Tutorial: Introduction to Vector Calculus

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Contents

Chapter 1

Basic vector operations

In this chapter, we revise some basic operations on vector.

1.1 Vectors in Cartesian co-ordinates

Using Cartesian co-ordinates, we denote the unit vectors along the x, y and z axis as $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ respectively. Some books use $\hat{x}$, $\hat{y}$ and $\hat{z}$ accordingly. For example, if we have a vector pointing along the positive x axis, we write it as $\vec{r} = 3\hat{\imath}$. Clearly, given a vector $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, its length or norm is determined by the Pythagoras Theorem:

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2} \tag{1.1}$$

You will immediately notice the advantages of writing vectors in algebraic way, rather than pictorial representation.

1.2 Addition and subtraction

Given any two vectors $\vec{A}_1 = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{B} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$, their sum (difference) can be obtained by adding (subtracting) the components separately:

$$\vec{A} \pm \vec{B} = (a_1 \pm b_1)\,\hat{\imath} + (a_2 \pm b_2)\,\hat{\jmath} + (a_3 \pm b_3)\,\hat{k} \tag{1.2}$$

Example: Electric force between charges

A positive charge $+4q$ is located at $(0, 0, 1)$, with four negative charges $-q$ located at $(-1, 0, 0)$, $(1, 0, 0)$, $(0, -1, 0)$ and $(0, -1, 0)$, find the electrostatic force acting on the positive charge due to the negative charges.

For convenience, we denote the positive charge as $Q = 4q$ and those negative charges as $q_i = -q$, with $i = 1, 2, 3, 4$. Clearly, the position vectors of the charges are $\vec{r}_Q = \hat{k}$, $\vec{r}_1 = -\hat{\imath}$, $\vec{r}_2 = \hat{\imath}$, $\vec{r}_3 = -\hat{\jmath}$ and $\vec{r}_4 = \hat{\jmath}$. By Coulomb's law, we know that the electrostatic force required is

$$\begin{aligned}
\vec{F} &= \sum_{i=1}^{4} \frac{Q \cdot q_i}{4\pi\epsilon_0 \left|\vec{r}_Q - \vec{r}_i\right|^3} \left(\vec{r}_Q - \vec{r}_i\right) \\
&= \frac{(4q)\,(-q)}{4\pi\epsilon_0 \left(\sqrt{2}\right)^3} \left[\left(\hat{k} + \hat{\imath}\right) + \left(\hat{k} - \hat{\imath}\right) + \left(\hat{k} + \hat{\jmath}\right) + \left(\hat{k} - \hat{\jmath}\right)\right] \\
&= -\frac{\sqrt{2}q^2}{\pi\epsilon_0}\hat{k}
\end{aligned} \tag{1.3}$$

1.3 Dot product

The dot product between any two vectors $\vec{A}$ and $\vec{B}$ is defined as

$$\vec{A} \cdot \vec{B} = \left|\vec{A}\right| \left|\vec{B}\right| \cos\theta, \tag{1.4}$$

where θ is the angle between the two vectors. Clearly, the dot product is commutative, that is $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$. From (1.4), we automatically have

$$\begin{aligned} \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \\ \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0 \end{aligned} \tag{1.5}$$

Therefore, if $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, the dot product becomes

$$\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{1.6}$$

Example: Work done

Suppose a box is pulled by a constant force $\vec{F} = \hat{i} + \hat{j}$ and moves from $(0,0)$ to $(0,5)$ along the x axis, the work done on the box due to the force is

$$\Delta W = \vec{F} \cdot \Delta\vec{s} = \left(\hat{i} + \hat{j}\right) \cdot \left(5\hat{j}\right) = 5 \tag{1.7}$$

1.4 Cross product

The cross product between two vectors $\vec{A}$ and $\vec{B}$ is defined as

$$\vec{A} \times \vec{B} = \left(\left|\vec{A}\right| \left|\vec{B}\right| \sin\theta\right) \hat{n}, \tag{1.8}$$

where $\hat{n}$ is a unit vector perpendicular to both A and B. Its direction is determined by the right hand grip rule. Geometrically, its magnitude gives the area of the parallel spanned by $\vec{A}$ and $\vec{B}$. It is reminded that the cross product is not commutative, but it satisfies

$$\vec{B} \times \vec{A} = -\left(\vec{A} \times \vec{B}\right) \tag{1.9}$$

From (1.8), we have

$$\begin{aligned} \hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j} \\ \hat{i} \times \hat{i} = \vec{0}, \quad \hat{j} \times \hat{j} = \vec{0}, \quad \hat{k} \times \hat{k} = \vec{0} \end{aligned} \tag{1.10}$$

If $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then we have

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{1.11}$$

Example: Torque acting on a particle

Suppose there is a particle at the point $(1,1,1)$, which is acted by a force given by $\vec{F} = 4\hat{i} + 4\hat{j}$, the torque acting on the particle about the origin is

$$\vec{\tau} = \vec{r} \times \vec{F} = \left(\hat{i} + \hat{j} + \hat{k}\right) \times \left(4\hat{i} + 4\hat{j}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 4 & 4 & 0 \end{vmatrix} = -4\hat{i} + 4\hat{j} \tag{1.12}$$

1.5 Triple product

Given any three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$, there are two legitimate triple products can be formed.

1.5.1 Scalar triple product

The first one is $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)$, which is called the scalar triple product. Geometrically, its magnitude gives the volume of the parallelepiped formed by $\vec{A}$, $\vec{B}$ and $\vec{C}$. In Cartesian co-ordinates, we have

$$\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} \tag{1.13}$$

The scalar triple product satisfies the following property:

$$\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=\vec{B}\cdot\left(\vec{C}\times\vec{A}\right)=\vec{C}\cdot\left(\vec{A}\times\vec{B}\right) \tag{1.14}$$

1.5.2 Vector triple product

The second one is $\vec{A}\times\left(\vec{B}\times\vec{C}\right)$, which is called the vector triple product. It is obtained by

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\vec{B}\left(\vec{A}\cdot\vec{C}\right)-\vec{C}\left(\vec{A}\cdot\vec{B}\right). \tag{1.15}$$

Fortunately, the vector triple product is rarely seen in Physics.

1.6 Differentiation

Using Cartesian co-ordinates, the differentiation of a vector can be easily performed. Suppose $\vec{s}=f\left(t\right)\hat{i}+g\left(t\right)\hat{j}+h\left(t\right)\hat{k}$, we have

$$\frac{d}{dt}\vec{s}=\frac{df\left(t\right)}{dt}\hat{i}+\frac{dg\left(t\right)}{dt}\hat{j}+\frac{dh\left(t\right)}{dt}\hat{k}, \tag{1.16}$$

as $\hat{i}$, $\hat{j}$ and $\hat{k}$ are constant vectors, which are no need to be differentiated.

Sometimes, it is more convenient to write vectors in other co-ordinates. For example, polar co-ordinates will be used when we study central force problem or rotating reference frame. Using polar co-ordinates, a vector can be written as $\vec{F}=F_r\hat{r}+F_\theta\hat{\theta}$, with the basis vectors defined as

$$\begin{aligned}\hat{r}&=\cos\theta\hat{i}+\sin\theta\hat{j},\\ \hat{\theta}&=-\sin\theta\hat{i}+\cos\theta\hat{j}\end{aligned} \tag{1.17}$$

Since the basis vectors are not constant vectors, extreme care is needed when we perform differentiation.

Example: Kinematics in polar co-ordinates

The position of a particle is given by $\vec{s}=r\hat{r}$, its velocity and acceleration are given by

$$\begin{aligned}\vec{v}&=\dot{r}\hat{r}+r\dot{\theta}\hat{\theta}\\ \vec{a}&=\left(\ddot{r}-r\dot{\theta}^2\right)\hat{r}+\left(2\dot{r}\dot{\theta}+r\ddot{\theta}\right)\hat{\theta}\end{aligned} \tag{1.18}$$

Chapter 2

Partial differentiation

In this chapter, we revise the technique of partial differentiation, which is important when we study vector calculus.

2.1 Definition

Given a multivariable function $f(x_1, x_2, \cdots, x_N)$, its partial derivative $\frac{\partial f}{\partial x_i}$, with $i \leq N$, is defined as

$$\frac{\partial f}{\partial x_i} = \lim_{\Delta x_i \to 0} \left[\frac{f(x_1, x_2, \cdots, x_{i-1}, x_i + \Delta x_i, x_{i+1}, \cdots, x_N) - f(x_i, x_2, \cdots, x_N)}{\Delta x_i} \right], \tag{2.1}$$

which is analogous to the first principle in ordinary differential. Geometrically, it gives the change of f due to the change in x_i, with other independent variables kept constant. The procedure for taking partial derivative is easy. Just differentiate f with respect to x_i, by treating other independent variables as constants, which is illustrated in the following example.

Example: Taking partial derivatives

Consider $f(x, y) = x^2 \sin y$, the first order partial derivatives are

$$\frac{\partial f}{\partial x} = 2x \sin y, \quad \frac{\partial f}{\partial y} = x^2 \cos y \tag{2.2}$$

If we consider the second order partial derivatives, there are four possible cases:

$$\frac{\partial^2 f}{\partial x^2} = 2 \sin y, \quad \frac{\partial^2 f}{\partial y \partial x} = 2x \cos y, \quad \frac{\partial^2 f}{\partial x \partial y} = 2x \cos y, \quad \frac{\partial^2 f}{\partial y^2} = -x^2 \sin y. \tag{2.3}$$

It is noticed that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$, which is generally valid for a twice differentiable function.

2.2 Chain rule for multivariable functions

The chain rule for multivariable functions is analogous to function with one independent variable. For simplicity, we assume $f = f(x, y)$ with $x = x(s, t)$ and $y = y(s, t)$. Then, we have

$$\begin{aligned} \frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}, \\ \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \end{aligned} \tag{2.4}$$

Chain rule for functions with more independent variables can be generalized in the similar way.

Example: Solution to wave equation

Given the one-dimensional wave equation as

$$\frac{\partial^2}{\partial x^2}u(x,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}u(x,t)\,, \tag{2.5}$$

we claim that $u(x,t) = f(kx - \omega t)$ is a solution to (2.5) if $\omega = ck$.

To prove that, we let $\xi = kx - ct$, so $f = f(\xi)$ with $\xi = \xi(x,t)$. Then, we have the first order partial derivatives from the chain rule:

$$\begin{aligned}\frac{\partial}{\partial x}f(kx-\omega t) &= \frac{df}{d\xi}\frac{\partial \xi}{\partial x} = k\frac{df}{d\xi}\\ \frac{\partial}{\partial t}f(kx-\omega t) &= \frac{df}{d\xi}\frac{\partial \xi}{\partial t} = -\omega\frac{df}{d\xi}\end{aligned} \tag{2.6}$$

The second order partial derivatives are

$$\begin{aligned}\frac{\partial^2}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{d}{d\xi}\left(k\frac{df}{d\xi}\right)\frac{\partial \xi}{\partial x} = k^2\frac{d^2 f}{d\xi^2}\\ \frac{\partial^2}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) = \frac{d}{d\xi}\left(-c\frac{df}{d\xi}\right)\frac{\partial \xi}{\partial t} = \omega^2\frac{d^2 f}{d\xi^2}\end{aligned} \tag{2.7}$$

Substitute into the wave equation can using $\omega = ck$, we complete the proof.

Chapter 3

Multiple integrals

In this chapter, we will revise the systematic procedures for handling multiple integrals, which is a generalization of the definite integral for single variable function you have learnt. This chapter serves as a preliminary for evaluating surface integral and volume integral.

3.1 Double and triple integrals

There are not much special things for double and triple integrals. The most difficult part should be writing the region for integration correctly. Most importantly, you should be careful of the differential area element and the differential volume element in different co-ordinate systems. For differential area element, we mainly consider the Cartesian co-ordinates and polar co-ordinates:

$$\begin{aligned} da &= dxdy \qquad \text{(Cartesian co-ordinates)} \\ da &= rdrd\theta \qquad \text{(Polar co-ordinates)} \end{aligned} \tag{3.1}$$

and the differential volume element:

$$\begin{aligned} d\tau &= dxdydz \qquad \text{(Cartesian co-ordinates)} \\ d\tau &= rdrd\theta dz \qquad \text{(Cylindrical co-ordinates)} \\ d\tau &= r^2 \sin\phi drd\theta d\phi \qquad \text{(Spherical co-ordinates)} \end{aligned} \tag{3.2}$$

with ϕ being the polar angle. In order to show you the standard procedures for multiple integrals, two examples will be used.

Example 1: Double integral on Cartesian co-ordinates

Suppose you are given the following double integral:

$$\int_0^2 \int_0^{4-x^2} 2x \mathrm{d}y \mathrm{d}x \tag{3.3}$$

and you are asked to evaluate it.

Region of Integration

Before evaluating the integral, it is suggested to plot the region of integration. This is very useful when you want to invert the order of integration. From the integral, we know that y is upper bounded by $4 - x^2$ and lower bounded by 0. Therefore, the upper bound of y depends on x. For x, it is upper bounded by 2 and lower bounded by 0. The region for the integration is given in FIG. 3.1.

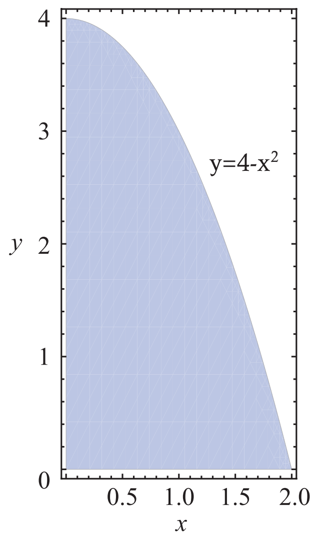

Figure 3.1: Region of the integration

Evaluate the integral

Suppose we integrate y first. The method is the same as single-variable integration by treating x as constant:

$$\int_0^2 \int_0^{4-x^2} 2x \mathrm{d}y \mathrm{d}x = \int_0^2 2xy|_{y=0}^{y=4-x^2} \mathrm{d}x = \int_0^2 2x\left(4-x^2\right) \mathrm{d}x \tag{3.4}$$

Then, we can integrate x by using what you have learnt in secondary school:

$$\int_0^2 2x\left(4-x^2\right)\mathrm{d}x = -\int_0^2 \left(4-x^2\right)\mathrm{d}\left(4-x^2\right) = -\frac{\left(4-x^2\right)^2}{2}|_{x=0}^{x=2} = 8 \tag{3.5}$$

Invert the order of integration

Suppose we want to invert the order of integration. Since $y = 4 - x^2$ for the curved boundary we have $x = \pm\sqrt{4-y^2}$. However, it is clear that we should choose the plus sign as seen from FIG.3.1. For y, it is clear that $0 < y < 4$. Therefore, the double integral becomes:

$$\int_0^4 \int_0^{\sqrt{4-y}} 2x \mathrm{d}x \mathrm{d}y \tag{3.6}$$

Using similar technique, we have:

$$\int_0^4 \int_0^{\sqrt{4-y}} 2x \mathrm{d}x \mathrm{d}y = \int_0^4 x^2|_{x=0}^{x=\sqrt{4-y}} \mathrm{d}y = \int_0^4 (4-y)\,\mathrm{d}y = \left(4y - \frac{y^2}{2}\right)|_{y=0}^{y=4} = 8 \tag{3.7}$$

The results of (3.3) and (3.6) are the same, which hold generally for well-behaved function.

Example 2: Triple integrals on cylindrical co-ordinates

You are given the following triple integral:

$$\int_{-1}^1 \int_0^{\sqrt{1-y^2}} \int_0^x \left(x^2+y^2\right) \mathrm{d}z \mathrm{d}x \mathrm{d}y \tag{3.8}$$

and you are asked to evaluate it in cylindrical co-ordinates.

Region of Integration

First of all, we know that x is lower bounded by 0 and upper bounded by $\sqrt{1-y^2}$. Also, we have $-1 < y < 1$. As a result, we know that the upper bound of x should be a semi-circular arc since $x^2 + y^2 = 1$ with x always positive. Lastly, we know that z is lower bounded by 0 and upper bounded by x. Therefore, the region of integration is obtained by sliding a half cylinder, as shown in FIG. 3.2.

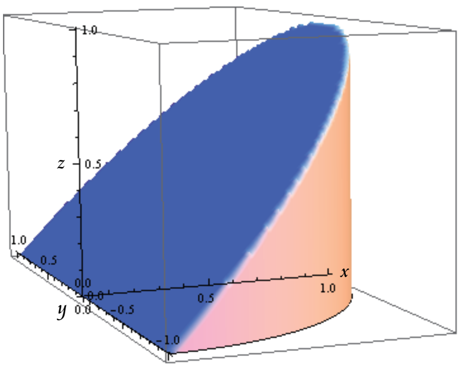

Figure 3.2: Region of the integration

Changing co-ordinates

Since we want to evaluate the integral in cylindrical co-ordinates, so we need to change variables from (x, y, z) to (ρ, θ, z). To recall your memory, the Cartesian co-ordinates and the cylindrical co-ordinates are related by:

$$x = \rho \cos\theta, \qquad y = \rho \sin\theta, \qquad z = z \tag{3.9}$$

Step 1: Rewriting the region of integration

From FIG. 3.2, it is easy to know that $0 < \rho < 1$ and $-\pi/2 < \theta < \pi/2$. For z, we have $0 < z < x$ in Cartesian co-ordinates. Since $x = \rho\cos\theta$ as given in (3.9), so the condition becomes $0 < z < \rho\cos\theta$.

Step 2: Rewriting the integrand

In cylindrical co-ordinates, we have $x^2 + y^2 = \rho^2$.

Step 3: The volume element

The volume element for cylindrical co-ordinates is given by $d\tau = \rho d\rho d\theta dz$. You can get this by drawing a graph or evaluating the Jacobian explicitly. For your reference, I will show you how to evaluate the Jacobian here:

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho \tag{3.10}$$

Therefore, the volume element is $\rho \mathrm{d}\rho \mathrm{d}\theta \mathrm{d}z$. You must include the Jacobian after changing co-ordinates, so that the volume element is correct!

Step 4: Putting all things together

After rewriting the region of integration, the integrand and finding out the volume element, we can finally rewrite the triple integral by using cylindrical co-ordinates:

$$\int_0^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\rho\cos\theta} \rho^2 \left(\rho \mathrm{d}z \mathrm{d}\theta \mathrm{d}\rho\right) \tag{3.11}$$

Step 5: Evaluating the integral

After transforming the integral into cylindrical co-ordinates, we need to actually calculate the triple integral. The calculation is very routine:

$$\int_0^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\rho\cos\theta} \rho^2 \left(\rho \mathrm{d}z \mathrm{d}\theta \mathrm{d}\rho\right) = \int_0^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho^4 \cos\theta \mathrm{d}\theta \mathrm{d}\rho = \int_0^1 2\rho^4 \mathrm{d}\rho = \frac{2}{5}. \tag{3.12}$$

Indeed, the same result can be obtained by evaluating (3.8) directly in Cartesian co-ordinates. I will leave it as your practice.

3.2 Applications in Physics

After knowing how to evaluate multiple integrals, we should know their applications in Physics. In mechanics, we usually it for finding the center of mass and the moment of inertia.

3.2.1 Center of mass

Given an object with density $\rho(x, y, z)$, its mass is given by the volume integral:

$$M = \int_{\mathcal{V}} \rho(x, y, z) \, \mathrm{d}\tau, \tag{3.13}$$

where V is the region for the object. The **center of mass** is given by:

$$\begin{aligned}
\bar{x}_{cm} &= \frac{1}{M} \int_{\mathrm{V}} \rho(x, y, z) \, x \mathrm{d}\tau \\
\bar{y}_{cm} &= \frac{1}{M} \int_{\mathrm{V}} \rho(x, y, z) \, y \mathrm{d}\tau \\
\bar{z}_{cm} &= \frac{1}{M} \int_{\mathrm{V}} \rho(x, y, z) \, z \mathrm{d}\tau
\end{aligned} \tag{3.14}$$

3.2.2 Moment of inertia

For the **moment of inertia**, we have:

$$\begin{aligned}
I_x &= \int_{\mathrm{V}} \rho(x, y, z) \left(y^2 + z^2\right) \mathrm{d}\tau \\
I_y &= \int_{\mathrm{V}} \rho(x, y, z) \left(x^2 + z^2\right) \mathrm{d}\tau \\
I_z &= \int_{\mathrm{V}} \rho(x, y, z) \left(x^2 + y^2\right) \mathrm{d}\tau
\end{aligned} \tag{3.15}$$

Example

Consider a uniform cone with density ρ, base radius R and height h. The base of the cone lies on the $x-y$ plane and the vertex of the cone is on the z-axis. We want to find its center of mass and moment of inertia along the z-axis.

1. Center of mass

Since the density is uniform, the mass of the cone is simply:

$$M = \frac{1}{3}\pi R^2 h\rho \tag{3.16}$$

By symmetry, $\bar{x}_{cm} = \bar{y}_{cm} = 0$. For z_{cm}, we have:

$$\bar{z}_{cm} = \frac{1}{M}\int_{\mathcal{V}} \rho z \mathrm{d}\tau \tag{3.17}$$

where $\mathcal{V}$ is the region bounded by the cone. Since the cone has azimuthal symmetry, the integral is much easier to evaluate if we use cylindrical co-ordinates. The volume integral becomes:

$$\bar{z}_{cm} = \frac{1}{M}\int_0^h \int_0^{2\pi} \int_0^{\frac{R}{h}(h-z)} \rho z r \mathrm{d}r\mathrm{d}\theta\mathrm{d}z = \frac{1}{M}\left(\frac{1}{12}\pi R^2 h^2 \rho\right) = \frac{h}{4} \tag{3.18}$$

Hence, the center of mass is $(0, 0, h/4)$.

2. Moment of inertia

The moment of inertia about the z-axis is given by:

$$I_z = \int_{\mathcal{V}} \rho\left(x^2 + y^2\right) \mathrm{d}\tau \tag{3.19}$$

Again, this volume integral is much easier to be evaluated on cylindrical co-ordinates. The volume integral becomes:

$$I_z = \int_0^h \int_0^{2\pi} \int_0^{\frac{R}{h}(h-z)} \rho r^2 \left(r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z\right) = \frac{1}{10}\pi R^4 h\rho = \frac{3}{10}MR^2 \tag{3.20}$$

Chapter 4

Vector Calculus

After learning all basic techniques from the previous chapters, it's time for us to really vector calculus. First, I will discuss the differential calculus by introducing the concept of gradient, divergence and curl. Then, I will introduce the del operator and several second order differential operators to end the discussion of differential calculus. I will go into the integral calculus by introducing the concept of surface integral and line integral. After learning the differential calculus and integral calculus separately, I will introduce the divergence theorem and Stokes' theorem. Through these two theorems, the two separate pieces can be connected together. Finally, I will further the use of Stokes' theorem to discuss conservative field and the related potential. For this chapter, [1] is a very good reference.

4.1 Gradient

Given a multi-variable function $f(x, y, z)$ on three dimensional Cartesian co-ordinates, we can construct a corresponding vector function, given by:

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \tag{4.1}$$

This vector function is called the **gradient** of f. Clearly, there is no reason for us to stick with Cartesian co-ordinates. However, I will leave the discussion of vector calculus on curvilinear co-ordinates to the end of the chapter.

4.1.1 Properties

There are several important properties for the gradient of a function. First, the gradients at different positions are different. At every position, the gradient has a direction and the associated magnitude. Geometrically, the direction corresponds to the direction in which f increases the most. On the other hand, the magnitude represents the rate of increase along that direction. Apart from these, the gradient must be perpendicular to f at every point. For physical example, consider the case of electrostatic potential $V(x, y, z)$ and the electrostatic field $\vec{E}(x, y, z) = -\nabla V$. In this case, the field is perpendicular to the equipotential surface at every point. If you plot the field out explicitly, you will see that the field points from a position with higher potential to a place with lower potential.

4.1.2 Directional derivatives

After introducing the concept of gradient, we can study the change of f at different directions. The idea is simple. Since we know the rate of maximum increase and its direction, we can find the change of f at different directions by projection. Suppose the direction is given by a unit vector $\hat{n}$, the rate of increase will be simply the dot product between the gradient and the unit vector:

$$D_n[f] = (\nabla f) \cdot \hat{n}, \tag{4.2}$$

which is called the **directional derivative** of f along the direction n.

Example

To conclude this section, I will use an example to illustrate the systematic procedures for finding gradient and directional derivatives. Given the height of a hill:

$$z = 32 - x^2 - 4y^2. \tag{4.3}$$

You are asked to sketch some of the contours and evaluate the rate of change along different directions at some particular points.

Sketching contours

For the contour $z = 16$, it satisfies the equation $x^2 + 4y^2 = 16$, which is an ellipse. Similarly, for the contour $z = 7$, it satisfies $x^2 + 4y^2 = 25$, so it is also an ellipse. For reference, the contours are shown in FIG.4.1.

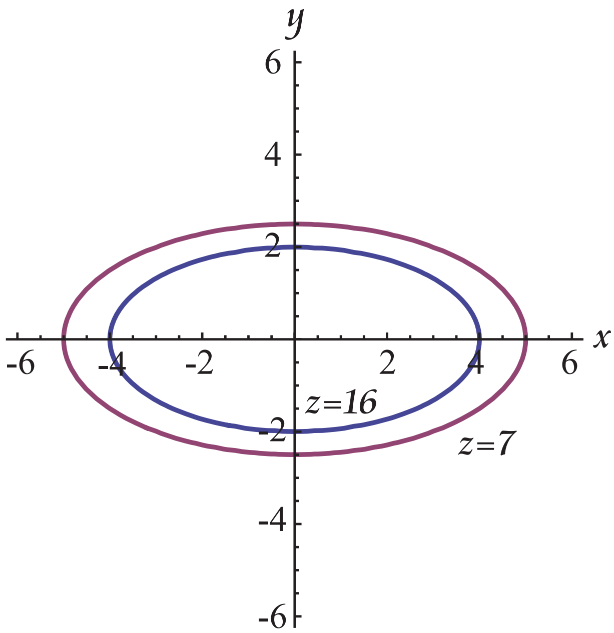

Figure 4.1: The contours for $z = 16$ and $z = 7$

Rate of change along different directions

In this part, we would like to know the rate of change along some directions at some particular points. Indeed, the procedures are very standard.

Step 1. Calculating the gradient

Since we want to know the rate of change, the directional derivatives are what we need. Before evaluating the directional derivatives, it is necessary for us to evaluate the gradient for this surface. Introducing a function $f(x, y) = 32 - x^2 - 4y^2$, the gradient for this scalar function is a vector field:

$$\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} = -2x\hat{\imath} - 8y\hat{\jmath}. \tag{4.4}$$

Step 2. The directional derivatives

The directional derivatives along a direction with unit vector $\hat{n}$ at point P is defined as:

$$D_n [f] |_P = \nabla f|_P \cdot \hat{n}. \tag{4.5}$$

If the directional derivative is positive, the rate of increase along $\hat{n}$ is positive and we say it is uphill. On the other hand, if the value is negative, we know that f decreases along $\hat{n}$ and we say it is downhill.

Some cases

Consider the point $P = (3, 2)$ and the direction we want to study is $\hat{\imath} + \hat{\jmath}$. Using (4.4), the directional derivative is given by:

$$[D_n] |_P = \left(-6\hat{\imath} - 16\hat{\jmath}\right) \cdot \left(\frac{\hat{\imath} + \hat{\jmath}}{\sqrt{2}}\right) = -11\sqrt{2}. \tag{4.6}$$

Therefore, it is downhill with rate of decrease $11\sqrt{2}$.

Next, for the point given by $P = (-3, 1)$ and the direction $4\hat{\imath}+3\hat{\jmath}$. The directional derivative is given by:

$$[D_n] |_P = \left(6\hat{\imath} - 8\hat{\jmath}\right) \cdot \left(\frac{4\hat{\imath} + 3\hat{\jmath}}{5}\right) = 0. \tag{4.7}$$

Therefore, it is neither uphill nor downhill.

Finally, for the point given by $P = (4, -2)$ and the direction $\hat{\imath}+\hat{\jmath}$. The directional derivative is given by:

$$[D_n] |_P = \left(-8\hat{\imath} + 16\hat{\jmath}\right) \cdot \left(\frac{\hat{\imath} + \hat{\jmath}}{\sqrt{2}}\right) = 4\sqrt{2}. \tag{4.8}$$

Therefore, it is uphill with rate of increase $4\sqrt{2}$.

4.2 Divergence

In physics, we always want to know how a vector field flows at a given point. The field lines can flow away from the point if the point is like a source. On the other hand, the field lines can flow into the point if the point is like a sink. Mathematically, we need to know the volume density of the outward flux of the field from an infinitestimal volume around the given point.

To quantify this outgoingness, the concept of divergence has been introduced. For a vector function $\vec{F} = F_x(x,y,z)\,\hat{i} + F_y(x,y,z)\,\hat{j} + F_z(x,y,z)\,\hat{k} = F_i\hat{e}_i$, its **divergence** is given by:

$$\nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{4.9}$$

If the divergence is positive, the field flow away from the point. On the other hand, the flow flow into the point if the divergence is negative.

Example

Given the following vector field:

$$\vec{F} = x\hat{i} + y\hat{j} \tag{4.10}$$

which is plotted in FIG.4.2. We want to determine its divergence. The divergence of the field is given by:

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y = 2 \tag{4.11}$$

Since the divergence of the vector field is positive at every position, so the field is outgoing at every point.

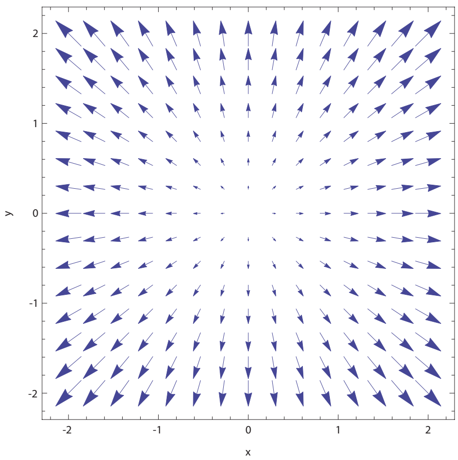

Figure 4.2: The plot of the vector field by Wolfram Mathematica 9, which shows that the field flows away from the point at every position

4.3 Curl

Similar to the case of outgoingness, we want to know how a vector field rotates at a given point. However, the rotation consists of the axis of rotation and the speed of rotation. Therefore, a vector quantity is needed to quantify the problem. Mathematically, we want to determine the circulation density of the vector field at that point. To do so, the concept of curl has been introduced. For a vector function $\vec{F} = F_x(x,y,z)\,\hat{i} + F_y(x,y,z)\,\hat{j} + F_z(x,y,z)\,\hat{k} = F_i\hat{e}_i$, its **curl** is given by:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \tag{4.12}$$

Clearly, this definition gives us a vector function. The direction of the curl specifies the axis of rotation determined by the right-hand grip rule. The magnitude of the curl specifies the speed of rotation.

Example

Consider the vector field:

$$\vec{F} = y\hat{i} - x\hat{j} \tag{4.13}$$

which is plotted as FIG.4.3. We want to determine the curl of the field. The curl of the field is given by:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = -2\hat{k} \tag{4.14}$$

Since the curl of the field is the same at different position, we know that $\vec{F}$ is a uniformly rotating field.

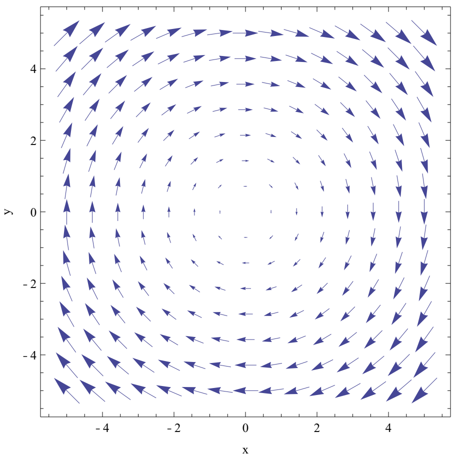

Figure 4.3: The plot of the vector field by Wolfram Mathematica 9, which shows that the field is rotating about the origin with $-\hat{k}$ as the axis of rotation

4.4 The del operator

In the previous sections, we have learnt different operations for differential calculus. Indeed, all those operations can be studied systematically if we introduce the **del operator** ∇. The exact from of the del operator depends on the co-ordinates we are working with.

4.4.1 Cartesian co-ordinates

For the easiest case, we consider the three dimensional Cartesian co-ordinates first. The del operator in Cartesian co-ordinates is defined as:

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} = \frac{\partial}{\partial x_i}\hat{e}_i \tag{4.15}$$

Using (4.15), the gradient of a scalar function f can be viewed as operating ∇ on the function:

$$\nabla f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \tag{4.16}$$

Similarly, the divergence on a vector function $\vec{F}$ can be viewed as the dot product between the del operator and the function:

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(F_x\hat{i} + F_y\hat{j} + F_z\hat{k}\right) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{4.17}$$

Finally, the curl of a vector function can be viewed as the cross product between the del operator and the function:

$$\nabla \times \vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \left(F_x\hat{i} + F_y\hat{j} + F_z\hat{k}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \tag{4.18}$$

This is the reason for the mathematical symbol of gradient, divergence and curl. However, we know that it is a misuse. Since the above procedures are valid for Cartesian co-ordinates only.

4.4.2 Curvilinear co-ordinates

The del operator for other co-ordinate systems are much more difficult. For the polar co-ordinates, we have:

$$\nabla = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} \tag{4.19}$$

For the cylindrical co-ordinates, we have:

$$\nabla = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{\partial}{\partial z}\hat{z} \tag{4.20}$$

and for the spherical co-ordinates, we have:

$$\nabla = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \phi}\hat{\phi} + \frac{1}{r\sin\phi}\frac{\partial}{\partial \theta}\hat{\theta} \tag{4.21}$$

where ϕ and θ are the polar angle and azimuthal angle respectively. The gradient of a function on curvilinear co-ordinates can be still obtained by operating ∇ on the corresponding function. However, the dot product and cross product analogy are invalid for divergence and curl on curvilinear co-ordinates. Indeed, the formulae are much terrible, which can be derived by using chain rule. However, I will not put them down as they can be found easily.

Example: Divergence on spherical co-ordinates

For a vector function $\vec{F}(r, \phi, \theta) = F_r\hat{r} + F_\phi\hat{\phi} + F_\theta\hat{\theta}$, its divergence is given by:

$$\nabla \cdot \vec{F} = \frac{1}{r^2}\frac{\partial}{\partial}\left(r^2 F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\sin\phi F_\phi\right)\frac{1}{r\sin\phi}\frac{\partial F_\theta}{\partial \theta} \tag{4.22}$$

which is clearly not equal to

$$\left[\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \phi}\hat{\phi} + \frac{1}{r\sin\phi}\frac{\partial}{\partial \theta}\hat{\theta}\right] \cdot \left(F_r\hat{r} + F_\phi\hat{\phi} + F_\theta\hat{\theta}\right) \tag{4.23}$$

From this example, you know the reason why the symbol of divergence and curl are misleading.

4.5 Second order differential operators

After studying the first order differential operators, it is normal to move one step further, that is studying the second order by combining two first order operators. At the first glance, there seem to be night possible second order differential operators can be formed from gradient, divergence and curl. However, only four second order differential operators are of interest in vector calculus. I will discuss them one by one. More importantly, some of the combination must give you zero.

4.5.1 Laplacian

The most commonly used second order differential operators must be the Laplacian. The **Laplacian** of a scalar function f is given by:

$$\nabla^2 f = \nabla \cdot (\nabla f)\,, \tag{4.24}$$

that is the divergence of gradient. Particularly, if the function is defined on three-dimensional Cartesian co-ordinates, we have:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{4.25}$$

Example: The Poisson equation

The Laplacian operator is commonly used in electrostatics. From the differential form of Gauss' Law, we have:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{4.26}$$

For electrostatics case, we know that the curl of $\vec{E}$ is zero from Faraday's Law. Hence, we define the electrostatic potential ϕ:

$$\vec{E} = -\nabla \phi \tag{4.27}$$

Therefore, we have for the electrostatic potential:

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0} \tag{4.28}$$

This equation is called the Poisson equation. If the region does not contain any charges, the Poisson equation reduces to the Laplace equation:

$$\nabla^2 \phi = 0. \tag{4.29}$$

4.5.2 Divergence of curl

Another possible combination is the divergence of a curl. However, this second order differential operator must give you zero:

$$\nabla \cdot \left(\nabla \times \vec{F}\right) = 0 \tag{4.30}$$

I will skip the proof here.

4.5.3 Curl of gradient

Another vanishing second order differential operator is the curl of a gradient:

$$\nabla \times (\nabla f) = \vec{0} \tag{4.31}$$

The vanishing for curl of gradient is very important, since it provides the construction of scalar potential for a conservative field, which will be discussed in the next section.

4.5.4 Curl of curl

The last second order differential operators commonly appear in Physics is the curl of curl. Given a vector vector $\vec{F}$, its curl of curl satisfies:

$$\nabla \times \left(\nabla \times \vec{F}\right) = \nabla\left(\nabla \cdot f\right) - \nabla^2 f \tag{4.32}$$

Example: Wave equation

In this example, we want to derive the wave equation from Maxwell's equations. Consider the Faraday's Law:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{4.33}$$

We take curl on both sides, then we get:

$$\nabla \times \left(\nabla \times \vec{E}\right) = -\frac{\partial}{\partial t}\left(\nabla \times \vec{B}\right) \tag{4.34}$$

Using (4.32) and Ampere-Maxwell's Law, we get:

$$\nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right) \tag{4.35}$$

For a region without charges and current, we have:

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \tag{4.36}$$

which is the **equation for electromagnetic wave in vacuum**. The speed of the wave is given by:

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c \tag{4.37}$$

Now, you see the elegance of classical electrodynamics. The conservation of charge, the wave equation are direct consequences of Maxwell's equations. But if you know nothing about vector calculus, you cannot see this!

4.6 Surface Integrals

Given a vector field $\vec{F}$, you may want to know the number of field lines passing through a surface $\mathcal{S}$. The **flux** is defined as:

$$\Phi = \int_{\mathcal{S}} \vec{F} \cdot \mathrm{d}\vec{a}. \tag{4.38}$$

I will use an examples to illustrate the systematic procedures for evaluating surface integrals.

4.6.1 Projection method

You are given the following vector field

$$\vec{F} = yz\hat{j} + z^2\hat{k} \tag{4.39}$$

and you want to evaluate the outward flux of $\vec{F}$ across the surface $\mathcal{S}$ cut from the semicircular cylinder $y^2 + z^2 = 1$, $z > 0$ by the planes $x = 0$ and $x = 1$. The surface is shown in FIG.4.4.

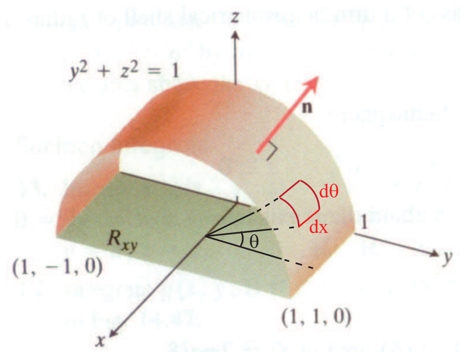

Figure 4.4: Region of the integration

Step 1: The normal vector for the surface

For a general surface $\mathcal{S}$, given in the form $g(x, y, z) = 0$, its normal vector is systematically given by:

$$\hat{n} = \frac{\nabla g}{|\nabla g|} \tag{4.40}$$

For this example, we have $g(x, y, z) = y^2 + z^2 - 1$. Hence:

$$\hat{n} = \pm\frac{2y\hat{j} + 2z\hat{k}}{\sqrt{(2y)^2 + (2z)^2}} = \pm\frac{y\hat{j} + z\hat{k}}{\sqrt{y^2 + z^2}} = y\hat{j} + z\hat{k}. \tag{4.41}$$

In the last step, I have used $y^2 + z^2 = 1$. The positive sign is chosen, since we know that it corresponds to the case of outward normal.

Step 2: Differential area for the surface

If the surface has simple geometric shape, we can write down the differential area directly. For general shape, we can project it on the $x - y$ plane, given by:

$$dA = \frac{|\nabla g|}{\left|\nabla g \cdot \hat{k}\right|}\mathrm{d}x\mathrm{d}y \tag{4.42}$$

For this case, we get $dA = 1/z\mathrm{d}x\mathrm{d}y$.

Step 3: Expressing the field only in x and y

Since the differential area depends on x and y only, we need to rewrite the vector field using x and y only. Using the equation of $\mathcal{S}$, we immediately get $z = \sqrt{1 - y^2}$ since $z > 0$. Then the vector field on $\mathcal{S}$ is rewritten as

$$\vec{F} = y\sqrt{1 - y^2}\hat{j} + \left(1 - y^2\right)\hat{k} \tag{4.43}$$

Step 4: Evaluating the surface integral

Putting all the materials together, we have:

$$\begin{aligned}\int_{\mathcal{S}} \vec{F} \cdot \mathrm{d}\vec{a} &= \int_{\mathcal{R}_{xy}} \left[y\sqrt{1-y^2}\hat{j} + \left(1-y^2\right)\hat{k}\right] \cdot \left(y\hat{j} + \sqrt{1-y^2}\hat{k}\right) \frac{1}{\sqrt{1-y^2}} \mathrm{d}x\mathrm{d}y \\ &= \int_{\mathcal{R}_{xy}} \left[y\sqrt{1-y^2}\hat{j} + \left(1-y^2\right)\hat{k}\right] \cdot \left(y\hat{j} + \sqrt{1-y^2}\hat{k}\right) \frac{1}{\sqrt{1-y^2}} \mathrm{d}x\mathrm{d}y \\ &= \int_{\mathcal{R}_{xy}} \mathrm{d}x\mathrm{d}y \\ &= 2,\end{aligned} \tag{4.44}$$

where $\mathcal{R}_{xy}$ is the projection of $\mathcal{S}$ on the $x-y$ plane. Be very carfeul that the multiple integral is evaluated on $\mathcal{R}_{xy}$, but not on $\mathcal{S}$ since we have projected everything on the $x-y$ plane.

4.6.2 Direct method

Indeed, the above surface integral can be tackled directly, without projection on the $x-y$ plane. It is because $\mathcal{S}$ has a cylindrical shape, such that the differential area can be obtained easily.

Step 1: The normal vector for the surface

The normal vector for the surface is given by:

$$\hat{n} = y\hat{j} + z\hat{k}, \tag{4.45}$$

which has been evaluated in the last section.

Step 2: The differential area for the surface

Referring to FIG.4.4, the differential area (in red) is given by:

$$dA = dx d\theta \tag{4.46}$$

Step 3: Writing the vector field and the normal vector

Since we have used x and θ to represent the surface, it is necessary for us to rewrite the vector field and the normal vector by the same set of parameters. Using θ, we have $y = \cos\theta$ and $z = \sin\theta$. Hence, the vector field is rewritten as:

$$\vec{F} = \sin\theta\cos\theta\hat{j} + \sin^2\theta\hat{k} \tag{4.47}$$

and the normal vector is given by:

$$\hat{n} = \cos\theta\hat{j} + \sin\theta\hat{k} \tag{4.48}$$

Step 4: Evaluating the surface integral

The surface integral can be directly evaluated as:

$$\int_{\mathcal{S}} \vec{F} \cdot \mathrm{d}\vec{A} = \int_0^1 \int_0^\pi \left(\sin\theta\cos\theta\hat{j} + \sin^2\theta\hat{k}\right) \cdot \left(\cos\theta\hat{j} + \sin\theta\hat{k}\right) \mathrm{d}\theta\mathrm{d}x = 2 \tag{4.49}$$

4.7 Line Integrals

When a particle is placed inside a force field, work is needed if we move the particle from one point to another point. Physically, we want to calculate the work done along a given path. In order to study this problem quantitatively, the concept of line integral has been introduced. Given a force field $\vec{F}$, the **work done** along a path $\mathcal{C}$ is given by:

$$W = -\int_{\mathcal{C}} \vec{F} \cdot \mathrm{d}\vec{r} \tag{4.50}$$

Example

You are given the following force field:

$$\vec{F} = \left(y - x^2\right)\hat{\imath} + \left(z - y^2\right)\hat{\jmath} + \left(x - z^2\right)\hat{k}, \tag{4.51}$$

we are asked to evaluate the work done for the particle moving along the path:

$$\mathcal{C} : y = (x-1)^2, \tag{4.52}$$

from $(1,0,0)$ to $(0,1,0)$. As the procedure is very systematic, I will show it clearly here.

Step 1: Parametrizing the path and field

The first step is parametrizing the path by a single parameter. For any point on $\mathcal{C}$, we have:

$$\vec{r} = t\hat{\imath} + (t-1)^2\hat{\jmath}, \tag{4.53}$$

where $\vec{r}$ is the position vector of any point on $\mathcal{C}$ and t runs from 1 to 0. Be very careful on how t changes since the path has direction! Using the parameter, the force field along the path is given by:

$$\vec{F} = \left[(t-1)^2 - t^2\right]\hat{\imath} - (t-1)^4\hat{\jmath} + t\hat{k} = (1-2t)\hat{\imath} - (t-1)^4\hat{\jmath} + t\hat{k} \tag{4.54}$$

Step 2: Writing the differential line element

The differential line element is systematically given by:

$$d\vec{r} = \frac{d\vec{r}}{dt}\mathrm{d}t = \left[\hat{\imath} + 2(t-1)\hat{\jmath}\right]\mathrm{d}t \tag{4.55}$$

Step 3: Evaluating the integral

Using the previous results, the line integral can be written as:

$$W = \int_1^0 \vec{F} \cdot \mathrm{d}\vec{r} = \int_1^0 \left[(1-2t) - 2(t-1)^5\right]\mathrm{d}t = -\frac{1}{3} \tag{4.56}$$

4.8 Divergence Theorem

After learning differential calculus and integral calculus separately, it's time to link them up together. Indeed, the surface integral over a closed surface can be turned into a volume integral through the divergence of the function. For a vector function $\vec{F}$, which has continuous first order partial derivatives. The surface integral of it over a closed orientable surface $\mathcal{S}$ is related to the volume integral on the region $\mathcal{D}$ by the **divergence theorem**:

$$\oint_{\mathcal{S}} \vec{F} \cdot \hat{n} \mathrm{d}\sigma = \int_{\mathcal{V}} \nabla \cdot \vec{F} \mathrm{d}\tau \tag{4.57}$$

where $\mathcal{V}$ is the region bounded by the surface $\mathcal{S}$. Let me take the problem from exercise class week 5 as an example.

Example

Consider the vector field:

$$\vec{V} = (3x - yz)\,\hat{\imath} + \left(z^2 - y^2\right)\hat{\jmath} + \left(2yz + x^2\right)\hat{k}. \tag{4.58}$$

The surface is given by:

$$\mathcal{S} : (x-2)^2 + (y+3)^2 + z^2 = 9. \tag{4.59}$$

We are asked to evaluate the flux of $\vec{V}$ through $\mathcal{S}$.

Step 1: The surface integral

In order to evaluate the flux, it is necessary to evaluate the surface integral:

$$\Phi = \oint_{\mathcal{S}} \vec{V} \cdot \mathrm{d}\vec{a}. \tag{4.60}$$

However, it is extremely tedious for us to evaluate the integral directly. If you do not trust me, do it! In fact, it is doable! Take it as an exercise. Although it is tedious to evaluate the surface integral directly, we can tackle it in an indirect way. It is clear that the surface is closed. Also, the vector field has continuous first order partial derivatives.

Step 2: Divergence Theorem

The divergence of a vector field $\vec{V}$ in Cartesian co-ordinates is defined as:

$$\nabla \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 3, \tag{4.61}$$

which is independent of the position. By divergence theorem, we get:

$$\Phi = \oint_{\mathcal{S}} \vec{V} \cdot \mathrm{d}\vec{a} = \int \left(\nabla \cdot \vec{V}\right) \mathrm{d}\tau = 3 \int \mathrm{d}\tau = 3\left[\frac{4}{3}\pi\,(3)^3\right] = 108\pi \tag{4.62}$$

4.9 Stokes' Theorem

Similar to the case of divergence theorem, the line integral of a vector function $\vec{F}$ over a closed path $\mathcal{C}$ can be turned into a surface integral on $\mathcal{S}$ by the curl of the function. For a vector function $\vec{F}$, which has continuous first order partial derivatives at every point in $\mathcal{S}$, we have the Stokes' theorem:

$$\oint_{\mathcal{C}} \vec{F} \cdot \mathrm{d}\vec{r} = \int_{\mathcal{S}} \left(\nabla \times \vec{F}\right) \cdot \hat{n} \mathrm{d}\sigma \tag{4.63}$$

where $\mathcal{S}$ is any orientable surface having the boundary $\mathcal{C}$.

Example

Consider the following velocity field:

$$\vec{v} = (x - y)\,\hat{\imath} + x\hat{\jmath} \tag{4.64}$$

and you are interested in the circulation of it around the path

$$\mathcal{C} : \vec{r} = \cos t\hat{\imath} + \sin t\hat{\jmath}, \quad 0 \le t \le 2\pi \tag{4.65}$$

It is obvious that $\mathcal{C}$ represents a unit circular loop, centered at the origin. Clearly, the circle enclosed by it is orientable and the velocity field has continuous first order partial derivatives.

Step 1: Curl of the field

The curl of a vector field in Cartesian co-ordinates is defined as:

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x - y & x & 0 \end{vmatrix} = \vec{0} \tag{4.66}$$

Step 2: Stokes' Theorem

By Stokes' Theorem, we have:

$$\oint_{\mathcal{C}} \vec{v} \cdot \mathrm{d}\vec{r} = \int_{\mathcal{S}} (\nabla \times \vec{v}) \cdot \mathrm{d}\vec{a}, \tag{4.67}$$

where $\mathcal{S}$ is the unit circle enclosed by $\mathcal{C}$. The area vector is determined by the **right hand grip rule**. In this case, it is $\hat{k}$. Nevertheless, since $\nabla \times \vec{v} = \vec{0}$, we know that the line integral must vanish.

4.10 Conservative field and potential

Generally, the values of the line integrals are path dependent. The values should be different despite having the same starting and the end points. For example in Physics, the heat absorbed and the hydrostatic work done in Thermodynamics are clearly path dependent. Fortunately, there is a kind of vector function, which the line integral is path-independent. In this section, we will discuss the checking of conservativeness and the construction of the associated potential.

4.10.1 Definition

A vector field is **conservative** if and only if its line integral is path-independent. There are many examples in Physics, like the electrostatic field and the gravitational field. However, using this definition, it is necessary for us to check all possible line integrals before concluding that the force is conservative. Clearly, this procedure is impractical. There are infinitely many paths, how can you check all of them?

4.10.2 Method for checking

Since the line integral is path-independent, its value depends on the starting point and the end point only. Consider the line integral along $\mathcal{C}_1$ and another line integral along the reverse path $\mathcal{C}_2$, we have:

$$\int_{\mathcal{C}_2} \vec{F} \cdot \mathrm{d}\vec{r} = -\int_{\mathcal{C}_1} \vec{F} \cdot \mathrm{d}\vec{r} \tag{4.68}$$

Hence, the line integral along the closed loop $\mathcal{C}$ from by $\mathcal{C}_1$ and $\mathcal{C}_2$ should vanish. If the vector field is well-behaved, we can apply the Stokes' theorem to achieve:

$$\oint_{\mathcal{C}} \vec{F} \cdot \mathrm{d}\vec{r} = \int_{\mathcal{S}} \left(\nabla \times \vec{F}\right) \cdot \hat{n} \mathrm{d}\sigma, \tag{4.69}$$

with $\mathcal{S}$ being the orientable surface with boundary $\mathcal{C}$. Since the line integral vanishes for every closed path, the curl of the vector field should be zero vector. Hence, we can conclude that the curl of a well-behaved conservative field must be zero:

$$\nabla \times \vec{F} = \vec{0} \tag{4.70}$$

This provides an effective way for us to determine if a vector field is conservative or not.

4.10.3 Scalar potential

From the previous section, we see that the curl of a gradient must be zero. As the curl of a conservative field is zero, we can write the field as:

$$\vec{F} = -\nabla\phi, \tag{4.71}$$

which ϕ is called the **scalar potential**. The negative sign is just convention. Physically, we want something flowing from a place with higher potential to a place with lower potential.

4.10.4 Gradient theorem

At the beginning of this section, we know that the line integral of a vector function is generally path dependent. We need to evaluate the line integral by parametrizing the path every time. However, if the vector function is conservative, we can introduce a scalar potential to it. Using the scalar potential, the line integral of a conservative vector function becomes extremely easy. For a conservative vector function $\vec{F}$ with scalar potential $\vec{F} = \nabla\phi$, its line integral can be easily evaluate by the **gradient theorem**:

$$\int_{\mathcal{C}} \vec{F} \cdot \mathrm{d}\vec{r} = \int_{\mathcal{C}} (\nabla\phi) \cdot \mathrm{d}\vec{r} = \phi(B) - \phi(A) \tag{4.72}$$

where A and B is the starting point and end point respectively.

Example

To show you the routine checking of conservative force field and show you the standard procedures for constructing scalar function, let us consider the following force field:

$$\vec{F} = y^2\hat{\imath} + \left(2xy + z^2\right)\hat{\jmath} + 2yz\hat{k} \tag{4.73}$$

First, we need to test the conservativeness of the force field. If the force field is conservative, we need to obtain its corresponding potential function V with the condition that $V(0) = 0$.

1. Checking conservativeness

To show that the force is conservative, we first notice that the force field is differentiable at any point. Then, we try to prove that its curl vanish at every point. Since we are working in Cartesian co-ordinates, the curl is simply given by:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y^2 & 2xy+z^2 & 2yz \end{vmatrix} = \vec{0} \tag{4.74}$$

Therefore, the curl of the field vanishes at every point. Hence, the force is conservative.

2. Construction of scalar potential

Since the force is conservative, we can construct a scalar potential to it. Recall that $\vec{F} = -\nabla V$, so the potential satisfies:

$$-\frac{\partial V}{\partial x} = y^2, \quad -\frac{\partial V}{\partial y} = 2xy + z^2, \quad -\frac{\partial V}{\partial z} = 2yz \tag{4.75}$$

Solving the first equation, we have $V(x, y, z) = -xy^2 + f(y, z)$, where $f(y, z)$ is an arbitrary function of y and z. Substitute into the second and last equation and compare terms, we obtain:

$$\frac{\partial f}{\partial y} = -z^2, \quad \frac{\partial f}{\partial z} = -2yz \tag{4.76}$$

Solving the first equation, we get $f(y, z) = -yz^2 + g(z)$. Substitute into the last equation, we get $dg/dz = 0$. Hence, the potential is given by $V(x, y, z) = -xy^2 - yz^2 + C$, where C is an arbitrary constant. Since $V(0, 0, 0) = 0$, we get $C = 0$. Therefore, the desired potential is $V(x, y, z) = -xy^2 - yz^2$.

3. Work done

If we want to know the work done from the point $(0, 0, 0)$ to $(1, 1, 1)$ along the path $\mathcal{C} : x = y = z$, we can use the gradient theorem:

$$W = V(1, 1, 1) - V(0, 0, 0) = -2 \tag{4.77}$$

Bibliography

[1] D. J. Griffiths. *Introduction to Electrodynamics*. See Chapter 1.