Physics Enhancement Programme Phase 2
Selection Test 3 (Total 50 points)
11 February 2017

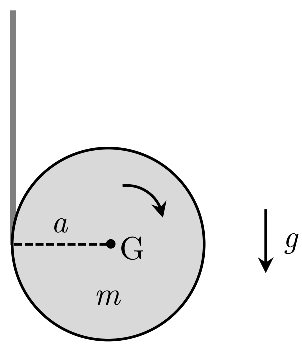
Physical Parameters

Planck's constant	$h = 4.136 \times 10^{-15}\text{eV} \cdot \text{s}$
Electron mass	$m_e = 9.109 \times 10^{-31}\text{kg}$
Proton mass	$m_p = 1.673 \times 10^{-27}\text{kg}$
Ground state energy of hydrogen	$E_0 = -13.6\ \text{eV}$
Bohr radius of hydrogen	$r_0 = 0.529\text{Å}$

1. Non-descending Yo-yo (6 points) 不降的摇摇（6 分）

Consider a yo-yo as a uniform cylinder of mass m, radius a, and moment of inertia mk^2 about the horizontal axis through its center of gravity G, where k is the radius of gyration. The free end of the string is held, and the yo-yo descends unwinding the string without slipping.

(a) Find the tension in the string in terms of a, g, k, and m, where g is the acceleration due to gravity. [3]



Equations of motion for G (with G descending):

$$m\ddot{y} = mg - T,$$

$$mk^2\dot{\omega} = Ta.$$ [2 points]

No slipping of point P: $\ddot{y} = a\dot{\omega}$.

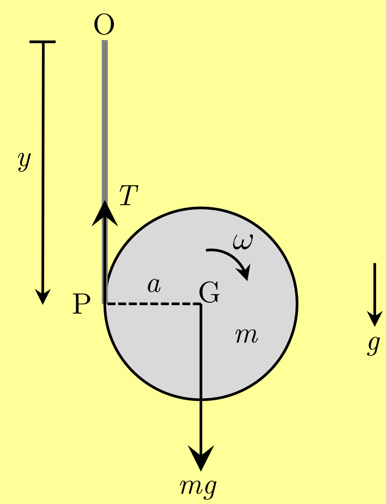
$$m\ddot{y}a = mga - mk^2\dot{\omega}$$

$$m\ddot{y}a = mga - \frac{mk^2}{a}\ddot{y}$$

$$\ddot{y}\left(a + \frac{k^2}{a}\right) = ga$$

$$\ddot{y} = \frac{ga^2}{a^2 + k^2}.$$

$$T = mg - m\ddot{y} = \frac{mgk^2}{a^2 + k^2}$$ [1 point]



(b) In order to maintain the spinning yo-yo at the same vertical level, the free end of the string is pulled upward. Find the acceleration of the free end in terms of a, g, and k. [3]

New equations of motion for G (with G now at rest):

$$\left.\begin{aligned} 0 &= mg - T' \\ mk^2\ddot{\omega} &= T'a \end{aligned}\right\} \Rightarrow \ddot{\omega} = \frac{ga^2}{k^2}. \quad \text{[2 points]}$$

Upward acceleration of point P is now

$$a\ddot{\omega} = \underline{\underline{\frac{ga^2}{k^2}}} \qquad \text{[1 point]}$$

2. Mirage (12 points) 海市蜃楼（12 分）

A light ray is travelling inside a medium with refractive index $n(y)$ varying with height y. At (x, y), the light ray makes an angle $\theta(y)$ with the horizontal direction, as shown in the figure below. Suppose the light ray passes through $(0, y_0)$, at which $\theta = \theta_0$.

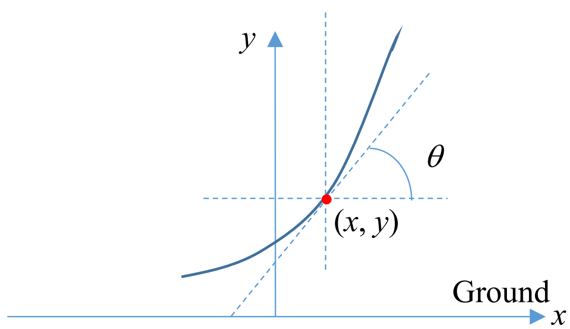
In this problem, you may find it convenient to use the following defined functions:

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) \text{ and } \sinh x = \frac{1}{2}(e^x - e^{-x}),$$

and their properties

$$\cosh^2 x - \sinh^2 x = 1, \frac{d}{dx}\cosh x = \sinh x, , \frac{d}{dx}\sinh x = \cosh x.$$

(a) Derive the differential equation satisfied by the trajectory $y = y(x)$ of the light ray. [3]



Using Snell's law, $n\cos\theta = \text{Constant} = n(y_0)\cos\theta_0$ [1 point]

$\cos\theta = \lim_{\Delta x \to 0} \frac{\Delta x}{\sqrt{\Delta x^2 + \Delta y^2}} = \frac{1}{\sqrt{1+y'^2}}$ [1 point]

Hence $\frac{n(y)}{\sqrt{1+y'(x)^2}} = \frac{n(y_0)}{\sqrt{1+y'(0)^2}} \Rightarrow y'^2 - \frac{y'(0)^2+1}{n^2(y_0)} n^2(y) + 1 = 0$

$$y'^2 - Kn^2(y) + 1 = 0 \qquad \text{[1 point]}$$

where $K = \frac{y'(0)^2+1}{n(y_0)^2} = \frac{1}{n(y_0)^2\cos^2\theta_0}$

(b) Suppose one can approximately let $n(y) = ay + b$ for $y \geq 0$, where $a, b > 0$. Find the equation of the trajectory. [Hint: The integral $\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u$ may be useful.] [5]

$$y'^2 = K(ay+b)^2 - 1$$

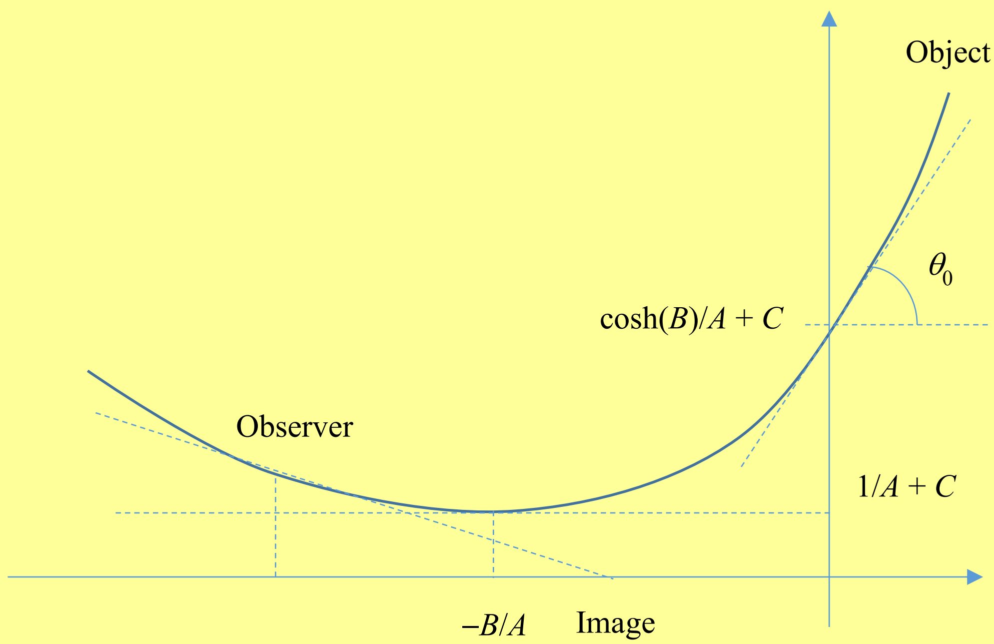
$$x = \int_{y_0}^{y} \frac{dy}{\sqrt{K(ay+b)^2 - 1}} \qquad \text{[2 points]}$$

Let $u = \sqrt{K}(ay + b)$

$$x = \frac{1}{\sqrt{K}a}\int_{\sqrt{K}(ay_0+b)}^{\sqrt{K}(ay+b)} \frac{du}{\sqrt{u^2 - 1}} = \frac{1}{\sqrt{K}a}\left\{\cosh^{-1}\left[\sqrt{K}(ay + b)\right] - \cosh^{-1}\left[\sqrt{K}(ay_0 + b)\right]\right\}$$

$y = \frac{1}{\sqrt{K}a}\cosh\left[\sqrt{K}ax + \cosh^{-1}\sqrt{K}(ay_0 + b)\right] - \frac{b}{a}$ [1 point for change of integration variable, 1 point for the integrated function, 1 point for the answer]

(c) Suppose the approximation in (c) is valid as a model of mirage. State the condition for the formation of mirage in terms of y_0, θ_0, a and b. [4]



$$y(x) = \frac{1}{A}\cosh(Ax + B) + C$$

For the mirage to form, the minimum of the optical path should be above the ground.

$$y(0) \geq 0 \qquad \text{[2 points]}$$

$$\frac{1}{A} + C = \frac{1}{\sqrt{K}a} - \frac{b}{a} \geq 0$$

$$\frac{1}{b} \geq \frac{\sqrt{{y'}_0^2 + 1}}{n(y_0)}$$

$$\cos\theta_0 \geq \frac{b}{ay_0+b} = \frac{n(0)}{n(y_0)} \qquad \text{[2 points]}$$

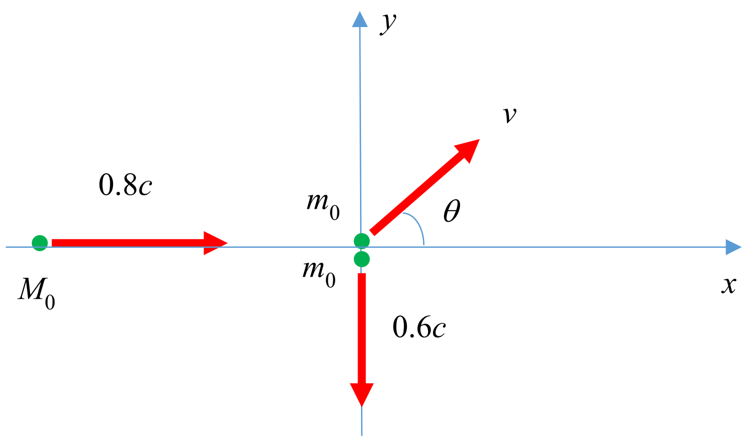
We will then see mirage.

3. Decay of a Relativistic Particle (12 points) 相对论粒子的衰变（12 分）

As shown in the figure below, a particle with rest mass M_0 initially moving with speed $0.8c$ in the positive x direction decays into two particles with equal rest mass m_0. One of the two particles moves with speed $0.6c$ in the $-y$ direction. Find

(a) the speed v and angle θ of the other particle,

(b) the ratio m_0/M_0.



(a) By conservation of energy we have

$$\frac{1}{\sqrt{1-0.8^2}}M_0c^2 = \frac{1}{\sqrt{1-0.6^2}}m_0c^2 + \frac{1}{\sqrt{1-v^2/c^2}}m_0c^2$$

$\frac{5}{3}M_0 = \left(\frac{5}{4} + \frac{1}{\sqrt{1-v^2/c^2}}\right)m_0$ [1 point for relativistic energy expression, 3 points for 3 terms correctly equated]

By conservation of momentum in the x direction we have

$$\frac{1}{\sqrt{1-v^2/c^2}}m_0 v\cos\theta = \frac{1}{\sqrt{1-0.8^2}}M_0 0.8c = \left(1 + \frac{4}{5}\frac{1}{\sqrt{1-v^2/c^2}}\right)m_0c$$

$\frac{1}{\sqrt{1-v^2/c^2}}v\cos\theta = \left(1 + \frac{4}{5}\frac{1}{\sqrt{1-v^2/c^2}}\right)c$ [1 point for relativistic momentum expression, 2 points for 2 terms correctly equated]

By conservation of momentum in the y direction we have

$$\frac{1}{\sqrt{1-v^2/c^2}}m_0 v\sin\theta = \frac{1}{\sqrt{1-0.6^2}}m_0 0.6c = \frac{3}{4}m_0c$$

$\frac{1}{\sqrt{1-v^2/c^2}}v\sin\theta = \frac{3}{4}c$ [2 points for 2 terms correctly equated]

Squaring the two equations and take sum, we have

$$\frac{v^2/c^2}{1-v^2/c^2} = \left(1 + \frac{4}{5}\frac{1}{\sqrt{1-v^2/c^2}}\right)^2 + \frac{9}{16}$$

$\gamma^2 - 1 = \left(1 + \frac{4}{5}\gamma\right)^2 + \frac{9}{16}$ [1 point for eliminating β or γ]

Solving the quadratic equation we get

$$\gamma = 5.694$$

Hence

$$v = 0.984c$$

Divide the y momentum equation by the x momentum equation, we have

$$\tan\theta = \frac{3/4}{1 + 4\gamma/5} = 0.135$$

Hence

$$\theta = 7.69° \qquad \text{[1 point for answer]}$$

(b) Substituting the value of γ back into the energy conservation equation, we have

$$\frac{5}{3}M_0 = \left(\frac{5}{4} + \gamma\right)m_0$$

$$\frac{m_0}{M_0} = \frac{\frac{5}{3}}{\frac{5}{4}+\gamma} = \left(\frac{3}{4} + \frac{3}{5}\gamma\right)^{-1} = 0.24 \qquad \text{[1 point for answer]}$$

4. A Tauon-Hydrogen Atom (8 points) Tau 粒子-氢原子（8 分）

The particle tauon (τ) has the same charge as that of an electron but is 210 times more massive. A tauon with initial speed of 8.0×10^5 m/s is captured by a proton to form a τ-hydrogen atom.

(a) When the atom decays to ground state, radiation will be emitted. What is the wavelength (in angstrom, Å) of the radiation?

(b) To excite a τ-hydrogen atom in ground state to excited states, what is the maximum possible wavelength (in Å)?

(c) What is the distance (in Å) between the tauon and the proton when the atom is in ground state?

Because tauon is much more massive than electron, we need to consider two-body effects.

The reduced mass is

$$\mu = \frac{m_\tau m_p}{m_\tau + m_p} = 188.5m_e \qquad \text{[1 point]}$$

Solving

$$\begin{cases} E = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} + \frac{1}{2}\mu v^2 \\ \mu\frac{v^2}{r} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} \\ \mu v r = \frac{nh}{2\pi} \end{cases} \qquad \text{[3 points for 3 equations]}$$

we get

$$E_n = -13.6\frac{1}{n^2}\frac{\mu}{m_e}\text{eV} \qquad \text{[1 point]}$$

$$r_n = 0.53n^2 \frac{m_e}{\mu} \text{Å}$$

(a) $\frac{hc}{\lambda} = \frac{1}{2}\mu v^2 - E_1 = \frac{1}{2}\mu v^2 + 13.6 \times 188.5 = 2735\text{eV}$

$$\lambda = 4.53\text{Å} \qquad \text{[1 point]}$$

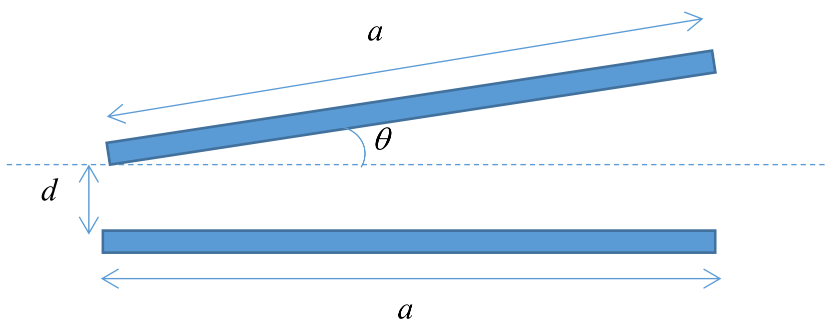
(b) $\frac{hc}{\lambda} = E_2 - E_1 = 13.6 \times 188.5 \times \frac{3}{4}\text{eV}$

$$\lambda = 6.45\text{Å} \qquad \text{[1 point]}$$

(c) $r_1 = 0.529 \frac{m_e}{\mu} = 2.81 \times 10^{-3}\text{Å}$ [1 point]

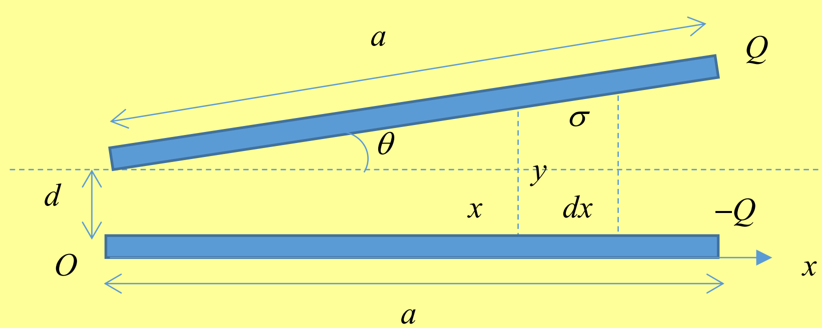
5. Inclined Capacitor (6 points) 斜面电容器（6 分）

As shown in the figure, a capacitor is formed by two square plates both with side length a, with minimum separation $d \ll a$. When the inclination angle θ is very small, derive the capacitance up to first order in θ.



Let the charges be Q and $-Q$ and the potential difference be V.

Consider a small strip with width dx at position x, as shown in the figure below.



The area of the strip is adx, and the distance between the two strips is y. The surface charge density is σ. Then we have

$$\int_{x=0}^{x=a} \sigma a dx = Q \qquad \text{[1 point]}$$

$$E = \frac{\sigma}{\varepsilon_0} \quad \text{[1 point]}$$

$$V = Ey = \frac{\sigma}{\varepsilon_0}(d + x\tan\theta) \quad \text{[1 point]}$$

The capacitance is hence

$$C = \frac{Q}{V} \quad \text{[1 point]}$$

$$C = \varepsilon_0 a \int_{x=0}^{x=a} \frac{dx}{d + x\tan\theta} = \frac{\varepsilon_0 a}{\tan\theta}\ln\left(1 + \frac{a\tan\theta}{d}\right) \quad \text{[1 point]}$$

For small θ, we have

$$C \approx \frac{\varepsilon_0 a}{\theta}\ln\left(1 + \frac{a\theta}{d}\right) \approx \frac{\varepsilon_0 a}{\theta}\left(\frac{a\theta}{d} - \frac{1}{2}\left(\frac{a\theta}{d}\right)^2\right) = \frac{\varepsilon_0 a^2}{d}\left(1 - \frac{a\theta}{2d}\right) \quad \text{[1 point]}$$

6. Newton's Rings(6 points) 牛顿环（6 分）

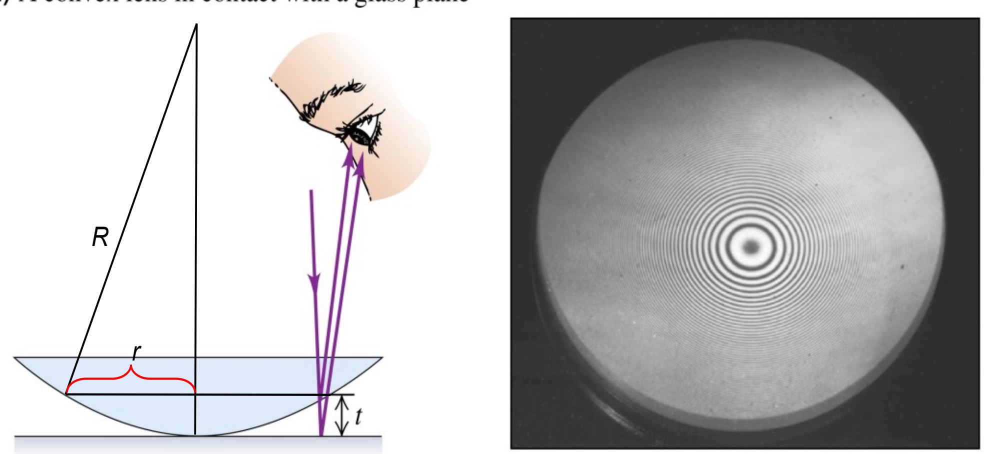
Figure (a) shows a convex lens with radius of curvature R lying on a flat glass plate (both glasses have the refractive index $n_g = 1.5$) and illuminated from above by light with wavelength λ. Figure (b) (a photograph taken from above the lens) shows that circular interference fringes (called Newton's rings) appear, associated with the variable thickness d of the air film between the lens and the plate.

(a) Find the radii r of the interference maxima assuming $r \ll R$. [3]

(b) Now, the entire setup is immersed in water (with refractive index $n_w = 1.33$). What are the new values of the radii r of the interference maxima assuming $r \ll R$. [3]

(a) A convex lens in contact with a glass plane

(b) Newton's rings: circular interference fringes



(a) path difference = $2t$ and maxima occurs at $2t = (m + \frac{1}{2})\lambda$ [1 point]

and

$$t = R - \sqrt{R^2 - r^2} \simeq \frac{1}{2}\frac{r^2}{R}$$ [1 point]

Remark: There is also a geometric theorem stating that when two chords of a circle intersect, the products of the lengths of the segments of the chords are the same. Hence $(2R-t)t = r^2 \Rightarrow$ $t \approx r^2 / 2R$.

Therefore

maxima located at $r_m = \sqrt{(m+\frac{1}{2})\lambda R}$ [1 point]

(b) When the setup is immersed in water, the path difference = $2t$ and maxima occurs at $2t = (m+\frac{1}{2})\lambda'$ where $\lambda' = \lambda/n_w$ [1 point]

Maxima located at $r_m = \sqrt{(m+\frac{1}{2})\frac{\lambda R}{n_w}}$. [2 points]