Tutorial 1
Differentiation

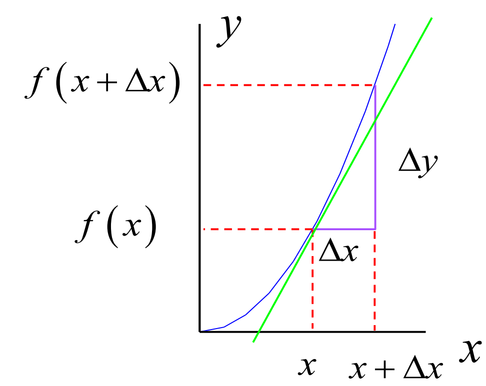
What is Calculus?

Calculus
微積分

Differential calculus
Differentiation
微分

The relation of very small changes of different quantities

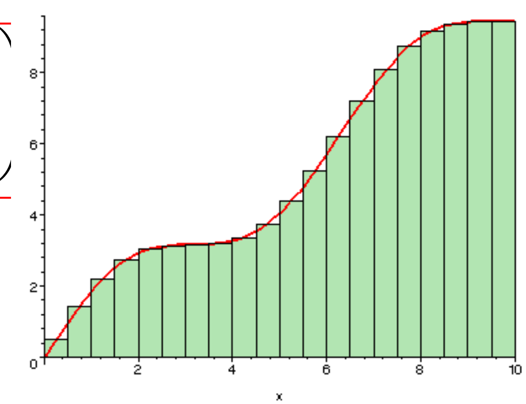
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$



Integral calculus
Integration
積分

Adding a large amount of small quantities to find the sum

$$y = \int f(x)\,dx = \lim_{N \to \infty} \left(\sum_{i=0}^{N} f(x_i)(x_{i+1} - x_i) \right)$$



Why we need these?

Limit

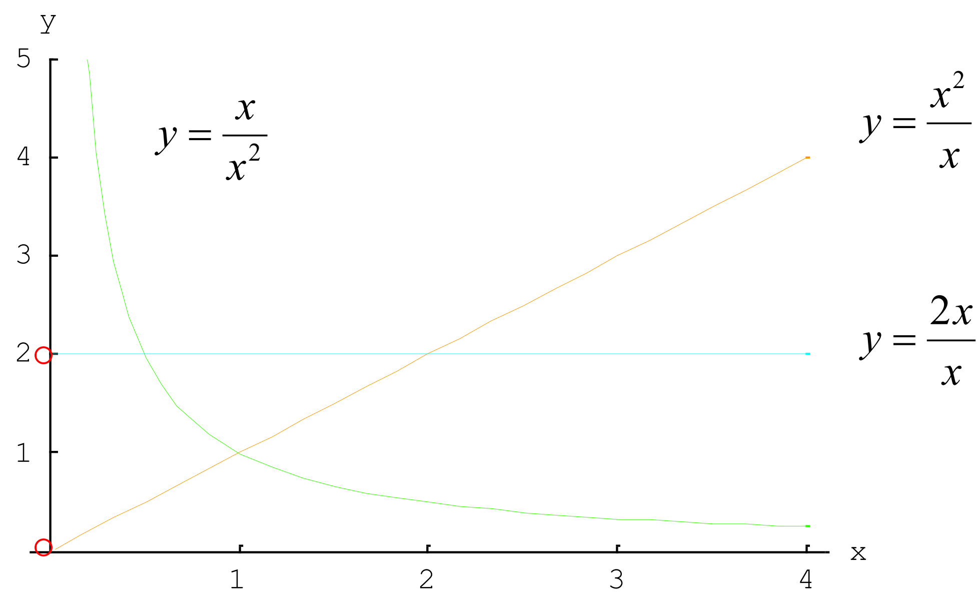
Consider the functions

$$f(x)=\frac{x^2}{x} \qquad g(x)=\frac{x}{x^2} \qquad h(x)=\frac{2x}{x}$$

What are the values of the functions when $x = 0$?

We cannot simply substitute $x = 0$ into the functions because in all three cases, this gives 0/0, which is undefined

In fact the functions are undefined at $x = 0$



However, we can still discuss what the functions *tend to* when x approaches 0

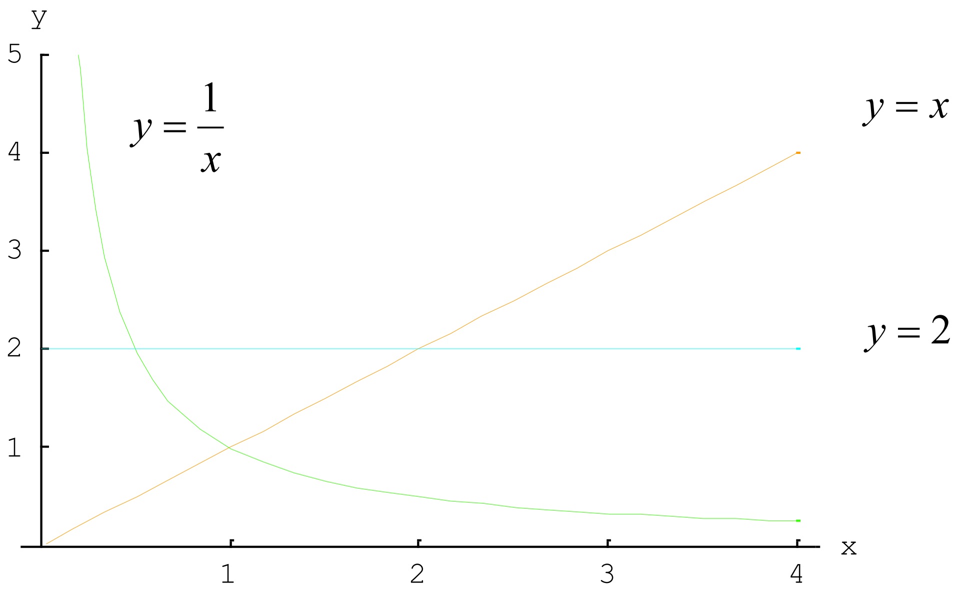
Notice that when $x \neq 0$, the functions are equivalent to

$$f(x) = \frac{x^2}{x} = x \qquad g(x) = \frac{x}{x^2} = \frac{1}{x} \qquad h(x) = \frac{2x}{x} = 2$$

So when $x \to 0$,

∞ = infinity

$$f(x) \to 0 \qquad g(x) \to +\infty \qquad h(x) \to 2$$



There are times when we need to discuss the value a fraction a/b tends to when both a and b approach zero

This form of 0/0 can be any value, depending on how fast a and b approach zero.

0/0 is called an *indeterminate form*

For example

faster (numerator), slower (denominator)

$$f(x) = \frac{x^2}{x} \longrightarrow 0$$

slower (numerator), faster (denominator)

$$g(x) = \frac{x}{x^2} \longrightarrow \infty$$

same rate

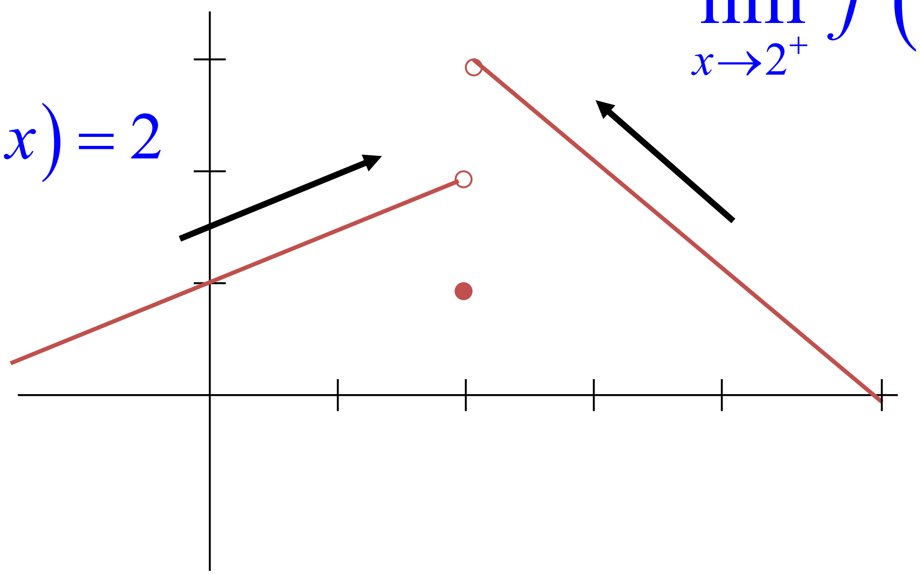
$$h(x) = \frac{2x}{x} \longrightarrow \text{Finite non-zero number}$$

The limit of a function refers to the value that the function approaches, not the actual value (if any)

Example:

$$\lim_{x \to 2^+} f(x) = 3$$

$$\lim_{x \to 2^-} f(x) = 2$$



$$f(x) = \begin{cases} \frac{1}{2}x + 1 & x < 2 \\ 1 & x = 2 \\ -x + 5 & x > 2 \end{cases}$$

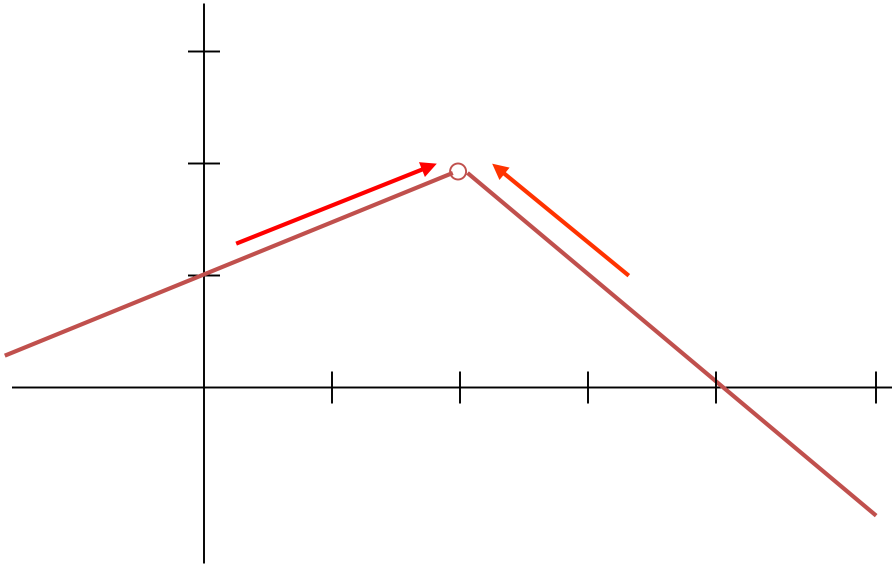
If $\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^-} f(x)$

we say that the limit of the function at $x = a$ exists and define

$$\lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$$

Example:

$$f(x) = \begin{cases} \frac{1}{2}x + 1 & x < 2 \\ \text{Not defined} & x = 2 \\ -x + 4 & x > 2 \end{cases}$$



$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = 2$$

Hence $\lim\limits_{x \to 2} f(x) = 2$

Most of the techniques of calculus require that functions be *continuous*. A function is continuous if you can draw it in one motion without picking up your pencil.

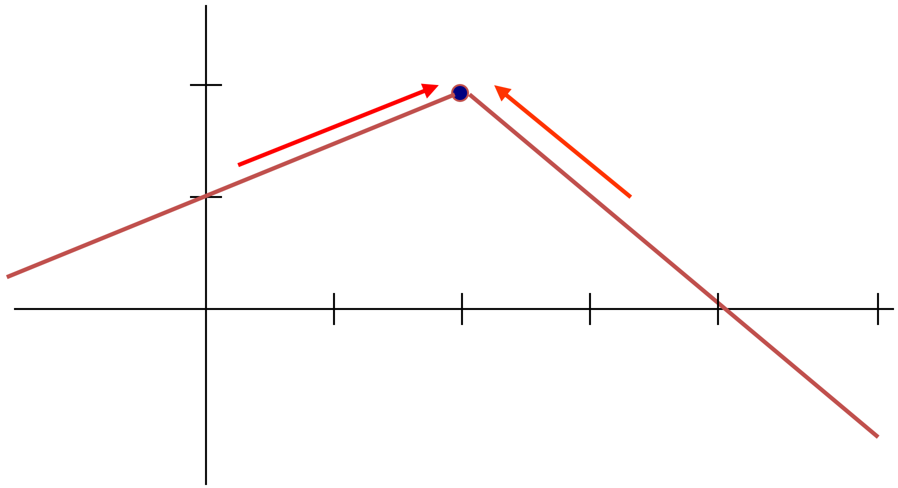
A function f is *continuous* at $x = a$ if

(1) $f(a)$ is defined

(2) the limit at $x = a$ exists

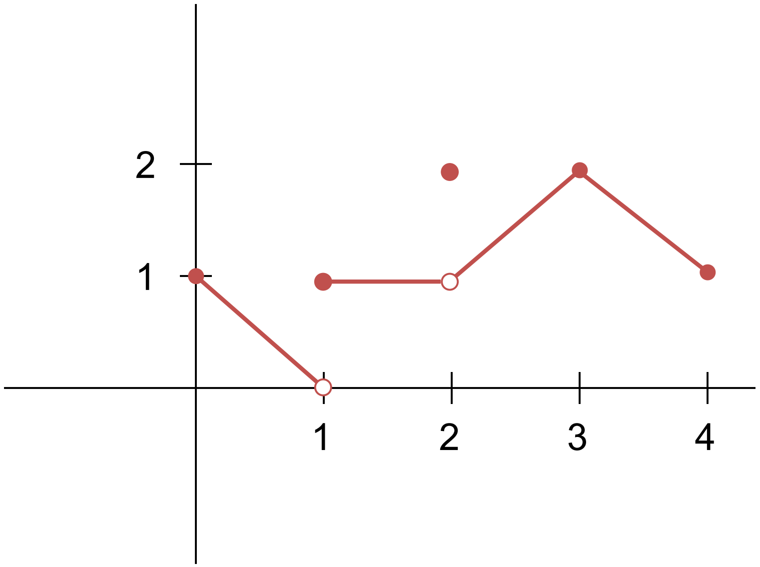
(3) the limit equals $f(a)$

If the function is continuous at every point inside a certain interval of x, we say that the it is a continuous function in that interval



Exercise:

Is the function continuous at x = 1, x = 2, and x = 3?



Properties of Limits

Limits can be added, subtracted, multiplied, multiplied by a constant, divided, and raised to a power

If $\lim_{x\to a} f(x) = P \ , \quad \lim_{x\to a} g(x) = Q$

Then

$$\lim_{x\to a}\left[f(x)+g(x)\right] = P+Q$$

$$\lim_{x\to a}\left[f(x)-g(x)\right] = P-Q$$

$$\lim_{x\to a}\left[f(x)g(x)\right] = PQ$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{P}{Q} \qquad \text{if } Q \neq 0$$

Note: Similar rules also hold for one-sided limits

The Squeeze (Sandwich) Theorem

If $g(x) \le f(x) \le h(x)$ for all $x \ne c$ in some interval around c

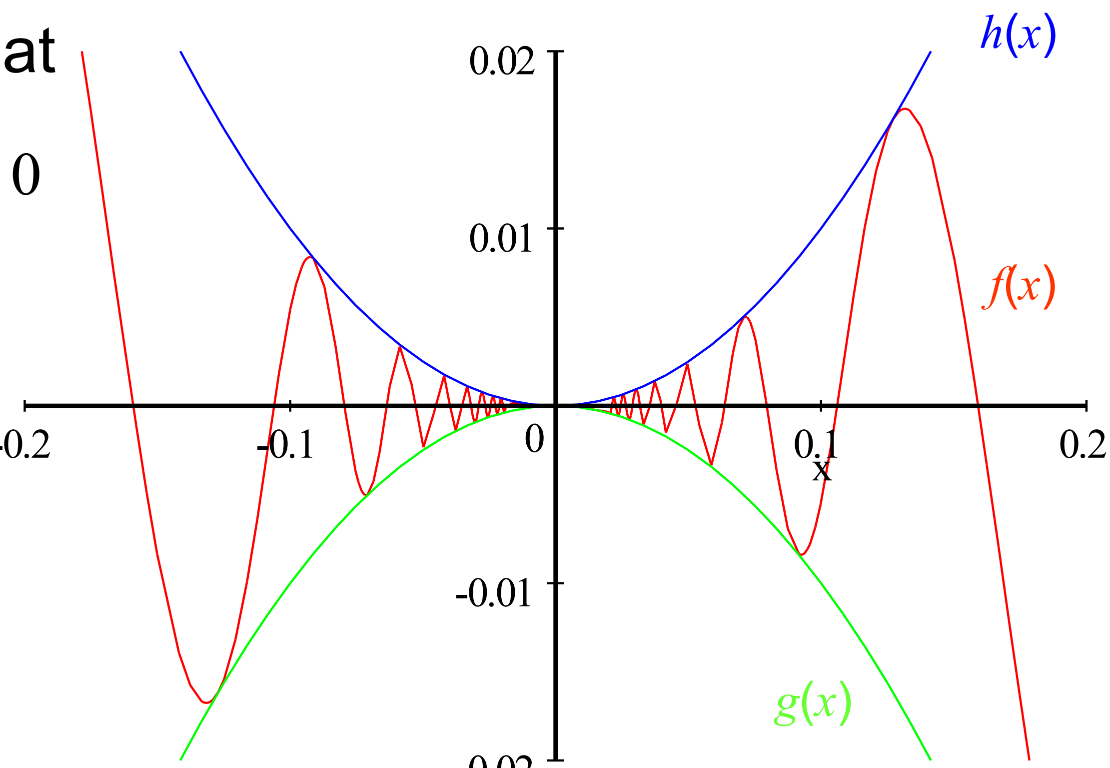
and $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x \to c} f(x) = L$.

Example: Given that

$$\lim_{x \to 0} h(x) = \lim_{x \to 0} g(x) = 0$$

Then

$$\lim_{x \to 0} f(x) = 0$$



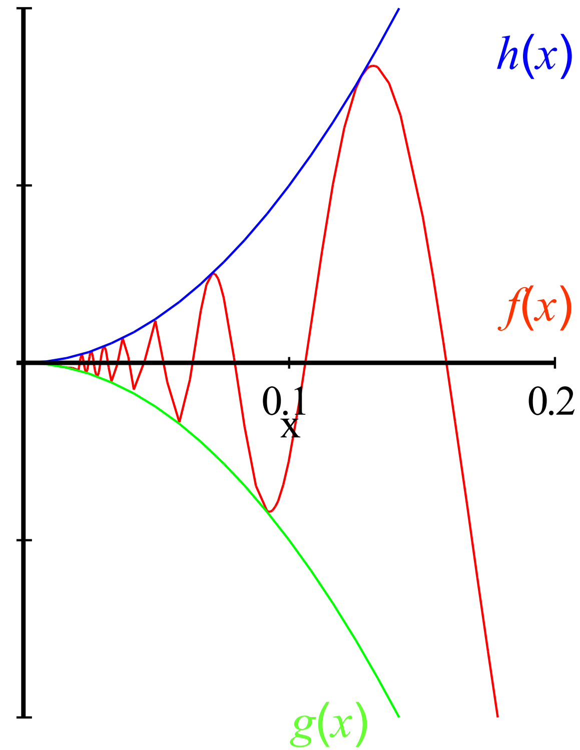
The Squeeze (Sandwich) Theorem

The theorem also holds when c is an end point of the interval
In this case, the limits become the one-sided limits

Example: Given that

$$\lim_{x\to 0^+} h(x) = \lim_{x\to 0^+} g(x) = 0$$

Then $\lim_{x\to 0^+} f(x) = 0$



Example:

$$f(\theta) = \frac{\sin\theta}{\theta}$$ when $\theta \to 0$

This function is defined everywhere except at $\theta = 0$
When $\theta = 0$, it is an *indeterminate form*
What is the limit of the function when $\theta \to 0$?

First, notice that f is an even function ($f(\theta) = f(-\theta)$)
Hence, the left-hand limit equals the right-hand limit, if it exists

To find the limit when $\theta \to 0$ from the positive side, let us use a geometric method

Consider a sector of a unit circle subtending an angle θ at the center:

Area of the sector = $\theta/2$
(*Note that this is true only when θ is measured in radian*)

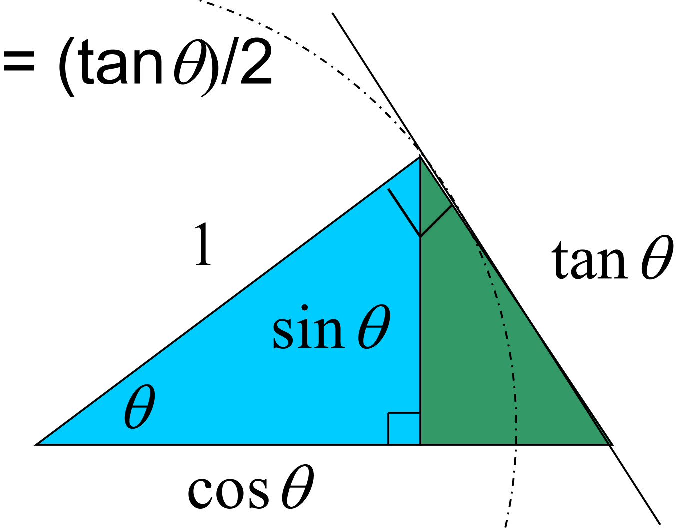
Area of the small (blue) triangle = $(\sin\theta\cos\theta)/2$

Area of the large (blue + green) triangle = $(\tan\theta)/2$

$$\frac{1}{2}\sin\theta\cos\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

□ $\cos\theta < \dfrac{\theta}{\sin\theta} < \dfrac{1}{\cos\theta}$

□ $\cos\theta < \dfrac{\theta}{\sin\theta} < \sec\theta$

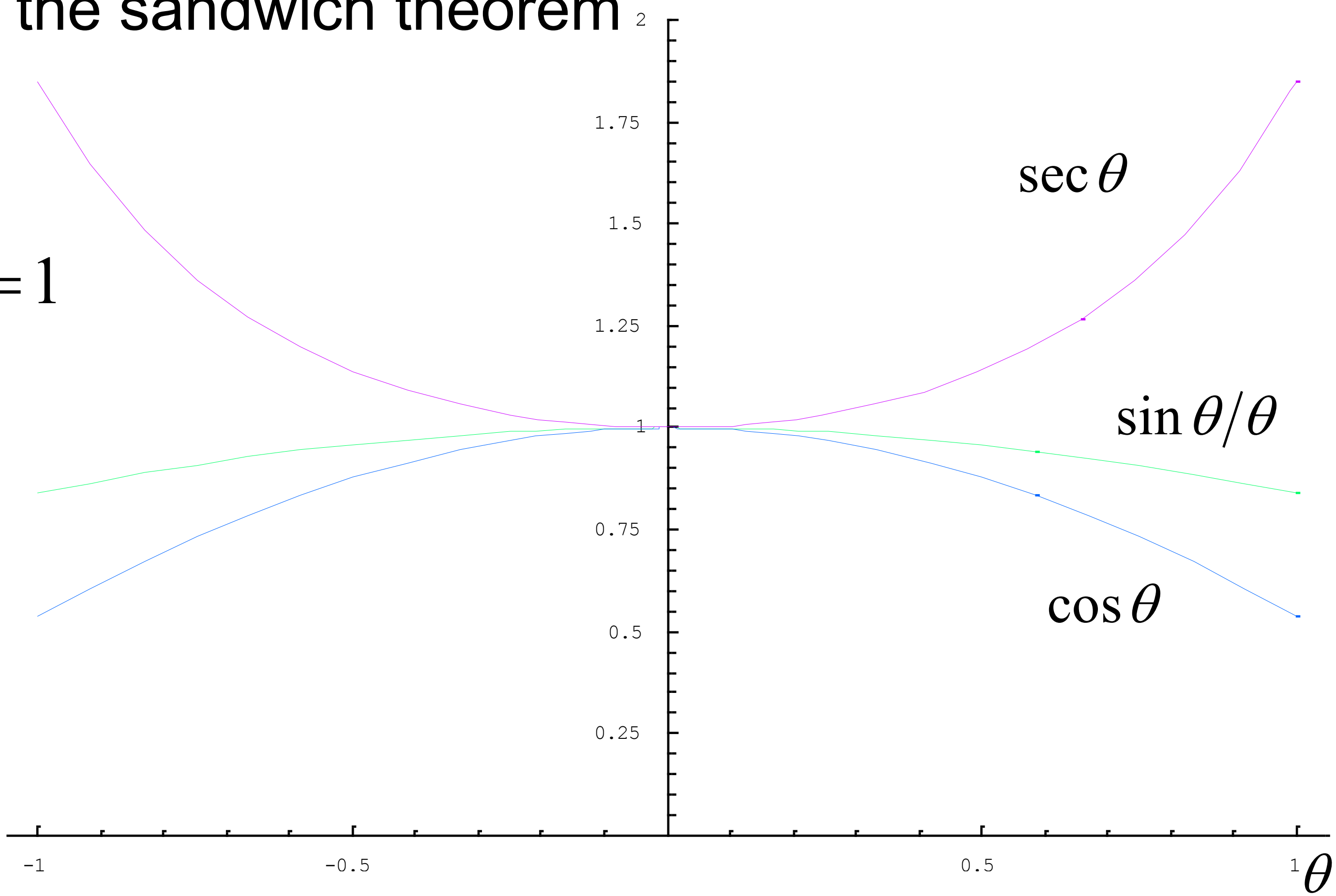


$$\frac{1}{\cos\theta} > \frac{\sin\theta}{\theta} > \cos\theta$$

Because $\lim_{\theta\to 0^+} \cos\theta = 1 \qquad \lim_{\theta\to 0^+} \frac{1}{\cos\theta} = 1$

Hence, by the sandwich theorem

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1$$



Since it is an even function

$$\lim_{\theta \to 0^+} \frac{\sin \theta}{\theta} = 1 \quad \Rightarrow \quad \lim_{\theta \to 0^-} \frac{\sin \theta}{\theta} = 1$$

So:

$$\boxed{\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1}$$

Exercise:

$$\lim_{x \to \infty} \frac{x^2}{x^2 + x + 1} = ?$$

$$\lim_{\theta \to 0} \frac{\theta}{\cos\theta} = ?$$

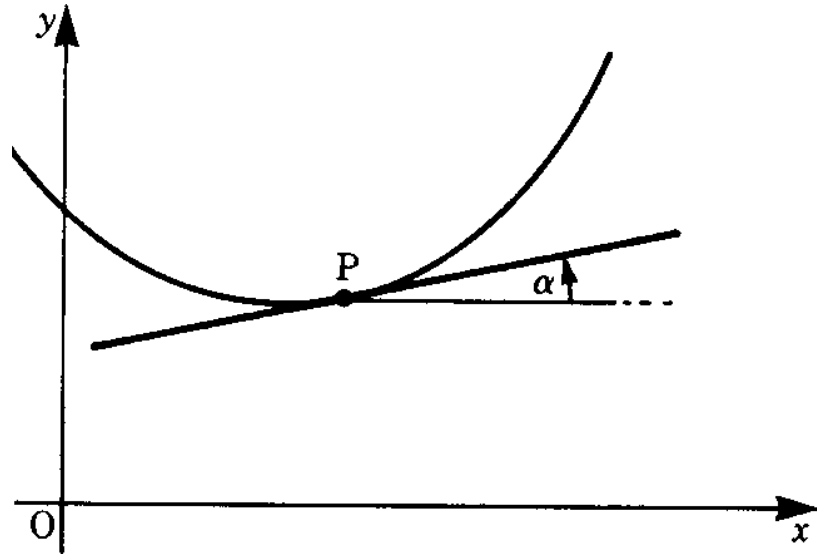
$$\lim_{\theta \to 0} \frac{\tan\theta}{\theta} = ?$$

Differentiation and Derivatives

Derivative and Differentiation

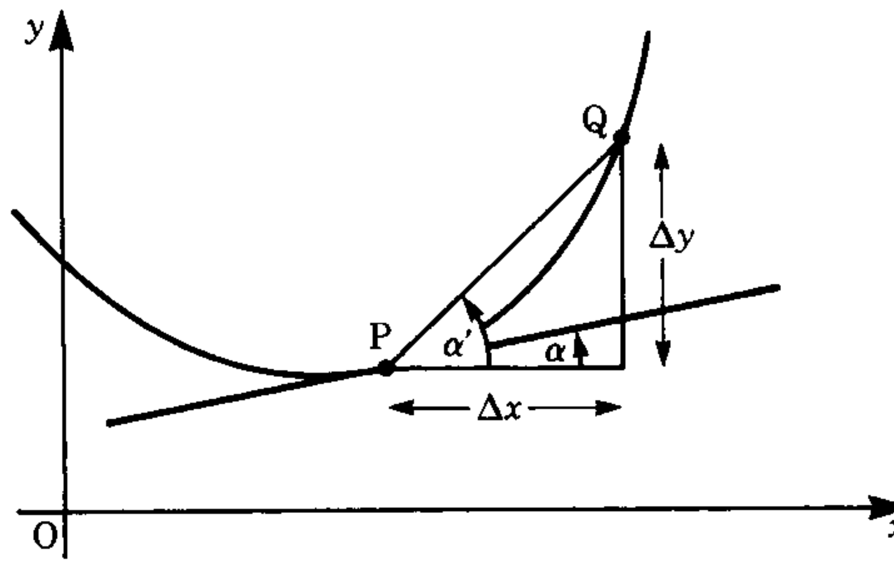
For a function $y = f(x)$

The slope of the tangent at point P $= \tan\alpha$



Taking a neighboring point Q, we have

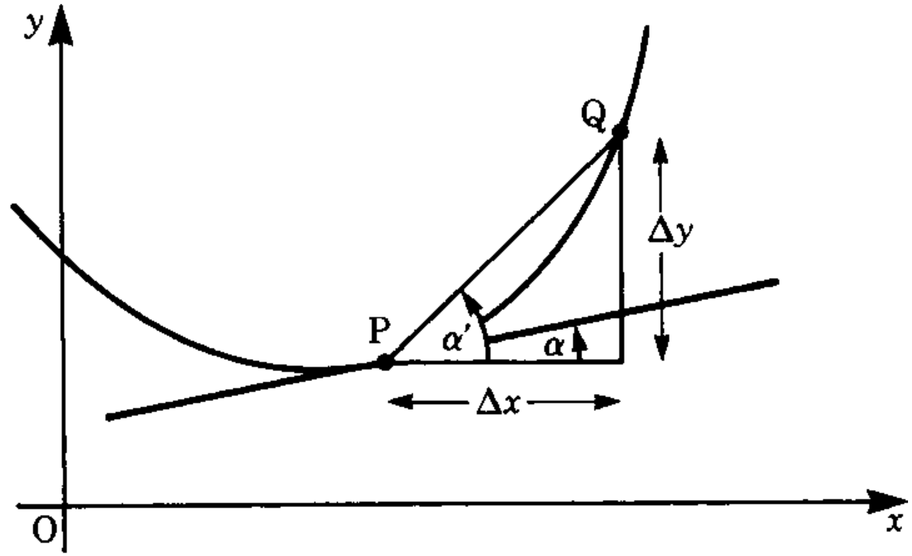
$$\tan\alpha' = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$



(a)

Let the point Q moves towards P. In the limit Q coincides with P, the angle α' is equal to α

$$\alpha' \to \alpha \quad \text{as} \quad Q \to P$$



(a)

The derivative at x is defined by

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

And we have

$$\tan\alpha = \frac{df}{dx}$$

Note:

$$\frac{dy}{dx}$$

dx does <u>not</u> mean d times x !

dy does <u>not</u> mean d times y !

$\frac{dy}{dx}$ does <u>not</u> mean the fraction "dy over dx" !

(except when it is convenient to think of it as division)

Note:

$\frac{dy}{dx}$ is better interpreted as a short hand of $\frac{d}{dx}y$

$\frac{d}{dx}$ refers to the operation of finding the derivative of the following function w.r.t x

Obtaining the derivative directly starting from the definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

is called derivation from *first principle*

Example: Obtain the derivative of

$$y = f(x) = \frac{1}{x}$$

from first principle

Answer:

$$\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$$

Solution:

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\ \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\ \frac{\dfrac{1\,x\cancel{(x+\Delta x)}}{\cancel{x+\Delta x}}-\dfrac{1\,\cancel{x}(x+\Delta x)}{\cancel{x}}}{\Delta x\ x(x+\Delta x)}$$

$$=\lim_{\Delta x\to 0}\ \frac{1}{\Delta x}\cdot\frac{x-(x+\Delta x)}{x(x+\Delta x)}$$

$$=\lim_{\Delta x\to 0}\ \frac{\cancel{x}-\cancel{x}-\cancel{\Delta x}}{\cancel{\Delta x}\cdot x(x+\underset{0}{\cancel{\Delta x}})}$$

$$=-\frac{1}{x^2}$$

If the derivative of a function is its slope, then for a constant function, the derivative must be zero

$$\frac{d}{dx}(c) = 0$$

Example: $y = 3$

$y' = 0$

The derivative of a constant is zero

Proof: Let $y = f(x) = c$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} = 0$$

Exercise:

Use the first principle method to calculate:

$$f(x) = \frac{1}{x-2} + 2$$

Differentiation Rules

- All the differentiation rules can be proved by the first principle
- You DON’T need to remember the proof
- BUT you need to know how to use the rules

Differentiation Rules

$u(x)$ and $v(x)$ are two functions of x

$$\frac{d}{dx}(x)=1$$

$$\frac{d}{dx}(cu)=c\frac{du}{dx}$$

constant multiple rule

$$\frac{d}{dx}(u\pm v)=\frac{du}{dx}\pm\frac{dv}{dx}$$

sum and difference rules

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

product rule

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

quotient rule

$$\frac{d}{dx}\left(x\right)=1$$

Proof: Let $y = f\left(x\right) = x$

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f\left(x+\Delta x\right)-f\left(x\right)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\left(x+\Delta x\right)-x}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x}=1$$

constant multiple rule:

$$\frac{d}{dx}\left(cu\right)=c\,\frac{du}{dx}$$

Examples:

$$\frac{d}{dx}cx^2=c\cdot 2x=2cx$$

$$\frac{d}{dx}\frac{7}{x}=7\cdot\left(-\frac{1}{x^2}\right)=-\frac{7}{x^2}$$

Proof:

$$\frac{d}{dx}\left(cu\right)=\lim_{\Delta x\to 0}\frac{cu\left(x+\Delta x\right)-cu\left(x\right)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\left[c\,\frac{u\left(x+\Delta x\right)-u\left(x\right)}{\Delta x}\right]$$

$$=c\lim_{\Delta x\to 0}\frac{u\left(x+\Delta x\right)-u\left(x\right)}{\Delta x}=c\,\frac{du}{dx}$$

sum and difference rules:

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

Examples:

$y = x^2 + 12x \qquad y' = 2x + 12$

$y = x^2 - 2x + 2 \qquad y' = 2x - 2$

(Each term is treated separately)

Proof:

$$\frac{d}{dx}(u \pm v) = \lim_{\Delta x \to 0} \frac{\left[u(x+\Delta x) \pm v(x+\Delta x)\right] - \left[u(x) \pm v(x)\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[u(x+\Delta x) - u(x)\right] \pm \left[v(x+\Delta x) - v(x)\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} \pm \lim_{\Delta x \to 0} \frac{v(x+\Delta x) - v(x)}{\Delta x}$$

$$= \frac{du}{dx} \pm \frac{dv}{dx}$$

product rule:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Notice that this is <u>not</u> just the product of two derivatives

This is sometimes memorized as: $d(uv) = u\ dv + v\ du$

Example:

$$\frac{d}{dx}\left[\left(\frac{1}{x}+3\right)\left(2x^2+5x\right)\right] = \left(\frac{1}{x}+3\right)(4x+5) + \left(2x^2+5x\right)\left(-\frac{1}{x^2}\right)$$

$$\frac{d}{dx}\left(6x^2+15x+2x+5\right)$$

$$\frac{d}{dx}\left(6x^2+17x+5\right)$$

$$12x+17$$

$$4+\frac{5}{x}+12x+15-2-\frac{5}{x}$$

$$12x+17$$

Proof: $\frac{d}{dx}\left[u(x)v(x)\right]$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x)v(x+\Delta x) - u(x)v(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(u+\Delta u)(v+\Delta v) - uv}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{uv + u\Delta v + v\Delta u + \Delta u \Delta v - uv}{\Delta x}$$

$$= \lim_{\Delta x \to 0} u\frac{\Delta v}{\Delta x} + \lim_{\Delta x \to 0} v\frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta u \Delta v}{\Delta x}$$

$$= u\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + v\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \Delta u \frac{\Delta v}{\Delta x} \quad \left(\text{or } + \lim_{\Delta x \to 0} \Delta v \frac{\Delta u}{\Delta x}\right)$$

$$= u\frac{dv}{dx} + v\frac{du}{dx} + \lim_{\Delta x \to 0} \Delta u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} \quad \left(\text{or } + \lim_{\Delta x \to 0} \Delta v \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}\right)$$

$$= u\frac{dv}{dx} + v\frac{du}{dx}$$

quotient rule:

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

or

$$d\left(\frac{u}{v}\right)=\frac{v\ du-u\ dv}{v^2}$$

Example:

$$\frac{d}{dx}\frac{2x^2+5}{x^2+3} \qquad =\frac{\left(x^2+3\right)\left(4x\right)-\left(2x^2+5\right)\left(2x\right)}{\left(x^2+3\right)^2}$$

Proof:

$$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right)$$

$$= \lim_{\Delta x \to 0} \frac{(u+\Delta u)/(v+\Delta v) - u/v}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{1}{\Delta x} \frac{v(u+\Delta u) - u(v+\Delta v)}{v(v+\Delta v)}\right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{1}{v(v+\Delta v)} \frac{v\Delta u - u\Delta v}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \frac{1}{v(v+\Delta v)} \lim_{\Delta x \to 0}\left(v\frac{\Delta u}{\Delta x} - u\frac{\Delta v}{\Delta x}\right)$$

$$= \frac{1}{v^2}\left(v \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} - u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}\right)$$

$$= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

We saw that if $y = x^2$, $y' = 2x$

This is part of a pattern

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

power rule

Examples:

$$f(x) = x^3$$

$$f'(x) = 3x^2$$

$$y = x^8$$

$$y' = 8x^7$$

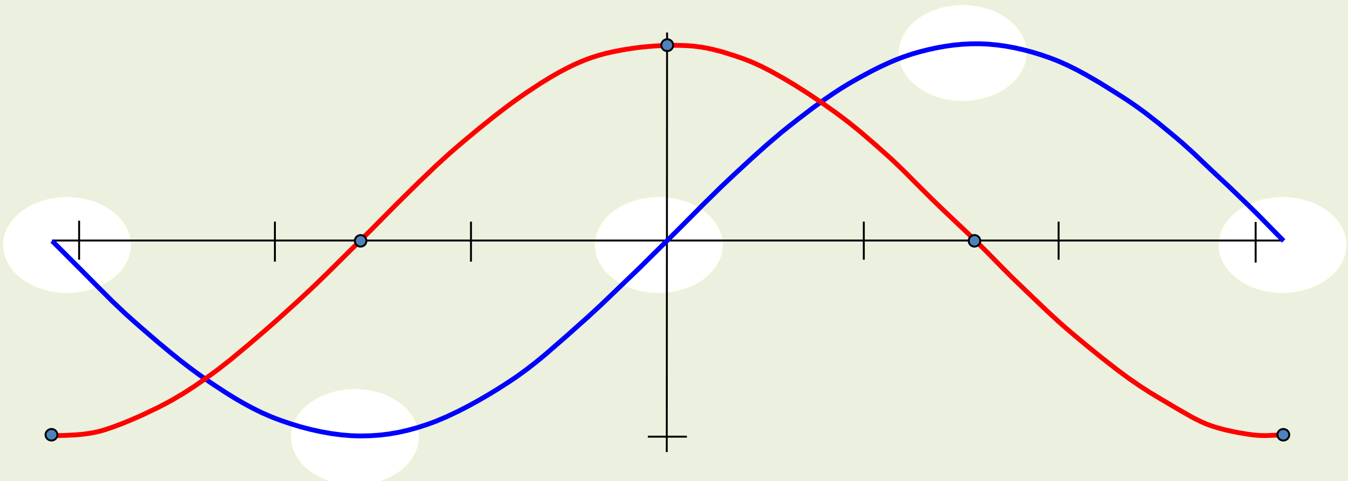
n is integer -> prove by induction

n is real -> prove by exponential and logarithm

Derivatives of Trigonometric Functions

Consider the function $y = \sin x$

We could make a graph of the slope:



x	slope
$-\pi$	-1
$-\dfrac{\pi}{2}$	0
0	1
$\dfrac{\pi}{2}$	0
π	-1

Now we connect the dots!

The resulting curve is a cosine curve

$$\frac{d}{dx}\sin(x) = \cos x$$

Proof:

From first principle

$$\frac{d}{dx}\sin x = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x}$$

Recall the identity

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

→

$$\sin(x+\Delta x) - \sin x = 2\cos\left(x + \frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}$$

→

$$\frac{d}{dx}\sin x = \lim_{\Delta x \to 0} \frac{2\cos\left(x+\frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0}\left[\cos(x+\Delta x/2)\frac{\sin(\Delta x/2)}{\Delta x/2}\right]$$

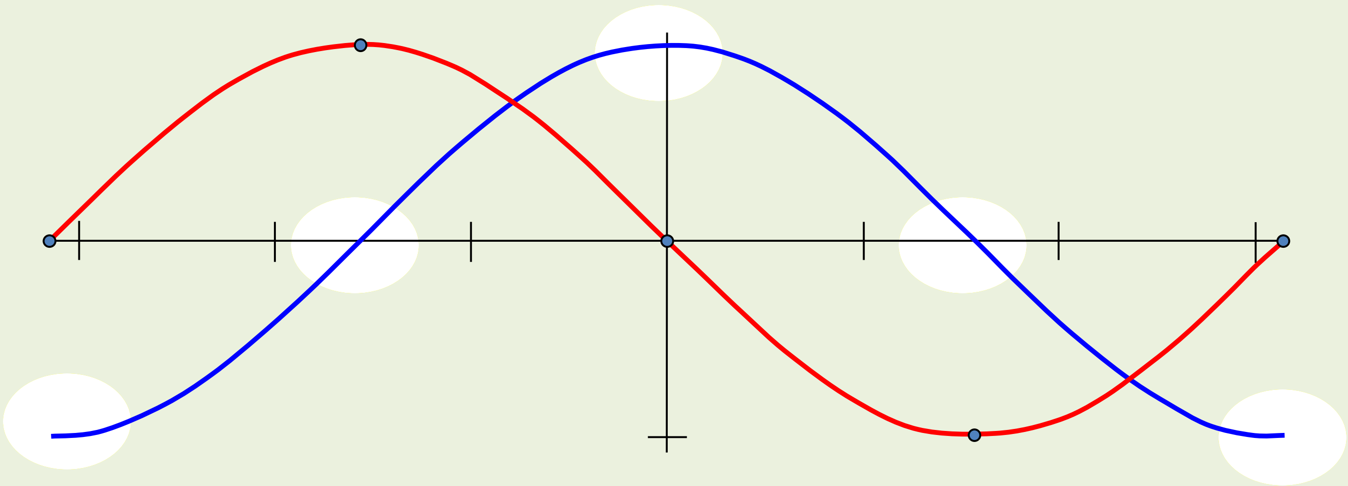
$$= \cos x \lim_{\Delta x/2 \to 0} \frac{\sin(\Delta x/2)}{\Delta x/2}$$

= 1

$$\lim_{\theta \to 0}\frac{\sin\theta}{\theta} = 1$$

$$\frac{d}{dx}\sin x = \cos x$$

We can do the same thing for $y = \cos x$



x	slope
$-\pi$	0
$-\dfrac{\pi}{2}$	1
0	0
$\dfrac{\pi}{2}$	-1
π	0

The resulting curve is a sine curve that has been reflected about the x-axis.

$$\frac{d}{dx}\cos(x) = -\sin x$$

Proof:

From first principle $$\frac{d}{dx}\cos x = \lim_{\Delta x \to 0} \frac{\cos(x+\Delta x) - \cos x}{\Delta x}$$

Recall the identity $$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

→ $$\cos(x+\Delta x) - \cos x = -2\sin\left(x+\frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}$$

→ $$\frac{d}{dx}\cos x = \lim_{\Delta x \to 0} \frac{-2\sin\left(x+\frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0}\left[-\sin(x+\Delta x/2)\frac{\sin(\Delta x/2)}{\Delta x/2}\right]$$

$$= -\sin x \lim_{\Delta x/2 \to 0} \frac{\sin(\Delta x/2)}{\Delta x/2}$$

Hence $$\frac{d}{dx}\cos x = -\sin x$$

We can find the derivative of tangent x by using the quotient rule

$$\frac{d}{dx}\tan x$$

$$\frac{d}{dx}\frac{\sin x}{\cos x}$$

$$\frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

$$\frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$\frac{1}{\cos^2 x}$$

$$\sec^2 x$$

$$\frac{d}{dx}\tan(x) = \sec^2 x$$

Derivatives of trigonometric functions:

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\sec x = \sec x \cdot \tan x$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\csc x = -\csc x \cdot \cot x$$

Note:

Remember that all these results are based on

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

which is correct only when the angle is measured in radian

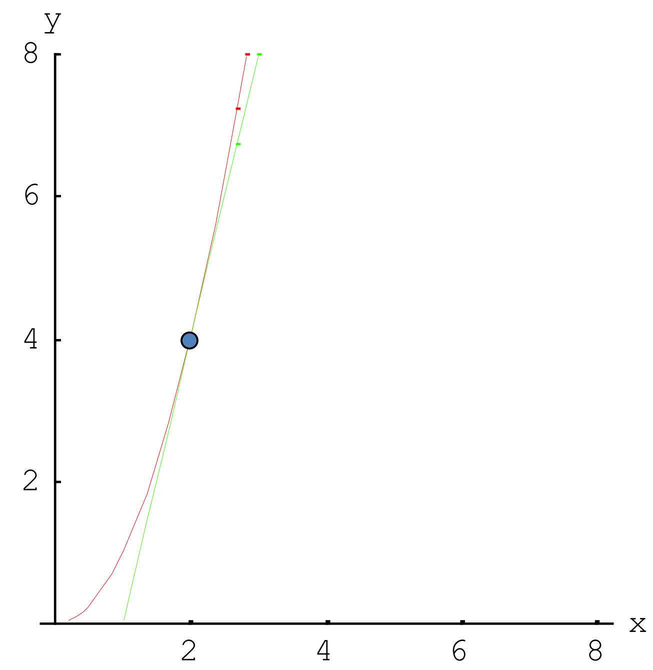
<u>In calculus, always use radian to measure angles</u>

Differentiation of Inverse Functions

As an example, consider

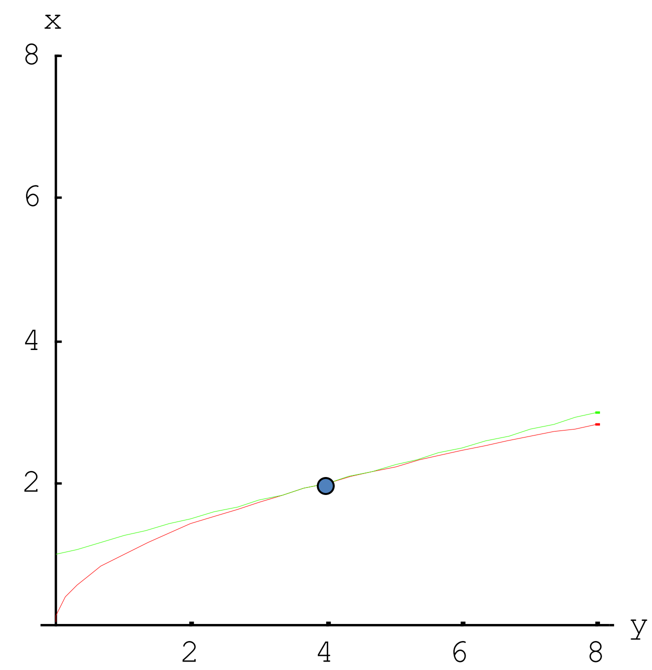
$$y = f(x) = x^2 \quad x \geq 0$$

$$\frac{dy}{dx} = 2x$$



Since the function is one-to-one, the inverse function exists:

$$x = f^{-1}(y) = \sqrt{y} \quad y \geq 0$$

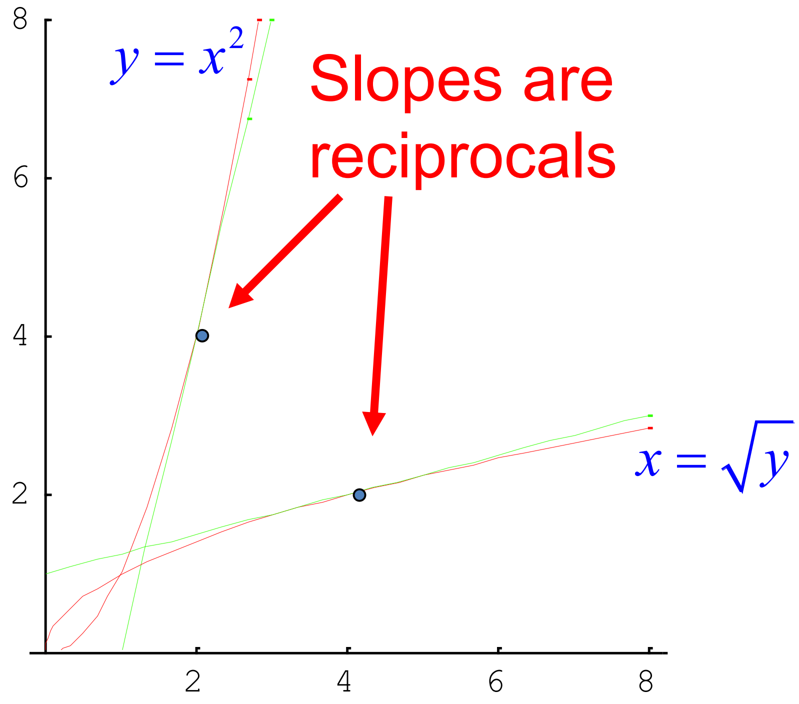


Use the power rule:

$$\frac{dx}{dy} = \frac{1}{2} y^{-1/2} = \frac{1}{2\sqrt{y}}$$

Notice that

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y}} = \frac{1}{2x} = 1\Big/\frac{dy}{dx}$$



Derivative Formula for Inverses:

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$

(NOT important)

Proof:

$$\frac{dx}{dy} = \lim_{\Delta y \to 0} \frac{\Delta x}{\Delta y} = \lim_{\Delta y \to 0} 1 \Big/ \frac{\Delta y}{\Delta x} = \frac{1}{\lim\limits_{\Delta y \to 0} \frac{\Delta y}{\Delta x}} = \frac{1}{\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}} = \frac{1}{dy / dx}$$

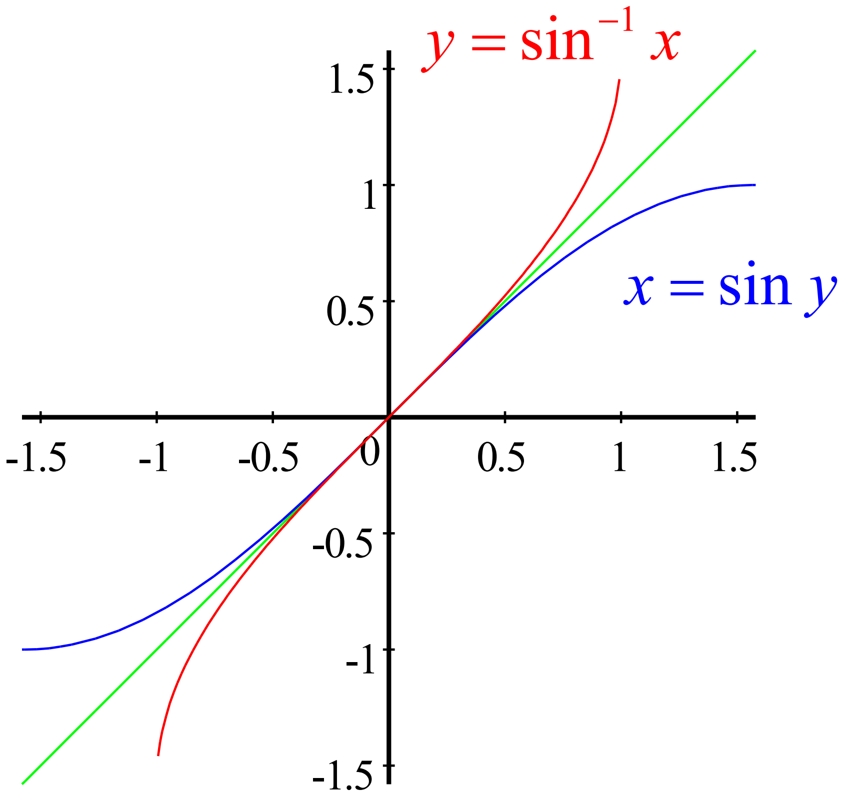
We assume the function y(x) is one-to-one and continuous of x

We can use this rule to find the derivatives of inverse trigonometric functions

Arcsine

$$y = \sin^{-1} x \quad \Leftrightarrow \quad x = \sin y$$

$$-1 < x \le 1 \qquad -\pi/2 < y < \pi/2$$



$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

Proof:

$$y = \sin^{-1} x$$

$$\sin y = x$$

$$\frac{dx}{dy} = \frac{d}{dy}\sin y = \cos y$$

$$\frac{dy}{dx} = 1 \Big/ \frac{dx}{dy} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\sin^2 y + \cos^2 y = 1$$

$$\cos^2 y = 1 - \sin^2 y$$

$$\cos y = \pm\sqrt{1-\sin^2 y}$$

But $-\frac{\pi}{2} < y < \frac{\pi}{2}$

so $\cos y$ is positive.

$$\therefore \cos y = \sqrt{1-\sin^2 y}$$

We could use the same technique to find the derivatives of other inverse trigonometric functions:

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\csc^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}}$$

The Chain Rule

It is undesirable to obtain the derivative of every function by first principle

With the rules we learned, we now have a pretty good list of “shortcuts” to find derivatives of simple functions

We will now learn another very powerful rule to calculate derivative of composite functions

Consider a simple composite function:

$$y = 2u$$

$$u = 3x - 5$$

then

$$y = 2(3x - 5)$$

$$= 6x - 10$$

$$y = 6x - 10 \quad\rightarrow\quad \frac{dy}{dx} = 6$$

$$y = 2u \quad\rightarrow\quad \frac{dy}{du} = 2$$

$$u = 3x - 5 \quad\rightarrow\quad \frac{du}{dx} = 3$$

$$6 = 2 \cdot 3$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

and another:

$$y = 5u - 2$$

where $u = 3t$

then $y = 5(3t) - 2$

$$y = 5(3t) - 2 \qquad y = 5u - 2 \qquad u = 3t$$

$$y = 15t - 2$$

$$\frac{dy}{dt} = 15 \qquad \frac{dy}{du} = 5 \qquad \frac{du}{dt} = 3$$

$$15 = 5 \cdot 3$$

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt}$$

Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example: $f(x) = \sin(x^2 - 4)$

Find: $\frac{df}{dx}$

Define $f(u) = \sin u$ and $u(x) = x^2 - 4$, we have $f(x) = f\big(u(x)\big) = \sin(x^2 - 4)$

$$\frac{df}{du} = \cos u \qquad \frac{du}{dx} = 2x$$

Chain rule:

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = \cos u \cdot 2x = 2x\cos(x^2 - 4)$$

After you become familiar with the rule, you can skip some steps:

$$y = \sin\left(x^2 - 4\right)$$

$$y' = \cos\left(x^2 - 4\right) \cdot \frac{d}{dx}\left(x^2 - 4\right)$$ Differentiate the outside function...

$$y' = \cos\left(x^2 - 4\right) \cdot 2x$$ …then the inside function

Another example:

$$\frac{d}{dx}\cos^2(3x)$$

$$\frac{d}{dx}\left[\cos(3x)\right]^2$$

$$2\left[\cos(3x)\right]\cdot\frac{d}{dx}\cos(3x)$$

It looks like we need to use the chain rule again!

derivative of the outside function

derivative of the inside function

Another example:

$$\frac{d}{dx}\cos^2(3x)$$

$$\frac{d}{dx}\left[\cos(3x)\right]^2$$

$$2\left[\cos(3x)\right]\cdot\frac{d}{dx}\cos(3x)$$

$$2\cos(3x)\cdot -\sin(3x)\cdot\frac{d}{dx}(3x)$$

← The chain rule can be used more than once.

$$-2\cos(3x)\cdot\sin(3x)\cdot 3$$

(That's what makes the "chain" in the "chain rule"!)

$$-6\cos(3x)\sin(3x)$$

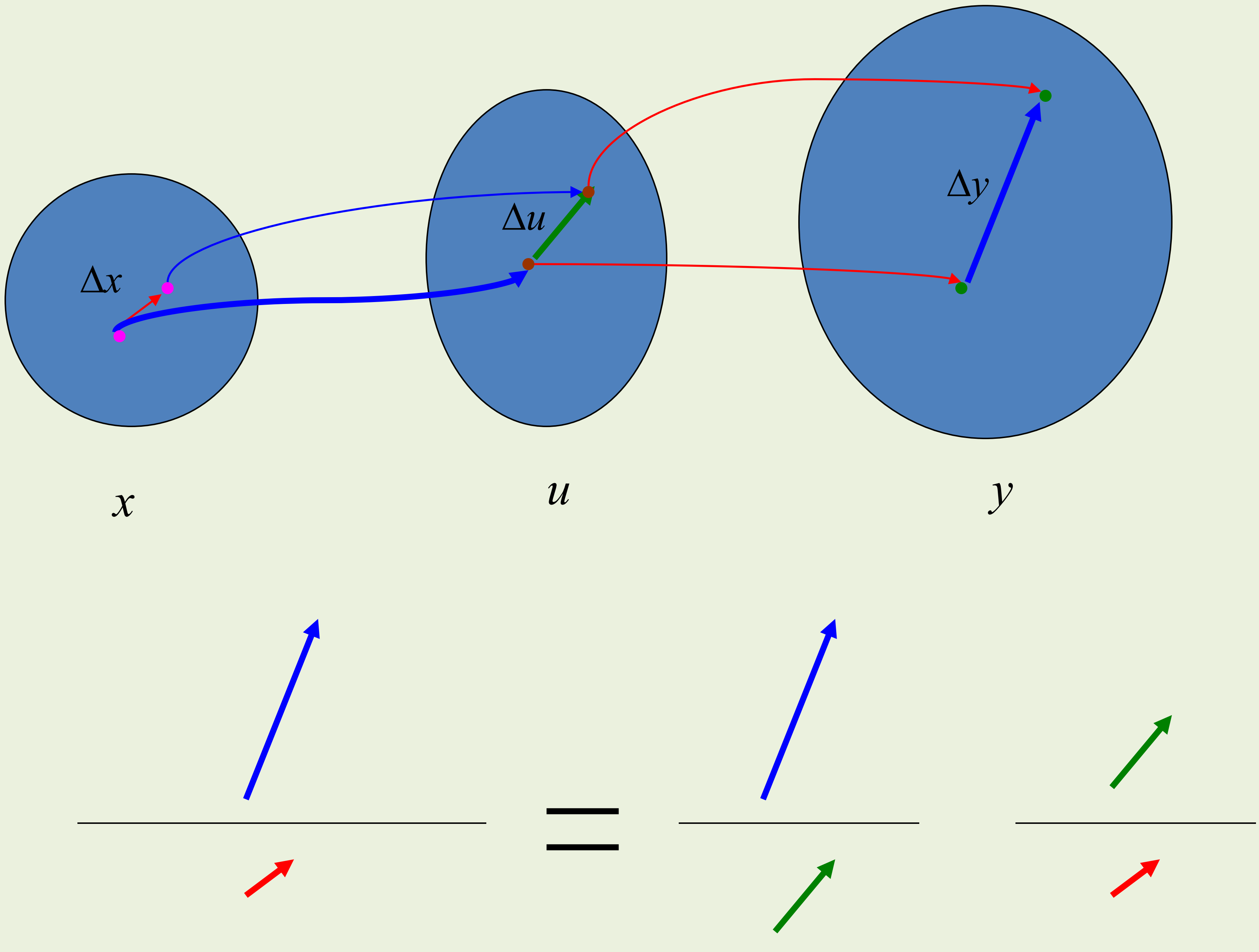
NOT important

Proof:

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}\right)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta u}\times\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}$$

$$=\lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u}\times\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}=\frac{dy}{du}\cdot\frac{du}{dx}$$

u(x) is a continuous function of x



Δx
Δu
Δy
x
u
y
=

Summary

$$\frac{d}{dx}(x)=1$$

$$\frac{d}{dx}(cu)=c\frac{du}{dx}$$

$$\frac{d}{dx}(u\pm v)=\frac{du}{dx}\pm\frac{dv}{dx}$$

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$\frac{d}{dx}\sin x=\cos x \qquad \frac{d}{dx}\cot x=-\csc^2 x$$

$$\frac{d}{dx}\cos x=-\sin x \qquad \frac{d}{dx}\sec x=\sec x\cdot\tan x$$

$$\frac{d}{dx}\tan x=\sec^2 x \qquad \frac{d}{dx}\csc x=-\csc x\cdot\cot x$$

$$\frac{dx}{dy}=\frac{1}{dy/dx}$$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

Exercise:

Find the derivative of the following functions:

1. $\tan^2 t$
2. $\sin x \cos x$
3. $\dfrac{\cos t}{t-1}$
4. $\sqrt{x+2}$
5. $\dfrac{1}{\sqrt{4-x}}$
6. $\cos \sqrt{x}$
7. $\sin^{3/2} x^2$

Higher Order Derivatives

The derivative of a function $y = f(x)$ is also called the first derivative:

$$y'(x) = f'(x) = \frac{dy}{dx}$$

Since it is still a function of x, one can differentiate it further

The second derivative is obtained when a function is differentiated twice:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Example:

$$y = 3x^2 + 2x + 5$$

$$\frac{dy}{dx} = 6x + 2$$

$$\frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}(6x + 2) = 6$$

Notation:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}=y''(x)=f''(x)$$

Example:

Consider the function $y = -5x^2 + 10x + 20$

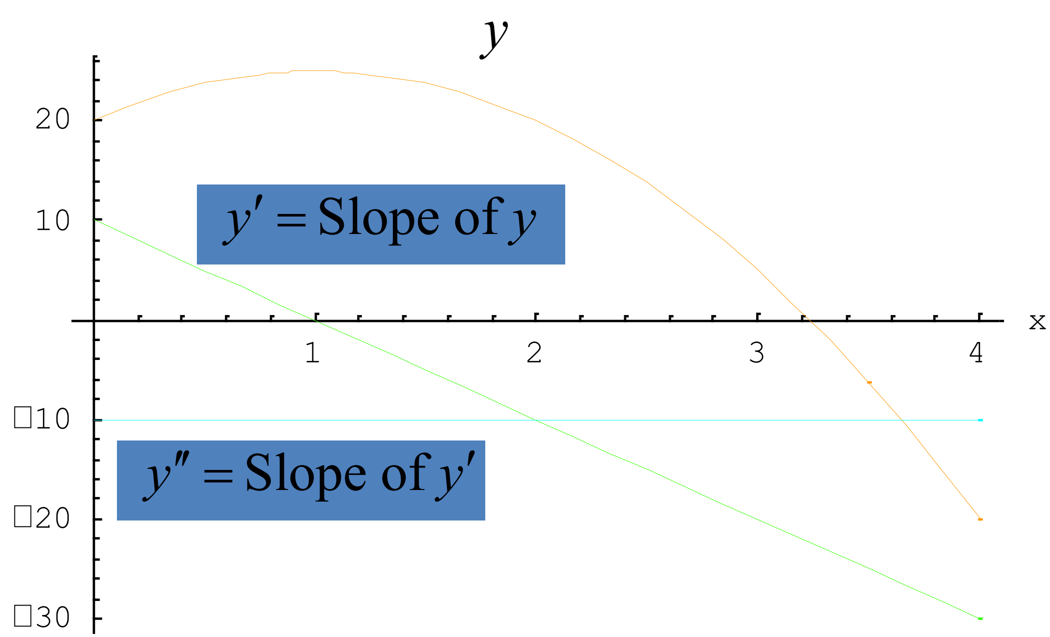
$$y'(x) = \frac{d}{dx}\left(-5x^2 + 10x + 20\right) = -10x + 10$$

$$y''(x) = \frac{d}{dx}\left(-10x + 10\right) = -10$$

$$y'''(x) = \frac{d}{dx}\left(-10\right) = 0$$

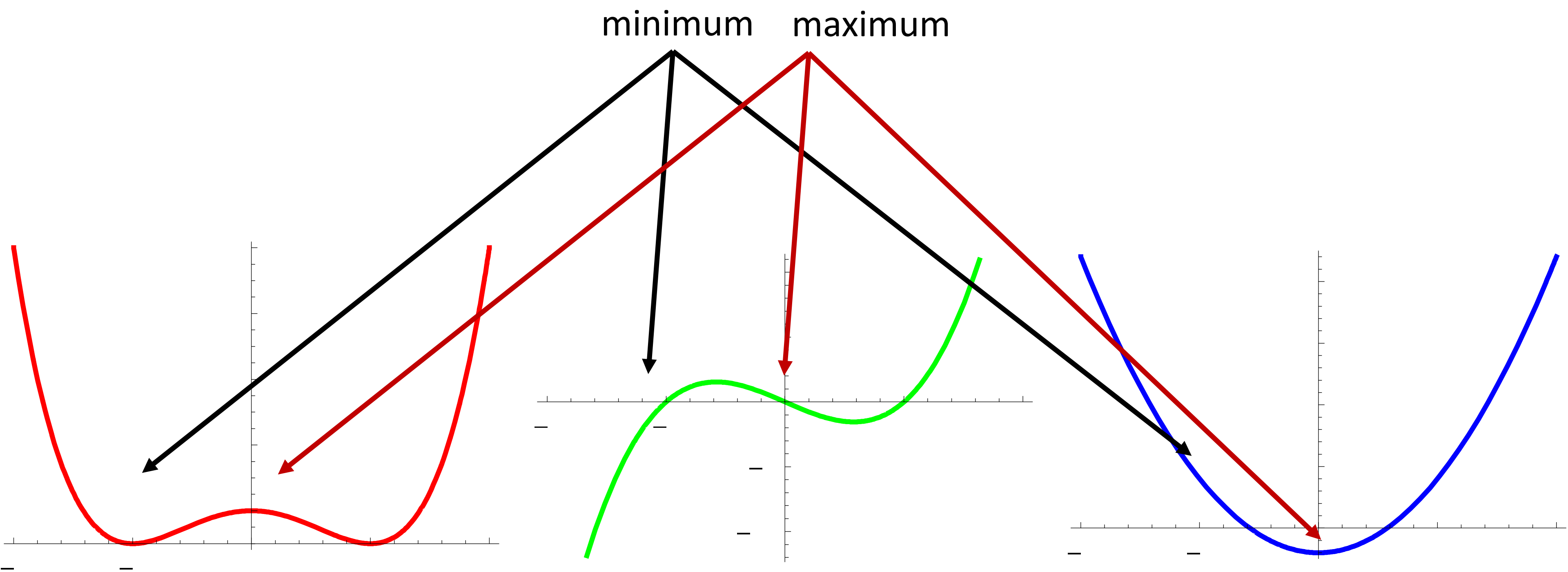
All further higher derivatives are zero:

$$y^{(n)}(x) = 0 \qquad n \geq 3$$



Application: Maximum or Minimum of a function f(x)

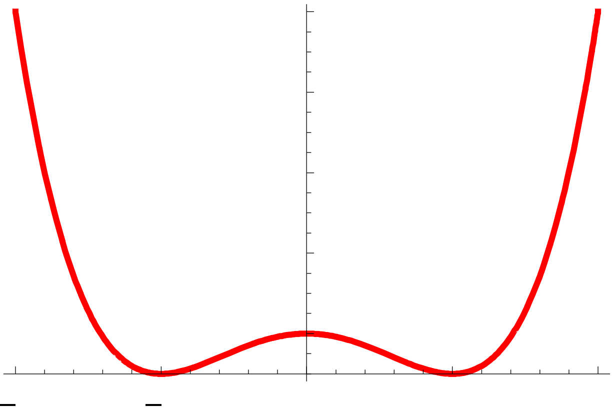
For a given function f(x), we can find the (local) maximum or minimum by inspecting the first and second order derivative respectively.

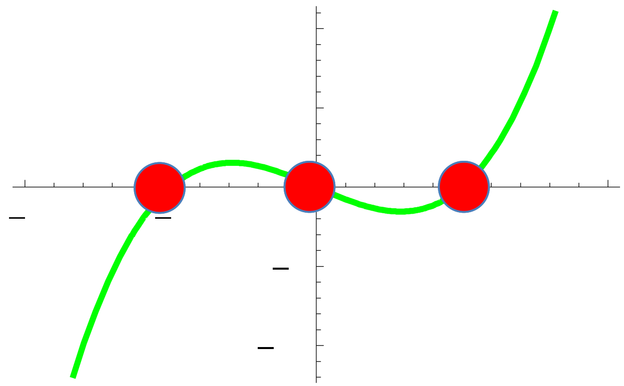


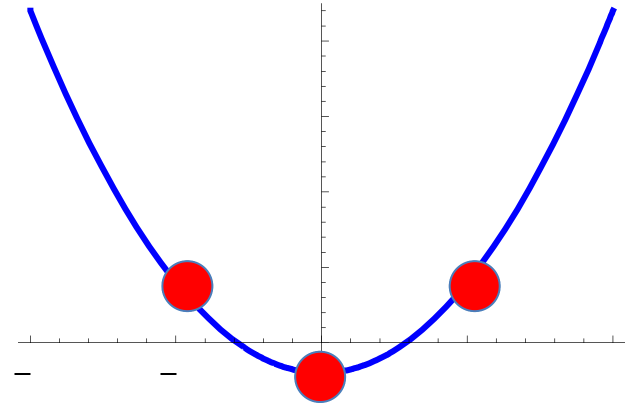
$$f(x) = x^4 - 2x^2 + 2$$

$$\frac{df(x)}{dx} = 4x^3 - 4x$$

$$\frac{d^2 f(x)}{dx^2} = 12x^2 - 4$$







$$f(x)=x^4-2x^2+2$$

$$\frac{df(x)}{dx}=4x^3-4x$$

$$\frac{d^2f(x)}{dx^2}=12x^2-4$$

f’(x)	f’’(x)	
0	<0	f(x) is maximum
0	>0	f(x) is minimum
0	=0	f(x) is a stationary point

Application in physics

Rates of Change

Speed, Velocity and Acceleration

A pilot ejected from his USAF Thunderbird aircraft due to an accident during an airshow
The pilot was not injured

Consider a function $y = f(x)$

The rate of change of y with respect to (w.r.t.) x is

Average rate of change = $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

Instantaneous rate of change = $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

These definitions are true for any function

(x does not have to represent time)

The first derivative $\dfrac{dy}{dx}$ is the rate of change of y w.r.t. x

Example:

The area of a circle is given by $A = \pi r^2$

The rate of change of A w.r.t. r is

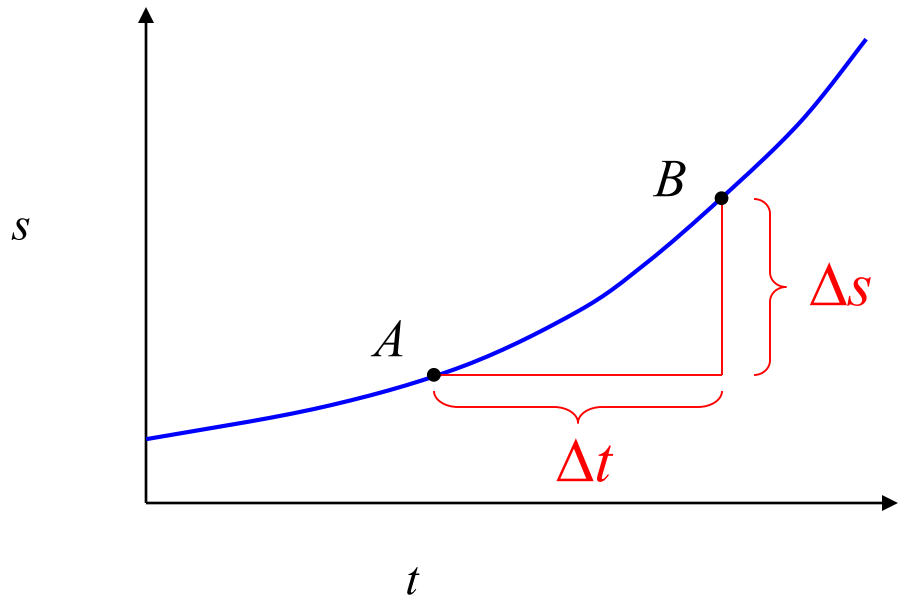
$$\frac{dA}{dr} = 2\pi r$$

which is the circumference of the circle

Rate of change increases with r

→ For larger circle, the area changes faster with respect to the same amount of change in r

Velocity



Average velocity can be found by taking:

$$\frac{\text{change in position}}{\text{change in time}} = \frac{\Delta s}{\Delta t}$$

$$v_{\text{av}} = \frac{\Delta s}{\Delta t} = \frac{s(t+\Delta t) - s(t)}{\Delta t}$$

The <u>instantaneous velocity</u> at t is the derivative obtained when $B \rightarrow A$

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t}$$

The velocity at one moment in time

Velocity is the *rate of change* of displacement with respect to time

$$v(t) = \frac{ds}{dt}$$

Velocity has both magnitude and direction

Speed is the *absolute value* of velocity

$$\text{Speed} = |v|$$

Speed has no direction

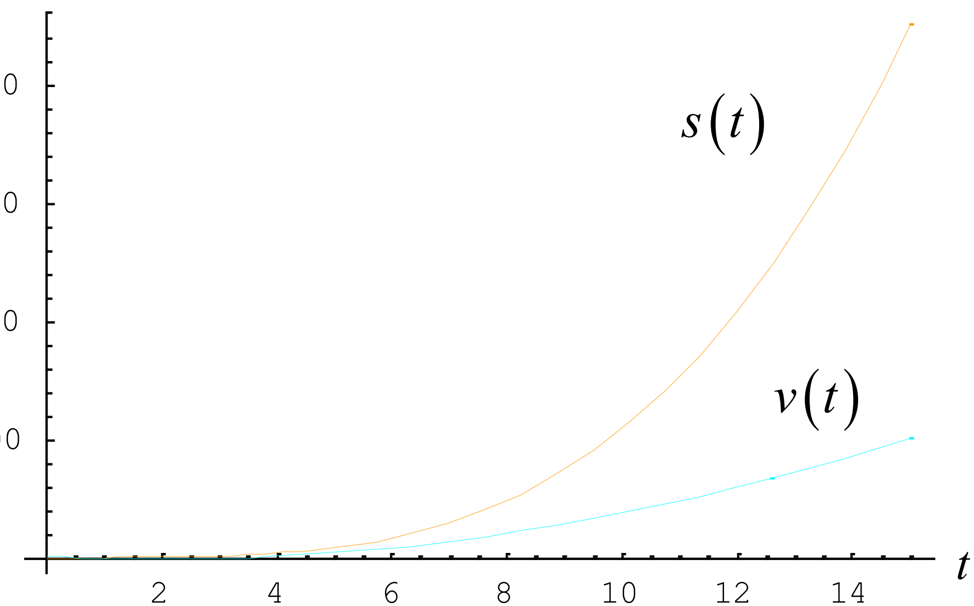
Example:

Suppose the position of a particle is given by

$$s(t) = t^3 - 6t^2 + 15t$$

Then the velocity is

$$v(t) = \frac{ds}{dt} = 3t^2 - 12t + 15$$

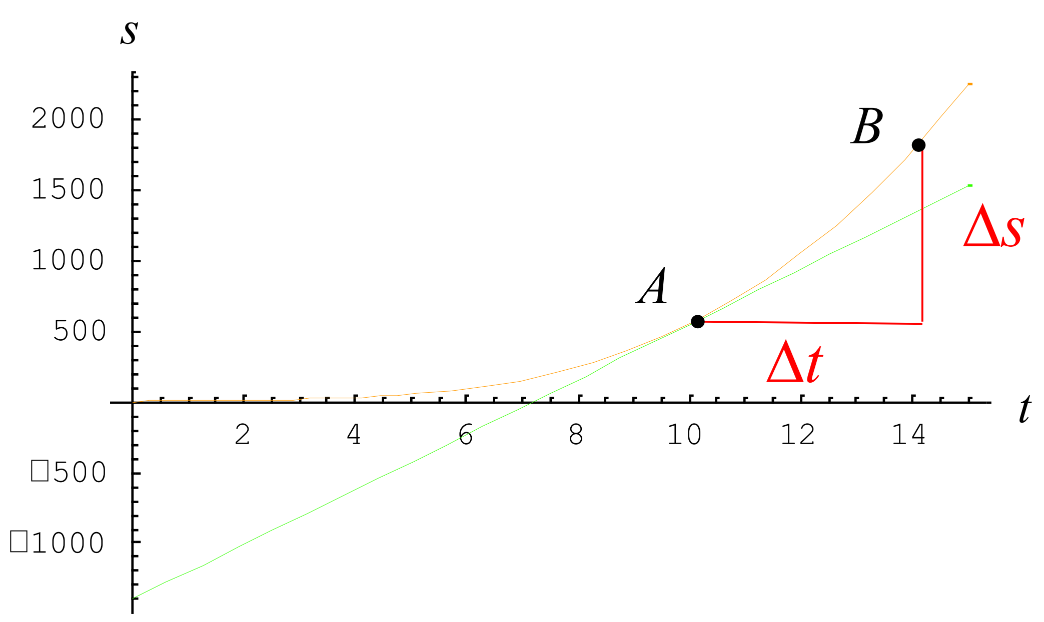


At t = 10, the velocity is

$$v(10) = 3 \cdot 10^2 - 12 \cdot 10 + 15 = 195$$

The tangent at t = 10, whose slope is 195, is shown by the green line in the figure

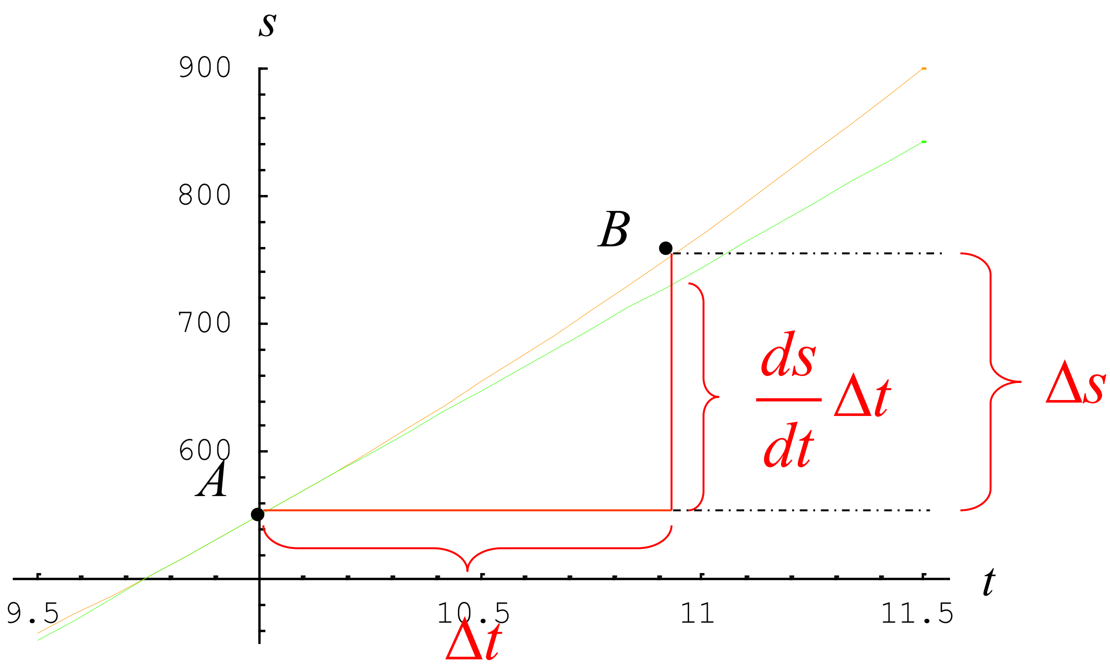
Now if you only know the velocity at t = 10, but not the position function $s(t)$, can you predict the change in position Δs after a very small time interval Δt?



As shown in the graph, if Δt is small enough, a good approximation to Δs is

$$\Delta s \approx \frac{ds}{dt}\Delta t = v\Delta t$$

The smaller the Δt, the better the approximation



Important trick:

When h is small, we have

$$\frac{\Delta f(x)}{\Delta x} = \frac{f(x+h) - f(x)}{h} \approx f'(x)$$

In general, we have

$$f(x+h) \approx f(x) + f'(x)h$$

Example:

Calculate $\tan 46^\circ$ without using a calculator.

Answer:

$$\tan 46^\circ = \tan\left(45^\circ + 1^\circ\right) = \tan\left(\frac{\pi}{4} + \frac{\pi}{180}\right)$$

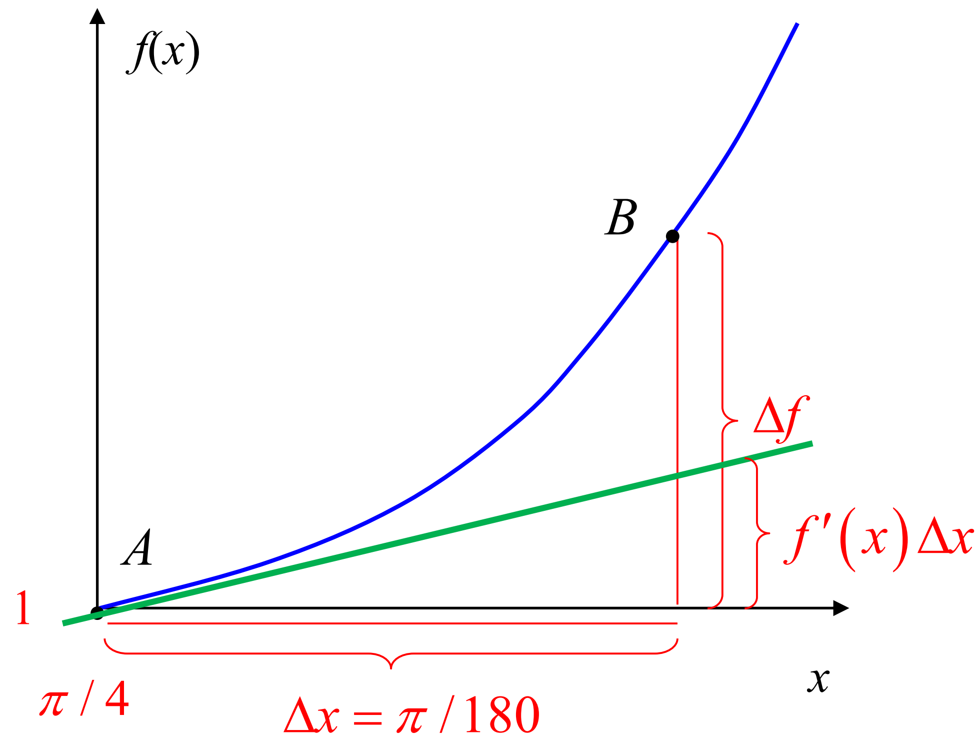
Consider the function

$$f(x) = \tan x$$

$$f(\pi/4) = 1$$

$$f'(x) = \sec^2 x$$

$$f'(\pi/4) = \sec^2(\pi/4) = 2$$



$$\Delta f \approx f'(\pi/4)\Delta x = 2 \times \pi/180 = \pi/90$$

$$\tan 46^\circ = \tan(\pi/4 + \pi/180) \approx \tan(\pi/4) + \pi/90 = 1 + \pi/90 \approx 1.035$$

Example:

Given that the instantaneous speed (km/h) of a car varies with time t (h) as $v(t) = 3t$.
At t = 5h, the car is heading towards a gas station at a distance of 3km
Estimate its distance from the station after 10 minutes.

Answer:

The instantaneous speed at t = 5h is 15 km/h

After Δt = 1/6, it has travelled approximately

$$\Delta s \approx 15 \times \frac{1}{6} = 2.5 \text{ km}$$

Hence it will be approximately at a distance of 0.5 km from the station

$$s(5h10min) \approx s(5h) + s'(5h) \times 10min$$

$$\Delta s = s(5h10min) - s(5h) = v(5h) \times \frac{1}{6}hr = 2.5km$$

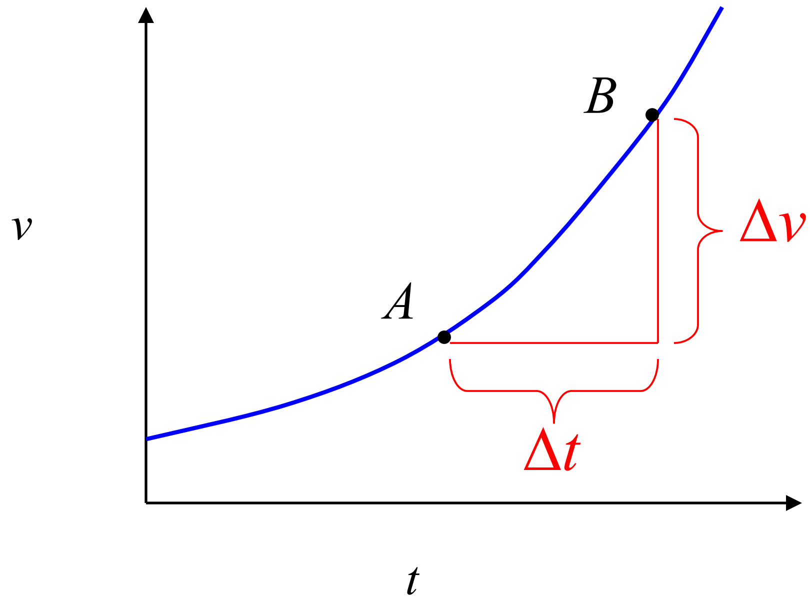
Acceleration

In general, the instantaneous velocity obtained is also a function of time

Acceleration means *how fast the velocity is changing*

Acceleration is the rate of change of velocity with respect to time

After you obtain the instantaneous velocity at all t, you can plot a graph of v vs. t



Average acceleration can be found by taking:

$$\frac{\text{change in velocity}}{\text{change in time}} = \frac{\Delta v}{\Delta t}$$

$$a_{\text{av}} = \frac{\Delta v}{\Delta t} = \frac{v(t+\Delta t) - v(t)}{\Delta t}$$

The <u>instantaneous acceleration</u> at t is the derivative obtained when $B \rightarrow A$

$$a(t) = \frac{dv}{dt} = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t}$$

The acceleration at one moment in time

Acceleration is the derivative of velocity

$$a = \frac{dv}{dt} = \frac{d}{dt}\frac{ds}{dt} = \frac{d^2 s}{dt^2}$$

Acceleration is the second derivative of displacement

Acceleration has both magnitude and direction

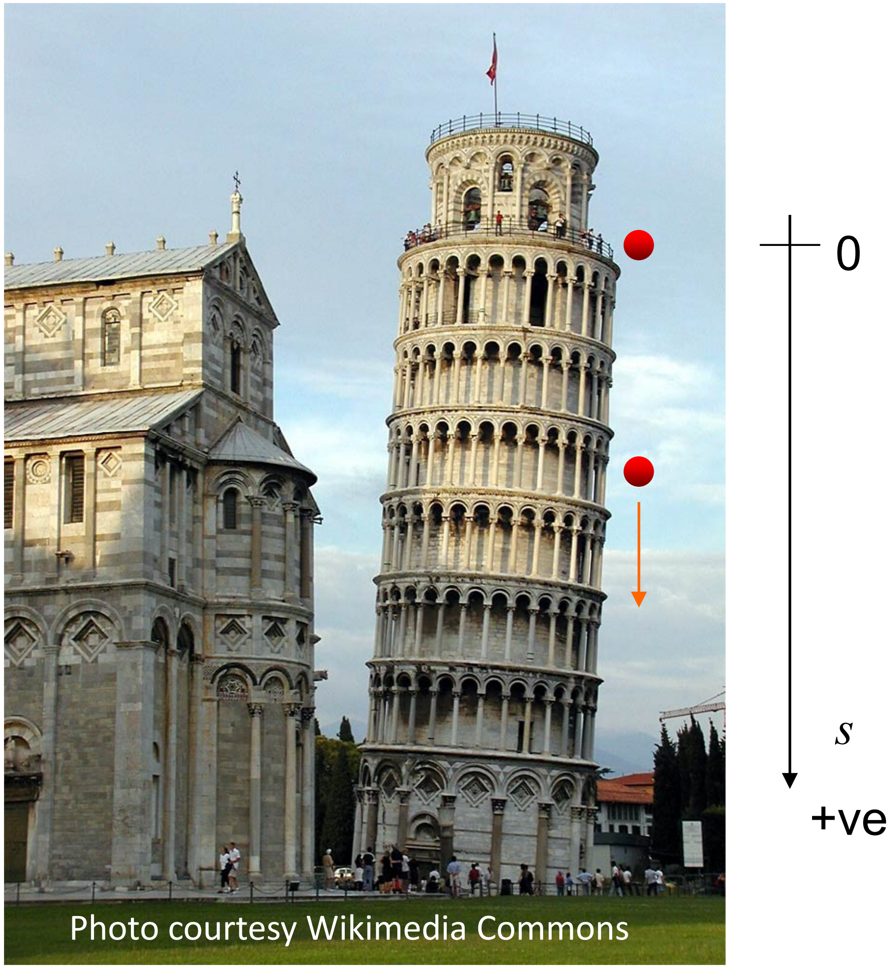
Example: Free Fall Equation

An object falling freely near the surface of the earth has a position function given by

$$s = \frac{1}{2} g\ t^2$$

Here g is a constant:

$$g \approx 9.8\ \text{ms}^{-2}$$



Velocity is the derivative of displacement:

$$v = \frac{ds}{dt} = \frac{d}{dt}\frac{1}{2}gt^2 = gt$$

Acceleration is the derivative of velocity:

$$a = \frac{dv}{dt} = \frac{d}{dt}gt = g$$

The constant g is called the *acceleration due to gravity*

All freely-falling objects near the earth's surface have the same constant downward acceleration of

$$g \approx 9.8 \text{ ms}^{-2}$$

Non-uniform Acceleration (SHM)

In general, the acceleration may not be constant

For example, consider an object moving on the x-axis, with position function given by

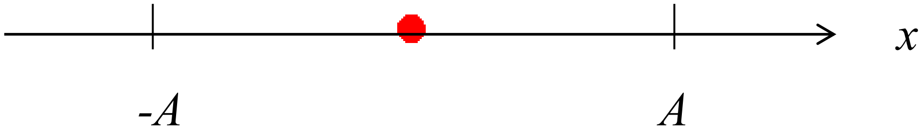
$$x(t) = A\sin(kt)$$

where A and k are constants

The acceleration is

$$a(t) = \frac{d^2}{dt^2}x(t) = \frac{d^2}{dt^2}A\sin(kt) = -k^2 A\sin(kt)$$

which is not a constant (Simple harmonic motion)



Non-uniform Acceleration (circular motion)

A particle moving according to the parametric equations

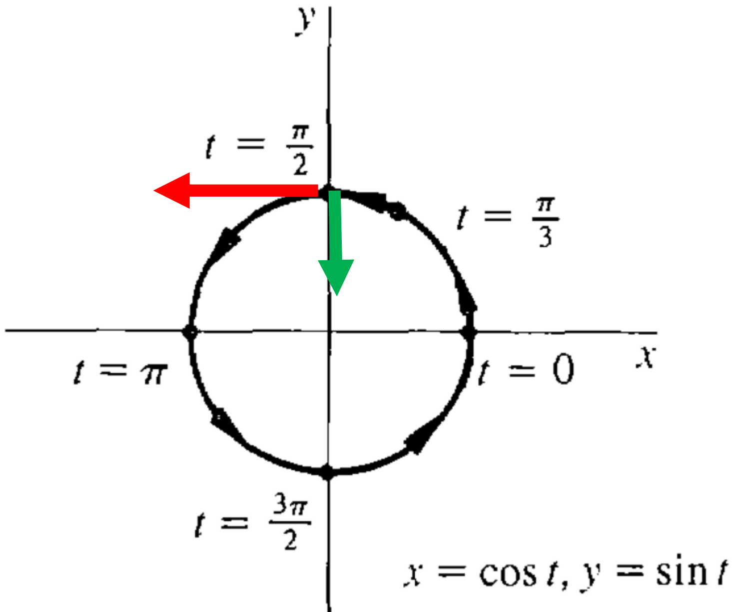
$$x(t) = \cos t\,, \qquad y(t) = \sin t$$

will move counterclockwise around the unit circle at one radian per second beginning at the point (1,0). Find the velocity and acceleration of its at time time t.

We use the vector notation $\vec{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath}$

Velocity $\vec{v}(t) = \dfrac{d\vec{r}}{dt} = -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath}$ and $\vec{r}(t) \cdot \vec{v}(t) = 0$

Acceleration $\vec{a}(t) = \dfrac{d\vec{v}}{dt} = -\cos t\,\hat{\imath} - \sin t\,\hat{\jmath} = -\vec{r}(t)$



Displacement, Velocity and Acceleration

Recall:

derivative of

Displacement Velocity

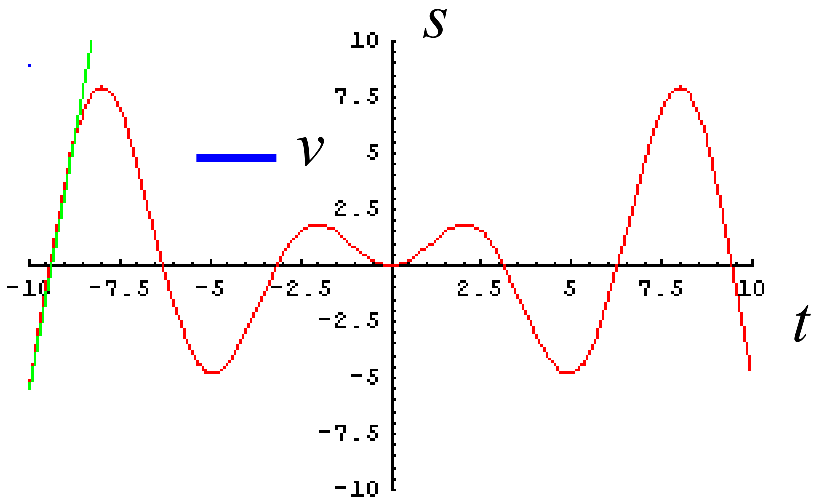
derivative of

Velocity Acceleration

Displacement, Velocity and Acceleration

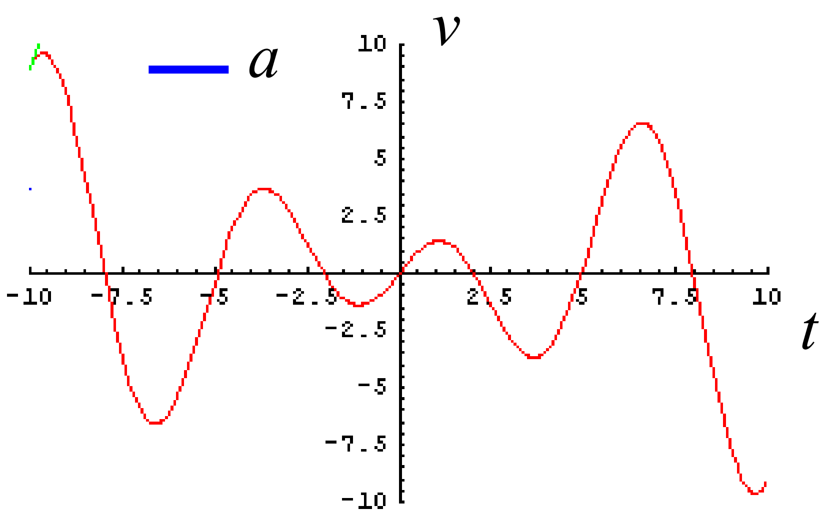
v is the slope of the s-t graph

$$v = \frac{ds}{dt}$$

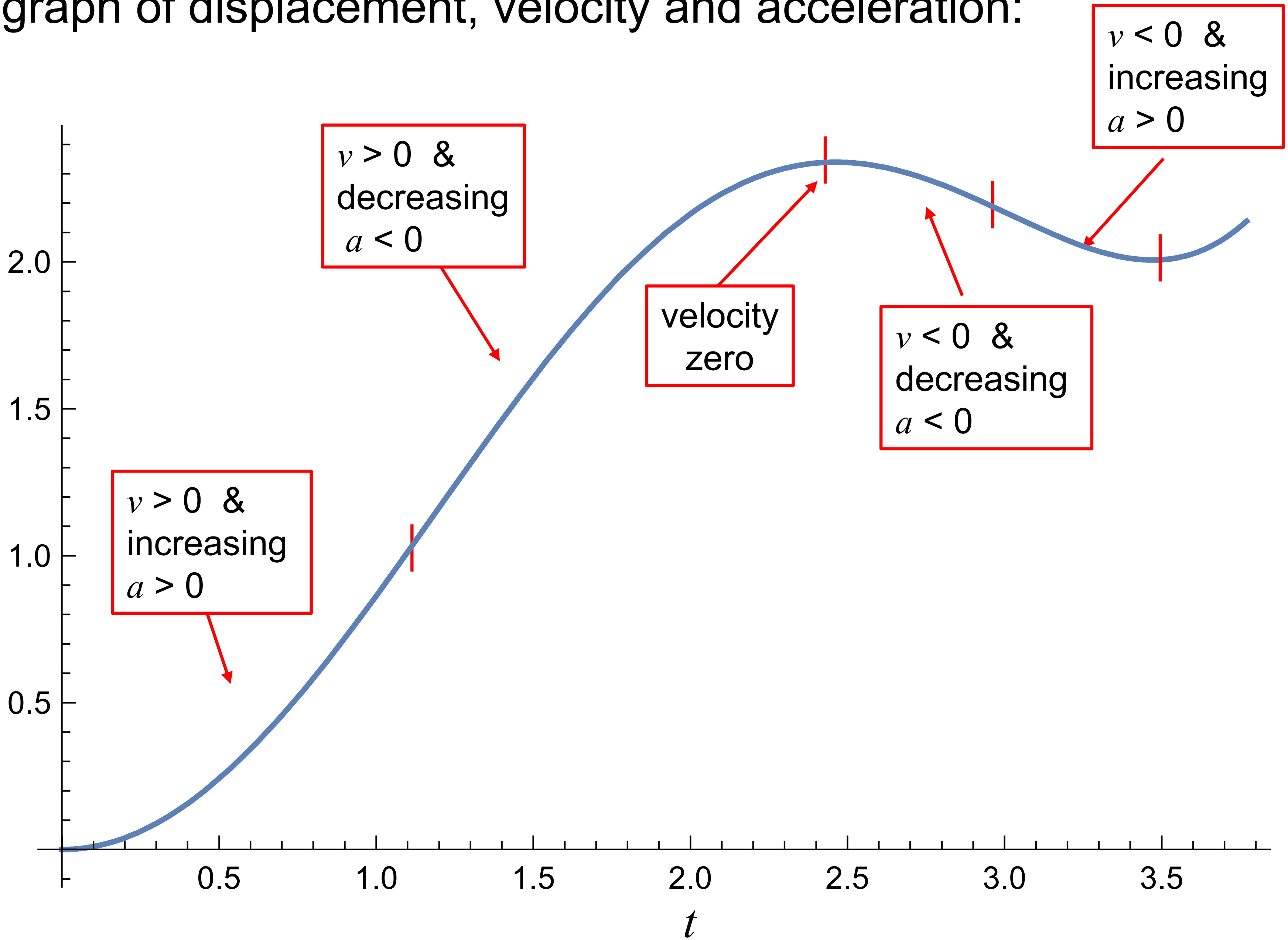


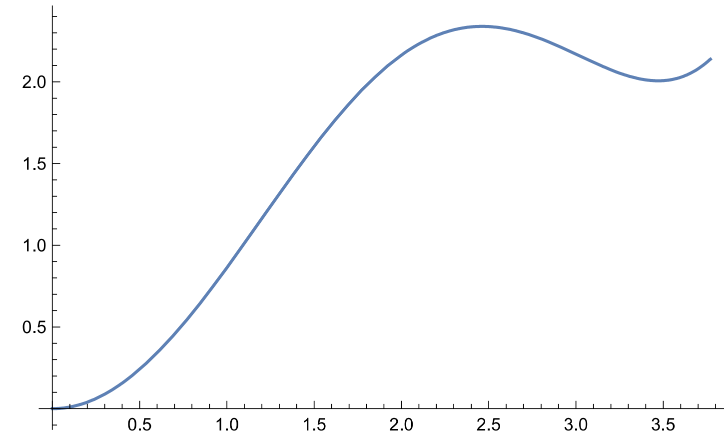
a is the slope of the v-t graph

$$a = \frac{dv}{dt}$$

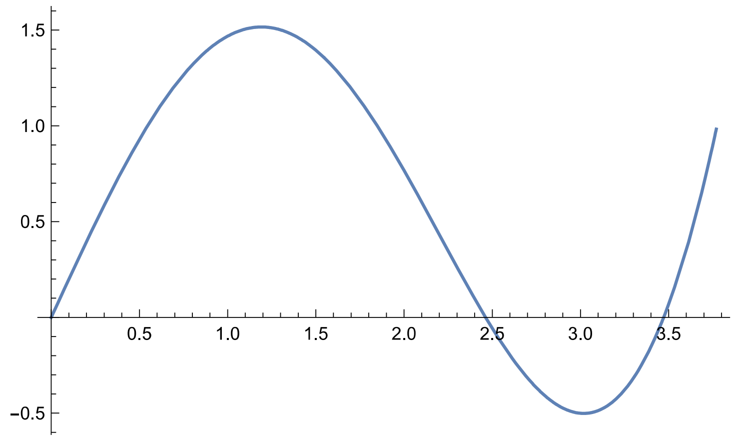


It is important to understand the relationship between the graph of displacement, velocity and acceleration:

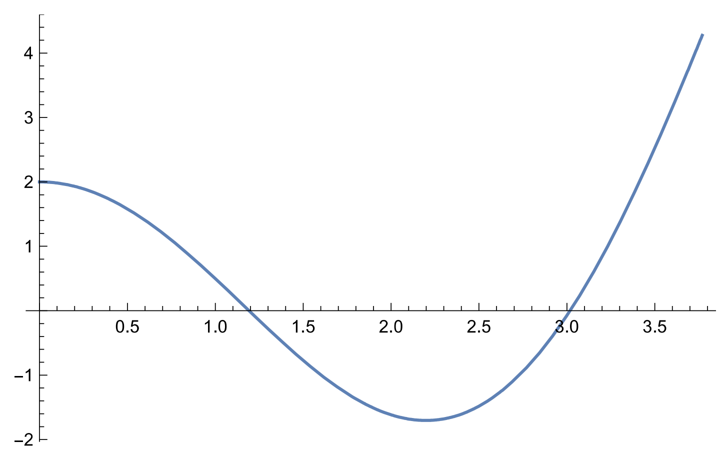




Displacement s(t)



Velocity v(t)



Acceleration a(t)