

1

(a)

$$PV = nkT$$

$$T = \frac{PV}{nk} = \frac{(101325)(0.002)}{(0.1 \times 6.02 \times 10^{23})(1.38 \times 10^{-23})} = 244 \text{ K}$$

[ Equation: 1 mark; Substitution of numeric values: 0.5 Mark; Answer: 0.5 Mark ]

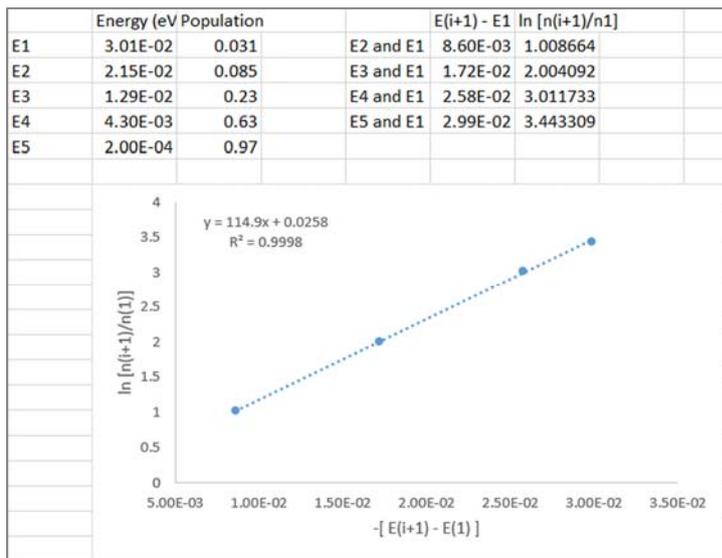
(b)

$$\frac{n_2}{n_1} = \exp\left(-\frac{E_2 - E_1}{kT}\right)$$

[ 2 Marks ]

$$\ln\left(\frac{n_2}{n_1}\right) = -\frac{E_2 - E_1}{kT}$$

Method #1: x-y graph with linear regression equation to find the slope. The best method, which deserves the highest mark.



- 4 Data points on graph: 0.5 Mark Each, Subtotal = 2 Marks
- Correct Plot  $\ln(n_2/n_1)$  against  $-(E_2 - E_1)$ : 2 Marks
- Linear regression line: 1 Mark
- Slope=112-118 (eV)<sup>-1</sup>: 2 Marks;  
Slope=109-121 (eV)<sup>-1</sup>: 1 Mark;  
Zero mark otherwise.
- Axis Titles: 1 Mark
- Axis Labels: 1 Mark
- x-y Graph > 70% area of graph paper: 1 Mark;  
> 50% area of graph paper: 0.5 Mark.

Slope = 114.9 (eV)<sup>-1</sup>

$$T = \frac{1.6 \times 10^{-19}}{114.9 \times 1.38 \times 10^{-23}} = 100.9 \text{ K}$$

Numeric Substitution: 0.5 Mark;  
 eV to J: 1 Mark  
 Answer: 1 Mark

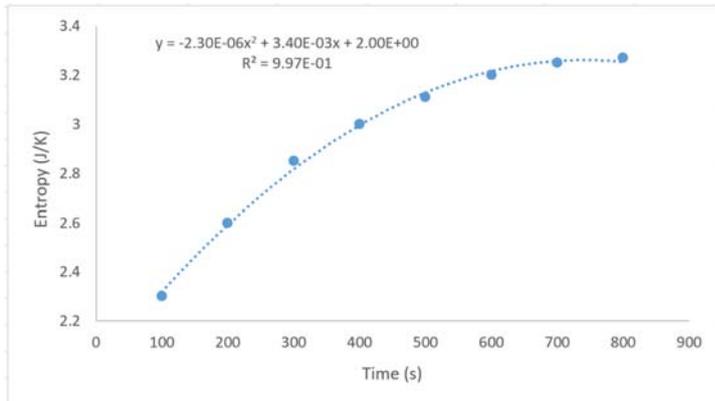
Method #2: Get the mean value of temperatures using the given  $n_1$  and  $n_2$  populations.

Calculation of mean temperature: 2 Marks  
 Numeric Substitution: 0.5 Mark  
 eV to J: 1 Mark  
 Answer: 1 Mark

(c) (i) and (ii)

$$\frac{dQ}{dt} = T \frac{dS}{dt} \quad [ 2 \text{ Marks }]$$

$$T = \frac{dQ/dt}{dS/dt}$$



- Data points on graph: 1 Mark
- Second-order curve: 1 Mark
- Axis Titles: 1 Mark
- Axis Labels: 1 Mark
- x-y Graph > 70% area of graph paper: 1 Mark;  
> 50% area of graph paper: 0.5 Mark.

$$S = -2.3 \times 10^{-6} t^2 + 3.4 \times 10^{-3} t + 2 \quad [ 2 \text{ Marks }]$$

$$\frac{dS}{dt} = -4.6 \times 10^{-6} t + 3.4 \times 10^{-3} \quad [ 1 \text{ Mark }]$$

$$T = \frac{dQ/dt}{dS/dt} = \frac{0.05}{-4.6 \times 10^{-6} t + 3.4 \times 10^{-3}} \quad [ 1 \text{ Mark }]$$

(iii)

$$T = \frac{0.05}{(-4.6 \times 10^{-6})(500) + 3.4 \times 10^{-3}} = 45.5 \text{ K} \quad [ 1 \text{ Mark }]$$

(iv)

Method #1

$$\frac{dS}{dt} = -4.6 \times 10^{-6} t + 3.4 \times 10^{-3} = 0 \quad [ 1 \text{ Mark for } dS/dt = 0 ]$$

$$t = 739 \text{ s} \quad [ 1 \text{ Mark }]$$

Method #2: By observation on the 2<sup>nd</sup> polynomial

$$t = 739 \pm 10 \text{ s} \quad [ 1 \text{ Mark; Zero Mark otherwise }]$$

(d)

Polyatomic [ 2 Marks ]

(e)

Thermal expansion  $\Rightarrow$  Sizes of both spheres increase.

CG of Sphere A rises, but CG of Sphere B sinks.  $\Rightarrow$  Potential energy of Sphere A increases, but Sphere B decreases.

According to the First Law of Thermodynamics, heat transferred to the spheres produces: i) increase in internal energy; ii) small amount of work done in expanding against the atmospheric pressure; and iii) change in gravitational potential energy.

Potential energy of Sphere A increases a small amount, while Sphere B decreases a small amount.

Sphere B: Decrease in potential energy contributes positively to the increase in its internal energy.

[ Internal Energy: 2 Marks; Potential Energy: 2 Marks ]

[ Kinetic Energy: Penalty -2 Marks, but ~~total~~ <sup>XXX</sup> mark of this subquestion is zero ]  
the minimum

Sphere B has the higher temperature.

[ 2 Marks given only if correctly answer i) higher temperature in Sphere B; and ii) Internal energy + potential energy ]

(a) Take zero potential energy at the lowest point on the pendulum path.

[1 Mark]

[1 Mark]

[1 Mark]

$$mg(L - L \cos \theta_0) = \frac{1}{2} mL^2 \left( \frac{d\theta}{dt} \right)^2 + mg(L - L \cos \theta)$$

[1 Mark for energy conservation equation]

$$\left( \frac{d\theta}{dt} \right)^2 = \frac{2g}{L} (\cos \theta - \cos \theta_0)$$

$$\frac{d\theta}{dt} = \pm \sqrt{\frac{2g}{L} (\cos \theta - \cos \theta_0)}$$

[1 Mark]

Consider the motion from  $\theta = \theta_0$  to  $\theta = 0$  (a quarter of period  $T$ ),

[0.5 Mark]

$$T = 4 \int_0^{\theta_0} \sqrt{\frac{L}{2g} \frac{1}{\cos \theta - \cos \theta_0}} d\theta$$

$$= \sqrt{\frac{8L}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}}$$

[1 Mark]

$$(b) \quad \cos \theta = 1 - 2 \sin^2 \left( \frac{\theta}{2} \right)$$

$$T = \sqrt{\frac{8L}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{[1 - 2 \sin^2(\frac{\theta}{2})] - [1 - 2 \sin^2(\frac{\theta_0}{2})]}}$$

$$= 2 \sqrt{\frac{L}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2(\frac{\theta_0}{2}) - \sin^2(\frac{\theta}{2})}}$$

[2 Marks]

①

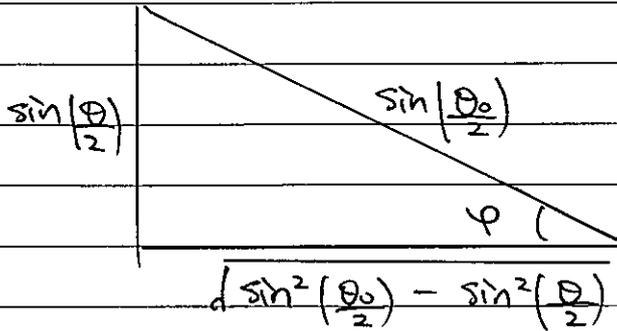
$$\sin \varphi = \frac{\sin(\frac{\theta}{2})}{\sin(\frac{\theta_0}{2})} \quad (\text{Change of variable})$$

$$\cos \varphi d\varphi = \frac{\frac{1}{2} \cos(\frac{\theta}{2}) d\theta}{\sin(\frac{\theta_0}{2})}$$

[1 Mark]

[ 1 Mark ]

(b)



$$\cos \varphi = \frac{\sqrt{\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}}{\sin\left(\frac{\theta_0}{2}\right)}$$

$$\frac{\sqrt{\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}}{\sin\left(\frac{\theta_0}{2}\right)} d\varphi = \frac{\cos\left(\frac{\theta}{2}\right) d\theta}{2 \sin\left(\frac{\theta_0}{2}\right)}$$

[ 1 Mark ]

$$2 \sqrt{\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)} d\varphi = \cos\left(\frac{\theta}{2}\right) d\theta$$

$$\Rightarrow d\theta = \frac{2 \sqrt{\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}}{\cos\left(\frac{\theta}{2}\right)} d\varphi \quad \text{--- (2)}$$

[ 2 Marks ]

① and ②, [ 1 Mark ]

$$T = \frac{2}{\sqrt{g}} \int_0^{\frac{\pi}{2}} \frac{2 \sqrt{\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}}{\cos\left(\frac{\theta}{2}\right) \sqrt{\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}} d\varphi$$

[ 1 Mark ]

$$= \frac{2}{\sqrt{g}} \int_0^{\frac{\pi}{2}} \frac{2 d\varphi}{\cos\left(\frac{\theta}{2}\right)} \quad \text{[ 2 Marks ]}$$

$$= \frac{4}{\sqrt{g}} \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - \sin^2\varphi \sin^2\left(\frac{\theta_0}{2}\right)}} \quad \text{[ 1 Mark ]}$$

$$= \frac{4}{\sqrt{g}} \int_0^{\frac{\pi}{2}} \left[ 1 + \frac{1}{2} \sin^2\left(\frac{\theta_0}{2}\right) \sin^2\varphi \right] d\varphi$$

[ 2 Marks for applying Taylor series correctly ]

$$\approx \frac{4}{\sqrt{g}} \int_0^{\frac{\pi}{2}} \left[ 1 + \frac{1}{2} \frac{\theta_0^2}{4} \sin^2\varphi \right] d\varphi$$

[ 1 Mark for small angle approximation ]

$$(b) \quad T = 4 \sqrt{\frac{L}{g}} \left[ \frac{\pi}{2} + \frac{\pi}{4} \frac{1}{2} \frac{\theta_0^2}{4} \right] \quad [2 \text{ Marks for solving integral}]$$

$$= 2\pi \sqrt{\frac{L}{g}} \left[ 1 + \frac{\theta_0^2}{16} \right] \quad [2 \text{ Marks}]$$

$$T \approx 2\pi \sqrt{\frac{L}{g}} \quad \bar{B} \quad \text{underestimated.} \quad [1 \text{ Mark}]$$

3.

$$(a) \quad dE = k_e \frac{dq}{x^2} = k_e \frac{\sigma dx}{x^2} \quad [1 \text{ Mark}]$$

$$E = \int_d^{L+d} k_e \sigma \frac{dx}{x^2} \quad [1 \text{ Mark}]$$

$$= \frac{k_e \sigma L}{d(L+d)} \quad [1 \text{ Mark}]$$

$$\text{Short Rod } (L \ll d), \quad E = \frac{k_e Q}{d^2} \quad [1 \text{ Mark}]$$

$$\text{Long Rod } (L \gg d), \quad E = \frac{k_e \sigma}{d} \quad [1 \text{ Mark}]$$

$$(b) \quad \text{From (a)}, \quad E = \frac{k_e Q}{d(L+d)} \quad (Q = \sigma L)$$

$$dF = \frac{k_e Q^2}{L} \frac{dx}{d(L+d)} \quad [1 \text{ Mark}]$$

$$= \frac{k_e Q^2}{L} \int_{d-L}^d \frac{dx}{x(L+x)} \quad [1 \text{ Mark}]$$

$$= \frac{k_e Q^2}{L} \left[ -\frac{1}{L} \ln \left( \frac{L+x}{x} \right) \right]_{d-L}^d \quad [1 \text{ Mark}]$$

$$= \frac{k_e Q^2}{2L} \left[ -\ln \left( \frac{L+d}{d} \right) + \ln \left( \frac{d}{d-L} \right) \right]$$

$$= \frac{k_e Q^2}{2L^2} \ln \left[ \frac{d^2}{(d-L)(d+L)} \right] \quad [1 \text{ Mark}]$$

$$= \frac{k_e Q^2}{2L^2} \ln \left( \frac{d^2}{d^2 - L^2} \right)$$

4.

(a) Let  $\Delta \mathbf{v} = \boldsymbol{\omega}_B \times \mathbf{r}_B - \boldsymbol{\omega}_A \times \mathbf{r}_A$  be the linear velocity difference between A and B,

$$\Delta v = |\Delta \mathbf{v}| = \omega_B(-r_B) - \omega_A r_A = -(r_B \omega_B + r_A \omega_A) \quad (\text{N.B. } r_i \text{ as scalars but } \omega_i \text{ vectors}) \quad [1]$$

Same material gives the same elastic deformation of teeth of the same size. The magnitude of the degree of deformation of each teeth as a vector quantity is half of the relative change in position of the teeth, so the rate of deformation is half of the relative velocity between A and B, i.e.

$$\begin{aligned} \dot{e}_A &= \frac{1}{2} \Delta v = -\frac{1}{2} (r_B \omega_B + r_A \omega_A) \\ \dot{e}_B &= -\dot{e}_A = \frac{1}{2} (r_B \omega_B + r_A \omega_A) \end{aligned} \quad [1]$$

Force acting on the tooth of wheel  $i$ ,  $\mathbf{f}_i = -k\mathbf{e}_j$  ( $i \neq j$ ), where  $k$  is the force constant [0.5], thus

$$\begin{aligned} \dot{f}_A &= -\frac{1}{2} k (r_B \omega_B + r_A \omega_A) \\ \dot{f}_B &= \frac{1}{2} k (r_B \omega_B + r_A \omega_A) \end{aligned} \quad [1]$$

$I_i \dot{\omega}_i = \boldsymbol{\tau}_i = \mathbf{r}_i \times \mathbf{f}_i$ ,  $I_i = \frac{1}{2} m r_i^2$ , so

$$\begin{aligned} \frac{1}{2} m r_A^2 \ddot{\omega}_A &= -\frac{1}{2} k r_A (r_B \omega_B + r_A \omega_A) \\ \frac{1}{2} m r_B^2 \ddot{\omega}_B &= \frac{1}{2} k (-r_B) (r_B \omega_B + r_A \omega_A) \end{aligned} \quad [1]$$

or

$$\begin{aligned} r_A \ddot{\omega}_A &= -\frac{k}{m} (r_B \omega_B + r_A \omega_A) \\ r_B \ddot{\omega}_B &= -\frac{k}{m} (r_B \omega_B + r_A \omega_A) \end{aligned} \quad [0.5]$$

(b) Method 1:

$$\begin{aligned} r_A \ddot{\omega}_A &= -\frac{k}{m} (r_B \omega_B + r_A \omega_A) \\ r_B \ddot{\omega}_B &= -\frac{k}{m} (r_B \omega_B + r_A \omega_A) \\ \Leftrightarrow -\frac{k}{m} \begin{pmatrix} 1 & \frac{r_B}{r_A} \\ \frac{r_A}{r_B} & 1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} &= \frac{d^2}{dt^2} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \\ \Rightarrow -\frac{k}{m} \begin{pmatrix} 1 & \frac{r_B}{r_A} \\ \frac{r_A}{r_B} & 1 \end{pmatrix} \begin{pmatrix} \alpha_A \\ \alpha_B \end{pmatrix} &= \beta^2 \begin{pmatrix} \alpha_A \\ \alpha_B \end{pmatrix} \quad (\text{Let } \omega_i = \alpha_i e^{\beta t}) \\ \Rightarrow \begin{pmatrix} \frac{k}{m} + \beta^2 & \frac{k}{m} \frac{r_B}{r_A} \\ \frac{k}{m} \frac{r_A}{r_B} & \frac{k}{m} + \beta^2 \end{pmatrix} \begin{pmatrix} \alpha_A \\ \alpha_B \end{pmatrix} &= \mathbf{0} \quad (*) [1] \end{aligned}$$

$$\det \begin{pmatrix} \frac{k}{m} + \beta^2 & \frac{k}{m} \frac{r_B}{r_A} \\ \frac{k}{m} \frac{r_A}{r_B} & \frac{k}{m} + \beta^2 \end{pmatrix} = 0 \Rightarrow \beta = 0 \text{ (repeated) or } \pm \sqrt{\frac{2k}{m}} i \quad [1]$$

$$\beta = 0, (*) \Rightarrow \alpha_A + \frac{r_A}{r_B} \alpha_B = 0 \Rightarrow \frac{\alpha_A}{\alpha_B} = \frac{-r_B}{r_A} \quad [1]$$

$$\beta = \pm \sqrt{\frac{2k}{m}}, (*) \Rightarrow \frac{\alpha_A}{\alpha_B} = \frac{r_B}{r_A} \quad [1]$$

$$\therefore \begin{pmatrix} \omega_A(t) \\ \omega_B(t) \end{pmatrix} = (C_1 + C_2 t) \begin{pmatrix} -r_B \\ r_A \end{pmatrix} + (C_3 \cos \omega t + C_4 \sin \omega t) \begin{pmatrix} r_B \\ r_A \end{pmatrix}, \quad \omega = \sqrt{\frac{2k}{m}}$$

$$\begin{pmatrix} \omega_A(0) \\ \omega_B(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \omega_0 \end{pmatrix} \Rightarrow C_1 = C_3 = \frac{\omega_0}{2r_A} \quad [0.5 + 0.5 + 0.5]$$

$$\text{No pressing each other at } t = 0 \Rightarrow \begin{pmatrix} \dot{\omega}_A(0) \\ \dot{\omega}_B(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow C_2 = C_4 = 0 \quad [0.5 + 0.5 + 0.5]$$

$$\therefore \omega_A = \frac{\omega_0 r_B}{2r_A} (\cos \omega t - 1) = -\frac{\omega_0 r_B}{r_A} \sin^2 \frac{\omega t}{2} \quad [1]$$

$$\omega_B = \frac{\omega_0}{2} (\cos \omega t + 1) = \omega_0 \cos^2 \frac{\omega t}{2}$$

Method 2:

Physical argument:

- 1) The resulting angular velocities comprise a macroscopic steady angular velocity and a microscopic oscillating term due to the elastic teeth pressing each other like a free-end mass-spring system [1<sup>a</sup>] with both teeth at the ends, with a spring constant  $k$  and effective (reduced) mass  $\frac{m}{2}$ . [1<sup>b</sup>]
- 2) The macroscopic term satisfies the no-slipping condition, i.e. moving in phase. [0.5<sup>c</sup>]
- 3) The oscillating term describes the teeth (wheels) moving anti-phase (same linear amplitude) with each other (as one of the normal mode of the free-end mass-spring system). [0.5<sup>d</sup>]

Then we have

$$\begin{cases} \omega_A(t) = C_1 R_1 + R_3 (C_3 \cos \omega t + C_4 \sin \omega t) \\ \omega_B(t) = C_1 R_2 + R_4 (C_3 \cos \omega t + C_4 \sin \omega t) \end{cases}, \quad \omega = \sqrt{\frac{2k}{m}} \quad [1^a, 1^b \text{ eqv}]$$

$$\frac{R_2 \omega_A}{R_1 \omega_B} = \frac{-r_A \omega_A}{r_B \omega_B} \Rightarrow \frac{R_1}{R_2} = \frac{-r_B}{r_A} \quad (\text{No slipping; } A \text{ and } B \text{ opposite rotation direction}) \quad [0.5^c \text{ eqv}]$$

$$\frac{R_3 \omega_A}{R_4 \omega_B} = \frac{r_A \omega_A}{r_B \omega_B} \Rightarrow \frac{R_3}{R_4} = \frac{r_B}{r_A} \quad [0.5^c \text{ eqv}] \quad [1 \text{ for correct magnitude of ratio}]$$

$$\therefore \begin{cases} \omega_A(t) = r_B(-C_1 + C_3 \cos \omega t + C_4 \sin \omega t) \\ \omega_B(t) = r_A(C_1 + C_3 \cos \omega t + C_4 \sin \omega t) \end{cases}, \quad \omega = \sqrt{\frac{2k}{m}} \quad [0.5 \text{ for implied } C_2 = 0]$$

$$\begin{cases} \omega_A(0) = 0 \\ \omega_B(0) = \omega_0 \end{cases} \Rightarrow C_1 = C_3 = \frac{\omega_0}{2r_A} \quad [0.5 \text{ for solving I.C., } 0.5 + 0.5 \text{ for ans}]$$

$$\text{No pressing each other at } t = 0 \Rightarrow \begin{cases} \dot{\omega}_A(0) = 0 \\ \dot{\omega}_B(0) = 0 \end{cases} \Rightarrow C_4 = 0 \quad [0.5 \text{ for solving I.C., } 0.5 \text{ for}$$

ans]

$$\therefore \omega_A = \frac{\omega_0 r_B}{2r_A} (\cos \omega t - 1) = -\frac{\omega_0 r_B}{r_A} \sin^2 \frac{\omega t}{2} \quad [1]$$

$$\omega_B = \frac{\omega_0}{2} (\cos \omega t + 1) = \omega_0 \cos^2 \frac{\omega t}{2}$$

(c) (1) We usually only observe the pure rotational motion [0.5]  $\omega_A = -\frac{\omega_0 r_B}{2r_A}, \omega_B = \frac{\omega_0}{2}$ . [0.5]

(2) The steady state oscillation behavior implies oscillation at the transient state. [0.5]

Such oscillation behavior cannot be seen even at transient state as the frequency is too

high, i.e.  $k = \frac{EA}{l_0}$  [0.5],  $E_{\text{steel}} \sim 10^{11} \text{ Pa}$ ,  $l_0 \sim 10^{-2}$  and even using a coarsest medium like

sandpaper P12, with grit size  $\sim 2 \text{ mm}$ ,  $k \sim 10^7$ ;  $m \sim 1$ , so  $\omega > 10^3$ . [1]

(3) In reality there is hysteretic effect and energy loss from the mass-spring system [0.5],

c.f. energy is conserved in this system ( $E_{\text{tot}} = \sum_i \frac{1}{2} \left( \frac{1}{2} m r_i^2 \right) \omega_i^2 = \frac{1}{2} \left( \frac{1}{2} m r_B^2 \right) \omega_0^2$ ) [0.5].

Therefore there should be a general decay term with the oscillation, i.e.

$$\begin{cases} \omega_A(t) = r_B \left[ -C_1 + e^{-\gamma t} (C_3 \cos \omega t + C_4 \sin \omega t) \right] \\ \omega_B(t) = r_A \left[ C_1 + e^{-\gamma t} (C_3 \cos \omega t + C_4 \sin \omega t) \right] \end{cases} \quad [1] \text{ which comes from some extra } \dot{\omega}_i \text{ terms}$$

(and/or higher order derivatives) in the equations  $\begin{cases} \dot{\omega}_A = -\frac{1}{2}(r_B \omega_B + r_A \omega_A) \\ \dot{\omega}_B = \frac{1}{2}(r_B \omega_B + r_A \omega_A) \end{cases}$ . [1]