boxed{\vec{L} = I\vec{\omega}}$$

$$c.f.\ \vec{p} = m\vec{v}$$



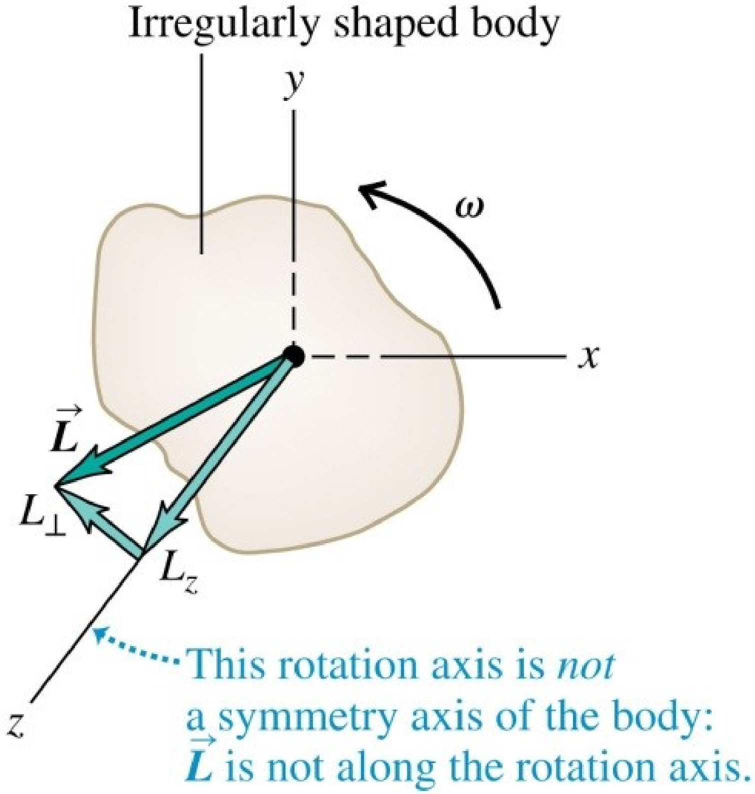
If you curl the fingers of your right hand in the direction of rotation ...

... your right thumb points in the direction of $\vec{\omega}$. If the rotation axis is an axis of symmetry, this is also the direction of $\vec{L}$.

$\vec{\omega}$ and $\vec{L}$ have the same direction

What if the rotation axis is not a symmetry axis? $\vec{\boldsymbol{L}} \neq I\vec{\boldsymbol{\omega}}$, but

$$L_z = I\omega$$

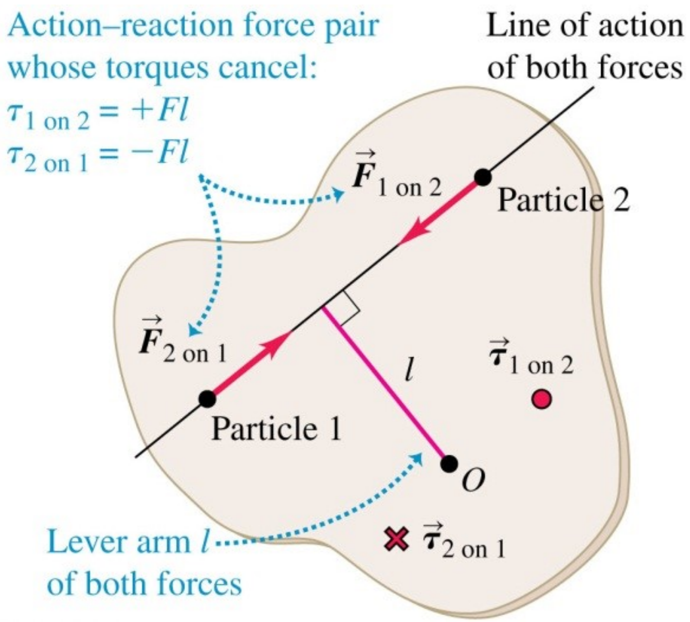


This rotation axis is *not* a symmetry axis of the body: $\vec{L}$ is not along the rotation axis.

For rotation about a fixed axis, “angular momentum” often means the component of $\vec{\boldsymbol{L}}$ along the axis of rotation, but not $\vec{\boldsymbol{L}}$ itself.

Internal forces (action and reaction pairs) have the same line of action → no net torque.

Therefore for a system of particles or a rigid body



$$\boxed{\frac{d\vec{\boldsymbol{L}}}{dt} = \sum \vec{\boldsymbol{\tau}}}$$

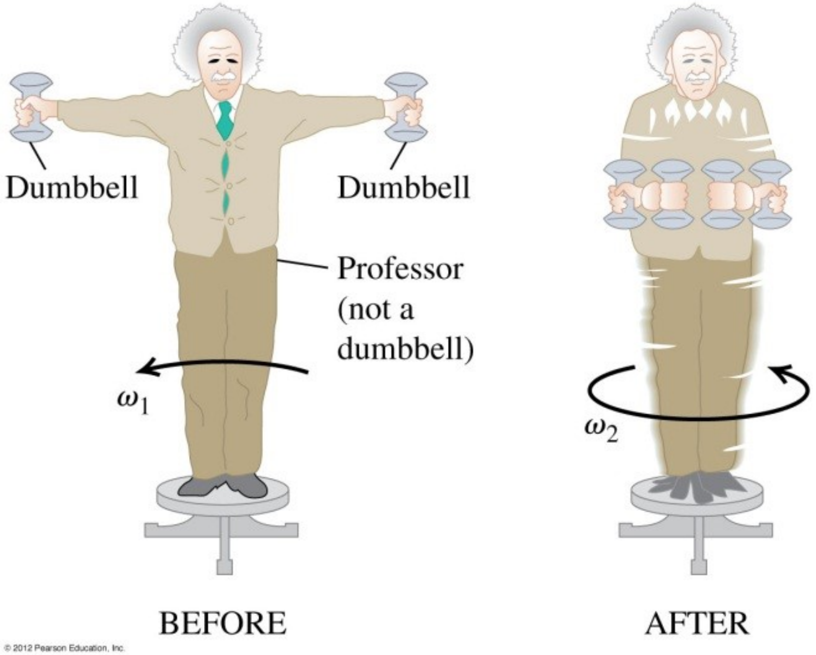
c.f. $\frac{d\vec{\boldsymbol{P}}}{dt} = \sum \vec{\boldsymbol{F}}_{\text{ext}}$

Under no external torque (⚠ *not* force)

$$\frac{d\vec{\boldsymbol{L}}}{dt} = 0$$

conservation of angular momentum

A spinning physics professor



Conservation of angular momentum

$$I_1\omega_1 = I_2\omega_2$$

If $I_2 = I_1/2$, then $\omega_2 = 2\omega_1$, and

$K_2 = \frac{1}{2} I_2\omega_2^2 =$ ____K_1.

Where comes the extra energy?

And in the reverse process $I_2 \rightarrow I_1$, where goes the energy?

Example

A bullet hits a door in a perpendicular direction, embeds in it and swings it open.

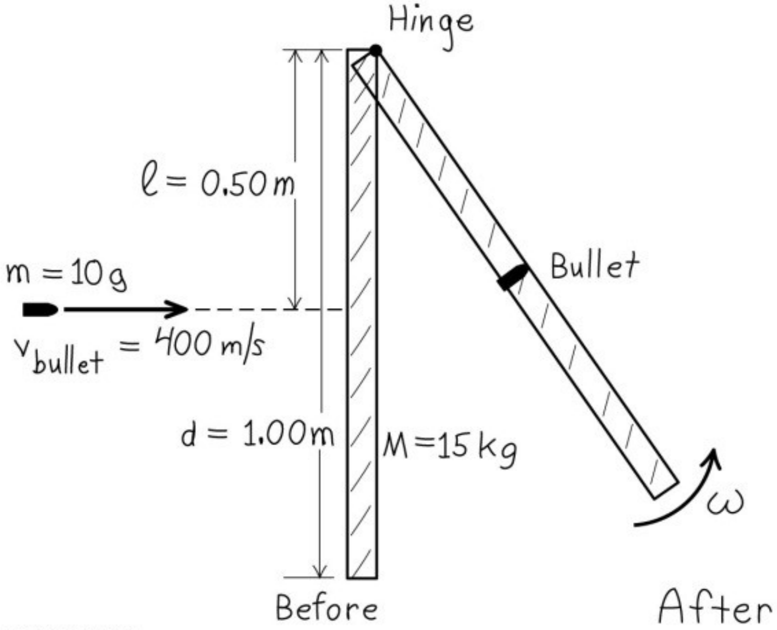
Linear momentum is not conserved because ______________________________

Angular momentum along the rotation axis is conserved because ______________

initial angular momentum of bullet about hinge

$$mvl = \underbrace{\left(\frac{Md^2}{3}\right)}_{\text{moment of inertia of door about hinge}}\omega + \underbrace{(ml^2)}_{\text{moment of inertia of bullet about hinge after embedded in door}}\omega$$

top view



© 2012 Pearson Education, Inc.

$$\Rightarrow \qquad \omega = \frac{mvl}{\frac{1}{3}Md^2 + ml^2}$$

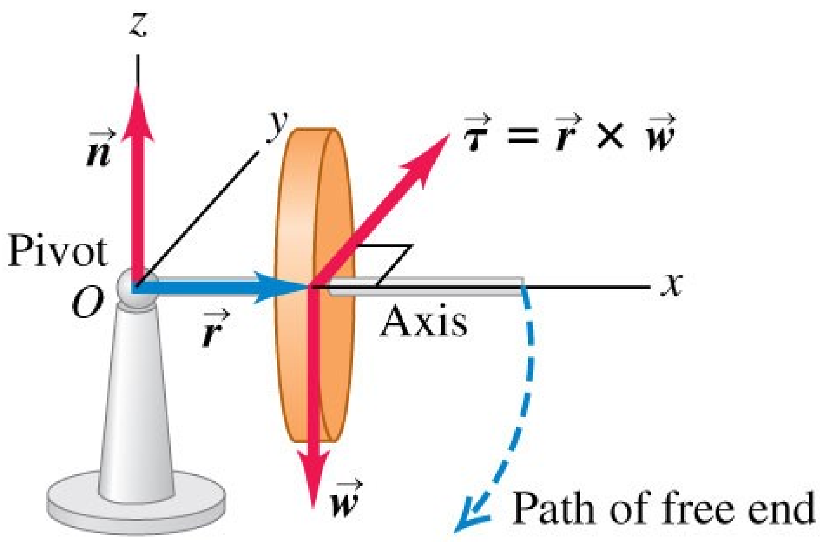
Question: If the polar ice caps were to completely melt due to global warming, the melted ice would redistribute itself over the earth. This change would cause the length of the day (the time needed for the earth to rotate once on its axis) to (increase / decrease / remain the same).

Gyroscope

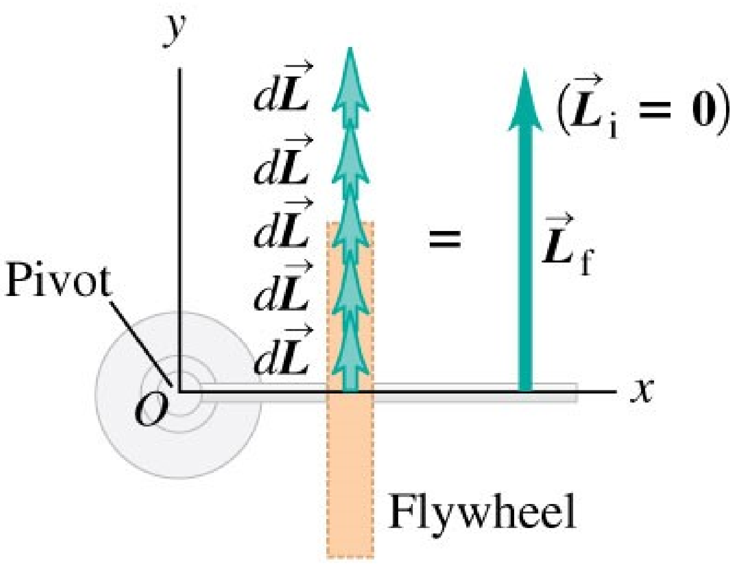
https://www.youtube.com/watch?v=cquvA_IpEsA&t=3s

Case 1: when the flywheel is not spinning – it falls down

(a) Nonrotating flywheel falls



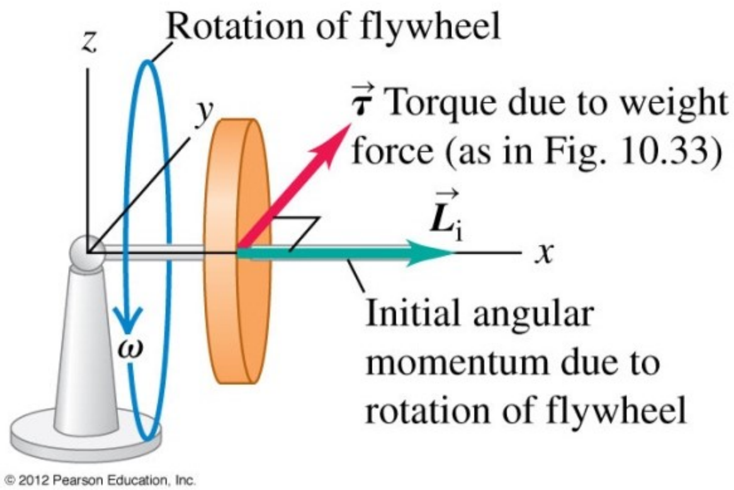
(b) View from above as flywheel falls



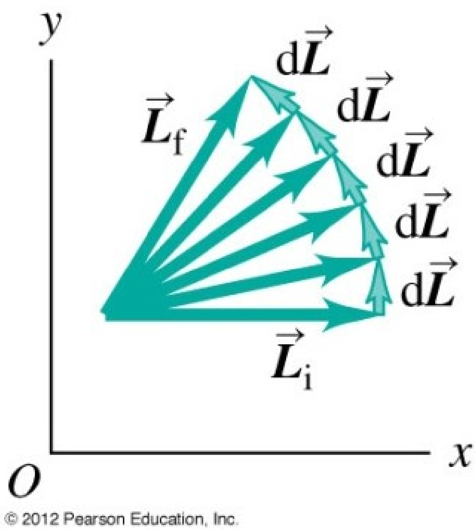
torque $\vec{\tau}$ due to weight of the flywheel $\vec{w}$ causes it to fall in the x-z plane

$\vec{L}$ increases as flywheel falls

Case 2: when flywheel spinning with initial angular moment $\vec{\boldsymbol{L}}_i$ – it **precesses**



© 2012 Pearson Education, Inc.



© 2012 Pearson Education, Inc.

Since $\vec{\boldsymbol{L}} \perp d\vec{\boldsymbol{L}}$, flywheel axis execute circular motion called **precession**, $|\vec{\boldsymbol{L}}|$ remains constant

faster spinning ω → slower precession Ω

Rotational motion of the angular momentum

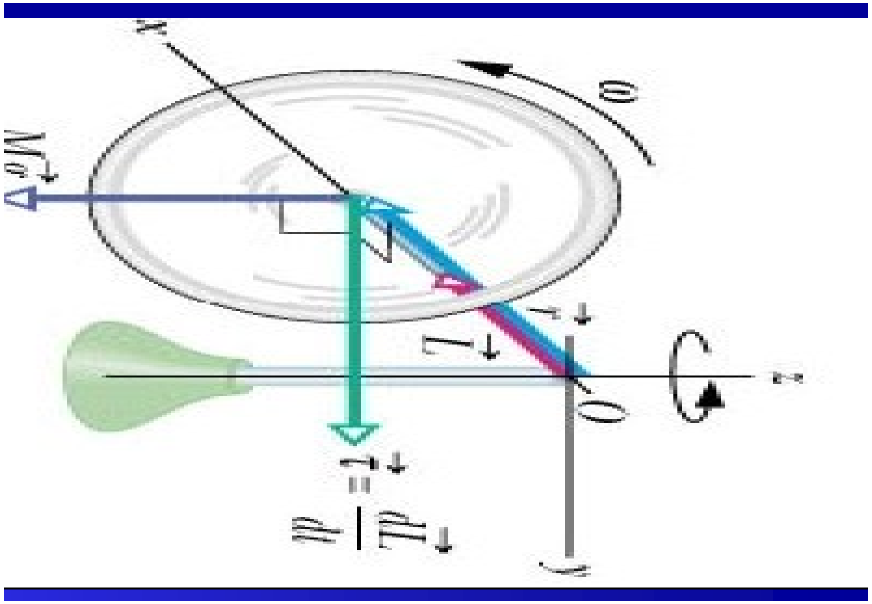
$$d\vec{L} = \vec{\tau}dt \rightarrow dL = \tau dt = Mgrdt$$

$d\vec{L} \perp \vec{L}$ ==> $\vec{L}$ can only change its direction, but NOT its magnitude

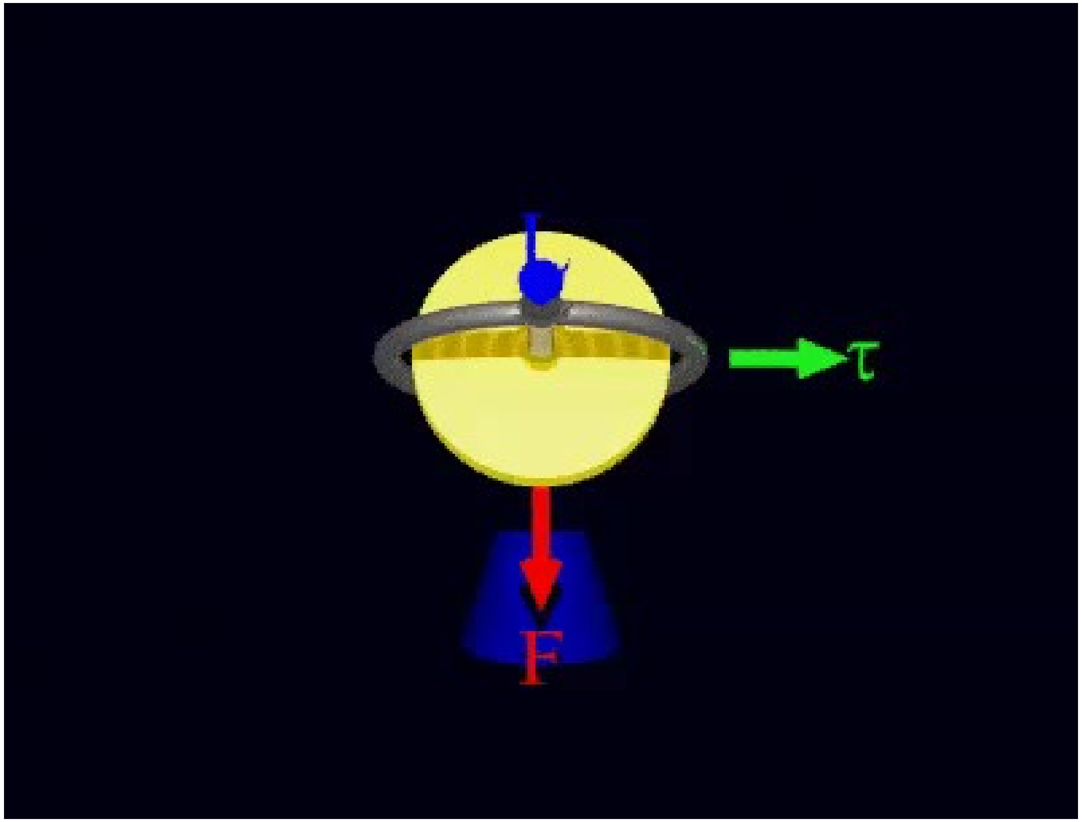
$$d\varphi = \frac{dL}{L} = \frac{Mgrdt}{I\omega}$$

Precession rate

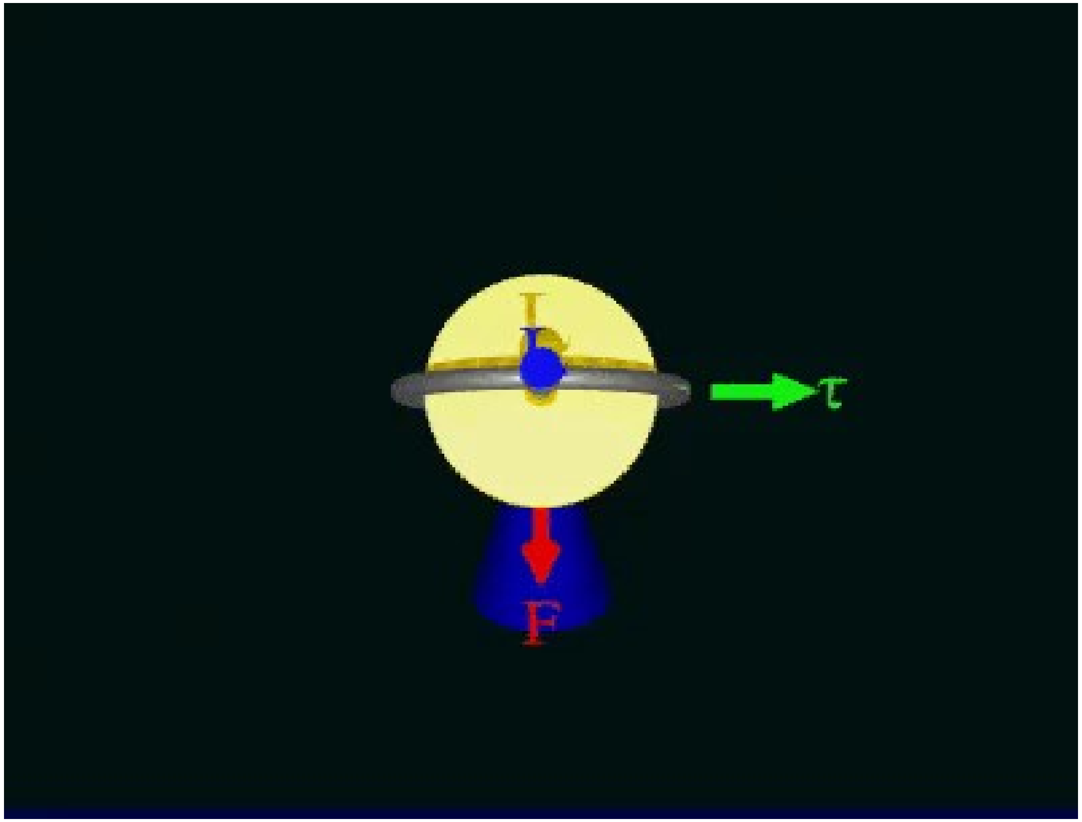
$$\boxed{\Omega = \frac{d\varphi}{dt} = \frac{Mgr}{I\omega}}$$



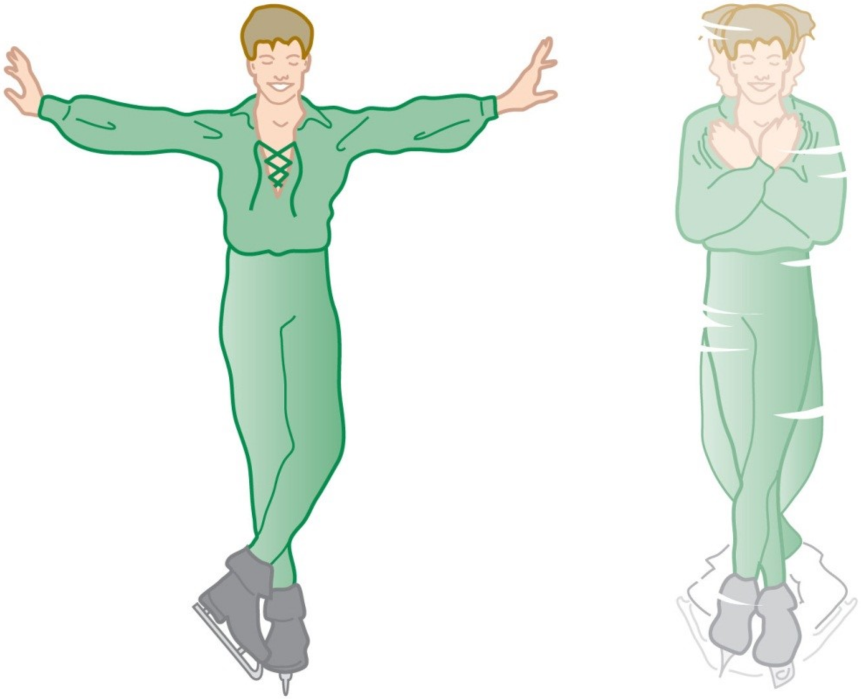
animation of the vectors $\vec{\boldsymbol{w}}$, $\vec{\boldsymbol{\tau}}$, and $\vec{\boldsymbol{L}}$ at
http://phys23p.sl.psu.edu/phys_anim/mech/gyro_s1_p.avi



If $\omega \gg \Omega$, can ignore angular momentum due to precession. Otherwise there is *nutation* of the flywheel axis – it wobbles up and down



A spinning figure skater pulls his arms in as he rotates on the ice. As he pulls his arms in, what happens to his angular momentum *L* and kinetic energy *K*?



A. *L* and *K* both increase.

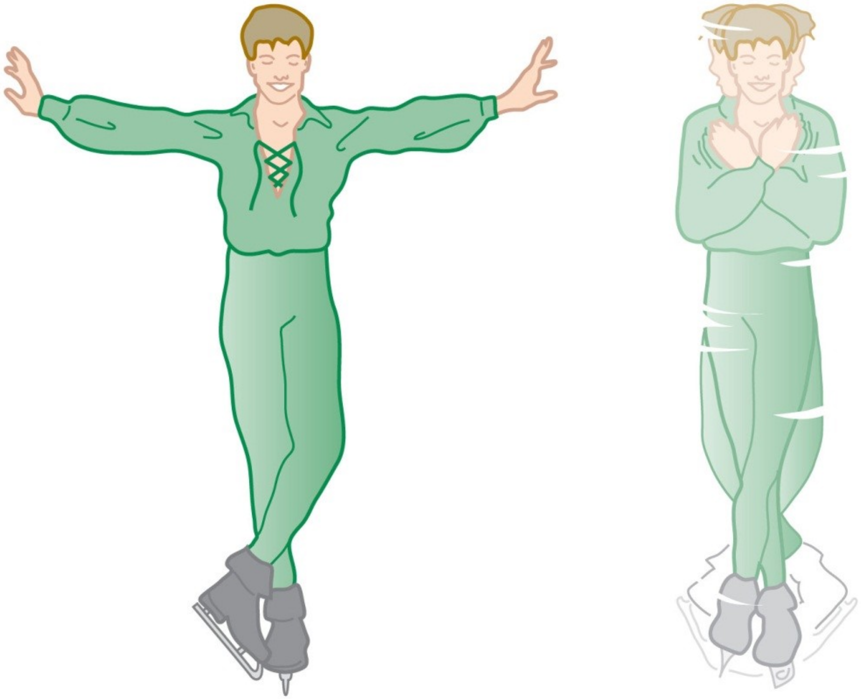
B. *L* stays the same; *K* increases.

C. *L* increases; *K* stays the same.

D. *L* and *K* both stay the same.

A10.11

A spinning figure skater pulls his arms in as he rotates on the ice. As he pulls his arms in, what happens to his angular momentum *L* and kinetic energy *K*?



A. *L* and *K* both increase.

✓ *L* stays the same; *K* increases.

C. *L* increases; *K* stays the same.

D. *L* and *K* both stay the same.

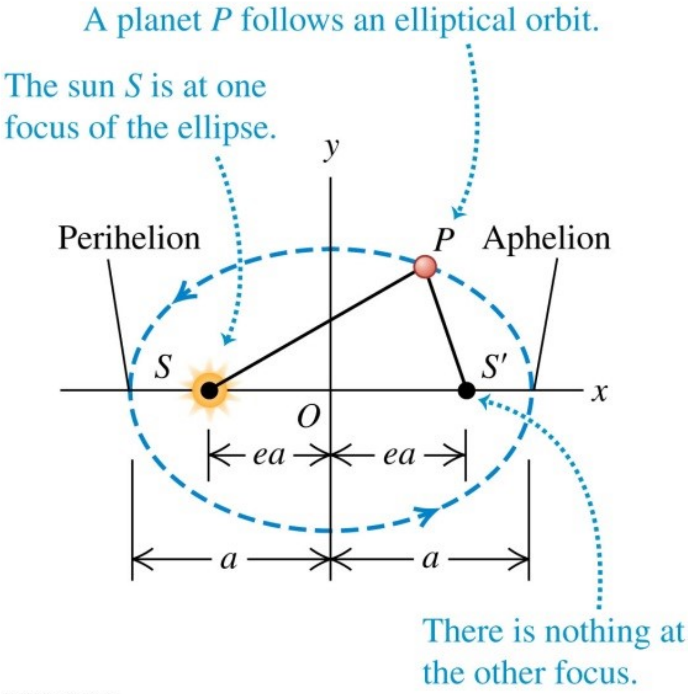
Kepler’s Laws of Planetary Motion

Purely phenomenological
– Kepler didn’t know why

Later derived by Newton using his laws of motion and gravitation

– significance: heavenly objects obey the same physical laws as terrestrial objects, don’t need, e.g., Greek myths!

First Law: Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse.



A planet P follows an elliptical orbit.

The sun S is at one focus of the ellipse.

There is nothing at the other focus.

© 2012 Pearson Education, Inc.

An **ellipse** is defined by the locus of a point P such that $|PS'| + |SP| = \text{constant}$

S and S' are the two **foci** of the ellipse

Semi-major axis a (⚠ a length, not an axis)

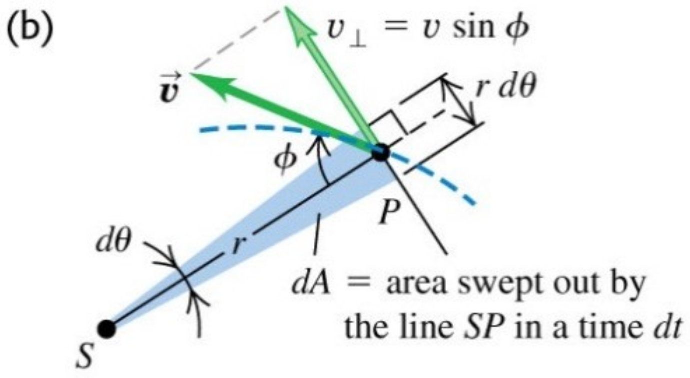
Eccentricity e ($e = 0$ for circle, $0 < e < 1$ for ellipse)

Aphelion – farthest $[(1 + e)a]$ point from sun

Perihelion – closest $[(1 - e)a]$ point to sun

Note: aphelion distance + perihelion distance = $2a$

Second Law: A line from the sun to a given planet sweeps out equal areas in equal times. See http://en.wikipedia.org/wiki/File:Kepler-second-law.gif

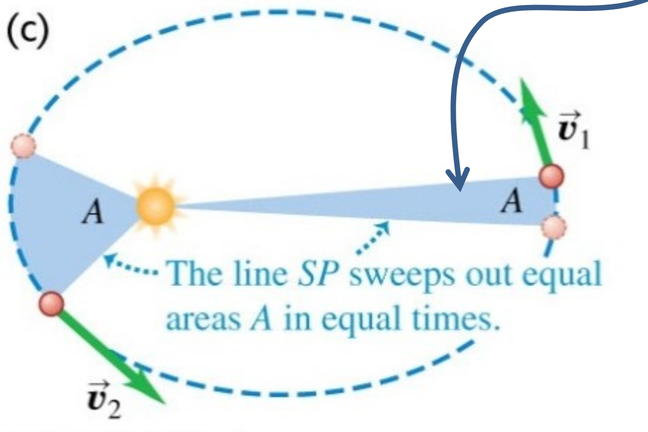


$$dA \approx \text{area of blue triangle} = \tfrac{1}{2}(r\,d\theta)r$$

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}$$

$$v_\perp = v\sin\phi = r\frac{d\theta}{dt}$$

$$\therefore \quad \frac{dA}{dt} = \frac{1}{2}rv\sin\phi = \frac{1}{2m}|\vec{\boldsymbol{r}} \times m\vec{\boldsymbol{v}}| = \frac{L}{2m}$$



i.e., Kepler's second law ⇔ conservation of angular momentum

- ⚠ Angular momentum is conserved because gravitational force (a central force) produces no torque
- ⚠ Another consequence of conservation of angular momentum – orbit lies in a plane

Third Law: The periods of the planets are proportional to the $\frac{3}{2}$ powers of the **major axis lengths** of their orbits.

$$\boxed{T = \frac{2\pi a^{3/2}}{\sqrt{Gm_S}}}$$

For the circular orbit, we have

$$\frac{Gm_s m}{a^2} = \frac{mv^2}{a}$$

$$T = \frac{2\pi a}{v}$$