Physics Enhancement Programme

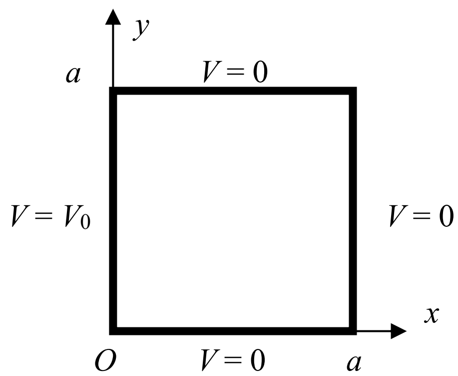
Phase 1 Selection Test 2

26 November 2016

[Time duration: 3 hours, Total points: 100]

1. **[12 points]**
   A uniformly charged sphere with radius R and volume charge density ρ has its center located at the origin. Outside the sphere there is an isolated spherical conducting shell. The shell carries no net charge and is also centered at the origin, with inner radius a and outer radius b, where $b > a > R$. Take infinity as the reference point. Evaluate the potential in all regions.

2. **[8 points]**
   Consider a very long cylinder with square cross-section as shown below.

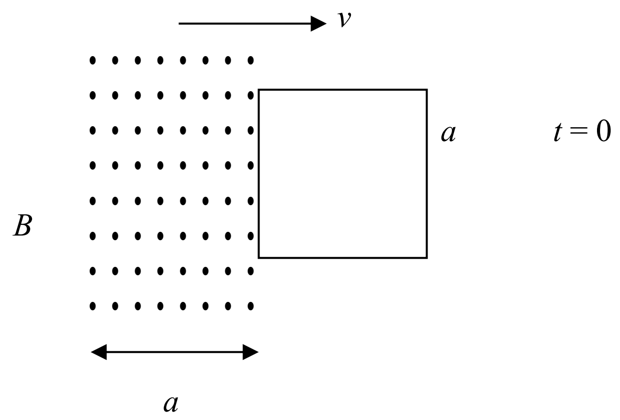


   The three plates $y = 0$, $y = a$, and $x = a$ are grounded, while the left plate $x = 0$ is kept at a constant potential V_0. The region inside the cylinder is charge-free. By symmetry and superposition, find the potential at the center of the square.

3. **[6 points]**
   The Maxwell's equations read

$$\begin{cases} \nabla \cdot \mathbf{E} = \rho / \varepsilon_0 & \text{Gauss' law} \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} & \text{Faraday's law} \\ \nabla \cdot \mathbf{B} = 0 & \text{No name} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} & \text{Ampere's law with Maxwell's correction} \end{cases}$$

(a) Consider a square wire with side length a. The wire is at rest. There is a region of uniform B field with width a moving towards the wire at speed v, as shown in the figure below. The B field has magnitude B and is perpendicular to the plane on which the wire lies, pointing outwards from the page. The wire enters the region at $t = 0$.



Use the integral form of the Faraday's law to obtain the emf along the wire as a function of t. Take emf in the counterclockwise direction as positive.

(b) By Gauss's law and the continuity equation $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$, show that

$$\nabla \cdot \left(\mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) = 0 .$$

4. **[10 points]**

An infinitely long wire carrying uniform free current I_f is located at the central axis of an infinitely long solid cylinder with radius R. The cylinder is made of linear magnetic material with permeability μ. Use cylindrical coordinates with the wire lying on the z-axis and the current flowing in the positive z direction.

(a) Find the B field (both inside and outside the cylinder).
(b) Find the bound current densities.

5. **[7 points]**

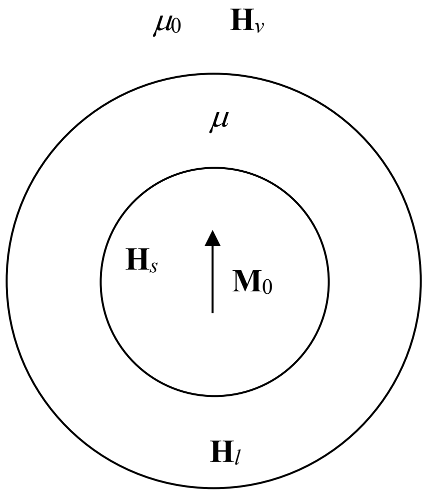
Consider a point dipole $\mathbf{p}$ located above the x-y plane on the positive z-axis at (0, 0, a), where $a > 0$. If the x-y plane is an infinite grounded conducting plate, find the surface charge density on the plate. It is given that the field due to a point dipole $\mathbf{p}$ at an observation point $\mathbf{r}$ is

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{\mathscr{r}^3}\left(3(\mathbf{p}\cdot\hat{\mathscr{r}})\hat{\mathscr{r}} - \mathbf{p}\right) = \frac{1}{4\pi\varepsilon_0}\frac{1}{\mathscr{r}^3}\left(\frac{3}{\mathscr{r}^2}(\mathbf{p}\cdot\boldsymbol{\mathscr{r}})\boldsymbol{\mathscr{r}} - \mathbf{p}\right),$$

where $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'$ and $\mathbf{r}'$ is the location of $\mathbf{p}$.

6. **[7 points]**

Consider a sphere with radius a carrying uniform magnetization $\mathbf{M}_0$ in the positive z-direction. Outside the sphere there is a spherical layer of linear material with inner radius a and outer radius b. The permeability of the material is μ. The H field inside the sphere, inside the layer, and outside in vacuum are $\mathbf{H}_s$, $\mathbf{H}_l$, and $\mathbf{H}_v$, respectively. There is no free current in the system.



(a) Write down the boundary condition of the parallel component of H field at the inner interface.

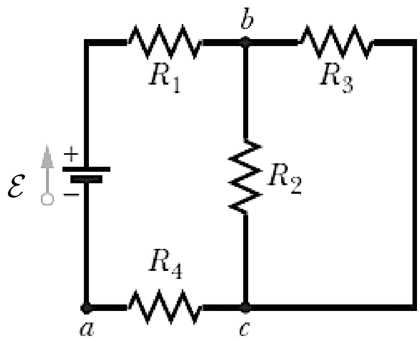
(b) Write down the boundary condition of the perpendicular component of H field at the inner interface, in terms of $\mathbf{M}_0$ and μ.

(c) Write down the boundary condition of the parallel component of H field at the outer interface.

(d) Write down the boundary condition of the perpendicular component of H field at the outer interface, in terms of μ and μ_0.

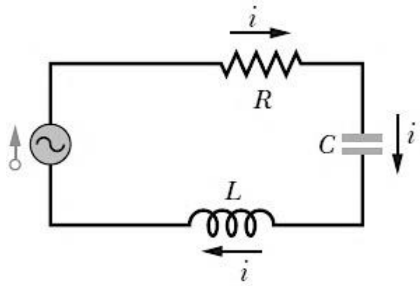
7. **[8 points]**
The figure below shows a multi-loop circuit containing one ideal battery and four resistances with the following values:

$$R_1 = 20\ \Omega,\quad R_2 = 20\ \Omega,\quad \mathcal{E} = 12\text{V},$$
$$R_3 = 30\ \Omega,\quad R_4 = 8.0\ \Omega.$$



(a) What is the current through the battery?
(b) What is the current i_2 through R_2?
(c) What is the current i_3 through R_3?

8. **[8 points]**
In the figure below, let $R = 200\ \Omega$, $C = 15.0\ \mu\text{F}$, $L = 230\ \text{mH}$, driving frequency $f_d = 60.0$ Hz, and voltage amplitude $\varepsilon_m = 36.0\ \text{V}$



(a) What is the current amplitude I ?
(b) What is the phase angle (with the sign indicated) of the current in the circuit relative to the emf?

9. **[9 points]**

Caught by surprise near a supernova, you race away from the explosion in your spaceship, hoping to outrun the high-speed materials ejected toward you. Your Lorentz factor relative to the inertial reference frame of the local stars is 22.4. The speed of light is c = 299,792,458 ms^{-1}.

(a) To reach a safe distance, you need to cover 9.00×10^{16} m as measured in the reference frame of the local stars. How long will the flight take, as measured in that frame?

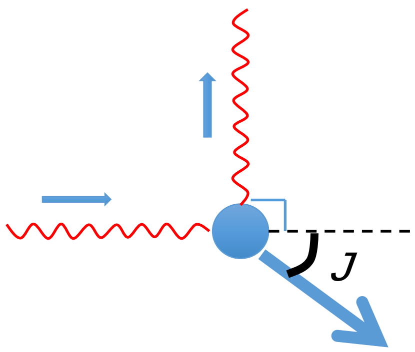
(b) How long does that trip take according to you (in your reference frame)?

10. **[5 points]**

A thermal neutron has a speed v at temperature T = 300K and kinetic energy $\frac{mv^2}{2} = \frac{3kT}{2}$ (Here $m = 1.6749\times10^{-27}$kg is the neutron mass and $k = 1.38\times10^{-23}\text{m}^2\text{kg s}^{-2}\text{K}^{-1}$ is the Boltzmann constant). Calculate its de Broglie wavelength. State whether a beam of these neutrons could be diffracted significantly by a crystal, and why? (Recall that the separation between atoms inside a crystal ~ 1nm and Planck constant $h = 6.626\times10^{-34}$Js).

11. **[10 points]**

X rays of wavelength 0.0100 nm are directed in the positive direction of an x axis onto a target containing loosely bound electrons. For Compton scattering from one of those electrons, at an angle of 90°, what are (a) the Compton shift, (b) the corresponding change in photon energy, (c) the kinetic energy of the recoiling electron, and (d) the angle between the positive direction of the x axis and the electron's direction of motion?



Hint: The formula may be useful.

Compton scattering:

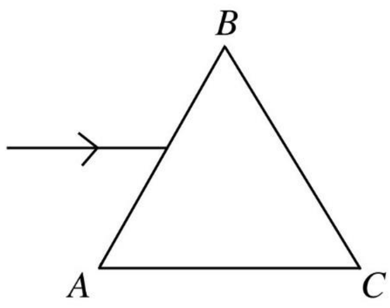
Wavelength of scattered radiation; Wavelength of incident radiation; Planck's constant; Scattering angle

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\phi)$$

Electron rest mass; Speed of light in vacuum

12. **[5 points]**
Two converging lenses, both of focal length 10 cm, are separated by 20 cm. An object is 15 cm to the left of the first lens. Find by calculation and construction of a ray diagram the position of the image. Find the magnification of the image and describe whether it is real or imaginary, erect or inverted, magnified or diminished.

13. **[5 points]**
Light in air is initially traveling parallel to the face *AC* of an equilateral triangular prism, as shown in the figure below. The prism is made of glass with an index of refraction of 1.52. If the light does not strike the face *AC*, what is the angle between the ray as it leaves the prism at face *BC* and the normal in air at that face?



[THE END]