Physical quantities and units

Three fundamental physical quantities are mass, length and time. The corresponding fundamental SI units are the kilogram, the meter and the second. Derived units for other physical quantities are products or quotients of the basic units. Equations must be dimensionally consistent; two terms can be added only when they have the same units.

Scalars, vectors, and vector addition

Scalar quantities are numbers and combine according to the usual rules of arithmetic. Vector quantities have direction as well as magnitude and combine according to the rules of vector addition. The negative of a vector has the same magnitude but points in the opposite direction.

Vector components and vector addition

Vectors can be added by using components of vectors. The x-component of $\vec{R} = \vec{A} + \vec{B}$ is the sum of the x-components of $\vec{A}$ and $\vec{B}$, and likewise for the y- and z- components.

$$R_x = A_x + B_x$$

$$R_y = A_y + B_y$$

$$R_z = A_z + B_z$$

Unit vectors

Unit vectors describe directions in space. A unit vector has a magnitude of 1, with no units. The unit vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$, aligned with the x-, y- and z-axes of a rectangular coordinate system, are especially useful.

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_k\hat{k}$$

Scalar product

The scalar product $C = \vec{A} \cdot \vec{B}$ of two vectors $\vec{A}$ and $\vec{B}$ is a scalar quantity. It can be expressed in terms of the magnitudes of $\vec{A}$ and $\vec{B}$. The scalar product is commutative; $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$. The scalar product of two perpendicular vectors is zero.

$$\vec{A} \cdot \vec{B} = AB\cos\phi = |\vec{A}||\vec{B}|\cos\phi$$

$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_y$$

Straight-line motion, average and instantaneous x-velocity

When a particle moves along a straight line, we describe its position with respect to an origin O by means of a co-ordinate such as x. The particle's average x-velocity $v_{\mathrm{av}-x}$ during a time interval $\Delta x = x_2 - x_1$ divided by Δt. The instantaneous x-velocity v_x at any time t is equal to the average x-velocity over the time interval from t to $t + \Delta t$ in the limit that Δt goes to zero. Equivalently, v_x is the derivative of the position function with respect to time.

$$v_{\text{av}-x} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}$$

$$v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

Average and instantaneous x-acceleration

The average x-acceleration $a_{\text{av}-x}$ during a time interval Δt is equal to the change in velocity $\Delta v_x = v_{2x} - v_{1x}$ during that time interval divided by Δt. The instantaneous x-acceleration a_x is the limit of $a_{\text{av}-x}$ as Δt goes to zero, or the derivative of v_x with respect to t

$$a_{\text{av}-x} = \frac{\Delta v_x}{\Delta t} = \frac{v_{2x} - v_{1x}}{t_2 - t_1}$$

$$v_x = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} = \frac{dv_x}{dt}$$

Straight-line motion with constant acceleration

When the x-acceleration is constant, four equations relate the position x and the x-veleocity v_x at any time t to the initial position x_0, the initial x-velocity v_{0x} (both measured at time $t = 0$), and the x-acceleration a_x.

$$v_x = v_{0x} + a_x t$$

$$x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2$$

$$v_x^2 = {v_{0x}}^2 + 2a_x(x - x_0)$$

$$x - x_0 = \frac{1}{2}(v_{0x} + v_x)t$$

Freely falling bodies

Free fall is a case of motion with constant acceleration. The magnitude of the acceleration due to gravity is a positive quantity, g. The acceleration of a body in free fall is always downward.

Straight-line motion with varying acceleration

When the acceleration is not constant but is a known function of time, we can find the velocity and position as functions of time by integrating the acceleration function.

$$v_x = v_{0x} + \int_0^t a_x \, dt$$

$$x_x = x_0 + \int_0^t v_x \, dt$$

Position, velocity, and acceleration vectors

The position vector $\vec{r}$ of a point P in space is the vector the origin to P. Its components are the coordinate x,y and z.

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

The average velocity vector $\vec{v}_{\text{av}}$ during the time interval Δt is the displacement $\Delta\vec{r}$ (the change in position vector $\vec{r}$) divided by Δt. The instantaneous velocity vector $\vec{v}$ is the time derivative of $\vec{r}$, and its components are the time derivatives of x,y and z.

$$\vec{v}_{\text{av}} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} = \frac{\Delta\vec{r}}{\Delta t}$$

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

$$v_x = \frac{dx}{dt} \quad v_y = \frac{dy}{dx} \quad v_z = \frac{dz}{dx}$$

The instantaneous speed is the magnitude of $\vec{v}$. The velocity $\vec{v}$ of a particle is always tangent to the particle's path.

The average acceleration vector $\vec{a}_{\text{av}}$ during the time interval Δt equals $\Delta\vec{v}$ (the change in velocity vector $\vec{v}$) divided by Δt. The instantaneous acceleration vector $\vec{a}$ is the time derivative of $\vec{v}$, and its components are the time derivatives of v_x, v_y and v_z.

$$\vec{a}_{\text{av}} = \frac{\vec{a}_2 - \vec{a}_1}{t_2 - t_1} = \frac{\Delta\vec{v}}{\Delta t}$$

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

$$a_x = \frac{dv_x}{dt} \quad a_y = \frac{dv_y}{dx} \quad a_z = \frac{dv_z}{dx}$$

The component of acceleration parallel to the direction of the instantaneous velocity affects the speed, while the component of $\vec{a}$ perpendicular to $\vec{v}$ affects the direction of motion.

Projectile motion

In projectile motion with no air resistance, $a_x = 0$ and $a_y = -g$. The coordinates and velocity components are simple functions of time, and the shape of the path is always a parabola. We usually choose the origin to be at the initial position of the projectile.

$$x = (v_0 \cos \alpha_0)t$$

$$y = (v_0 \sin \alpha_0)t - \frac{1}{2}gt^2$$

$$v_x = v_0 \cos \alpha_0$$

$$v_x = v_0 \sin \alpha_0 - gt$$

Uniform and nonuniform circular motion

When a particle moves in a circular path of radius R with constant speed v (uniform circular motion), its acceleration $\vec{a}$ is directed toward the centre of the circle and perpendicular to $\vec{v}$. The magnitude a_{rad} of the acceleration can be expressed in terms of v and R or in terms of R and the period T (the time for the one revolution), where $v = 2\pi R/T$.

If the speed is not constant in circular motion (nonuniform circular motion), there is still a radial component of $\vec{a}$ given by

$$a_{\text{rad}} = \frac{v^2}{R}$$

$$a_{\text{rad}} = \frac{4\pi^2 R}{T^2}$$

But there is also a component of $\vec{a}$ parallel 9tangential) to the path. This tangential component is equal to the rate of change of speed, dv/dt.

Relative velocity

When a body P moves relative to a body (or reference frame) B, and B moves relative to a body (or reference frame) A, we denote the velocity to A by $\vec{v}_{P/B}$, the velocity of P relative to A by $\vec{v}_{P/A}$, and B relative to A by the velocity $\vec{v}_{B/A}$. If these velocities are all along the same line, their component along that line are related by

$$v_{P/A-x} = v_{P/B-x} + v_{B/A-x}$$

More generally, these velocities are related by

$$\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A}$$

Force as a vector

Force is quantitative measure of the interaction between two bodies. It is a vector quantity. When several forces act on a body, the effect on its motion is the same as when a single force, equal to the vector sum (resultant) of the forces, acts on the body.

$$\vec{R} = \sum \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots$$

The net force on a body and Newton's first law

Newton's first law states that when the vector sum of all forces acting on a body (the net force) is zero, the body is in equilibrium and has zero acceleration. If the body is initially at rest, it remains at rest; if it is initially in motion, it continues to move with constant velocity. This is valid in inertial frames of reference only.

$$\sum \vec{F} = 0$$

Mass, acceleration, and Newton's second law

The inertial properties of a body are characterized by its mass. The acceleration of a body under the action of a given set of forces is directly proportional to the vector sum of the forces (the net force) and inversely proportional to the mass of the body. This relationship is Newton' second law. Like Newton's first law, this law is valid in inertial frames of reference only. The unit of force is defined in terms of the units of mass and acceleration. In Si units, the unit of force is the newton (N), equal to 1 kg . m/s^2

$$\sum \vec{F} = m\vec{a}$$

$$\sum F_x = ma_x \quad \sum F_y = ma_y \quad \sum F_z = ma_z$$

Weight

The weight $\vec{w}$ of a body is the gravitational force exerted on it by the earth. Weight is a vector quantity. The magnitude of the weight of a body at any specific location is equal to the product of its mass m and the magnitude of the acceleration due to gravity g at that location. While the weight of a body depends on its location, the mass is independent of location.

$$w = mg$$

Newton's third law and action-reaction pairs

Newton's third law states that when two bodies interact, they exert forces on each other that are equal in magnitude and opposite in direction. These forces are called action and reaction forces. Each of these two forces acts on only one of the two bodies; they never act on the same body.

$$\vec{F}_{A\ on\ B} = -\vec{F}_{B\ on\ A}$$

Using Newton's first law

When a body is in equilibrium in an inertial frame of reference – that is, either at rest or moving with constant velocity – the vector sum of forces acting on it must be zero (Newton's first law). Free body diagrams are essential in identifying the forces that act on the body being considered.

$$\sum \vec{F} = 0 \text{ (Vector form)}$$

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \text{(Component form)}$$

Newton's third law (action and reaction) is also frequently needed in equilibrium problems. The two forces in an action-reaction pair never act on the same body. The normal force exerted on a body by a surface is not always equal to the body's weight.

Using Newton's second law

If the vector sum of forces on a body is not zero, the body accelerates. The acceleration is related to the net force by Newton's second law. Just as for equilibrium problems, free-body diagrams are essential for solving problems involving Newton's second law, and the normal force exerted on a body is not always equal to its weight.

$$\sum \vec{F} = m\vec{a} \text{ (Vector form)}$$

$$\sum F_x = ma_x \quad \sum F_y = ma_y \quad \text{(Component form)}$$

Friction and fluid resistance

The contact force between two bodies can always be represented in terms of a normal force $\vec{n}$ perpendicular to the surface of contact and a friction force f parallel to the surface.

When a body is sliding over the surface, the fiction force is called kinetic friction. Its magnitude f_k is approximately equal to the normal force magnitude n multiplied by the coefficient of kinetic friction μ_k. When a body is not moving relative to a surface, the fiction force is called static friction. The maximum possible static friction force is approximately equal to the magnitude n of the normal force multiplied by the coefficient of static friction μ_s. The actual static friction force may be anything from zero to this maximum value, depending on the situation. Usually μ_s is greater than μ_k for a given pair of surfaces in contact.

$$f_k = \mu_k n \text{ (Magnitude of kinetic friction force)}$$

$$f_s \le (f_s)_{\max} = \mu_s n \text{ (Magnitude of static friction force)}$$

Rolling friction is similar to kinetic friction, but the force of fluid resistance depends on the speed of an object through a fluid.

Forces in circular motion

In uniform circular motion, the acceleration vector is directed toward the center of the circle. The motion is governed by Newton's second law, $\sum \vec{F} = m\vec{a}$.

Acceleration in uniform circular motion:

$$a_{rad} = \frac{v^2}{R} = \frac{4\pi^2 R}{T^2}$$

Work done by a force

When a constant force $\vec{F}$ acts on a particle that undergoes a straight-line displacement $\vec{s}$, the work done by the force on the particle is defined to be the scalar product of $\vec{F}$ and $\vec{s}$. The unit of work in SI units is 1 joule = 1 Newton-meter. Work is a scalar quantity; it can be positive or negative, but it has no direction in space.

$$W = \vec{F} \cdot \vec{s} = Fs\cos\phi$$

$$\phi = \text{angle between } \vec{F} \text{ and } \vec{s}$$

Kinetic energy

The kinetic energy K of a particle equals the amount of work required to accelerate the particle from rest to speed v. It is also equal to the amount of work the particle can do in the process of being brought the rest. Kinetic energy is a scalar that has no direction is space; it is always positive or zero. Its units are the same as the units of work.

$$K = \frac{1}{2}mv^2$$

The work-energy theorem

When forces act on a particle while it undergoes a displacement, the particle's kinetic energy changes by an amount equal to the total work done on the particle by all forces. This relationship, called the work-energy theorem, is valid whether the forces are constant or varying and whether the particle moves along a straight or curved path. It is applicable only to bodies that can be treated as particles.

$$W_{tot} = K_2 - K_1 = \Delta K$$

Work done by a varying or on a curved path

When a force varies during a straight-line displacement, the work done by the force is given by an integral

$$\mathrm{W} = \int_{x_1}^{x_2} F_x \, dx$$

$$\mathrm{W} = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{l}$$

When a particle follows a curved path, the work done on it by a force $\vec{F}$ is given by an integral that involves the angle ϕ between the force and the displacement. The expression is valid even if the force magnitude and the angle ϕ varying during the displacement.

Power

Power is the time rate of doing work. The average power P_{av} is the amount of work ΔW done in time Δt divided by that time. The instantaneous power is the limit of the average power as Δt goes to zero. When a force $\vec{F}$ acts on a particle moving with velocity $\vec{v}$, the instantaneous power (the rate at which the force does work) is the scalar product of $\vec{F}$ and $\vec{v}$. Like work and kinetic energy, power is a scalar quantity. The SI unit of power is 1 watt = 1 joule/second.

$$P_{\mathrm{av}} = \frac{\Delta W}{\Delta t}$$

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}$$

$$P = \vec{F} \cdot \vec{v}$$

Gravitational potential energy and elastic potential energy

The work done on a particle by a constant gravitational force can be represented as a change in the gravitational potential energy, $U_{\mathrm{grav}} = mgy$. This energy is a shared property of the particle and the earth.

$$W_{\mathrm{grav}} = mgy_1 - mgy_1$$
$$W_{\mathrm{grav}} = U_{\mathrm{grav},1} - U_{\mathrm{grav},2}$$

$$W_{\mathrm{grav}} = -\Delta U_{\mathrm{grav}}$$

A potential energy is also associated with the elastic force $F_x = -kx$ exerted by an ideal spring, where x is the amount of stretch or compression. The work done by this force can be represented as a change in the elastic potential energy of the spring, $U_{\mathrm{el}} = \frac{1}{2}kx^2$.

$$W_{\mathrm{el}} = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2$$
$$W_{\mathrm{el}} = U_{\mathrm{el},1} - U_{\mathrm{el},2}$$

$$W_{\mathrm{el}} = -\Delta U_{\mathrm{el}}$$

When the total mechanical energy is conserved:

The total potential energy U is the sum of the gravitational and elastic potential energies

$$U = U_{\mathrm{grav}} + U_{el}$$

If no forces other than the gravitational and elastic forces do work on a particle, the sum of kinetic and potential energies is conserved. This sum $E = K + U$ is called the total mechanical energy.

$$K_1 + U_1 = K_2 + U_2$$

When the total mechanical energy is not conserved:

When forces other than the gravitational and elastic forces do work on particle, the work W_{other} done by these other forces equals the change in total mechanical energy (kinetic energy plus total potential energy).

$$K_1 + U_1 + W_{\mathrm{other}} = K_2 + U_2$$

Conservative forces, non-conservative forces, and the law of conservation energy

All forces war either conservative or non-conservative. A conservative force is one for which the work-kinetic energy relationship is completely reversible. The work of a conservative force can always be represented by a potential-energy function, but the work of a non-conservative force cannot. The work done by non-conservative forces manifests itself as changes in the internal energy of bodies. The sum of kinetic, potential, and internal energies is always conserved.

$$\Delta K + \Delta U + \Delta U_{\mathrm{int}} = 0$$

Determining force from potential energy

For motion along a straight line, a conservative force $F_x(x)$ is the negative derivative of its associated potential-energy function U. In three dimensions, the components of a conservative force are negative partial derivatives of U.

Momentum of particle

The momentum $\vec{p}$ of a particle is a vector quantity equal to the product of the particle's mass m and velocity $\vec{v}$. Newton's second law says that the net force on a particle is equal to the rate of change of the particle's momentum.

$$\vec{p} = m\vec{v}$$

$$\sum \vec{F} = \frac{d\vec{p}}{dt}$$

Impulse and momentum

If a constant net force $\sum \vec{F}$ acts on a particle for time interval Δt from t_1 to t_2, the impulse $\vec{J}$ of the net force is the product of the net force and the time interval. If $\sum \vec{F}$ varies with time, $\vec{J}$ is the integral of the net force over the time interval. In any case, the change in a particle's momentum during a time interval equals the impulse of the net force that acted on the particle during that interval. The momentum of a particle equals the impulse that accelerated it from rest to its present speed.

$$\vec{J} = \sum \vec{F}(t_2 - t_1) = \sum \vec{F}\ \Delta t$$

$$\vec{J} = \int_{t_1}^{t_2} \sum \vec{F}\ dt$$

$$\vec{J} = \vec{p}_2 - \vec{p}_1$$

Conservation of momentum

An internal force is a force exerted by one part of a system on another. An external force is a force exerted on any part of a system by something outside the system. If the net external force on a system is zero, the total momentum of the system $\vec{P}$ (the vector sum of the momenta of the individual particles that make up the system) is constant, or conserved. Each component of total momentum is separately conserved.

$$\vec{p} = \vec{p}_A + \vec{p}_A + \cdots$$

$$\vec{p} = m_A\vec{v}_A + m_B\vec{v}_B + \cdots$$

$$\text{If} \sum \vec{F} = 0, \text{then } \vec{P} = \text{constant}$$

Collisions

In collisions of all kinds, the initial and final total momenta are equal. In an elastic collision between two bodies, the initial and final total kinetic energies are also equal, and the initial and final relative velocities have the same magnitude. In an inelastic two-body collision, the total kinetic energy is less after the collision than before. If the two bodies have the same final velocity, the collision is completely inelastic.

Center of mass

The position vector of the center of mass of a system of particles, $\vec{r}_{\text{cm}}$, is a weighted average of the positions $\vec{r}_1, \vec{r}_2, \ldots$ of the individual particles. The total momentum $\vec{P}$ of a system equals the system's total mass M multiplied by the velocity of its center of mass, $\vec{v}_{\text{cm}}$. The center of mass moves as though all the mass M were concentrated at that point. If the net external force on the system is zero, the center-of-mass velocity $\vec{v}_{\text{cm}}$ is constant. If the net external force is not zero, the center of mass accelerates as though it were a particle of mass M being acted on by the same net external force.

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_2\vec{r}_2 + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

$$\vec{r}_{cm} = \frac{\sum_i m_i\vec{r}_{\text{i}}}{\sum_i m_i}$$

$$\vec{p} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots$$

$$\vec{p} = M\vec{v}_{\text{cm}}$$

$$\sum \vec{F}_{\text{ext}} = M\vec{a}_{\text{cm}}$$

Rotational kinematics

When a rigid body rotates about a stationary axis (usually called the z-axis), the body's position is described by an angular coordinate θ.The angular velocity ω_z is the time derivative of θ, and the angular acceleration α_z is the time derivative ω_z or the second derivative of θ.

$$\omega_z = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

$$\alpha_z = \lim_{\Delta t \to 0} \frac{\Delta\omega_z}{\Delta t} = \frac{d\omega_z}{dt}$$

If the angular acceleration is constant, then θ, ω_z, and α_z are related by simple kinematic equations analogous to those for straight-line motion with constant linear acceleration.

Constant α_z only:

$$\theta = \theta_0 + \omega_{0z} t + \frac{1}{2}\alpha_z t^2$$

$$\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t$$

$$\omega_z = \omega_{0z} + \alpha_z t$$

$${\omega_z}^2 = {\omega_{0z}}^2 + 2\alpha_z(\theta - \theta_0)$$

Relating linear and angular kinematics

The angular speed ω of a rigid body is the magnitude of the body's angular velocity. The rate of change of ω is $\alpha = d\omega/dt$. For a particle in the body at a distance r from the rotation axis, the speed v and the components of the acceleration $\vec{a}$ are related to ω and α.

$$v = r\omega$$

$$a_{\text{tan}} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha$$

$$a_{\text{rad}} = \frac{v^2}{r} = \omega^2 r$$

Moment of inertia and rotational kinetic energy

The moment of inertia I of a body about a given axis is a measure of its rotational inertia. The greater the value of I, the more difficult it is to change the state of the body's rotation. The moment of inertia can be expressed as a sum over the particles m_i that make up the body, each of which is at its own perpendicular distance $r_i$from the axis. The rotational kinetic energy of a rigid body rotating about a fixed axis depends on the angular speed ω and the moment of inertia I for that rotation axis.

$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots$$

$$I = \sum_i m_i r_i^2$$

$$K = \frac{1}{2} I \omega^2$$

Calculating the moment of inertia

The parallel-axis theorem relates the moments of inertia of a rigid body of mass M about two parallel axes; an axis through the center of mass (moment of inertia I_{cm}) and a parallel axis a distance d from the first axis (moment of inertia I_p) . If the body has a continuous mass distribution, the moment of inertia can be calculated by integration.

$$I_p = I_{\mathrm{cm}} + Md^2$$

Torque

When a force $\vec{F}$ acts on a body, the torque of that force with respect to a point O has a magnitude given by the product of the force magnitude F and the lever arm l. More generally, torque is a vector $\vec{\tau}$ equal to the vector product of $\vec{r}$ (the position vector of the point at which the force acts) and $\vec{F}$.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Rotational dynamics

The rotational analog of newton's second law says that the net torque acting on a body equals the product of the body's moment of inertia and its angular acceleration.

$$\sum \tau_z = I \alpha_z$$

Combined translation and rotation

If a rigid body is both moving through space and rotating, its motion can be regarded as translational motion of the center of mass plus rotational motion about an axis through the center of mass. Thus the kinetic energy is a sum of translational and rotational kinetic energies. For dynamics, Newton's second law decribes the motion of the center of mass, and the rotational equivalent of Newton's second law describes rotation about the center of mass. In the case of rolling without slipping, there is a special relationship between the motion of the center of mass and the rotational motion.

$$K = \frac{1}{2} M {v_{\mathrm{cm}}}^2 + \frac{1}{2} I_{\mathrm{cm}} \omega^2$$

$$\sum \vec{F}_{\mathrm{ext}} = M \vec{a}_{\mathrm{cm}}$$

$$\sum \tau_z = I_{\mathrm{cm}} \alpha_z$$

$$v_{\mathrm{cm}} = R\omega \ \text{(Rolling without slipping)}$$

Work done by a torque

A torque that acts on a rigid body as it rotates does work on that body. The work can be expressed as an integral of the torque. The work-energy theorem says that the total rotational work done on a rigid body is equal to the change in rotational kinetic energy. The power, or rate at which the torque does work, is the product of the torque and the angular velocity.

$$W = \int_{\theta_1}^{\theta_2} \tau_z \, d\theta$$

$$W = \tau_z(\theta_2 - \theta_1) = \tau_z \Delta\theta \text{ (Constant torque only)}$$

$$W_{\text{tot}} = \frac{1}{2} I {\omega_2}^2 - \frac{1}{2} I {\omega_1}^2$$

$$P = \tau_z \omega_z$$

Angular momentum

The angular momentum of a particle with respect to point O is the vector product of the particle's position vector $\vec{r}$ relative to O and its momentum $\vec{p} = m\vec{v}$. When a symmetrical body rotates about a stationary axis of symmetry, its angular momentum is the product of its momentum of inertia and its angular velocity vector $\vec{\omega}$. If the body is not symmetrical or the rotation (z) axis is not an axis of symmetry, the component of angular momentum along the rotation axis is $I\omega_z$.

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} \text{ (Particle)}$$

$$\vec{L} = I\vec{\omega} \text{ (Rigid body rotating about axis of symmetry)}$$

Rotational dynamics and angular momentum

The new external torque on a system is equal to the rate of change of its angular momentum. If the net external torque on a system is zero, the total angular momentum of the system is constant (conserved).

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt}$$

Density and pressure

Density is mass per unit volume. If a mass m of homogeneous material has volume V, its density ρ is the ratio m/V. Specific gravity is the ratio of the density of the material to the density of water.

$$\rho = \frac{m}{V}$$

Pressure is normal force per unit area. Pascal's law states that pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid. Absolute pressure is the total pressure in a fluid; gauge pressure is the difference between absolute pressure and atmospheric pressure. The SI unit of pressure is the pascal (Pa): 1 Pa = 1 N/m^2

$$P = \frac{dF_{\perp}}{dA}$$

Pressures in a fluid at rest

The pressure difference between points 1 and 2 in a static fluid of uniform density ρ (an incompressible fluid) is proportional to the difference between the elevations y_1 and y_2. If the pressure at the surface of an incompressible liquid at rest is ρ_0, then the pressure at a depth *h* is greater by an amount $\rho g h$.

Pressure in a fluid of uniform density:

$$P_2 - P_1 = -\rho g (y_2 - y_1)$$

$$P = P_0 + \rho g h$$

Buoyancy

Archimedes's principle states that when a body is immersed in a fluid, the fluid exerts an upward buoyant force on the body equal to the weight of the fluid that the body displaces.

Fluid flow

An ideal fluid is incompressible and has no viscosity (no internal friction). A flow line is the path of a fluid particle; a streamline is a curve tangent at each point to the velocity vector at that point. A flow tube is a tube bounded at its sides by flow lines. In laminar flow, layers of fluid slide smoothly past each other. In turbulent flow, there is great disorder and a constantly changing flow pattern. Conservation of mass in an incompressible fluid is expressed by the continuity equation, which relates the flow speeds v_1 and v_2 for two cross sections A_1 and A_2 in a flow tube. The product Av equals the volume flow rate, dV/dt, the rate at which volume crosses a section of the tube.

Continuity equation, incompressible fluid:

$$A_1 v_1 = A_2 v_2$$

Volume flow rate:

$$\frac{dV}{dt} = Av$$

Bernoulli's equation:

$$P + \rho g y + \frac{1}{2}\rho v^2 = \text{Constant}$$

Bernoulli's equation states that a quantity involving the pressure P, flow speed v, and elevation y has the same value anywhere in a flow tube, assuming steady flow in an ideal fluid. This equation can be used to relate the properties of the flow at any two points.

Newton's law of gravitation

Any two particles with masses m_1 and m_2, a distance r apart, attract each other with forces inversely proportional to r^2. These forces form an action–reaction pair and obey Newton's third law. When two or more bodies exert gravitational forces on a particular body, the total gravitational force on that individual body is the vector sum of the forces exerted by the other bodies. The gravitational interaction between spherical mass distributions, such as planets or stars, is the same as if all the mass of each distribution were concentrated at the center.

$$F_g = \frac{Gm_1m_2}{r^2}$$

Gravitational force, weight, and gravitational potential energy

The weight w of a body is the total gravitational force exerted on it by all other bodies in the universe. Near the surface of the earth (mass m_E and radius R_E), the weight is essentially equal to the gravitational force of the earth alone. The gravitational potential energy U of two masses m and m_E separated by a distance r is inversely proportional to r. The potential energy is never positive; it is zero only when the two bodies are infinitely far apart.

Weight at Earth's surface:

$$w = F_g = \frac{Gm_Em}{{R_E}^2}$$

Acceleration due to gravity at Earth's surface:

$$g = -\frac{Gm_E}{{R_E}^2}$$

Also

$$U = -\frac{Gm_Em}{r}$$

Orbits

When a satellite moves in a circular orbit, the centripetal acceleration is provided by the gravitational attraction of the earth. Kepler's three laws describe the more general case: an elliptical orbit of a planet around the sun or a satellite around a planet.

Speed in circular orbit:

$$v = \sqrt{\frac{Gm_E}{r}}$$

Period in circular orbit:

$$T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{Gm_E}} = \frac{2\pi r^{3/2}}{\sqrt{Gm_E}}$$

Black holes

If a nonrotating spherical mass distribution with total mass M has a radius less than its Schwarzschild radius R_s, it is called a black hole. The gravitational interaction prevents anything, including light, from escaping from within a sphere with radius R_s.

Schwarzschild radius:

$$R_s = \frac{2GM}{c^2}$$

Periodic motion

Periodic motion is motion that repeats itself in a definite cycle. It occurs whenever a body has a stable equilibrium position and a restoring force that acts when the body is displaced from equilibrium. Period T is the time for one cycle. Frequency f is the number of cycles per unit time. Angular frequency ω is 2π times the frequency.

$$f = \frac{1}{T} \qquad T = \frac{1}{f}$$

$$\omega = 2\pi f = \frac{2\pi}{T}$$

Simple harmonic motion

If the restoring force F_x in periodic motion is directly proportional to the displacement x, the motion is called simple harmonic motion (SHM). In many cases this condition is satisfied if the displacement from equilibrium is small. The angular frequency, frequency, and period in SHM do not depend on the amplitude but on only the mass m and force constant k. The displacement, velocity, and acceleration in SHM are sinusoidal functions of time; the amplitude A and phase angle ϕ of the oscillation are determined by the initial displacement and velocity of the body.

$$F_x = -kx$$

$$a_x = \frac{F_x}{m} = -\frac{k}{m}x$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}}$$

$$x = A\cos(\omega t + \phi)$$

Energy in simple harmonic motion

Energy is conserved in SHM. The total energy can be expressed in terms of the force constant k and amplitude A.

$$E = \frac{1}{2}m{v_x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 = \text{Constant}$$

Angular simple harmonic motion

In angular SHM, the frequency and angular frequency are related to the moment of inertia I and the torsion constant κ.

$$\omega = \sqrt{\frac{\kappa}{I}}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{\kappa}{I}}$$

Simple pendulum

A simple pendulum consists of a point mass m at the end of a massless string of length L. Its motion is approximately simple harmonic for sufficiently small amplitude; the angular frequency, frequency, and period then depend on only g and L, not on the mass or amplitude.

$$\omega = \sqrt{\frac{g}{L}}$$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$$

$$T = \frac{2\pi}{\omega} = \frac{1}{f} = 2\pi\sqrt{\frac{L}{g}}$$

Physical pendulum

A physical pendulum is any body suspended from an axis of rotation. The angular frequency and period for small-amplitude oscillations are independent of amplitude but depend on the mass m, distance d from the axis of rotation to the center of gravity, and moment of inertia I about the axis.

$$\omega = \sqrt{\frac{mgd}{I}}$$

$$T = 2\pi\sqrt{\frac{I}{mgd}}$$

Damped oscillations

When a force $F_x = -bv_x$ is added to a simple harmonic oscillator, the motion is called a damped oscillation. If $b < 2\sqrt{km}$ (called underdamping), the system oscillates with a decaying amplitude and an angular frequency ω' that is lower than it would be without damping. If $b = 2\sqrt{km}$ (called critical damping) or $b > 2\sqrt{km}$ (called overdamping), when the system is displaced it returns to equilibrium without oscillating.

$$x = Ae^{-\left(\frac{b}{2m}\right)t}\cos(\omega' t + \phi)$$

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

Forced oscillations and resonance

When a sinusoidally varying driving force is added to a damped harmonic oscillator, the resulting motion is called a forced oscillation or driven oscillation. The amplitude is a function of the driving frequency ω_d and reaches a peak at a driving frequency close to the natural frequency of the system. This behavior is called resonance.

$$A = \frac{F_{\max}}{(k - m{\omega_d}^2)^2 + b^2\omega_d^2}$$

[END OF SUMMARY]