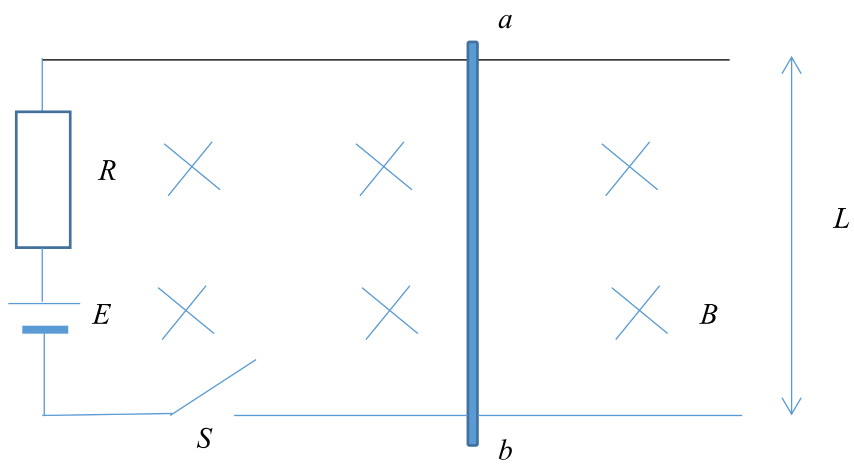
Physics Enhancement Programme Phase 2
Selection Test 4 (Total 50 points)
18 February 2017

1. (9 points) Consider the system as shown in the figure below,



The electromotive force of the battery is E, with internal resistance r. The metal rod ab and the rails have negligible resistance. The metal rod can move on the rail. The kinetic friction is a constant force f. The circuit is inside a region of uniform magnetic field B pointing into the page.

(a) Find the terminal velocity v_T of the metal rod after the switch S is closed.
(b) The terminal velocity depends on B. Find the magnitude of the magnetic field so that v_T is maximum. What is the current in this case?
(c) If $r > R$, and when the velocity of the metal rod is v_p, the power output external to the battery is maximum $P_{\max}$, find the relation between $P_{\max}$ and v_p.

(a) After closing the switch, there will be current flowing through the rod and the rod will move to the right under Lorentz force. The induced emf in the due to the rod's cutting the magnetic field line is $-BLv$. Hence the equation of motion is

$$\frac{E - BLv}{R + r} BL - f = ma$$

The terminal velocity satisfies

$$\frac{E - BLv_T}{R + r} BL - f = 0$$

$$v_T = \frac{E}{BL} - \frac{f(R + r)}{B^2 L^2}$$

(b) Write in the form of quadratic equations in B:

$$v_T L^2 B^2 - LEB + f(R + r) = 0$$

In order to have real solutions of B, the discriminant should be non-negative:

$$L^2E^2 \geq 4v_T L^2 f(R+r)$$

Hence

$$v_T \leq \frac{E^2}{4f(R+r)}$$

Therefore the maximum value is

$$v_{T\max} = \frac{E^2}{4f(R+r)}$$

At this value, the discriminant vanishes and hence

$$B = \frac{E}{2Lv_{T\max}} = \frac{2f(R+r)}{EL}$$

Alternatively,

$$\frac{d}{dB}v_T = \frac{d}{dB}\left(\frac{E}{BL} - \frac{f(R+r)}{B^2L^2}\right) = -\frac{E}{B^2L} + \frac{2f(R+r)}{B^3L^2} = 0$$

$$B = \frac{2f(R+r)}{EL}$$

$$\frac{d^2}{dB^2}v_T = \frac{2E}{B^3L} - \frac{6f(R+r)}{B^4L^2}$$

Substitute the value of B obtained, we have

$$\frac{d^2}{dB^2}v_T = \frac{2E}{L}\left(\frac{EL}{2f(R+r)}\right)^3 - \frac{6f(R+r)}{L^2}\left(\frac{EL}{2f(R+r)}\right)^4$$
$$= \left(\frac{EL}{2f(R+r)}\right)^3\left(\frac{2E}{L} - \frac{6f(R+r)}{L^2}\frac{EL}{2f(R+r)}\right)$$

$$= \left(\frac{EL}{2f(R+r)}\right)^3\left(\frac{2E}{L} - \frac{3E}{L}\right) = -\frac{E}{L}\left(\frac{EL}{2f(R+r)}\right)^3 < 0$$

The current at this value is

$$I = \frac{E - BLv_{T\max}}{R+r} = \frac{E}{R+r}\left[1 - \frac{BLE}{4f(R+r)}\right] = \frac{E}{2(R+r)} < \frac{E}{R+r}$$

(c) There are two ways of writing the output power of the battery

$$P = EI - I^2r$$

or

$$P = BLvI + I^2R$$

Substitute

$$I = \frac{E - BLv}{R + r}$$

both will lead to

$$P = \frac{1}{(R+r)^2}[-B^2L^2rv^2 + BLEv(r-R) + E^2R]$$

$$= \frac{B^2L^2r}{(R+r)^2}\left\{-\left[v - \frac{E(r-R)}{2BLr}\right]^2 + \frac{E^2(R+r)^2}{4B^2L^2r^2}\right\}$$

So P is maximum when

$$v_p = \frac{E(r-R)}{2BLr}$$

and

$$P_{\max} = \frac{B^2L^2r}{(R+r)^2}\frac{E^2(R+r)^2}{4B^2L^2r^2} = \frac{E^2}{4r}$$

Combining the above two equations we have

$$P_{\max} = \frac{B^2L^2r}{(r-R)^2}v_p^2$$

2. **(8 points)** An elastic string has a natural length of 1 m. It is assumed that when it is stretched, the restoring force obeys Hooke's law. One end of the string *A* is fixed at the ceiling, while the other end *B* is attached to a small ball with mass m = 0.2 kg. It is measured that after the ball is attached, the length of the string is stretched to 2 m at equilibrium. Now if the ball is released from rest at point *A*, after how long will it hit the ceiling? ($g = 9.8$ ms^{-2})

Denote the original length by $l_0 = 1$ m.

Let the original length be at *B*, and the new equilibrium position be *O*.

The ball will first free fall from *A* to *B*, and the time taken is

$$t_1 = \sqrt{\frac{2l_0}{g}} = 0.452\text{s}$$

Below *B*, the ball is not only under gravity but also the elastic force.

Let the vertically downward direction be the positive *x* axis, and take *O* as the origin.

Also denote the stretched length by $l' = 1$ m. Then when the ball is at x, the stretched length is $l' + x$ and the restoring force is $-k(l' + x)$, where k is the force constant in the Hooke's law. So the force acting on the ball is

$$F = mg - k(l' + x) = -kx.$$

Therefore below B the ball will perform SHM with equilibrium position at O:

$$x = A\cos(\omega t + \varphi),$$

where

$$\omega = \sqrt{k/m}$$

From the initial conditions at point B, we have

$$A\cos\varphi = -l'$$

$$-\omega A\sin\varphi = \sqrt{2gl_0}$$

And hence

$$A = \sqrt{l'^2 + 2gl_0 m/k} = \sqrt{3}$$

Also φ is in the third quadrant and therefore

$$\varphi = \arctan\sqrt{2} + \pi$$

When the ball moves from B to the lowest point C, the phase angle changes from $\arctan\sqrt{2} + \pi$ to 2π and hence the time taken is

$$t_2 = \frac{2\pi - \left(\arctan\sqrt{2} + \pi\right)}{\omega} = \frac{\pi - \arctan\sqrt{2}}{\sqrt{k/m}} = 0.698\text{s}$$

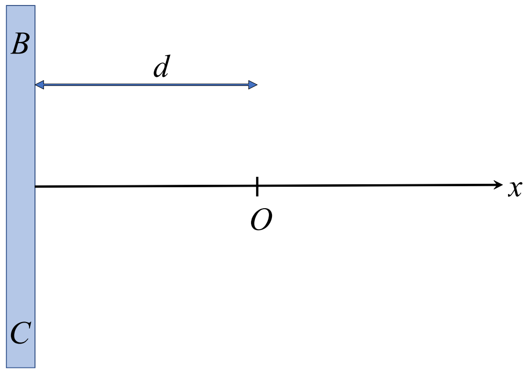
The total time taken from A to C is

$$t_1 + t_2 = 1.15\text{s}$$

The time taken for the ball to go back from C to A is also 1.15 s.

Hence the ball will hit the ceiling again after 2.3 s.

3. **(9 points)** In the figure, the origin O is a source which oscillates with amplitude A and angular frequency ω, that is, $y(0,t) = A\cos\omega t$. The oscillation generates a one-dimensional sinusoidal wave with wavelength λ along the x-axis of the stretched string with linear density ρ. BC is a dense reflecting plane where all incoming waves get reflected. $d = 5\lambda/4$ where λ is the wavelength.



(a) Find the wave function for $0 > x > -d$ and $x > 0$.

The 1d sinusoidal wave generated by a point source at origin is

$$y_1(x,t) = A\cos\omega(t - x/v) \quad \text{for } x \geq 0$$

$$y_1(x,t) = A\cos\omega(t + x/v) \quad \text{for } d \leq x \leq 0$$

The reflected wave from the dense plane is

$$y_2(x,t) = A\cos[\omega(t - x/v) + \alpha] \quad \text{for } x \geq 0$$

where α is the angle such that

$$y_1(-d,t) + y_2(-d,t) = 0 \quad \Rightarrow \alpha = \omega\left(t + \tfrac{-d}{v}\right) - \omega\left(t - \tfrac{-d}{v}\right) + (2n+1)\pi \text{ for and integer n.}$$

$$\Rightarrow \alpha = -\frac{2\omega d}{v} + (2n+1)\pi = -5\pi + (2n+1)\pi$$

Therefore, the resultant wave becomes

$$y(x,t) = y_1(x,t) + y_2(x,t) = 2A\cos\omega(t - x/v) \quad \text{for } x \geq 0$$

$$y(x,t) = A\cos\omega(t + x/v) + A\cos\omega(t - x/v) = 2A\cos\omega x/v\cos\omega t \quad \text{for } -d \leq x \leq 0$$

(b) Find the time-averaged mechanical energy per unit length for $0 > x > -d$. (Hint: You can calculate the kinetic energy and potential energy of a small segment at x separately.)

Consider the kinetic energy of a segment of length dx.

$$dT = \frac{1}{2}(\rho dx)\left(\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2}(\rho dx)\left(-2A\omega\cos\frac{\omega x}{v}\sin\omega t\right)^2 = 2\rho\omega^2 A^2\cos^2\frac{\omega x}{v}\sin^2\omega t\, dx$$

Consider the potential energy of the segment. Extension of the segment:

$$ds = dx\left[\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2} - 1\right] \approx \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx$$

$$dV = \tau ds = \frac{\tau}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx = \frac{\tau}{2}\left(-\frac{2\omega A}{v}\sin\frac{\omega x}{v}\cos\omega t\right)^2 dx = \frac{2\tau\omega^2 A^2}{v^2}\sin^2\frac{\omega x}{v}\cos^2\omega t\, dx$$

Since $v^2 = \tau/\rho$, $dV = 2\rho\omega^2 A^2 \sin^2\frac{\omega x}{v}\cos^2\omega t\, dx$.

Averaging over time, $\langle dT\rangle = \rho\omega^2 A^2\cos^2\frac{\omega x}{v}dx$ and $\langle dV\rangle = \rho\omega^2 A^2 \sin^2\frac{\omega x}{v}dx$.

Total mechanical energy: $\langle dE\rangle = \langle dT\rangle + \langle dV\rangle = \rho\omega^2 A^2 dx$.

Mechanical energy per unit length: $\langle\frac{dE}{dx}\rangle = \rho\omega^2 A^2$.

(c) Find the time-averaged mechanical power transmitted by the oscillator.

For $x = 0^-$, $\frac{\partial y}{\partial x} = -\frac{2\omega A}{v}\sin\frac{\omega x}{v}\cos\omega t = 0$.

For $x = 0^+$, $\frac{\partial y}{\partial x} = -\frac{2\omega A}{v}\sin\omega\left(t - \frac{x}{v}\right) = \frac{2\omega A}{v}\sin\omega t$.

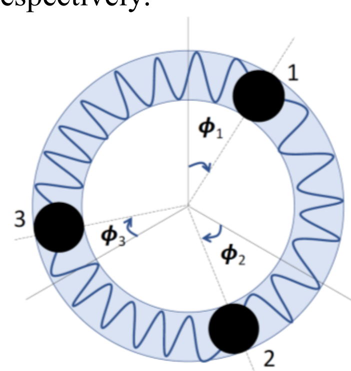
Force exerted by the oscillator: $F = -\tau\left.\frac{\partial y}{\partial x}\right|_{0^+} + \tau\left.\frac{\partial y}{\partial x}\right|_{0^-} = -\frac{2\tau\omega A}{v}\sin\omega t$.

Velocity: $v = \frac{\partial y}{\partial t} = -2\omega A\sin\omega t$

Transmitted power: $P = Fv = \frac{4\tau\omega^2 A^2}{v}\sin^2\omega t$

Time-averaged transmitted power: $\langle P\rangle = \frac{2\tau\omega^2 A^2}{v} = 2\rho\omega^2 A^2 v$

4. **(8 points)** Three identical particles, each of mass m, are constrained to lie on a horizontal circle of radius R and are connected by identical springs lying on the circle, each of spring constant k and natural length $l = 2\pi R/3$. Initially, three particles are located at the vertices of an equilateral triangle. In the figure, ϕ_1, ϕ_2, ϕ_3 are the angular displacements of particles from their equilibrium position respectively.



(a) With small angular displacement, find the angular frequencies of oscillations in this system.
(b) Derive the general forms of $\{\phi_1(t), \phi_2(t), \phi_3(t)\}$ for these small oscillations.
(c) At $t = 0$, $\{\phi_1(0), \phi_2(0), \phi_3(0)\} = \{A, 2A, -3A\}$ and $\{\dot{\phi}_1(0), \dot{\phi}_2(0), \dot{\phi}_3(0)\} = \{0,0,0\}$. Derive $\{\phi_1(t), \phi_2(t), \phi_3(t)\}$ at any time $t > 0$.

(a) The length of string a (between mass 1 & 2) = $R\left(\frac{2\pi}{3} - \phi_1 + \phi_2\right)$

The length of string b (between mass 2 & 3) = $R\left(\frac{2\pi}{3} - \phi_2 + \phi_3\right)$

The length of string c (between mass 3 & 1) = $R\left(\frac{2\pi}{3} - \phi_3 + \phi_1\right)$

For particle 1, the equation of motion becomes:

$mR\ddot{\phi}_1 = kR\left(\frac{2\pi}{3} - \phi_1 + \phi_2\right) - kR\left(\frac{2\pi}{3} - \phi_3 + \phi_1\right) = kR(-2\phi_1 + \phi_2 + \phi_3).$

Similarly, we have

$$m\ddot{\phi}_1 = k(-2\phi_1 + \phi_2 + \phi_3)$$
$$m\ddot{\phi}_2 = k(-2\phi_2 + \phi_3 + \phi_1)$$
$$m\ddot{\phi}_3 = k(-2\phi_3 + \phi_1 + \phi_2)$$

Equation 1 – Equation 2, we have

$$(\ddot{\phi}_1 - \ddot{\phi}_2) = -\frac{3k}{m}(\phi_1 - \phi_2)$$

i.e. the angular frequency is $\omega = \sqrt{3k/m}$.

(b) The general forms of the solution are:

$$\{\phi_1(t), \phi_2(t), \phi_3(t)\} = \{C\cos(\omega t + \alpha), -C\cos(\omega t + \alpha), 0\}.$$

By symmetry, another independent solution is

$$\{\phi_1(t), \phi_2(t), \phi_3(t)\} = \{C\cos(\omega t + \alpha), 0, -C\cos(\omega t + \alpha)\}.$$

(c) Let the general solution be

$$\begin{cases} \phi_1(t) = C_1\cos(\omega t + \alpha_1) + C_2\cos(\omega t + \alpha_2) \\ \phi_2(t) = -C_1\cos(\omega t + \alpha_1) \\ \phi_3(t) = -C_2\cos(\omega t + \alpha_2) \end{cases}$$

$\{\dot{\phi}_1(0), \dot{\phi}_2(0), \dot{\phi}_3(0)\} = \{0,0,0\} \Rightarrow \alpha_1 = \alpha_2 = 0$

$\{\phi_1(0), \phi_2(0), \phi_3(0)\} = \{A, 2A, -3A\}$

$$\Rightarrow \begin{cases} C_1 + C_2 = A \\ -C_1 = 2A \\ -C_2 = -3A \end{cases} \Rightarrow \begin{cases} \phi_1(t) = A\cos(\omega t) \\ \phi_2(t) = 2A\cos(\omega t) \\ \phi_3(t) = -3A\cos(\omega t) \end{cases}$$

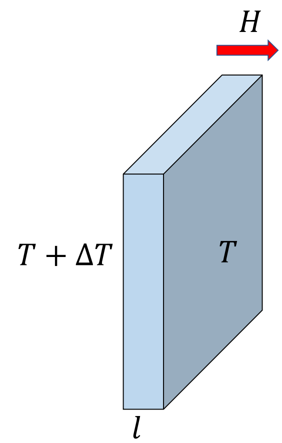
5. Heat Transfer (8 points)

An air-conditioner is a machine that extracts heat from indoor (cold-environment) and dumps into outdoor (hot environment) by doing work. Assume that the absolute temperatures of hot- and cold-environment are T_1 and T_2 respectively. If the heat extracted from the cold-environment is Q and the work done by the air-conditioner is W, we have

$$\frac{Q}{W} \leq \frac{T_2}{T_1 - T_2}$$

where the equality sign holds for the ideal air-conditioner.

An ideal air-conditioner is operating during the summer when the outdoor temperature is 35ºC and the indoor temperature is maintained at 20ºC. As the outdoor temperature is higher, heat is transferred from outdoor to indoor via conduction which satisfies the following condition: Consider a layer of material with thickness l, surface area S. The temperature difference between two sides of the material is ΔT. The heat transfer rate (heat energy transferred per second) through this conducting layer from high temperature to low temperature is



$$H = \kappa \frac{\Delta T}{l} S$$

where κ is the thermal conductivity which is a constant.

(a) If the heat can only be transferred into the indoor via the window glass with surface area $S = 5\text{m}^2$ and $l = 2\text{mm}$. And the thermal conductivity of glass is $\kappa = 0.75\text{W}/(\text{m} \cdot \text{K})$. Calculate the amount of work done by the air-conditioner in 12 hours.

(b) If the window glass is replaced by the double-layer glass where the thickness of each glass is $l = 2\text{mm}$. Between two glasses, there is a layer of air with thickness $l_0 = 0.5\text{mm}$ and the thermal conductivity of air is $\kappa_0 = 0.025\text{W}/(\text{m} \cdot \text{K})$. Calculate the amount of work-done by the air-conditioner in this case.

(c) In Hong Kong, the cost of electricity is \$0.8 per kilowatt hour (kW·h), calculate how much we can save by replacing the window with double-layer glass in every 12 hours.

(a)

$$H = \kappa \frac{\Delta T}{l} S = \kappa \frac{T_1 - T_2}{l} S$$

Therefore, the rate of heat extracted (power) from indoor to outdoor is

$$Q = H$$

And the work done of the ideal air conditioner per second (power) should be

$$W = \frac{T_1 - T_2}{T_2} Q = \kappa \frac{(T_1 - T_2)^2}{T_2} \frac{S}{l} = 1440W$$

The energy consumed in 12 hours is

$$W = 1.4kW \times 12 hours = 16.8kW \cdot hr$$

(b)

The heat resistivity becomes

$$R=\frac{l}{\kappa S}+\frac{l_0}{\kappa_0 S}+\frac{l}{\kappa S}=0.005K/W$$

Rate of heat transferred via conduction

$$H=\frac{T_1-T_2}{R}=Q$$

And the work done of the ideal air conditioner per second (power) should be

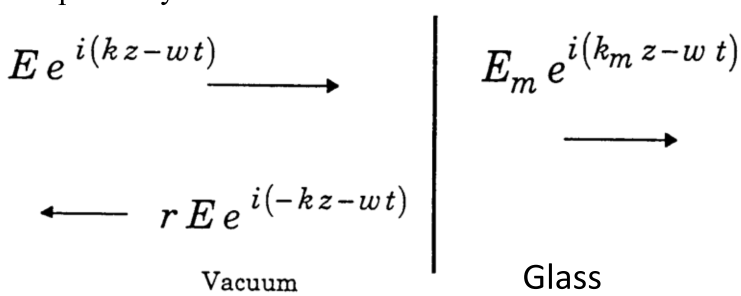
$$W=\frac{T_1-T_2}{T_2}=\frac{(T_1-T_2)^2}{T_2}\frac{1}{R}=153.6W$$

The energy consumed in 12 hours is

$$W=0.15kW\times 12hours=1.8kW\cdot hour$$

(c) Money saved = (16.8-1.8)*0.8=$12.

6. **(8 points)** A plane of electromagnetic wave of frequency ω is normally incident on a thick glass of unit area where the permittivity and permeability are ε and μ respectively. The incident electric field has amplitude E and wavenumber k and the reflected field has amplitude rE. The transmitted electric field has amplitude E_m and wavenumber k_m.
   (a) Find E_m/B_m where B_m is the magnetic field amplitude in the glass. Assume that the glass is nonmagnetic so that $\mu=\mu_0$.
   (b) Find r by considering the boundary conditions at the plate and the result of (a).
   (c) Calculate the fraction of time-averaged power carried by the reflected and transmitted waves respectively.



(a) $B_m=\frac{E_m}{v}=\sqrt{\varepsilon\mu_0}E_m$

(b)
$$B_1=B_2\Longrightarrow\frac{E-rE}{c}=\frac{E_m}{v}$$
$$E_1=E_2\Longrightarrow E+rE=E_m$$

Solving together, we have $r=\frac{\sqrt{\varepsilon_0}-\sqrt{\varepsilon}}{\sqrt{\varepsilon_0}+\sqrt{\varepsilon}}$

(c) From (b), $E_m=(1+r)E=\left(\frac{2\sqrt{\varepsilon_0}}{\sqrt{\varepsilon_0}+\sqrt{\varepsilon}}\right)E=tE$

Time-averaged power carried by reflected wave = $\frac{1}{\mu_0}\langle \vec{E}_r \times \vec{B}_r \rangle = \frac{r^2 E^2}{2c\mu_0}$

Time-averaged power carried by transmitted wave = $\frac{1}{\mu_0}\langle \vec{E}_t \times \vec{B}_t \rangle = \frac{t^2 E^2}{2c\mu_0}$