and $f'''(0) = -1$ miles/min^3. Predict the location of the plane at time $t = 2$ min.

52. Suppose that an astronaut is at (0, 0) and the moon is represented by a circle of radius 1 centered at (10, 5). The astronaut's capsule follows a path $y = f(x)$ with current position $f(0) = 0$, slope $f'(0) = 1/5$, concavity $f''(0) = -1/10$, $f'''(0) = 1/25$, $f^{(4)}(0) = 1/25$ and $f^{(5)}(0) = -1/50$. Graph a Taylor polynomial approximation of $f(x)$. Based on your current information, do you advise the astronaut to change paths? How confident are you in the accuracy of your approximation?

53. Find the Taylor series for e^x about a general center c.

54. Find the Taylor series for $\sqrt{x}$ about a general center $c = a^2$.

Exercises 55–58 involve the binomial expansion.

55. Show that the Maclaurin series for $(1+x)^r$ is $1 + \sum_{k=1}^{\infty} \frac{r(r-1)\cdots(r-k+1)}{k!} x^k$, for any constant r.

56. Simplify the series in exercise 55 for $r = 2$; $r = 3$; r is a positive integer.

57. Use the result of exercise 55 to write out the Maclaurin series for $f(x) = \sqrt{1+x}$.

58. Use the result of exercise 55 to write out the Maclaurin series for $f(x) = (1+x)^{3/2}$.

59. Find the Maclaurin series of $f(x) = \cosh x$ and $f(x) = \sinh x$. Compare to the Maclaurin series of $\cos x$ and $\sin x$.

60. Use the Maclaurin series for $\tan x$ and the result of exercise 59 to conjecture the Maclaurin series for $\tanh x$.

EXPLORATORY EXERCISES

1. Almost all of our series results apply to series of complex numbers. Defining $i = \sqrt{-1}$, show that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ and so on. Replacing x with ix in the Maclaurin series for e^x, separate terms containing i from those that don't contain i (after the simplifications indicated above) and derive **Euler's formula:** $e^{ix} = \cos x + i \sin x$.

2. Using the technique of exercise 1, show that $\cos(ix) = \cosh x$ and $\sin(ix) = i \sinh x$. That is, the trig functions and their hyperbolic counterparts are closely related as functions of complex variables.

3. The method used in examples 7.3, 7.5, 7.6 and 7.7 does not require us to actually find $R_n(x)$, but to approximate it with a worst-case bound. Often this approximation is fairly close to $R_n(x)$, but this is not always true. As an extreme example of this, show that the bound on $R_n(x)$ for $f(x) = \ln x$ about $c = 1$ (see exercise 23) increases without bound for $0 < x < \frac{1}{2}$. Explain why this does not necessarily mean that the actual error increases without bound. In fact, $R_n(x) \to 0$ for $0 < x < \frac{1}{2}$ but we must show this using some other method. Use integration of an appropriate power series to show that $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k}$ converges to $\ln x$ for $0 < x < \frac{1}{2}$.

4. Verify numerically that if a_1 is close to π, the sequence $a_{n+1} = a_n + \sin a_n$ converges to π. (In other words, if a_n is an approximation of π, then $a_n + \sin a_n$ is a better approximation.) To prove this, find the Taylor series for $\sin x$ about $c = \pi$. Use this to show that if $\pi < a_n < 2\pi$, then $\pi < a_{n+1} < a_n$. Similarly, show that if $0 < a_n < \pi$, then $a_n < a_{n+1} < \pi$.

9.8 APPLICATIONS OF TAYLOR SERIES

In section 9.7, we developed the concept of a Taylor series expansion and gave many illustrations of how to compute these. In this section, we expand on our earlier presentation, by giving a few examples of how Taylor series are used to approximate the values of transcendental functions, evaluate limits and integrals and define important new functions. These represent but a small sampling of the important applications of Taylor series.

First, consider how calculators and computers might calculate values of transcendental functions, such as sin(1.234567). We illustrate this in example 8.1.

EXAMPLE 8.1 Using Taylor Polynomials to Approximate a Sine Value

Use a Taylor series to approximate sin(1.234567) accurate to within 10^{-11}.

Solution In section 9.7, we left it as an exercise to show that the Taylor series expansion for $f(x) = \sin x$ about $x = 0$ is

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots,$$

where the interval of convergence is $(-\infty, \infty)$. Notice that if we take $x = 1.234567$, the series representation of $\sin 1.234567$ is

$$\sin 1.234567 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(1.234567)^{2k+1},$$

which is an alternating series. We can use a partial sum of this series to approximate the desired value, but how many terms will we need for the desired accuracy? Recall that for alternating series, the error in a partial sum is bounded by the absolute value of the first neglected term. (Note that you could also use the remainder term from Taylor's Theorem to bound the error.) To ensure that the error is less than 10^{-11}, we must find an integer k such that $\frac{1.234567^{2k+1}}{(2k+1)!} < 10^{-11}$. By trial and error, we find that

$$\frac{1.234567^{17}}{17!} \approx 1.010836 \times 10^{-13} < 10^{-11},$$

so that $k = 8$ will do. This says that the first neglected term corresponds to $k = 8$ and so, we compute the partial sum

$$\begin{aligned}\sin 1.234567 &\approx \sum_{k=0}^{7} \frac{(-1)^k}{(2k+1)!}(1.234567)^{2k+1} \\ &= 1.234567 - \frac{1.234567^3}{3!} + \frac{1.234567^5}{5!} - \frac{1.234567^7}{7!} + \cdots - \frac{1.234567^{15}}{15!} \\ &\approx 0.94400543137.\end{aligned}$$

Check your calculator or computer to verify that this matches your calculator's estimate. ■

In example 8.1, while we produced an approximation with the desired accuracy, we did not do this in the most efficient fashion, as we simply grabbed the most handy Taylor series expansion of $f(x) = \sin x$. You should try to resist the impulse to automatically use the Taylor series expansion about $x = 0$ (i.e., the Maclaurin series), rather than making a more efficient choice. We illustrate this in example 8.2.

EXAMPLE 8.2 Choosing a More Appropriate Taylor Series Expansion

Repeat example 8.1, but this time, make a more appropriate choice of the Taylor series.

Solution Recall that Taylor series converge much faster close to the point about which you expand, than they do far away. Given this and the fact that we know the exact value of $\sin x$ at only a few points, you should quickly recognize that a series expanded about $x = \frac{\pi}{2} \approx 1.57$ is a better choice for computing $\sin 1.234567$ than one expanded about $x = 0$. (Another reasonable choice is the Taylor series expansion about $x = \frac{\pi}{3}$.) In example 7.5, recall that we had found that

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(x - \frac{\pi}{2}\right)^{2k} = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \cdots,$$

where the interval of convergence is $(-\infty, \infty)$. Taking $x = 1.234567$ gives us

$$\sin 1.234567 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(1.234567 - \frac{\pi}{2}\right)^{2k}$$
$$= 1 - \frac{1}{2}\left(1.234567 - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(1.234567 - \frac{\pi}{2}\right)^4 - \cdots,$$

which is again an alternating series. Using the remainder term from Taylor's Theorem to bound the error, we have that

$$|R_n(1.234567)| = \left|\frac{f^{(2n+2)}(z)}{(2n+2)!}\right|\left|1.234567 - \frac{\pi}{2}\right|^{2n+2}$$
$$\le \frac{\left|1.234567 - \frac{\pi}{2}\right|^{2n+2}}{(2n+2)!}.$$

(Note that we get the same error bound if we use the error bound for an alternating series.) By trial and error, you can find that

$$\frac{\left|1.234567 - \frac{\pi}{2}\right|^{2n+2}}{(2n+2)!} < 10^{-11}$$

for $n = 4$, so that an approximation with the required degree of accuracy is

$$\sin 1.234567 \approx \sum_{k=0}^{4} \frac{(-1)^k}{(2k)!}\left(1.234567 - \frac{\pi}{2}\right)^{2k}$$
$$= 1 - \frac{1}{2}\left(1.234567 - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(1.234567 - \frac{\pi}{2}\right)^4$$
$$- \frac{1}{6!}\left(1.234567 - \frac{\pi}{2}\right)^6 + \frac{1}{8!}\left(1.234567 - \frac{\pi}{2}\right)^8$$
$$\approx 0.94400543137.$$

Compare this result to example 8.1, where we needed many more terms of the Taylor series to obtain the same degree of accuracy. ■

We can also use Taylor series to quickly conjecture the value of difficult limits. Be careful, though: the theory of when these conjectures are guaranteed to be correct is beyond the level of this text. However, we can certainly obtain helpful hints about certain limits.

EXAMPLE 8.3 Using Taylor Polynomials to Conjecture the Value of a Limit

Use Taylor series to conjecture $\lim_{x \to 0} \frac{\sin x^3 - x^3}{x^9}$.

Solution Again recall that the Maclaurin series for $\sin x$ is

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots,$$

where the interval of convergence is $(-\infty, \infty)$. Substituting x^3 for x gives us

$$\sin x^3 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(x^3)^{2k+1} = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots.$$

This gives us

$$\frac{\sin x^3 - x^3}{x^9} = \frac{\left(x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots\right) - x^3}{x^9} = -\frac{1}{3!} + \frac{x^6}{5!} + \cdots$$

and so, we conjecture that

$$\lim_{x\to 0} \frac{\sin x^3 - x^3}{x^9} = -\frac{1}{3!} = -\frac{1}{6}.$$

You can verify that this limit is correct using l'Hôpital's Rule (three times, simplifying each time). ■

Since Taylor polynomials are used to approximate functions on a given interval and since polynomials are easy to integrate, we can use a Taylor polynomial to obtain an approximation of a definite integral. It turns out that such an approximation is often better than that obtained from the numerical methods developed in section 4.7. We illustrate this in example 8.4.

EXAMPLE 8.4 Using Taylor Series to Approximate a Definite Integral

Use a Taylor polynomial with $n = 8$ to approximate $\int_{-1}^{1} \cos(x^2)\, dx$.

Solution Since we do not know an antiderivative of $\cos(x^2)$, we must rely on a numerical approximation of the integral. Since we are integrating on the interval $(-1, 1)$, a Maclaurin series expansion (i.e., a Taylor series expansion about $x = 0$) is a good choice. We have

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots,$$

which converges on all of $(-\infty, \infty)$. Replacing x by x^2 gives us

$$\cos(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{4k} = 1 - \frac{1}{2}x^4 + \frac{1}{4!}x^8 - \frac{1}{6!}x^{12} + \cdots,$$

so that

$$\cos(x^2) \approx 1 - \frac{1}{2}x^4 + \frac{1}{4!}x^8.$$

This leads us to the approximation

$$\begin{aligned}\int_{-1}^{1} \cos(x^2)\, dx &\approx \int_{-1}^{1} \left(1 - \frac{1}{2}x^4 + \frac{1}{4!}x^8\right) dx \\ &= \left(x - \frac{x^5}{10} + \frac{x^9}{216}\right)\Bigg|_{x=-1}^{x=1} \\ &= \frac{977}{540} \approx 1.809259.\end{aligned}$$

Our CAS gives us $\int_{-1}^{1} \cos(x^2)\, dx \approx 1.809048$, so our approximation appears to be very accurate. ■

You might reasonably argue that we don't need Taylor series to obtain approximations like those in example 8.4, as you could always use other, simpler numerical methods like Simpson's Rule to do the job. That's often true, but just try to use Simpson's Rule on the integral in example 8.5.

EXAMPLE 8.5 Using Taylor Series to Approximate the Value of an Integral

Use a Taylor polynomial with $n = 5$ to approximate $\int_{-1}^{1} \frac{\sin x}{x}\, dx$.

Solution Note that you do not know an antiderivative of $\frac{\sin x}{x}$. Further, while the integrand is discontinuous at $x = 0$, this does *not* need to be treated as an improper integral, since $\lim_{x\to 0} \frac{\sin x}{x} = 1$. (This says that the integrand has a removable discontinuity at $x = 0$.) From the first few terms of the Maclaurin series for $f(x) = \sin x$, we have the Taylor polynomial approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!},$$

so that

$$\frac{\sin x}{x} \approx 1 - \frac{x^2}{3!} + \frac{x^4}{5!}.$$

Consequently,

$$\begin{aligned}\int_{-1}^{1} \frac{\sin x}{x}\, dx &\approx \int_{-1}^{1} \left(1 - \frac{x^2}{6} + \frac{x^4}{120}\right) dx \\ &= \left(x - \frac{x^3}{18} + \frac{x^5}{600}\right)\Bigg|_{x=-1}^{x=1} \\ &= \left(1 - \frac{1}{18} + \frac{1}{600}\right) - \left(-1 + \frac{1}{18} - \frac{1}{600}\right) \\ &= \frac{1703}{900} \approx 1.89222.\end{aligned}$$

Our CAS gives us $\int_{-1}^{1} \frac{\sin x}{x}\, dx \approx 1.89216$, so our approximation is quite good. On the other hand, if you try to apply Simpson's Rule or Trapezoidal Rule, the algorithm will not work, as they will attempt to evaluate $\frac{\sin x}{x}$ at $x = 0$. ■

While you have now calculated Taylor series expansions of many familiar functions, many other functions are actually *defined* by a power series. These include many functions in the very important class of **special functions** that frequently arise in physics and engineering applications. One important family of special functions are the Bessel functions, which arise in the study of fluid mechanics, acoustics, wave propagation and other areas of applied mathematics. The **Bessel function of order *p*** is defined by the power series

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+p}}{2^{2k+p} k!(k+p)!}, \tag{8.1}$$

for nonnegative integers p. Bessel functions arise in the solution of the differential equation $x^2 y'' + xy' + (x^2 - p^2)y = 0$. In examples 8.6 and 8.7, we explore several interesting properties of Bessel functions.

EXAMPLE 8.6 The Radius of Convergence of a Bessel Function

Find the radius of convergence for the series defining the Bessel function $J_0(x)$.

Solution From equation (8.1) with $p = 0$, we have $J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}$. The Ratio Test gives us

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{x^{2k+2}}{2^{2k+2}[(k+1)!]^2}\frac{2^{2k}(k!)^2}{x^{2k}}\right| = \lim_{k\to\infty}\left|\frac{x^2}{4(k+1)^2}\right| = 0 < 1,$$

for all x. The series then converges absolutely for all x and so, the radius of convergence is ∞. ■

In example 8.7, we explore an interesting relationship between the zeros of two Bessel functions.

EXAMPLE 8.7 The Zeros of Bessel Functions

Verify graphically that on the interval [0, 10], the zeros of $J_0(x)$ and $J_1(x)$ alternate.

Solution Unless you have a CAS with these Bessel functions available as built-in functions, you will need to graph partial sums of the defining series:

$$J_0(x) \approx \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2} \quad \text{and} \quad J_1(x) \approx \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!}.$$

Before graphing these, you must first determine how large n should be in order to produce a reasonable graph. Notice that for each fixed $x > 0$, both of the defining series are alternating series. Consequently, the error in using a partial sum to approximate the function is bounded by the first neglected term. That is,

$$\left|J_0(x) - \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}\right| \le \frac{x^{2n+2}}{2^{2n+2}[(n+1)!]^2}$$

and

$$\left|J_1(x) - \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!}\right| \le \frac{x^{2n+3}}{2^{2n+3}(n+1)!(n+2)!},$$

with the maximum error in each occurring at $x = 10$. Notice that for $n = 12$, we have that

$$\left|J_0(x) - \sum_{k=0}^{12} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}\right| \le \frac{x^{2(12)+2}}{2^{2(12)+2}[(12+1)!]^2} \le \frac{10^{26}}{2^{26}(13!)^2} < 0.04$$

and

$$\left|J_1(x) - \sum_{k=0}^{12} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!}\right| \le \frac{x^{2(12)+3}}{2^{2(12)+3}(12+1)!(12+2)!} \le \frac{10^{27}}{2^{27}(13!)(14!)} < 0.04.$$

So, in either case, using a partial sum with $n = 12$ results in an approximation that is within 0.04 of the correct value for each x in the interval [0, 10]. This is plenty of accuracy for our present purposes. Figure 9.43 shows graphs of partial sums with $n = 12$ for $J_0(x)$ and $J_1(x)$.

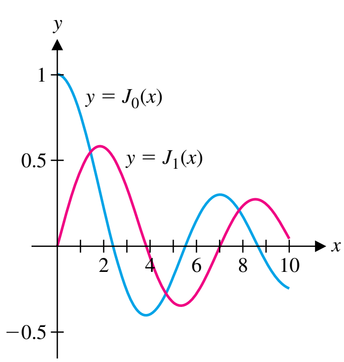

FIGURE 9.43
$y = J_0(x)$ and $y = J_1(x)$

Notice that $J_1(0) = 0$ and in the figure, you can clearly see that $J_0(x) = 0$ at about $x = 2.4$, $J_1(x) = 0$ at about $x = 3.9$, $J_0(x) = 0$ at about $x = 5.6$, $J_1(x) = 0$ at about $x = 7.0$ and $J_0(x) = 0$ at about $x = 8.8$. From this, it is now apparent that the zeros of $J_0(x)$ and $J_1(x)$ do indeed alternate on the interval [0, 10]. ■

It turns out that the result of example 8.7 generalizes to any interval of positive numbers and any two Bessel functions of consecutive order. That is, between consecutive zeros of

$J_p(x)$ is a zero of $J_{p+1}(x)$ and between consecutive zeros of $J_{p+1}(x)$ is a zero of $J_p(x)$. We explore this further in the exercises.

The Binomial Series

You are already familiar with the Binomial Theorem, which states that for any positive integer n,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \cdots + nab^{n-1} + b^n.$$

We often write this as
$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k,$$

where we use the shorthand notation $\binom{n}{k}$ to denote the binomial coefficient, defined by

$$\binom{n}{0} = 1, \quad \binom{n}{1} = n, \quad \binom{n}{2} = \frac{n(n-1)}{2} \quad \text{and}$$
$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}, \quad \text{for } k \geq 3.$$

For the case where $a = 1$ and $b = x$, the Binomial Theorem simplifies to

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Newton discovered that this result could be extended to include values of n other than positive integers. What resulted is a special type of power series known as the *binomial series,* which has important applications in statistics and physics. We begin by deriving the Maclaurin series for $f(x) = (1+x)^n$, for some constant $n \neq 0$. Computing derivatives and evaluating these at $x = 0$, we have

$$\begin{aligned} f(x) &= (1+x)^n & f(0) &= 1 \\ f'(x) &= n(1+x)^{n-1} & f'(0) &= n \\ f''(x) &= n(n-1)(1+x)^{n-2} & f''(0) &= n(n-1) \\ &\vdots & &\vdots \\ f^{(k)}(x) &= n(n-1)\cdots(n-k+1) & f^{(k)}(0) &= n(n-1)\cdots(n-k+1). \end{aligned}$$

We call the resulting Maclaurin series the **binomial series,** given by

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k &= 1 + nx + n(n-1)\frac{x^2}{2!} + \cdots + n(n-1)\cdots(n-k+1)\frac{x^k}{k!} + \cdots \\ &= \sum_{k=0}^{\infty} \binom{n}{k} x^k. \end{aligned}$$

From the Ratio Test, we have

$$\begin{aligned} \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty} \left|\frac{n(n-1)\cdots(n-k+1)(n-k)x^{k+1}}{(k+1)!} \frac{k!}{n(n-1)\cdots(n-k+1)x^k}\right| \\ &= |x| \lim_{k\to\infty} \frac{|n-k|}{k+1} = |x|, \end{aligned}$$

so that the binomial series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. By showing that the remainder term $R_k(x)$ tends to zero as $k \to \infty$, we can confirm that the binomial series converges to $(1+x)^n$ for $|x| < 1$. We state this formally in Theorem 8.1.

THEOREM 8.1 (Binomial Series)

For any real number r, $(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k$, for $-1 < x < 1$.

As seen in the exercises, for some values of the exponent r, the binomial series also converges at one or both of the endpoints $x = \pm 1$.

EXAMPLE 8.8 Using the Binomial Series

Using the binomial series, find a Maclaurin series for $f(x) = \sqrt{1+x}$ and use it to approximate $\sqrt{17}$ accurate to within 0.000001.

Solution From the binomial series with $r = \frac{1}{2}$, we have

$$\sqrt{1+x} = (1+x)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} x^k = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \cdots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots,$$

for $-1 < x < 1$. To use this to approximate $\sqrt{17}$, we first rewrite it in a form involving $\sqrt{1+x}$, for $-1 < x < 1$. Observe that we can do this by writing

$$\sqrt{17} = \sqrt{16 \cdot \frac{17}{16}} = 4\sqrt{\frac{17}{16}} = 4\sqrt{1 + \frac{1}{16}}.$$

Since $x = \frac{1}{16}$ is in the interval of convergence, $-1 < x < 1$, the binomial series gives us

$$\sqrt{17} = 4\sqrt{1+\frac{1}{16}} = 4\left[1 + \frac{1}{2}\left(\frac{1}{16}\right) - \frac{1}{8}\left(\frac{1}{16}\right)^2 + \frac{1}{16}\left(\frac{1}{16}\right)^3 - \frac{5}{128}\left(\frac{1}{16}\right)^4 + \cdots\right].$$

Since this is an alternating series, the error in using the first n terms to approximate the sum is bounded by the first neglected term. So, if we use only the first three terms of the series, the error is bounded by $\frac{1}{16}\left(\frac{1}{16}\right)^3 \approx 0.000015 > 0.000001$. Similarly, if we use the first four terms of the series to approximate the sum, the error is bounded by $\frac{5}{128}\left(\frac{1}{16}\right)^4 \approx 0.0000006 < 0.000001$, as desired. So, we can achieve the desired accuracy by summing the first four terms of the series:

$$\sqrt{17} \approx 4\left[1 + \frac{1}{2}\left(\frac{1}{16}\right) - \frac{1}{8}\left(\frac{1}{16}\right)^2 + \frac{1}{16}\left(\frac{1}{16}\right)^3\right] \approx 4.1231079,$$

where this approximation is accurate to within the desired accuracy. ■

EXERCISES 9.8

WRITING EXERCISES

1. In example 8.2, we showed that an expansion about $x = \frac{\pi}{2}$ is more accurate for approximating $\sin(1.234567)$ than an expansion about $x = 0$ with the same number of terms. Explain why an expansion about $x = 1.2$ would be even more efficient, but is not practical.

2. Assuming that you don't need to rederive the Maclaurin series for $\cos x$, compare the amount of work done in example 8.4 to the work needed to compute a Simpson's Rule approximation with $n = 16$.

3. In equation (8.1), we defined the Bessel functions as series. This may seem like a convoluted way of defining a function, but compare the levels of difficulty doing the following with a Bessel function versus $\sin x$: computing $f(0)$, computing $f(1.2)$, evaluating $f(2x)$, computing $f'(x)$, computing $\int f(x)\,dx$ and computing $\int_0^1 f(x)\,dx$.

4. Discuss how you might estimate the error in the approximation of example 8.4.

In exercises 1–6, use an appropriate Taylor series to approximate the given value, accurate to within 10^{-11}.

1. $\sin 1.61$
2. $\sin 6.32$
3. $\cos 0.34$
4. $\cos 3.04$
5. $e^{-0.2}$
6. $e^{0.4}$

In exercises 7–12, use a known Taylor series to conjecture the value of the limit.

7. $\lim\limits_{x\to 0} \dfrac{\cos x^2 - 1}{x^4}$
8. $\lim\limits_{x\to 0} \dfrac{\sin x^2 - x^2}{x^6}$
9. $\lim\limits_{x\to 1} \dfrac{\ln x - (x-1)}{(x-1)^2}$
10. $\lim\limits_{x\to 0} \dfrac{\tan^{-1} x - x}{x^3}$
11. $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x}$
12. $\lim\limits_{x\to 0} \dfrac{e^{-2x} - 1}{x}$

In exercises 13–18, use a known Taylor polynomial with n nonzero terms to estimate the value of the integral.

13. $\displaystyle\int_{-1}^{1} \frac{\sin x}{x}\,dx,\ n = 3$
14. $\displaystyle\int_{-\sqrt{\pi}}^{\sqrt{\pi}} \cos x^2\,dx,\ n = 4$
15. $\displaystyle\int_{-1}^{1} e^{-x^2}\,dx,\ n = 5$
16. $\displaystyle\int_{0}^{1} \tan^{-1} x\,dx,\ n = 5$
17. $\displaystyle\int_{1}^{2} \ln x\,dx,\ n = 5$
18. $\displaystyle\int_{0}^{1} e^{\sqrt{x}}\,dx,\ n = 4$

19. Find the radius of convergence of $J_1(x)$.

20. Find the radius of convergence of $J_2(x)$.

21. Find the number of terms needed to approximate $J_2(x)$ within 0.04 for x in the interval [0, 10].

22. Show graphically that the zeros of $J_1(x)$ and $J_2(x)$ alternate on the interval (0, 10].

23. Einstein's theory of relativity states that the mass of an object traveling at velocity v is $m(v) = m_0/\sqrt{1 - v^2/c^2}$, where m_0 is the rest mass of the object and c is the speed of light. Show that $m \approx m_0 + \left(\dfrac{m_0}{2c^2}\right)v^2$. Use this approximation to estimate how large v would need to be to increase the mass by 10%.

24. Find the fourth-degree Taylor polynomial expanded about $v = 0$, for $m(v)$ in exercise 23.

25. The weight (force due to gravity) of an object of mass m and altitude x miles above the surface of the earth is $w(x) = \dfrac{mgR^2}{(R+x)^2}$, where R is the radius of the earth and g is the acceleration due to gravity. Show that $w(x) \approx mg(1 - 2x/R)$. Estimate how large x would need to be to reduce the weight by 10%.

26. Find the second-degree Taylor polynomial for $w(x)$ in exercise 25. Use it to estimate how large x needs to be to reduce the weight by 10%.

27. Based on your answers to exercises 25 and 26, is weight significantly different at a high-altitude location (e.g., 7500 ft) compared to sea level?

28. The radius of the earth is up to 300 miles larger at the equator than it is at the poles. Which would have a larger effect on weight, altitude or latitude?

In exercises 29–32, use the Maclaurin series expansion $\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \cdots$.

29. The tangential component of the space shuttle's velocity during reentry is approximately $v(t) = v_c \tanh\left(\dfrac{g}{v_c}t + \tanh^{-1}\dfrac{v_0}{v_c}\right)$, where v_0 is the velocity at time 0 and v_c is the terminal velocity (see Long and Weiss, *The American Mathematical Monthly,* February 1999). If $\tanh^{-1}\dfrac{v_0}{v_c} = \dfrac{1}{2}$, show that $v(t) \approx gt + \dfrac{1}{2}v_c$. Is this estimate of $v(t)$ too large or too small?

30. Show that in exercise 29, $v(t) \to v_c$ as $t \to \infty$. Use the approximation in exercise 29 to estimate the time needed to reach 90% of the terminal velocity.

31. The downward velocity of a sky diver of mass m is $v(t) = \sqrt{40mg}\ \tanh\left(\sqrt{\dfrac{g}{40m}}\,t\right)$. Show that $v(t) \approx gt - \dfrac{g^2}{120m}t^3$.

32. The velocity of a water wave of length L in water of depth h satisfies the equation $v^2 = \dfrac{gL}{2\pi}\tanh\dfrac{2\pi h}{L}$. Show that $v \approx \sqrt{gh}$.

In exercises 33–36, use the Binomial Theorem to find the first five terms of the Maclaurin series.

33. $f(x) = \dfrac{1}{\sqrt{1-x}}$
34. $f(x) = \sqrt[3]{1+2x}$
35. $f(x) = \dfrac{6}{\sqrt[3]{1+3x}}$
36. $f(x) = (1+x^2)^{4/5}$

In exercises 37 and 38, use the Binomial Theorem to approximate the value to within 10^{-6}.

37. (a) $\sqrt{26}$ (b) $\sqrt{24}$
38. (a) $\dfrac{2}{\sqrt[3]{9}}$ (b) $\sqrt[4]{17}$

39. Apply the Binomial Theorem to $(x+4)^3$ and $(1-2x)^4$. Determine the number of nonzero terms in the binomial expansion for any positive integer n.

40. If n and k are positive integers with $n > k$, show that $\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

41. Use exercise 33 to find the Maclaurin series for $\dfrac{1}{\sqrt{1-x^2}}$ and use it to find the Maclaurin series for $\sin^{-1} x$.

42. Use the Binomial Theorem to find the Maclaurin series for $(1+2x)^{4/3}$ and compare this series to that of exercise 34.

43. Use a Taylor polynomial to estimate $\int_0^{\pi} \frac{\sin x}{x}\,dx$ accurate to within 0.00001. (This value will be used in the next section.)

44. Use a Taylor polynomial to conjecture the value of $\lim_{x\to 0} \dfrac{e^x + e^{-x} - 2}{x^2}$ and then confirm your conjecture using l'Hôpital's Rule.

45. The energy density of electromagnetic radiation at wavelength λ from a black body at temperature T (degrees Kelvin) is given by **Planck's law** of black body radiation: $f(\lambda) = \dfrac{8\pi hc}{\lambda^5(e^{hc/\lambda kT} - 1)}$, where h is Planck's constant, c is the speed of light and k is Boltzmann's constant. To find the wavelength of peak emission, maximize $f(\lambda)$ by minimizing $g(\lambda) = \lambda^5(e^{hc/\lambda kT} - 1)$. Use a Taylor polynomial for e^x with $n = 7$ to expand the expression in parentheses and find the critical number of the resulting function. (Hint: Use $\frac{hc}{k} \approx 0.014$.) Compare this to **Wien's law:** $\lambda_{\max} = \dfrac{0.002898}{T}$. Wien's law is accurate for small λ. Discuss the flaw in our use of Maclaurin series.

46. Use a Taylor polynomial for e^x to expand the denominator in Planck's law of exercise 45 and show that $f(\lambda) \approx \dfrac{8\pi kT}{\lambda^4}$. State whether this approximation is better for small or large wavelengths λ. This is known in physics as the **Rayleigh-Jeans law.**

47. The power of a reflecting telescope is proportional to the surface area S of the parabolic reflector, where $S = \dfrac{8\pi}{3}c^2\left[\left(\dfrac{d^2}{16c^2}+1\right)^{3/2} - 1\right]$. Here, d is the diameter of the parabolic reflector, which has depth k with $c = \dfrac{d^2}{4k}$. Expand the term $\left(\dfrac{d^2}{16c^2}+1\right)^{3/2}$ and show that if $\dfrac{d^2}{16c^2}$ is small, then $S \approx \dfrac{\pi d^2}{4}$.

48. A disk of radius a has a charge of constant density σ. Point P lies at a distance r directly above the disk. The **electrical potential** at point P is given by $V = 2\pi\sigma(\sqrt{r^2+a^2} - r)$. Show that for large r, $V \approx \dfrac{\pi a^2 \sigma}{r}$.

EXPLORATORY EXERCISES

1. The Bessel functions and **Legendre polynomials** are examples of the so-called special functions. For nonnegative integers n, the Legendre polynomials are defined by

$$P_n(x) = 2^{-n} \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{(n-k)!\,k!\,(n-2k)!} x^{n-2k}.$$

Here, $[n/2]$ is the greatest integer less than or equal to $n/2$ (for example, $[1/2] = 0$ and $[2/2] = 1$). Show that $P_0(x) = 1$, $P_1(x) = x$ and $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$. Show that for these three functions,

$$\int_{-1}^{1} P_m(x) P_n(x)\,dx = 0, \quad \text{for } m \neq n.$$

This fact, which is true for all Legendre polynomials, is called the **orthogonality condition.** Orthogonal functions are commonly used to provide simple representations of complicated functions.

2. Use the Ratio Test to show that the radius of convergence of $\sum_{k=0}^{\infty} \binom{n}{k} x^k$ is 1. (a) If $n \le -1$, show that the interval of convergence is $(-1, 1)$. (b) If $n > 0$ and n is not an integer, show that the interval of convergence is $[-1, 1]$. (c) If $-1 < n < 0$, show that the interval of convergence is $(-1, 1]$.

3. Suppose that p is an approximation of π with $|p - \pi| < 0.001$. Explain why p has at least two digits of accuracy and has a decimal expansion that starts $p = 3.14\ldots$. Use Taylor's Theorem to show that $p + \sin p$ has six digits of accuracy. In general, if p has n digits of accuracy, show that $p + \sin p$ has $3n$ digits of accuracy. Compare this to the accuracy of $p - \tan p$.

9.9 FOURIER SERIES

Many phenomena we encounter in the world around us are periodic in nature. That is, they repeat themselves over and over again. For instance, light, sound, radio waves and x-rays are all periodic. For such phenomena, Taylor polynomial approximations have shortcomings. As x gets farther away from c (the point about which you expanded), the difference between the function and a given Taylor polynomial grows. Such behavior is illustrated in Figure 9.44 for the case of $f(x) = \sin x$ expanded about $x = \frac{\pi}{2}$.