Physics Enhancement Programme

Selection Test 2

Solution

1. Construct spherical Gaussian surface with radius r and centered at the origin. By symmetry, the field should be radial and with constant magnitude on the surface. Hence Gauss's law implies

$$E \times 4\pi r^2 = \frac{Q_{\text{enc}}}{\varepsilon_0} \Rightarrow \mathbf{E} = E\hat{\mathbf{r}} = \frac{Q_{\text{enc}}}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$$

For $R \geq r \geq 0$, $Q_{\text{enc}} = \rho \times \frac{4}{3}\pi r^3 \Rightarrow \mathbf{E} = \frac{\rho \times 4\pi r^3/3}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} = \frac{\rho r}{3\varepsilon_0}\hat{\mathbf{r}} = \frac{\rho}{3\varepsilon_0}\mathbf{r}$.

For $a > r > R$, $Q_{\text{enc}} = \rho \times \frac{4}{3}\pi R^3 \Rightarrow \mathbf{E} = \frac{\rho \times 4\pi R^3/3}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} = \frac{\rho R^3}{3\varepsilon_0 r^2}\hat{\mathbf{r}}$.

For $b > r > a$, $\mathbf{E} = \mathbf{0}$, since it is inside a conductor. <u>(2)</u>

The volume charge density is zero because there can be no volume charge inside a conductor in electrostatics. <u>(0.5)</u>

Construct spherical Gaussian surface centered at the origin and with radius r, where $b > r > a$. Since the E field is zero inside the conductor, by Gauss's law, the total charge enclosed by this surface should be zero. Hence the total amount of charge on the inner surface is $-\frac{4}{3}\pi\rho R^3$. By symmetry, the charges should be uniformly distributed over the surface. Hence $\sigma = \frac{-4\pi\rho R^3/3}{4\pi a^2} = -\frac{\rho R^3}{3a^2}$. <u>(2.5)</u>

Since the conductor is isolated and carries no net charge, the total amount of charge on the outer surface must be $\frac{4}{3}\pi\rho R^3$. Again, by symmetry, the distribution is uniform. Hence $\sigma = \frac{4\pi\rho R^3/3}{4\pi b^2} = \frac{\rho R^3}{3b^2}$. <u>(1)</u>

Construct spherical Gaussian surface centered at the origin and with radius r, where $r > b$. Since $Q_{\text{enc}} = \frac{4}{3}\pi\rho R^3$, Gauss's law implies

$$\mathbf{E}=\frac{\rho\times 4\pi R^3/3}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}=\frac{\rho}{3\varepsilon_0}\frac{R^3}{r^2}\hat{\mathbf{r}}\,. \qquad \underline{(1.5)}$$

By definition, $V(\mathbf{r})=-\int_\infty^{\mathbf{r}}\mathbf{E}\cdot d\mathbf{l}=-\int_\infty^r E dr$.

For $r>b$, $V(r)=-\int_\infty^r \frac{\rho}{3\varepsilon_0}\frac{R^3}{r^2}dr=\frac{\rho R^3}{3\varepsilon_0 r}$.

For $b\ge r\ge a$, $V(r)=\frac{\rho R^3}{3\varepsilon_0 b}$.

For $a>r>R$,

$$V(r)=\frac{\rho R^3}{3\varepsilon_0 b}-\int_a^r \frac{\rho}{3\varepsilon_0}\frac{R^3}{r^2}dr=\frac{\rho R^3}{3\varepsilon_0 b}-\frac{\rho R^3}{3\varepsilon_0 a}+\frac{\rho R^3}{3\varepsilon_0 r}=\frac{\rho R^3}{3\varepsilon_0}\left(\frac{1}{b}-\frac{1}{a}+\frac{1}{r}\right).$$

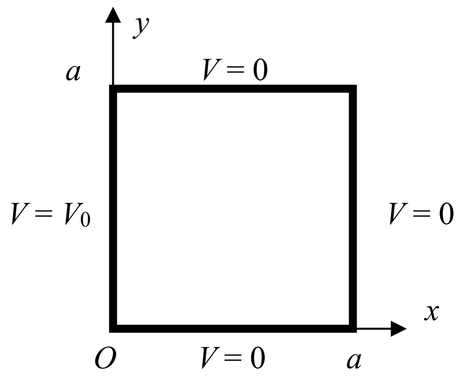
For $R\ge r\ge 0$,

$$V(r)=\frac{\rho R^3}{3\varepsilon_0}\left(\frac{1}{b}-\frac{1}{a}+\frac{1}{R}\right)-\int_R^r \frac{\rho r}{3\varepsilon_0}dr=\frac{\rho R^3}{3\varepsilon_0}\left(\frac{1}{b}-\frac{1}{a}+\frac{1}{R}\right)+\frac{\rho}{6\varepsilon_0}\left(R^2-r^2\right)$$

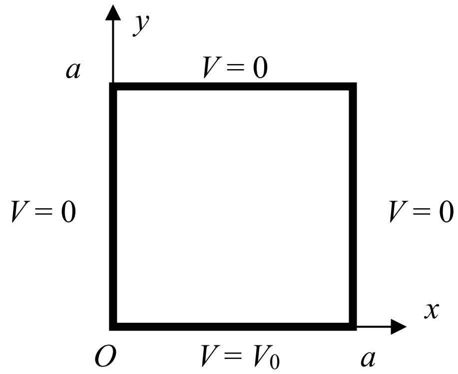
$$=\frac{\rho}{6\varepsilon_0}\left(\frac{2R^3}{b}-\frac{2R^3}{a}+3R^2-r^2\right)$$

$\underline{(4.5)}$

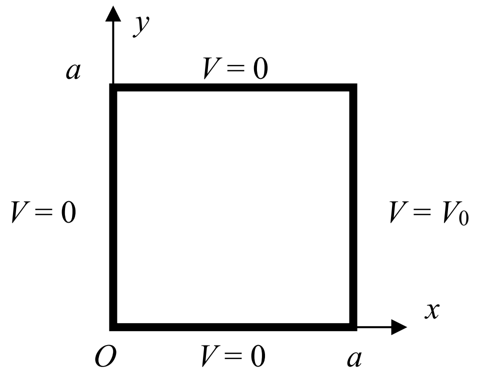
2. Let the solution of the following configuration be $\Phi_1(x,y)$.



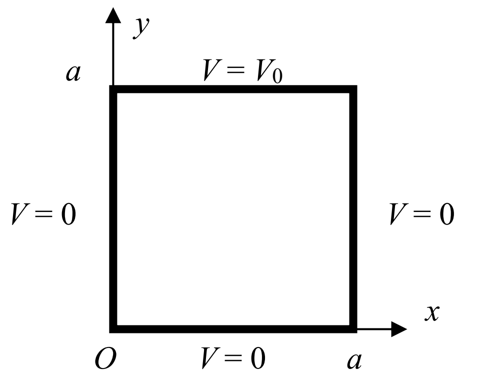
Let the solution of the following configuration be $\Phi_2(x,y)$.



Let the solution of the following configuration be $\Phi_3(x, y)$.

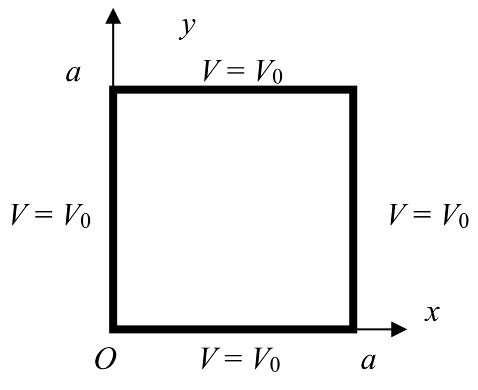


Let the solution of the following configuration be $\Phi_4(x, y)$.



Then the solution of the following configuration is

$$\Phi(x, y) = \Phi_1(x, y) + \Phi_2(x, y) + \Phi_3(x, y) + \Phi_4(x, y)$$



But it is trivial that $\Phi(x, y) = V_0$.

Besides, by symmetry we have

$$\Phi_1(a/2, a/2) = \Phi_2(a/2, a/2) = \Phi_3(a/2, a/2) = \Phi_4(a/2, a/2)$$

Hence

$$\Phi_1\left(a/2, a/2\right) = \frac{V_0}{4} \qquad \underline{(8)}$$

3. (a)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \Rightarrow \int_S \left(\nabla \times \mathbf{E}\right) \cdot d\mathbf{a} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}$$

$$\Rightarrow \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{a}$$

$$\Rightarrow \mathcal{E} = -\frac{d\Phi}{dt}$$

For $t < 0$, $\Phi = 0$, $\mathcal{E} = -\dfrac{d\Phi}{dt} = 0$.

For $a/v > t > 0$, $\Phi = Bavt$, $\mathcal{E} = -\dfrac{d\Phi}{dt} = -vBa$.

For $2a/v > t > a/v$, $\Phi = Ba^2 - Bavt$, $\mathcal{E} = -\dfrac{d\Phi}{dt} = vBa$.

For $t > 2a/v$, $\Phi = 0$, $\mathcal{E} = -\dfrac{d\Phi}{dt} = 0$. <u>(4)</u>

(b)

$$\nabla \cdot \left(\mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) = \mu_0\left(\nabla \cdot \mathbf{J} + \varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \mathbf{E}\right)$$

$$= \mu_0\left(\nabla \cdot \mathbf{J} + \varepsilon_0 \frac{\partial}{\partial t}\frac{\rho}{\varepsilon_0}\right) = \mu_0\left(\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}\right) = 0$$

<u>(2)</u>

4. (a) Construct circular Amperian loop with radius *s*. The loop is centered at the wire and lies on a plane perpendicular to it.
By symmetry, the H field should be along the tangential direction and with constant magnitude along the wire. Hence, by Ampere's law

$$H \times 2\pi s = I_f \Rightarrow \mathbf{H} = \frac{I_f}{2\pi s}\hat{\boldsymbol{\phi}}.$$

For $R > s > 0$, $\mathbf{H} = \dfrac{\mathbf{B}}{\mu} \Rightarrow \mathbf{B} = \dfrac{\mu I_f}{2\pi s}\hat{\boldsymbol{\phi}}$.

For $s > R$, $\mathbf{H} = \dfrac{\mathbf{B}}{\mu_0} \Rightarrow \mathbf{B} = \dfrac{\mu_0 I_f}{2\pi s}\hat{\boldsymbol{\phi}}$. <u>(4)</u>

(b) Since $\mathbf{J}_b = \left(\frac{\mu}{\mu_0} - 1\right)\mathbf{J}_f$, inside the cylinder, $\mathbf{J}_b$ is everywhere zero except at the location of the wire, at which $\mathbf{I}_b = \left(\frac{\mu}{\mu_0} - 1\right)\mathbf{I}_f = \left(\frac{\mu}{\mu_0} - 1\right)I_f\hat{\mathbf{z}}$. (2)

$$\mathbf{M} = \left(\frac{\mu}{\mu_0} - 1\right)\mathbf{H} = \left(\frac{\mu}{\mu_0} - 1\right)\frac{I_f}{2\pi s}\hat{\boldsymbol{\phi}}. \quad \underline{(2)}$$

$$\mathbf{K}_b = \mathbf{M}(R)\times\hat{\mathbf{s}} = \left(\frac{\mu}{\mu_0} - 1\right)\frac{I_f}{2\pi R}\hat{\boldsymbol{\phi}}\times\hat{\mathbf{s}} = -\left(\frac{\mu}{\mu_0} - 1\right)\frac{I_f}{2\pi R}\hat{\mathbf{z}}. \quad \underline{(2)}$$

5. Let the dipole be

$$\mathbf{p} = p_x\hat{\mathbf{x}} + p_y\hat{\mathbf{y}} + p_z\hat{\mathbf{z}}$$

then the image dipole is

$$\mathbf{p}' = -p_x\hat{\mathbf{x}} - p_y\hat{\mathbf{y}} + p_z\hat{\mathbf{z}}$$

located at $(0,0,-a)$. (2)

The total electric field just above the conducting plate is

$$\begin{aligned}
\mathbf{E}(x,y,0) &= \frac{1}{4\pi\varepsilon_0}\frac{1}{\left(x^2+y^2+a^2\right)^{3/2}} \\
&\quad \left[\frac{3}{x^2+y^2+a^2}\left(p_x x + p_y y - p_z a\right)\left(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - a\hat{\mathbf{z}}\right) - p_x\hat{\mathbf{x}} - p_y\hat{\mathbf{y}} - p_z\hat{\mathbf{z}}\right. \\
&\quad \left. + \frac{3}{x^2+y^2+a^2}\left(-p_x x - p_y y + p_z a\right)\left(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + a\hat{\mathbf{z}}\right) + p_x\hat{\mathbf{x}} + p_y\hat{\mathbf{y}} - p_z\hat{\mathbf{z}}\right] \\
&= \frac{1}{2\pi\varepsilon_0}\frac{1}{\left(x^2+y^2+a^2\right)^{3/2}}\left[\frac{3a}{x^2+y^2+a^2}\left(p_z a - p_x x - p_y y\right) - p_z\right]\hat{\mathbf{z}}
\end{aligned}$$

(3)

The surface charge density can be obtained by

$$\sigma = \varepsilon_0 E = \frac{1}{2\pi}\frac{1}{\left(x^2+y^2+a^2\right)^{3/2}}\left[\frac{3a}{x^2+y^2+a^2}\left(p_z a - p_x x - p_y y\right) - p_z\right]. \quad \underline{(2)}$$

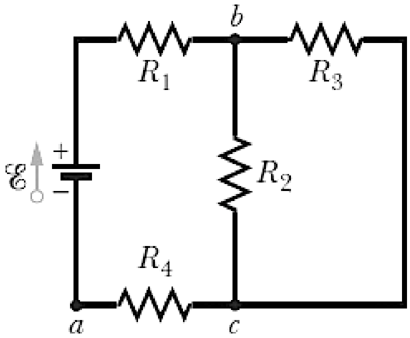
6. (a) $\mathbf{H}_s^{//}(a) = \mathbf{H}_l^{//}(a)$. (1)

(b) $B_s^{\perp}(a) = B_l^{\perp}(a) \;\Rightarrow\; \mu_0\left(H_s^{\perp}(a) + M_0\cos\theta\right) = \mu H_l^{\perp}(a)$. (3)

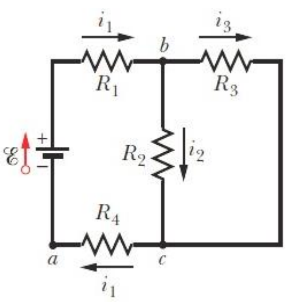
(c) $\mathbf{H}_l^{//}(b) = \mathbf{H}_v^{//}(b)$. (1)

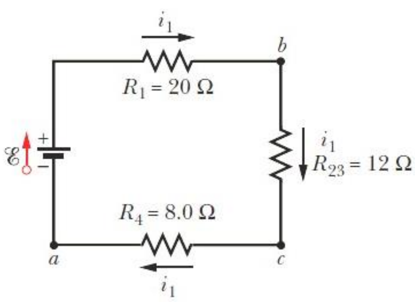
(d) $B_l^{\perp}(b) = B_v^{\perp}(b) \;\Rightarrow\; \mu H_l^{\perp}(b) = \mu_0 H_v^{\perp}(b)$. (2)

7.



(a) $R_{23} = \dfrac{R_2R_3}{R_2 + R_3} = \dfrac{(20)(30)}{20+30} = 12\ \Omega.$ [2 points]

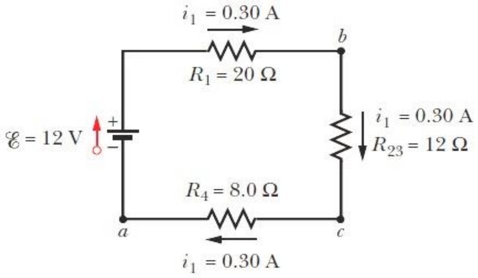




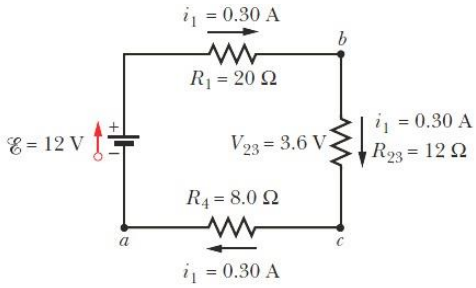
$$
\begin{aligned}
+E - i_1R_1 - i_1R_{23} - i_1R_4 &= 0 \\
(12) - i_1(20) - i_1(12) - i_1(8) &= 0 \\
i_1 = \frac{12}{40} &= 0.30\ \text{A.}
\end{aligned}
$$

[2 points]

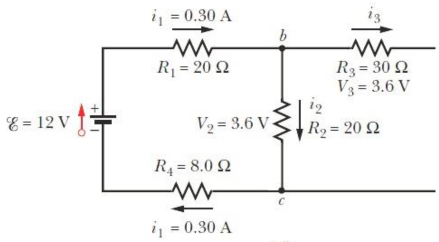
(b)



(d)



(e)

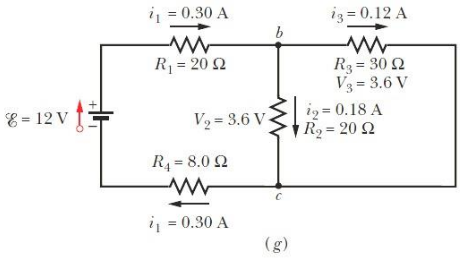


(f)

$$V_{23} = i_1 R_{23} = (0.3)(12) = 3.6 \text{ V.}$$
$$i_2 = \frac{V_2}{R_2} = \frac{3.6}{20} = 0.18 \text{ A.}$$

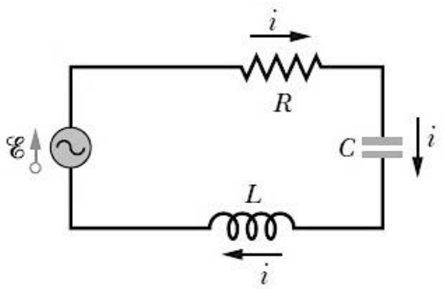
[3 points]

(c)



$$i_3 = i_1 - i_2 = 0.30 - 0.18 = 0.12 \text{ A.}$$ [1 point]

8.



(a)

$$X_C = \frac{1}{2\pi f_d C} = \frac{1}{2\pi(60.0)(15.0\times 10^{-6})} = 177\ \Omega.$$ [1.5 points]

$$X_L = 2\pi f_d L = 2\pi(60.0)(230\times 10^{-3}) = 86.7\ \Omega.$$ [1.5 points]

$$\begin{aligned} Z &= \sqrt{R^2 + (X_L - X_C)^2} \\ &= \sqrt{(200)^2 + (86.7 - 177)^2} \\ &= 219\ \Omega. \end{aligned}$$

$$I = \frac{E_m}{Z} = \frac{36.0}{219} = 0.164 \text{ A.}$$ [1.5 points]

(b)

$$\phi = \tan^{-1}\frac{X_L - X_C}{R} = \tan^{-1}\frac{86.7 - 177}{200} = -24.3° = -0.424 \text{ rad.}$$

[2 points; deduct 1 point without "-" sign]

9.

(a)

$$\begin{pmatrix}\text{time interval}\\ \text{relative to stars}\end{pmatrix} = \frac{\text{distance relative to stars}}{c} = \frac{9.00\times 10^{16}}{299{,}792{,}458} = 3.00\times 10^{8} \text{ s} = 9.51 \text{ y.}$$

[4.5 points]

(b)

$$\begin{pmatrix}\text{time interval}\\ \text{relative to you}\end{pmatrix} = \frac{\text{distance relative to you}}{c} = \frac{L_0/\gamma}{c} = \frac{9.00\times 10^{16}/22.4}{299{,}792{,}458} = 1.340\times 10^{7} \text{ s} = 0.425 \text{ y.}$$

[4.5 points]

10.

$$\frac{mv^2}{2} = \frac{3kT}{2} \Rightarrow p = mv = \sqrt{3mkT} = \frac{h}{\lambda} \qquad \text{(2 points)}$$

$$\Rightarrow \lambda = 1.45\times 10^{-10} m \qquad \text{(1 points)}$$

Since the wavelength of the thermal neutrons is comparable with the atom separation, they can be diffracted by the crystal. (2 point)

11.

(a) $\lambda' - \lambda = \frac{h}{mc}(1-\cos\phi) \Rightarrow \lambda' - \lambda = 2.4\times 10^{-12} m$ (2 points)

$\Rightarrow \lambda' = 1.24\times 10^{-11} m$ (1 points)

(b) $\Delta E = hf' - hf$ (1 point)

$= hc\left(\frac{1}{\lambda'} - \frac{1}{\lambda}\right) = -3.88\times 10^{-15} J$ (1 points)

(c) the kinetic energy of the recoiling electron = $3.88\times 10^{-15} J$ (1 points)

(d) $\tan\theta = \frac{p}{p'} = \frac{\lambda'}{\lambda} \Rightarrow \theta = 51.2^\circ$ (3 points)

The electron make an angle $90 - 51.2 = 38.8^\circ$ with the x-axis. (1 point)

12.

$\frac{1}{s_1} + \frac{1}{s_1'} = \frac{1}{f} \Rightarrow \frac{1}{s_1'} = \frac{1}{10} - \frac{1}{15} \Rightarrow s_1' = 30cm$ (1 point)

(i.e. the image is 30cm to the right of the first lens
or it is 10cm to the right of the second lens)

$s_2 = -10cm$ (1 point)

$\frac{1}{s_2} + \frac{1}{s_2'} = \frac{1}{f} \Rightarrow \frac{1}{s_2'} = \frac{1}{10} + \frac{1}{10} \Rightarrow s_2' = 5cm$ (1 point)

the image is 5cm to the right of the second lens which is real (1 point).

The magnification $m = \left(-\frac{s_1'}{s_1}\right)\left(-\frac{s_2'}{s_2}\right) = -1.5$ (the image is magnified and inverted) (1 point)

13.

$\theta_0 = 30^\circ$ (1 point)

$\Rightarrow n\sin\theta_1 = \sin\theta_0 \Rightarrow \theta_1 = 19.2$ (1 point)

$60 + (90 - \theta_1) + (90 - \theta_2) = 180 \Rightarrow \theta_2 = 60 - \theta_1 = 40.8$ (2 point)

$\sin\theta_3 = n\sin\theta_2 \Rightarrow \theta_3 = 83^\circ$ (1 point)