Tutorial 2

Integration

香港科技大學
THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY

香港資優教育學苑
The Hong Kong Academy for Gifted Education

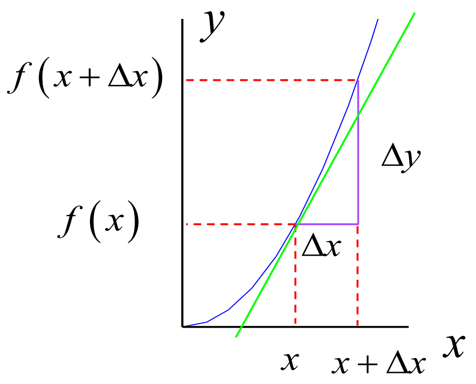
What is Calculus?

Calculus
微積分

Differential calculus
Differentiation
微分

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

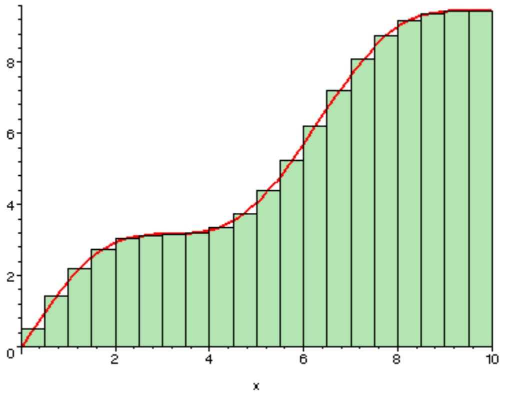
The relation of very small changes of different quantities



Integral calculus
Integration
積分

$$y = \int f(x)\,dx = \lim_{N \to \infty} \left(\sum_{i=0}^{N} f(x_i)(x_{i+1} - x_i) \right)$$

Adding a large amount of small quantities to find the sum



Why we need these?

Antiderivatives, Indefinite Integrals and Slope Fields

Differential Equations (微分方程)

A differential equation is an equation to solve for an unknown function which involves derivatives of the function

Differential equations play a prominent role in physics and many other disciplines

Example: Find y as a function of x which satisfies

$$y' = 2x$$

Solution:

$$y = x^2 \qquad y = x^2 - 3 \qquad y = x^2 + 2$$

General solution: $y = x^2 + C$

where C is an arbitrary constant

If we have some more information we can find C

Example:

Given $y' = 2x$ and y = -2 when x = 1, find y as a function of x

$$y = x^2 + C$$

$$-2 = 1^2 + C$$

$$C = -3$$

$$y = x^2 - 3$$

This is called an *initial value problem*
We need the *initial condition* to find C

A differential equation becomes an initial value problem when you are given the *initial condition* and asked to find the solution

Are we sure that this family of functions already includes all the solutions to the differential equation?

Yes, it does include all solutions when the right hand side depends on x only (but not y)

Proof:

For differential equation of the form $\frac{dy}{dx} = f(x)$

where f is an arbitrary function

we consider any two functions $F(x)$ and $G(x)$, both being the solution of the differential equation:

$$\frac{d}{dx}F(x) = f(x) \qquad \frac{d}{dx}G(x) = f(x)$$

Construct the function $H(x) = F(x) - G(x)$

then $\frac{d}{dx}H(x) = \frac{d}{dx}F(x) - \frac{d}{dx}G(x) \equiv 0$

Therefore $H(x) = C$

Any two solutions of the equation differ at most by a constant

Q.E.D.

Antiderivative

For differential equations of the form:

$$y' = f(x)$$

the slope depends on x only

The equation asks for a solution y such that

$$\frac{dy}{dx} = f(x)$$

y is called the *antiderivative* of $f(x)$

Indefinite Integral

Another name for antiderivative is *indefinite integral*

The notation of antiderivative and indefinite integral is

$$\int f(x)\,dx$$

$$y = \int f(x)\,dx \quad \Leftrightarrow \quad \frac{dy}{dx} = f(x)$$

Indefinite Integral

It is called "indefinite" integral because we the answer is not unique: $\int 2x\,dx = x^2 + C$

Integration Symbol
An elongated "S"

$$\int f(x)\,dx$$

integrand

Examples:

$$\frac{d}{dx}x = 1 \quad \Rightarrow \quad \int dx = x + C$$

$$\frac{d}{dx}\frac{x^2}{2} = x \quad \Rightarrow \quad \int x dx = \frac{x^2}{2} + C$$

$$\frac{d}{dx}\frac{x^3}{3} = x^2 \quad \Rightarrow \quad \int x^2 dx = \frac{x^3}{3} + C$$

$$\frac{d}{dx}\left(-\frac{1}{x}\right) = \frac{1}{x^2} \quad \Rightarrow \quad \int \frac{1}{x^2} dx = -\frac{1}{x} + C$$

$$\frac{d}{dx}\frac{x^n}{n} = x^{n-1} \quad \Rightarrow \quad \int x^{n-1} dx = \frac{x^n}{n} + C \qquad \text{Integer } n \neq 0$$

$$\frac{d}{dx}\sin x = \cos x \quad \Rightarrow \quad \int \cos x dx = \sin x + C$$

$$\frac{d}{dx}(-\cos x) = \sin x \quad \Rightarrow \quad \int \sin x dx = -\cos x + C$$

Integration Rules

Integration Rules

- Because indefinite integration is just the inverse operation of differentiation, each differentiation rule has correspondingly an integration rule:

Differentiation	Integration
Constant-multiple rule	Constant-multiple rule
Sum-and-difference rule	Sum-and-difference rule
Chain rule	Method of substitution
Product rule	Integration by parts

Constant-Multiple Rule

For arbitrary constant k,

$$\int kf(x)\,dx = k\int f(x)\,dx$$

Proof:

$$\frac{d}{dx}\left[k\int f(x)\,dx\right] = k\frac{d}{dx}\int f(x)\,dx$$

$$= kf(x)$$

$$\Leftrightarrow k\int f(x)\,dx = \int kf(x)\,dx$$

Constant-multiple rule of differentiation

Definition of indefinite integral

Sum-and-Difference Rule

$$\int \left[f(x) \pm g(x) \right] dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Sum-and-difference rule of differentiation

Proof:

$$\frac{d}{dx}\left[\int f(x)\,dx \pm \int g(x)\,dx\right] = \frac{d}{dx}\int f(x)\,dx \pm \frac{d}{dx}\int g(x)\,dx$$

$$= f(x) \pm g(x)$$

$$\Leftrightarrow \int \left[f(x) \pm g(x) \right] dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Definition of indefinite integral

Method of Substitution

Suppose you want to find the indefinite integral of an integrand $f(x)$

$$y = \int f(x)\,dx$$

which means

$$\frac{dy}{dx} = f(x)$$

Unfortunately $f(x)$ is not a recognizable form that you can find its antiderivative in the integration table

Method of Substitution

Try to see if there exists a new variable u (which is a function of x) such that $f(x)$ can be written as

$$f(x) = h(u)\frac{du}{dx}$$

and the antiderivative of $h(u)$

$$\int h(u)\,du$$

is known

Then the answer is

$$\int f(x)\,dx = \int h(u)\frac{du}{dx}\,dx = \int h(u)\,du$$

Proof:

If $$H(u) = \int h(u)\,du$$

Then $$\frac{dH}{du} = h(u)$$

By chain-rule $$\frac{dH}{dx} = \frac{dH}{du}\frac{du}{dx} = h(u)\frac{du}{dx} = f(x)$$

Therefore $$\square f(x)\,dx = \square h(u)\,du$$

It helps you remember the rule if you treat the derivative du/dx *as if it were* a fraction, and part of the integration symbol dx *as if it were* a number multiplied to du/dx

$$\int f(x)\,dx \;\rightarrow \int h(u)\frac{du}{\cancel{dx}}\,\cancel{dx} \;\rightarrow \int h(u)\,du$$

Example:

$$\int (x+2)^5 dx$$

Let $u = x + 2$

$$\frac{du}{dx} = 1$$

$$\int u^5 \frac{du}{dx} dx$$

Usually skipped

$$\int u^5 du$$

$$\frac{1}{6}u^6 + C$$

Don't forget to substitute the value for u back into the problem!

$$\frac{(x+2)^6}{6} + C$$

Example:

$$\int \sqrt{4x-1}\ dx \qquad \text{Let } u = 4x-1$$

$$\frac{1}{4}\int u^{1/2} \cdot 4dx \qquad \frac{du}{dx} = 4$$

$$\frac{1}{4}\int u^{1/2} \cdot \frac{du}{dx} dx$$

$$\frac{1}{4}\int u^{1/2} du$$

$$\frac{1}{4}\cdot\frac{2}{3}u^{3/2} + C \rightarrow \frac{1}{6}u^{3/2} + C$$

$$\frac{1}{6}(4x-1)^{3/2} + C$$

Example:

$$\int \cos(2x)\ dx$$

Let $u = 2x$

$$\frac{du}{dx} = 2$$

$$\frac{1}{2}\int \cos u \cdot 2dx$$

$$\frac{1}{2}\int \cos u du$$

$$\frac{1}{2}\sin u + C$$

$$\frac{1}{2}\sin(2x) + C$$

Exercise:

$$\int \sqrt{1+x^2} \cdot 2x \, dx$$

$$\int x^2 \sin\left(x^3\right) \, dx$$

$$\int \sin^4 x \cdot \cos x \, dx$$

Integration by Parts

The rule of substitution is a result of the chain rule of differentiation

Similarly, the product rule of differentiation leads to a rule in integration called Integration by Parts

$$uv = \int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

It is easier to be remembered when written in the following form

$$\int udv = uv - \int vdu$$

Proof of integration by parts:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\Rightarrow \int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

$$\Rightarrow uv = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

$$\Rightarrow \int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

Example: Find $\int x \cos x dx$

Let $u = x,\ v = \sin x$

$$\int x \cos x dx = \int u \frac{dv}{dx} dx$$

$$= uv - \int v \frac{du}{dx} dx$$

$$= x \sin x - \int \sin x \frac{dx}{dx} dx$$

$$= x \sin x - \int \sin x dx$$

$$= x \sin x + \cos x + C$$

Exercise:

$$\int x \sin x dx$$

$$\int x^2 \cos x dx$$

Derivative of function

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\sec x = \sec x \cdot \tan x$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\csc x = -\csc x \cdot \cot x$$

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Summary for integration

$$\int 1dx = x + C$$

$$\int kf(x)dx = k\int f(x)dx$$

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

$$\int f(x)dx = \int h(u)\frac{du}{dx}dx = \int h(u)du$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \sec^2 x dx = \tan x + C$$

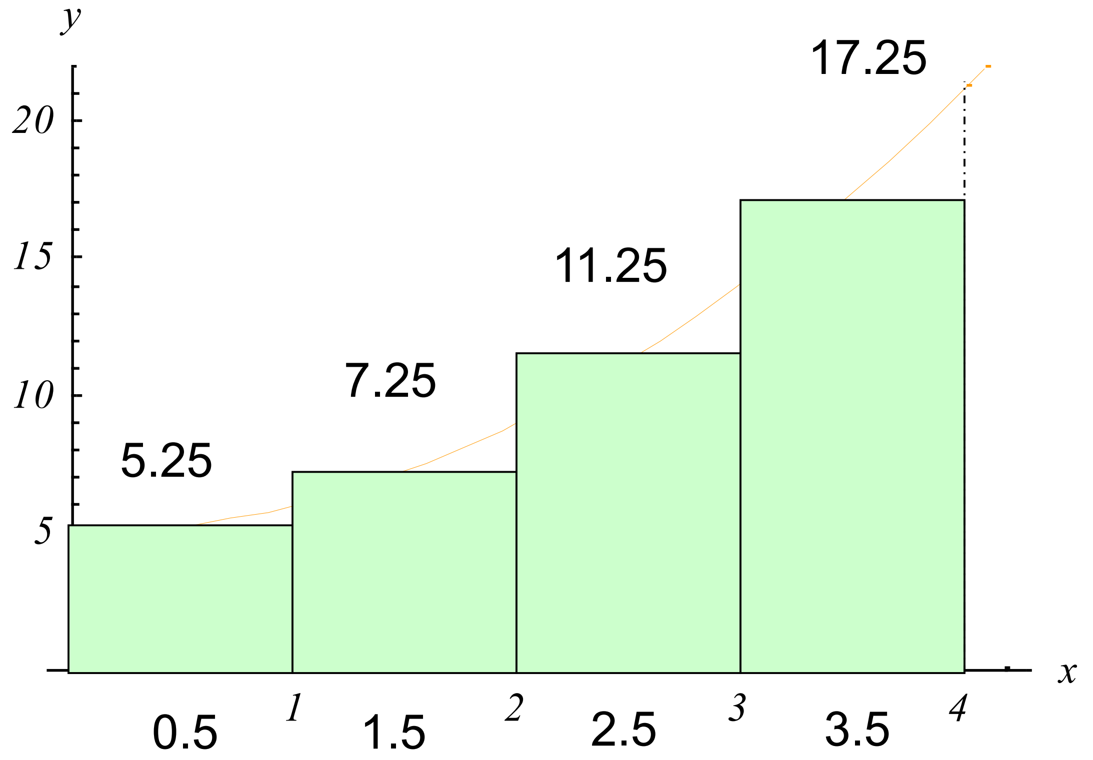
Definite Integrals

We can estimate the area under a curve by drawing rectangles touching the curve at some points

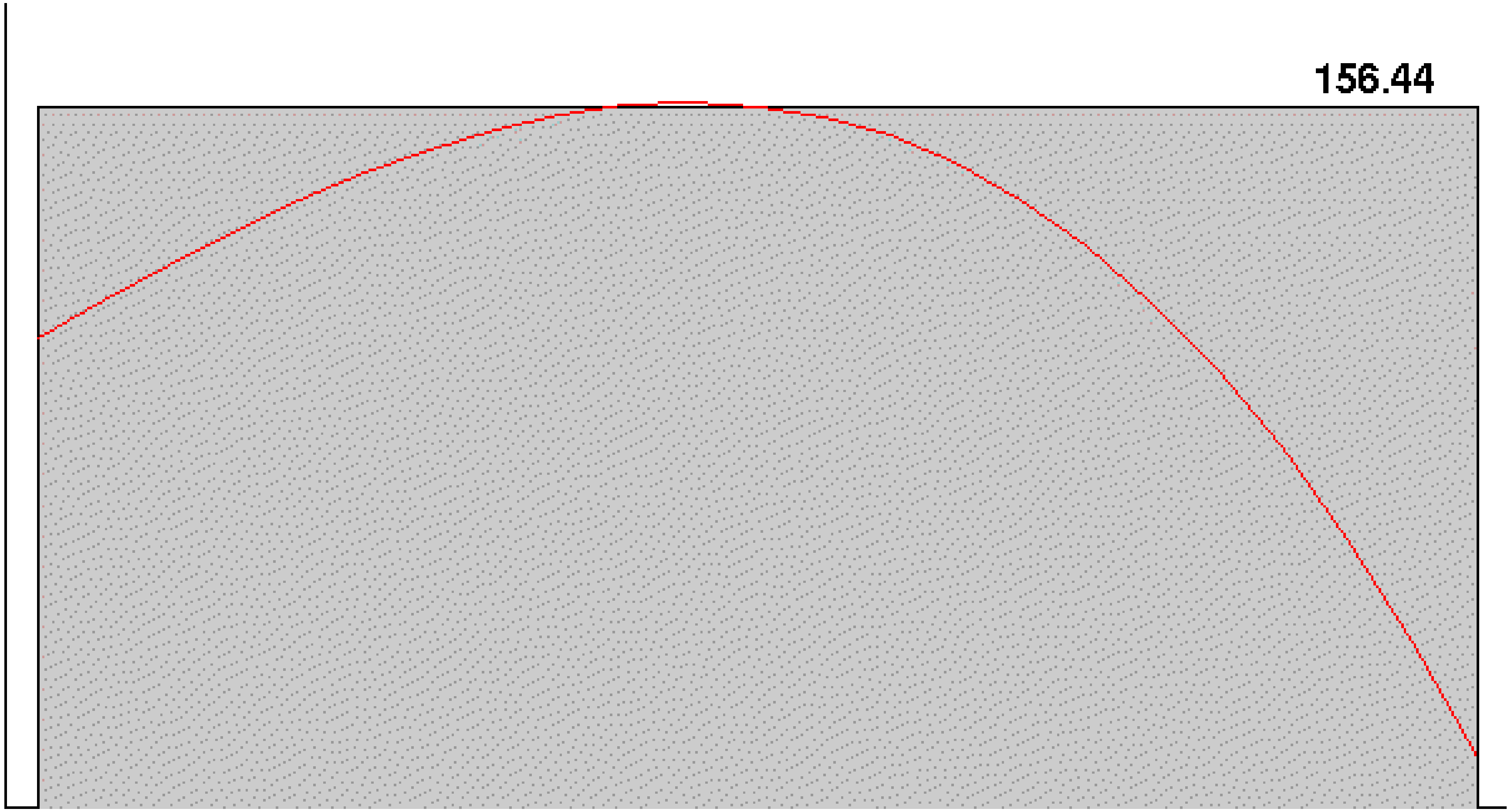
Example: Area under $y = x^2 + 5$
inside the interval $0 < x < 4$

Use rectangles with equal width that touch the curve at the mid-point

Approximate area: $5.25 + 7.25 + 11.25 + 17.25 = 41$



In this example there are four subintervals
As the number of subintervals increases, so does the accuracy



This is how the ancient people calculated area of some regions with curved boundaries

They stopped at certain stage when the approximation is already good enough for practical purpose

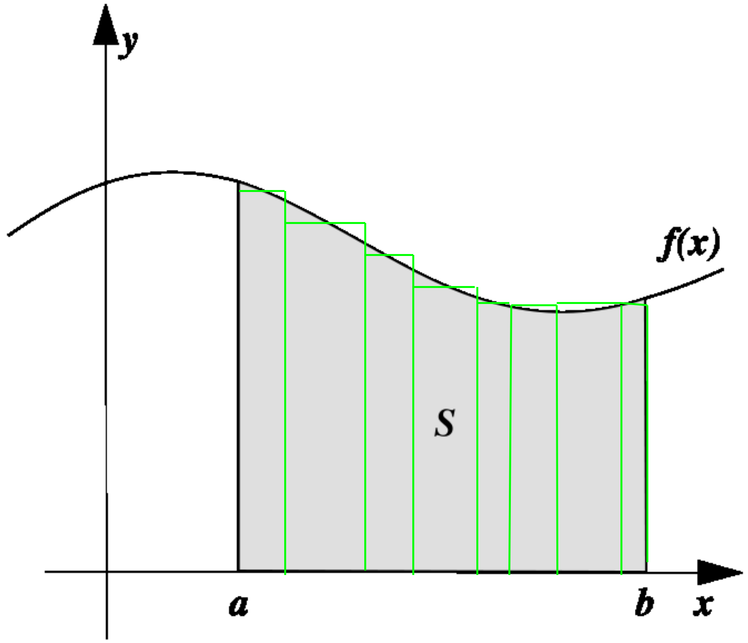
Sometimes (but rarely) it is possible to deduce the exact area when the number of rectangles tends to infinity

Is there a more systematic method to find these areas?

Definition of Definite Integral

When we find the area under a curve by adding rectangles, the answer is called a **Riemann sum**

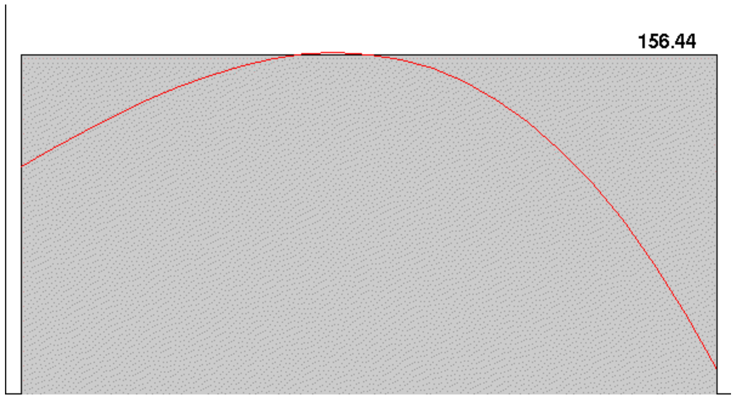
Note: Subintervals do not all have to be the same size



$$\sum_{\text{A set of points in the interval}} f\left(x\right)\Delta x$$

Definition of Definite Integral

The **definite integral** of f over $[a, b]$, where $b > a$, is the limit of the Riemann sum when the number of subintervals tends to infinity and the length of each subinterval tends to zero



$$\lim_{\substack{\text{Number of points}\to\infty \\ \text{Width of each subinterval}\to 0}} \left(\sum_{\text{A set of points in the interval}} f(x)\Delta x \right) \equiv \int_a^b f(x)dx$$

Notation of Definite Integral

Leibniz and Joseph Fourier introduced a simpler notation for the definite integral:

upper limit of integration

Integration Symbol
An elongated "*S*"

$$\int_a^b f(x)\,dx$$

integrand

lower limit of integration

variable of integration (dummy variable)

It is called a dummy variable because the answer does not depend on the symbol chosen

1. Reversing the limits changes the sign

$$\int_b^a f(x)\,dx \qquad \text{where} \quad a < b$$

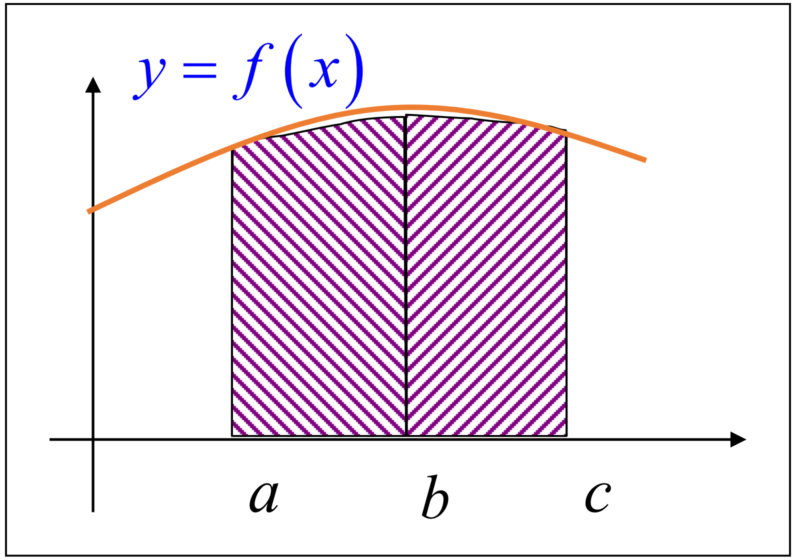
is *defined* by $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

2. When the upper limit and the lower limit are the same, the definite integral is zero

$$\int_a^a f(x)\,dx = 0$$

3. Intervals can be added

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



Note

Although I have been using the word "area" to refer to definite integrals, remember that the fundamental definition is a limit of the Riemann sum:

$$\lim_{\substack{\text{Number of points}\to\infty \\ \text{Width of each subinterval}\to 0}} \left(\sum_{\text{A set of points in the interval}} f(x)\,\Delta x \right)$$

It is defined using algebra and does not rely on any geometric interpretations

For example, obviously it does not represent "area" when $f(x) < 0$

Differentiation and Definite Integrals

In the 17th Century, Newton and Leibniz discovered the fundamental theorem of calculus independently (?)

The theorem connects integration and differentiation
Through this connection, differentiation can be exploited to calculate definite integrals and hence allows one to solve a much broader class of problems

Finding derivatives is comparatively much easier than calculating the Riemann sum, especially when supplied with a lot of rules that we learned, making it unnecessary to start from the first principle every time

Fundamental Theorem of Calculus

Let

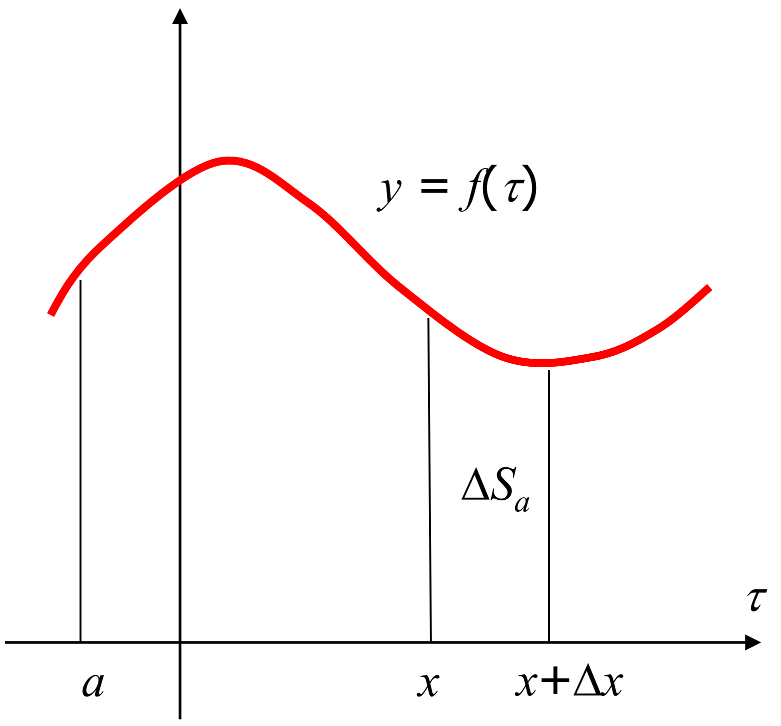
$$S_a(x) = \int_a^x f(\tau)\,d\tau$$

Refer to the supp. notes

Then

$$\frac{d}{dx}S_a(x) = \lim_{\Delta x \to 0} \frac{S_a(x+\Delta x) - S_a(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta S_a}{\Delta x} = f(x)$$



$$\frac{d}{dx}\int_a^x f(\tau)\,d\tau = f(x)$$

Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_a^x f(\tau)\,d\tau = f(x)$$

The above result means $\int_a^x f(\tau)\,d\tau$ is an antiderivative of $f(x)$

Suppose we know a particular antiderivative $F(x)$, then

$$\int_a^x f(\tau)\,d\tau = F(x) + C$$

To determine the integration constant, notice that

$$\int_a^a f(\tau)\,d\tau = F(a) + C = 0 \Rightarrow C = -F(a)$$

$$\int_a^x f(\tau)\,d\tau = F(x) - F(a)$$

Example: Evaluate $\int_1^2 \cos\tau d\tau$

$$\frac{d}{dx}\int_1^x \cos\tau d\tau = \cos x$$

$$\int_1^x \cos\tau d\tau = \int \cos x dx = \sin x + C$$

$$\int_1^1 \cos\tau d\tau = \sin 1 + C = 0 \Rightarrow C = -\sin 1$$

$$\int_1^x \cos\tau d\tau = \sin x - \sin 1$$

$$\int_1^2 \cos\tau d\tau = \sin 2 - \sin 1$$

Note about notation:

$$\int_a^x f(\tau)\,d\tau = F(x) - F(a)$$

If $x = b$

$$\int_a^b f(\tau)\,d\tau = F(b) - F(a)$$

Dummy variable doesn't matter, replace τ by x

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

This is what you usually see in textbooks

Exercise:

$$\int_{2}^{1} \frac{1}{x^2}\, dx$$

$$\int_{0}^{\pi} \cos^2 x\, dx$$

The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a,b]$, then the function

$$S_a(x) = \int_a^x f(\tau)\,d\tau$$

has a derivative at every point in $[a,b]$, and

$$\frac{d}{dx} S_a(x) = \frac{d}{dx} \int_a^x f(\tau)\,d\tau = f(x)$$

The Fundamental Theorem of Calculus, Part 2

If f is continuous at every point of $[a,b]$, and if F is any antiderivative of f on $[a,b]$, then

$$S_a(b) = \int_a^b f(x)\,dx = F(b) - F(a)$$

(Also called the **Integral Evaluation Theorem**)

To evaluate a definite integral, take the anti-derivatives and subtract

Example:

$$\int_0^{\frac{\pi}{4}} \tan x \ \sec^2 x \ dx$$

The method of substitution is a little different for definite integrals

new limit

$$\int_0^1 u \ du$$

new limit

$$\text{Let } u = \tan x$$

$$du = \sec^2 x \ dx$$

$$\frac{1}{2}u^2 \Big|_0^1$$

$$u(0) = \tan 0 = 0$$

$$u\left(\frac{\pi}{4}\right) = \tan \frac{\pi}{4} = 1$$

$$\frac{1}{2}$$

We could have substituted back and used the original limits

Example:

$$\int_0^{\frac{\pi}{4}} \tan x \ \sec^2 x \, dx$$

Using the original limits:

$$\text{Let } u = \tan x$$

$$du = \sec^2 x \, dx$$

~~$$\int_0^{\frac{\pi}{4}} u \, du$$~~

$$\int u \, du$$

Leave the limits out until you substitute back.

$$= \frac{1}{2} u^2$$

$$= \frac{1}{2} (\tan x)^2 \Big|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(\tan \frac{\pi}{4}\right)^2 - \frac{1}{2}(\tan 0)^2$$

$$= \frac{1}{2} \cdot 1^2 - \frac{1}{2} \cdot 0^2 \qquad = \frac{1}{2}$$

This is usually better than finding new limits

Example:

$$\frac{d}{dx}\int_0^{x^2} \cos\tau \, d\tau$$

The upper limit of integration does not match the derivative, but we could use the **chain rule**

$$\cos\left(x^2\right)\cdot\frac{d}{dx}x^2$$

$$\cos\left(x^2\right)\cdot 2x$$

$$2x\cos\left(x^2\right)$$

Example:

$$\frac{d}{dx}\int_{x}^{5} 3\tau \sin\tau \; d\tau$$

The lower limit of integration is not a constant, but the upper limit is

We can **change the sign** of the integral and **reverse the limits**

$$-\frac{d}{dx}\int_{5}^{x} 3\tau \sin\tau \; d\tau$$

$$-3x\sin x$$

Exercise:

$$\frac{d}{dx}\int_{2x}^{x^2}\frac{1}{\sqrt{1-\tau^2}}\,d\tau$$

Supplementary Notes

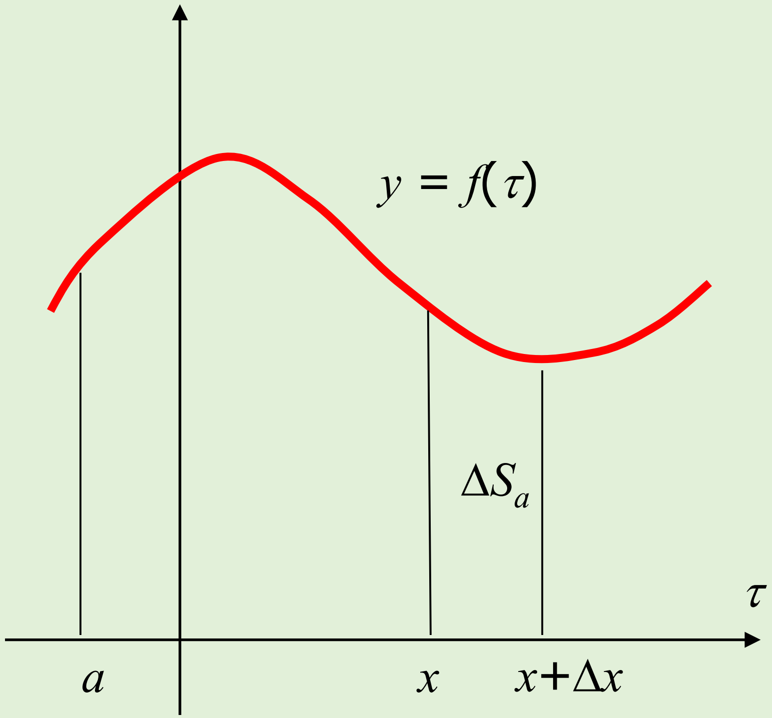
To prove $$\lim_{\Delta x \to 0} \frac{\Delta S_a}{\Delta x} = f(x)$$

$$S_a(x) = \int_a^x f(\tau)\,d\tau$$

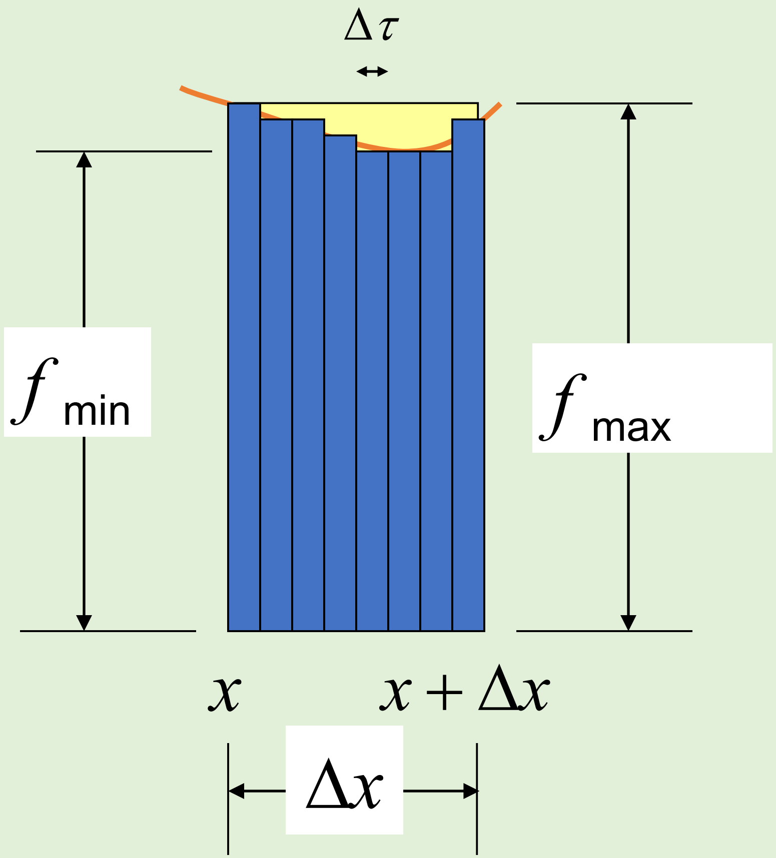
$$\Delta S_a = S_a(x+\Delta x) - S_a(x)$$

$$= \int_a^{x+\Delta x} f(\tau)\,d\tau - \int_a^x f(\tau)\,d\tau$$

$$= \int_x^{x+\Delta x} f(\tau)\,d\tau$$



$$\Delta S_a = \int_x^{x+\Delta x} f(\tau)\, d\tau = \lim_{\Delta\tau \to 0} \sum f(\tau)\Delta\tau$$



$$\lim_{\Delta\tau \to 0} \sum f(\tau)\Delta\tau \le f_{\max}\Delta x$$

$$f_{\min}\Delta x \le \lim_{\Delta\tau \to 0} \sum f(\tau)\Delta\tau$$

$$f_{\min}\Delta x \le \Delta S_a \le f_{\max}\Delta x$$

$$f_{\min} \le \frac{\Delta S_a}{\Delta x} \le f_{\max}$$

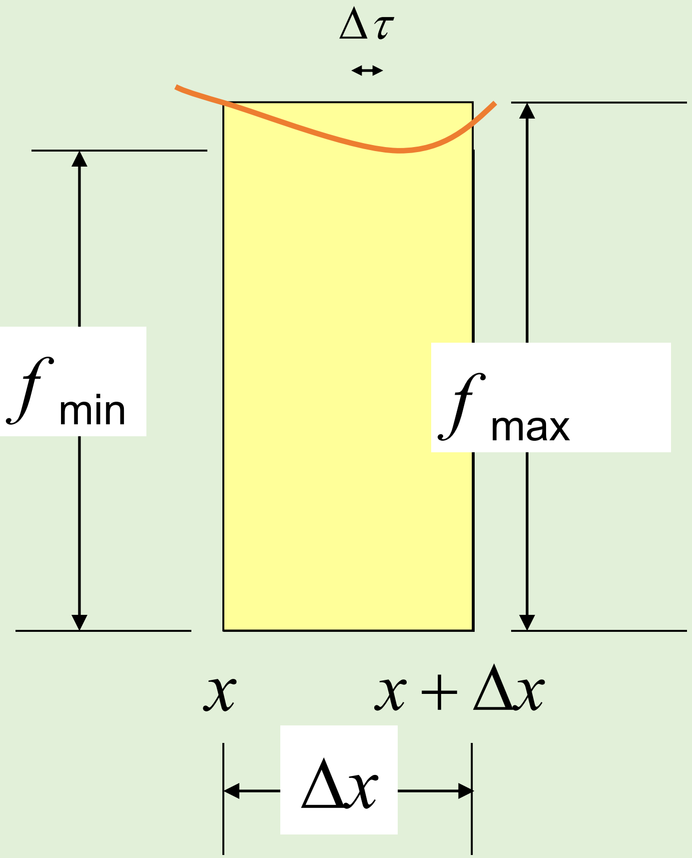
$$f_{\min} \le \frac{\Delta S_a}{\Delta x} \le f_{\max}$$

As Δx gets smaller, $f_{\min}$ and $f_{\max}$ get closer together

$$\lim_{\Delta x \to 0} f_{\min} = \lim_{\Delta x \to 0} f_{\max} = f(x)$$

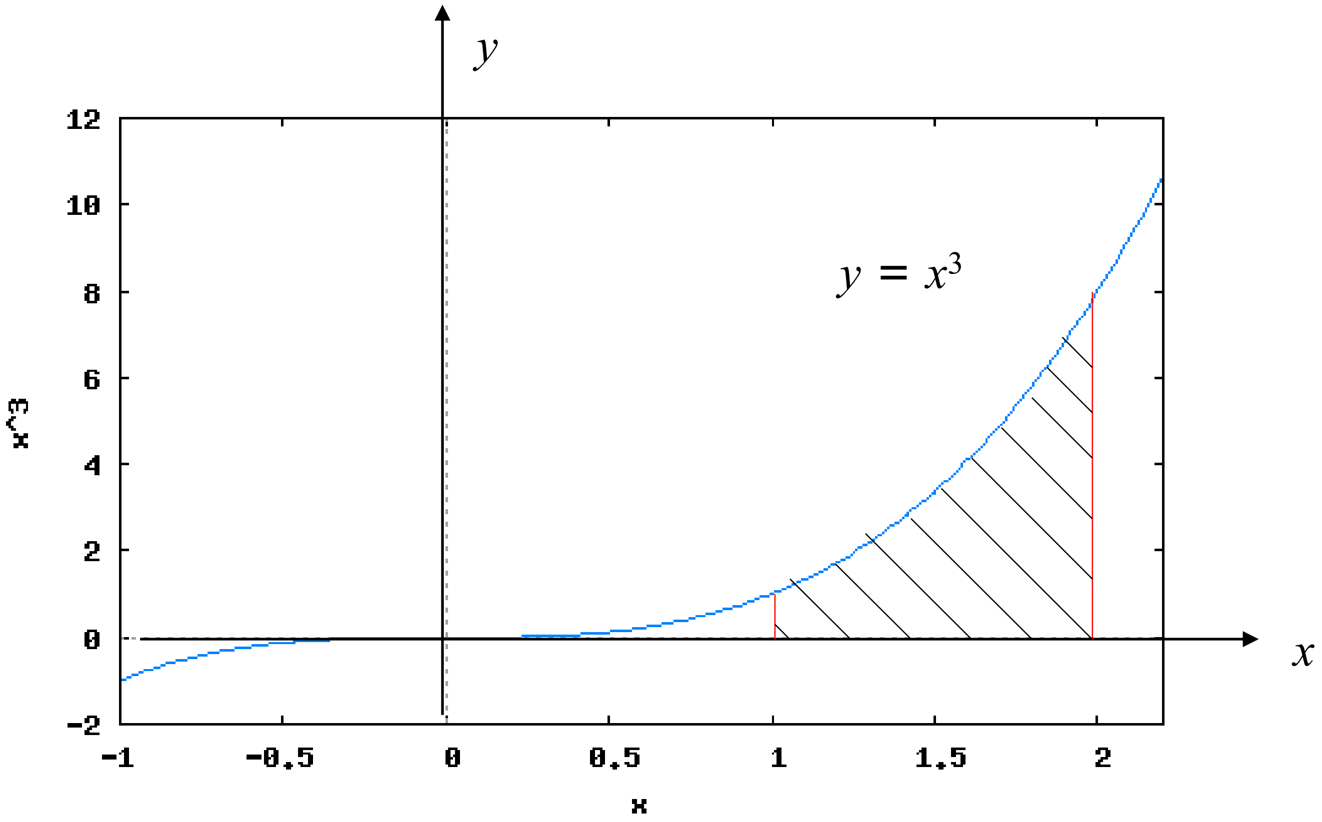
By Squeeze-Sandwich Theorem:

$$\lim_{\Delta x \to 0} \frac{\Delta S_a}{\Delta x} = f(x)$$



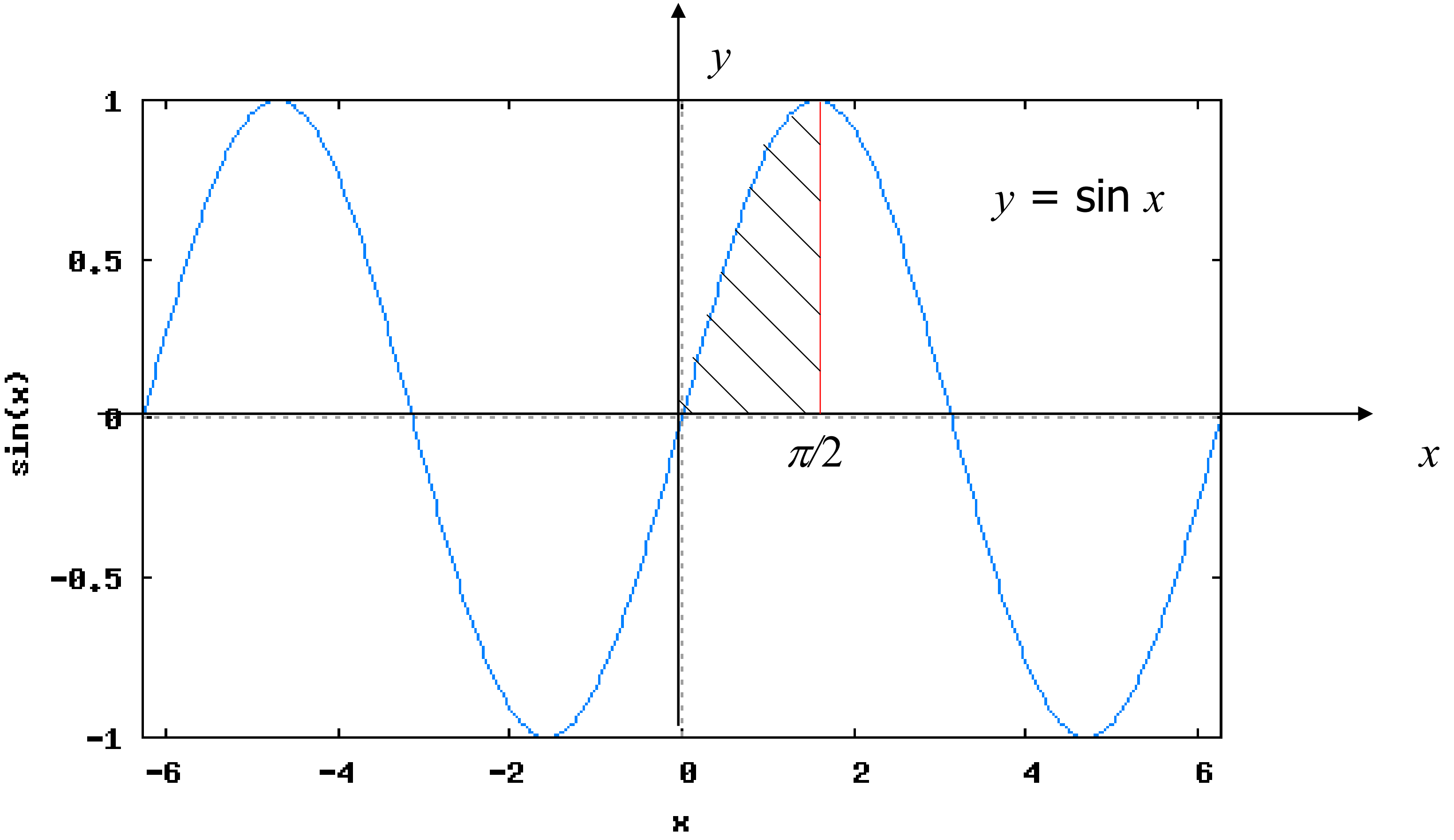
Example: Find the area of bounded by the curve $y = x^3$, the x-axis, and the two vertical lines $x = 1$ and $x = 2$

$$\int_1^2 x^3 dx = \left.\frac{x^4}{4}\right|_1^2 = \frac{2^4}{4} - \frac{1^4}{4} = \frac{15}{4}$$



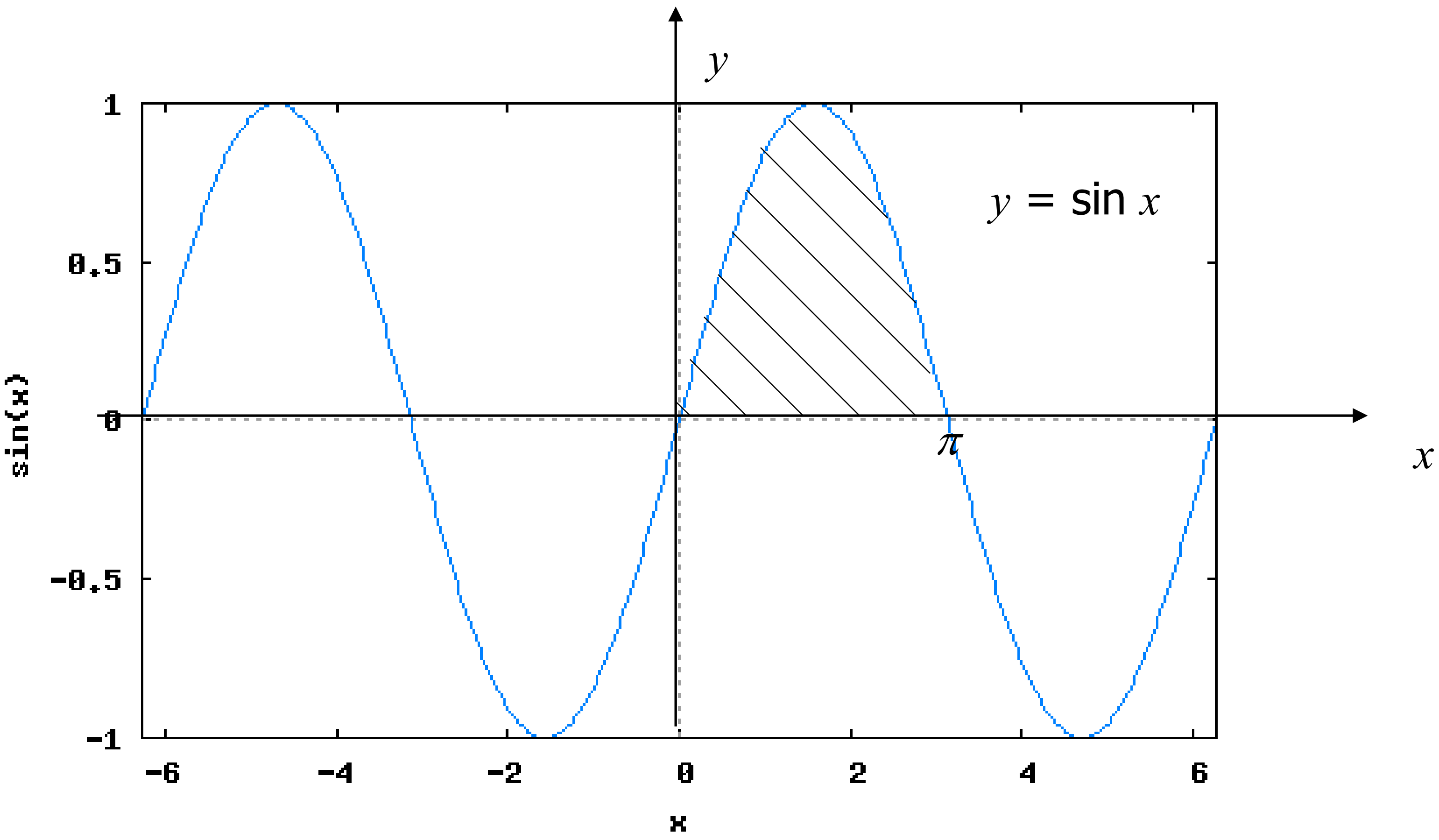
Example: Find the area of bounded by the curve $y = \sin x$ and the x-axis, from $x = 0$ to $x = \pi/2$

$$\int_0^{\pi/2} \sin x dx = \left[-\cos x\right]_0^{\pi/2} = \left(-\cos\frac{\pi}{2}\right) - \left(-\cos 0\right) = 1$$



Example: Find the area of bounded by the curve $y = \sin x$ and the x-axis, from $x = 0$ to $x = \pi$

$$\int_0^{\pi} \sin x dx = \left[-\cos x\right]_0^{\pi} = (-\cos \pi) - (-\cos 0) = 2$$



Definite Integral and Area

Definite integral defined by the Riemann sum

$$\lim_{\substack{\text{Number of points}\to\infty \\ \text{Width of each subinterval}\to 0}} \left(\sum_{\text{A set of points in the interval}} f(x)\Delta x \right)$$

has the geometrical meaning of area under the curve $y = f(x)$ when $f(x) \geq 0$

When $f(x) < 0$, the Riemann sum yields the negative value of the area bounded by the curve $y = f(x)$ and the x-axis

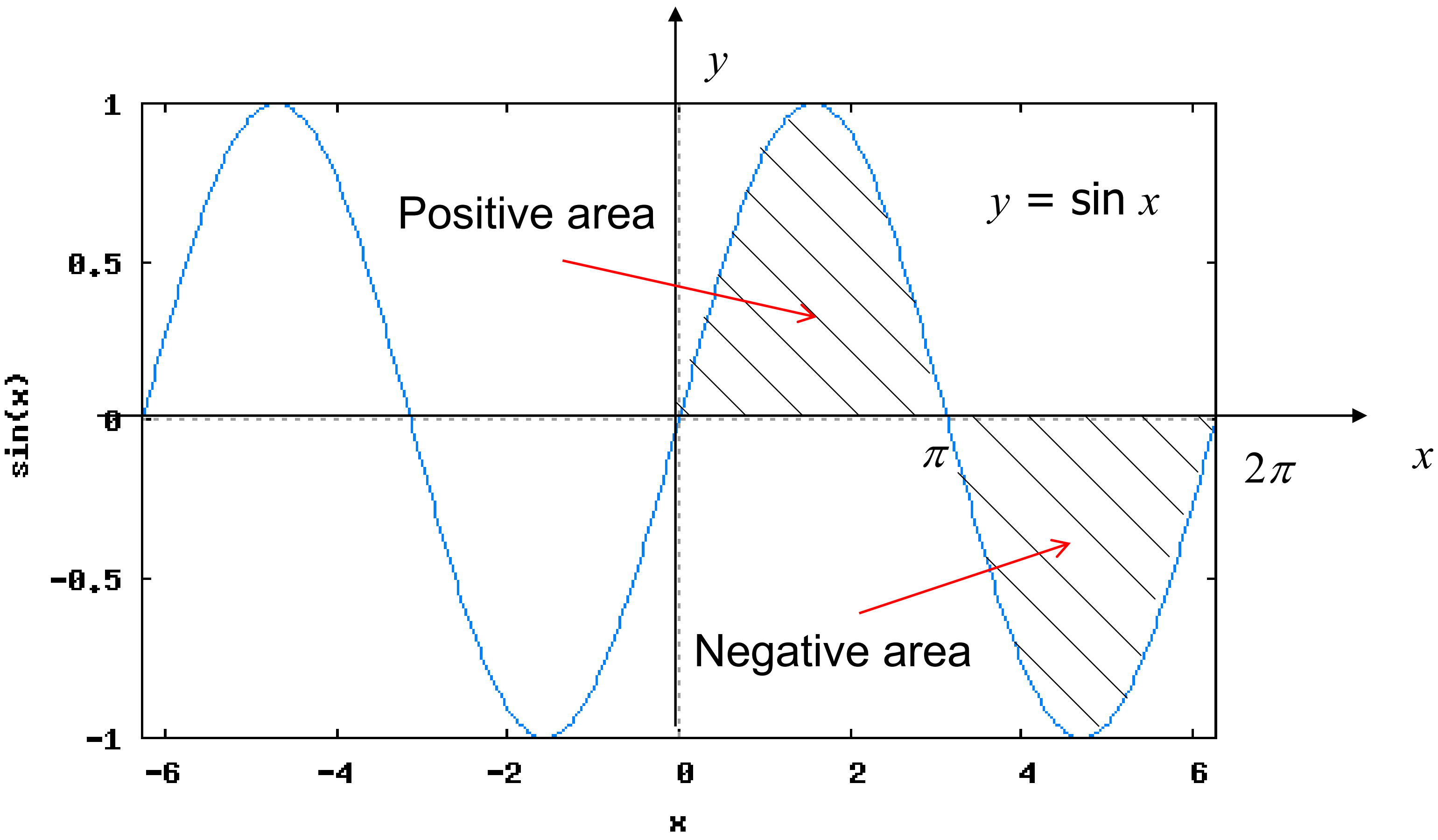
In general, the geometrical meaning of

$$\int_a^b f(x)\,dx$$

is the algebraic sum of the positive and negative areas bounded by the curve and the x-axis

Example: Find the "area" of bounded by the curve y = $\sin x$ and the x-axis, from x = 0 to x = 2π

$$\int_0^{2\pi} \sin x dx = \left[-\cos x\right]_0^{2\pi} = \left(-\cos 2\pi\right) - \left(-\cos 0\right) = 0$$



Integration by Parts of Definite Integrals

Recall: Integration by parts of indefinite integrals

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

For definite integrals,

$$\begin{aligned}\int_a^b u \frac{dv}{dx} dx &= [u(x)v(x)]_a^b - \int_a^b \frac{du}{dx} v dx \\ &= u(b)v(b) - u(a)v(a) - \int_a^b \frac{du}{dx} v dx\end{aligned}$$

Example:

$$\begin{aligned}\int_0^{\pi} x\sin x dx &= -\int_0^{\pi} xd\cos x \\ &= -[x\cos x]_0^{\pi} + \int_0^{\pi} \cos x dx \\ &= \pi + [\sin x]_0^{\pi} \\ &= \pi\end{aligned}$$

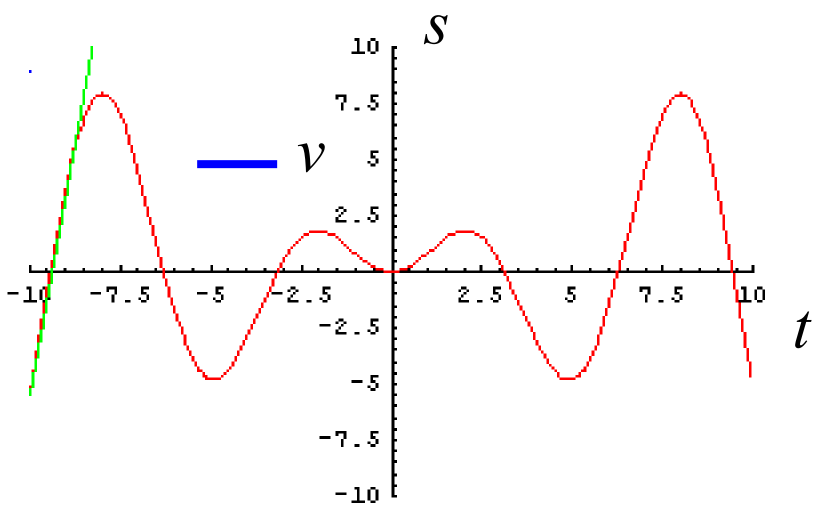
Applications in Physics

Displacement, Velocity and Acceleration

Recall:

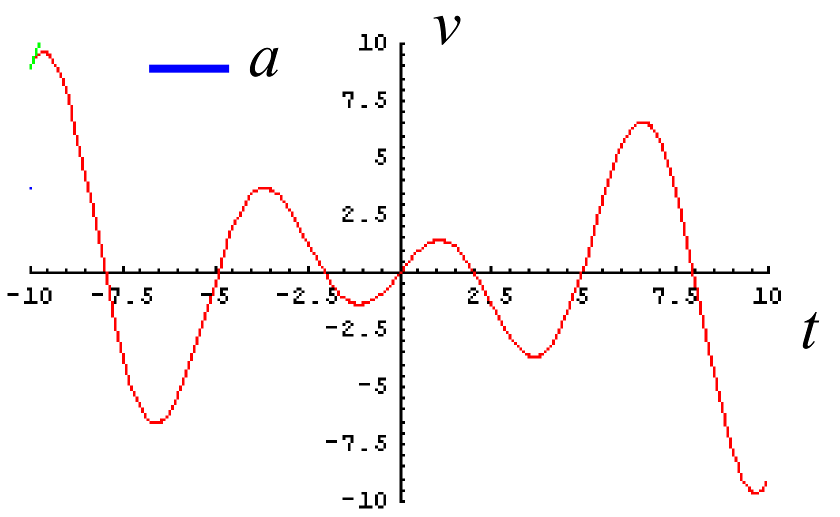
v is the slope of the s-t graph

$$v(t) = \frac{d}{dt} s(t)$$



a is the slope of the v-t graph

$$a(t) = \frac{d}{dt} v(t)$$



Velocity as Antiderivative of Acceleration

Given velocity as a function of time $v = v(t)$

we can find the acceleration by differentiation $a(t) = \frac{d}{dt} v(t)$

This implies $v(t)$ is an antiderivative of $a(t)$

By the fundamental theorem of calculus, we know that for arbitrary t_0,

$$\int_{t_0}^{t} a(\tau)\, d\tau$$

is also an antiderivative of $a(t)$

Hence

$$v(t) = \int_{t_0}^{t} a(\tau)\, d\tau + C$$

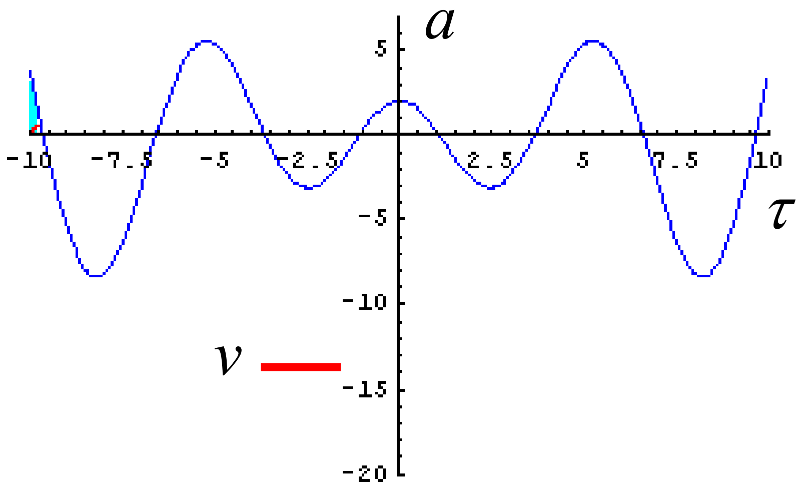
Velocity as Antiderivative of Acceleration

To fix C, notice that when $t = t_0$,

$$v(t_0) = \int_{t_0}^{t_0} a(\tau)\, d\tau + C = C$$

Hence

$$v(t) = v(t_0) + \int_{t_0}^{t} a(\tau)\, d\tau$$



$v(t) - v(t_0)$ is the "area" under the a-t graph from t_0 to t

Example:

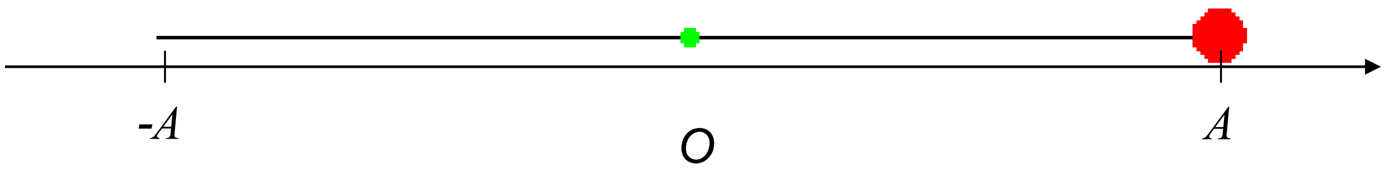
An object attached to a spring is initially (t = 0) displaced by a distance A = 0.5 m from the equilibrium position and at rest
Its acceleration is given by $a(t) = -2\cos(2t)$

Find its velocity at time t.

$$v(t) = v(0) + \int_0^t -2\cos(2\tau)\, d\tau$$

Initial condition: $v(0) = 0$ → $v(t) = \int_0^t -2\cos(2\tau)\, d\tau$

Method of substitution → $v(t) = \left[-\sin(2\tau)\right]_0^t = -\sin(2t)$



Displacement as Antiderivative of Velocity

Given displacement as a function of time $s = s(t)$

we can find the velocity by differentiation $v(t) = \frac{d}{dt} s(t)$

This implies $s(t)$ is an antiderivative of $v(t)$

By the fundamental theorem of calculus, we know that for arbitrary t_0,

$$\int_{t_0}^{t} v(\tau)\, d\tau$$

is also an antiderivative of $v(t)$

Hence

$$s(t) = \int_{t_0}^{t} v(\tau)\, d\tau + C$$

Displacement as Antiderivative of Velocity

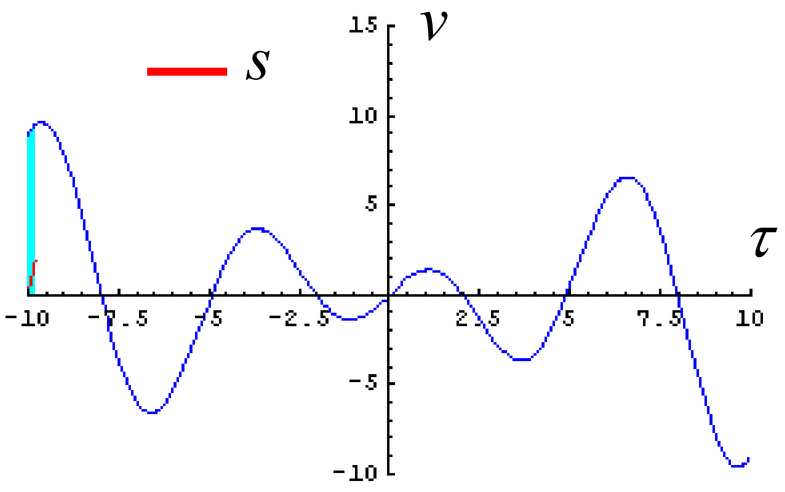
To fix C, notice that when $t = t_0$,

$$s(t_0) = \int_{t_0}^{t_0} v(\tau)\,d\tau + C = C$$

Hence

$$s(t) = s(t_0) + \int_{t_0}^{t} v(\tau)\,d\tau$$

$s(t) - s(t_0)$ is the "area" under the v-t graph from t_0 to t



Example:

An object attached to a spring is initially (t = 0) displaced by a distance $A = 0.5$ m from the equilibrium position and at rest

Its acceleration is given by

$$a(t) = -2\cos(2t)$$

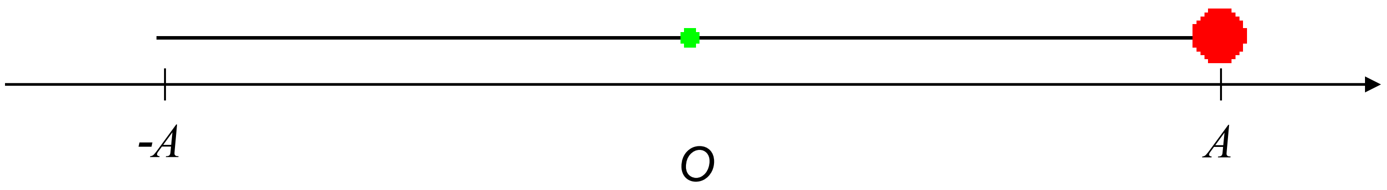
Find its displacement at time t.

$$v(t) = -\sin(2t)$$

$$s(t) = s(0) + \int_0^t -\sin(2\tau)\,d\tau$$

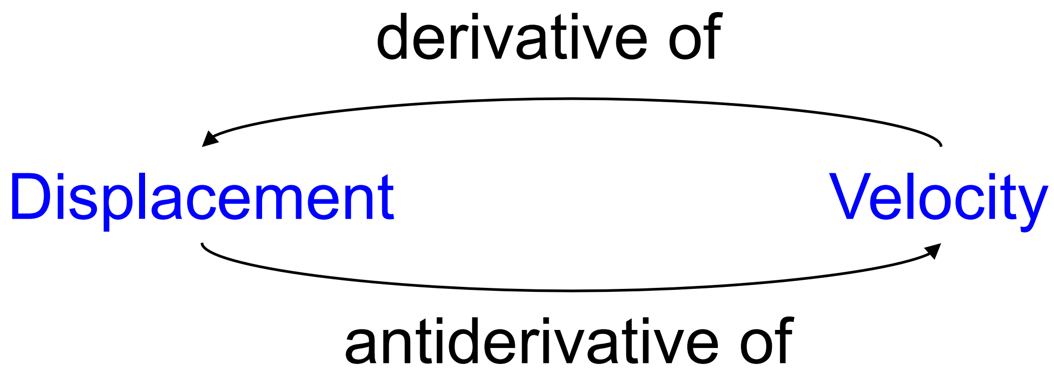
Initial condition: $s(0)$ = 1/2 → $s(t) = 1/2 + \int_0^t -\sin(2\tau)\,d\tau$

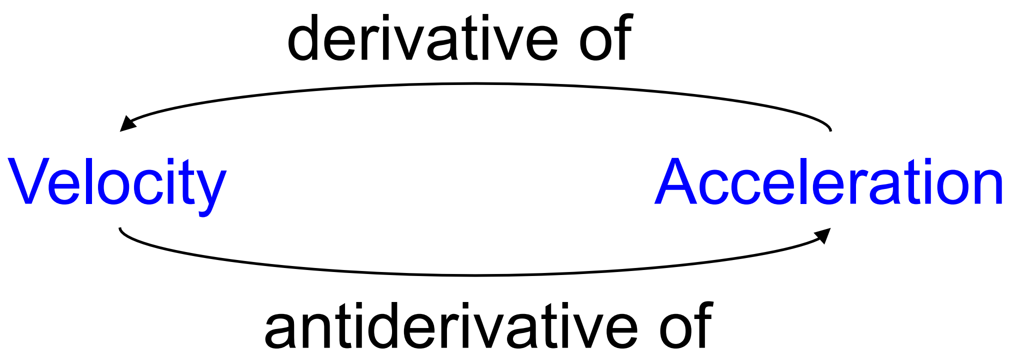
Method of substitution → $s(t) = 1/2 + \left[\cos(2\tau)/2\right]_0^t = \cos(2t)/2$



Summary

Displacement, Velocity and Acceleration





Motions under Constant Acceleration

When the object is under a constant acceleration

$$\frac{d^2s}{dt^2} = a$$

one can solve the motion in two steps:

First, solve for velocity v:

$$v(t) = v(t_0) + \int_{t_0}^{t} a\,d\tau$$

$$v(t) = v_0 + a(t - t_0)$$

where $v_0 = v(t_0)$ is the initial velocity

In particular, if $t_0 = 0$ $$v(t) = v_0 + at$$

Then one can find s by integrating v

$$s(t) = s(t_0) + \int_{t_0}^{t} \left[v_0 + a(\tau - t_0) \right] d\tau$$

$$s(t) = s_0 + v_0(t - t_0) + \frac{1}{2} a (t - t_0)^2$$

where $s_0 = s(t_0)$ is the initial displacement

In particular, if t_0 is taken to be zero, at the initial position of the object is set as the origin, then

$$s(t) = v_0 t + \frac{1}{2} a t^2$$

One can eliminate t using the two equations

$$v = v_0 + a\left(t - t_0\right)$$

$$s = s_0 + v_0\left(t - t_0\right) + \frac{1}{2}a\left(t - t_0\right)^2$$

From the 1st eq.: $\left(t - t_0\right) = \frac{v - v_0}{a}$

Sub. into the 2nd eq.: $s - s_0 = v_0 \frac{v - v_0}{a} + \frac{1}{2}a\left(\frac{v - v_0}{a}\right)^2$

$$= \frac{v - v_0}{a}\frac{v + v_0}{2} = \frac{v^2 - v_0^2}{2a}$$

$$v^2 - v_0^2 = 2a\left(s - s_0\right)$$

In particular, if $s_0 = 0$: $v^2 - v_0^2 = 2as$

Equations of Constant Acceleration Motions

General equations	$t_0 = s_0 = 0$
$v = v_0 + a(t - t_0)$	$v = v_0 + at$
$s = s_0 + v_0(t - t_0) + \frac{1}{2}a(t - t_0)^2$	$s = v_0 t + \frac{1}{2}at^2$
$v^2 - v_0^2 = 2a(s - s_0)$	$v^2 - v_0^2 = 2as$

Example – Projectile motion

We consider the motion of a particle with constant acceleration

$$\vec{\boldsymbol{a}} = a_x\hat{\imath} + a_y\hat{\jmath} = -g\hat{\jmath}$$

$$\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}_0 + \int_0^t \vec{\boldsymbol{a}}\, dt$$

$$= v_{0x}\hat{\imath} + \left(v_{0y} - gt\right)\hat{\jmath}$$

$$= v_0 \cos\alpha_0\, \hat{\imath} + (v_0 \sin\alpha_0 - gt)\hat{\jmath}$$

Example – Projectile motion

$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_0 + \int_0^t \vec{\boldsymbol{v}} dt$$

$$= \int_0^t (v_0 \cos \alpha_0 \, \hat{\imath} + (v_0 \sin \alpha_0 - gt)\hat{\jmath})\, dt$$

$$= v_0 \cos \alpha_0 \, t\hat{\imath} + \left(v_0 \sin \alpha_0 \, t - \frac{1}{2} g t^2 \right) \hat{\jmath}$$

Example – Projectile motion

Trajectory: $x(t) = v_0 \cos\alpha_0\, t,\ y(t) = v_0 \sin\alpha_0\, t - \frac{1}{2}gt^2$

Eliminate $t \Longrightarrow \quad y = (\tan\alpha_0)x - \frac{g}{2v_0^2\cos^2\alpha_0}x^2$

i.e. $y = bx - cx^2$ a parabola 抛物線

Exercise: Hong Kong Physics Olympiad Exam 2007

A rocket is launched vertically upwards from ground and moves at a constant acceleration of 19.6 ms^{-2}. Due to an accident, the engine stops 10 seconds after launch. To escape, the astronauts must eject at least 3 seconds before the rocket hits the ground. Neglect air resistance. How long do the astronauts have before ejection?

Physical Interpretation

Let us try to explain why

(1) $s(t) - s(t_0)$ is the "area" under the v-t graph from t_0 to t

(2) $v(t) - v(t_0)$ is the "area" under the a-t graph from t_0 to t

Here we shall discuss (1) in detail. The explanation for (2) is similar.

Consider an object moving at a constant velocity v

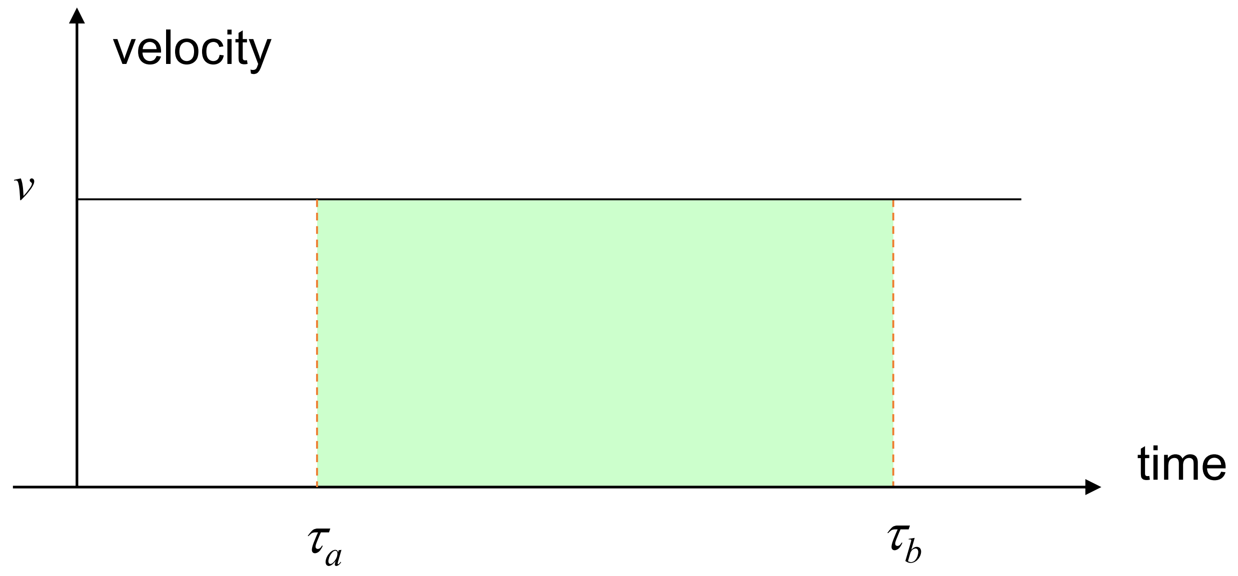
Consider the time interval $\tau_a < \tau < \tau_b$

For constant v:
Displacement = Velocity X Time

The change in position of the object from τ_a to τ_b is

$$s(\tau_b) - s(\tau_a) = v \cdot (\tau_b - \tau_a)$$

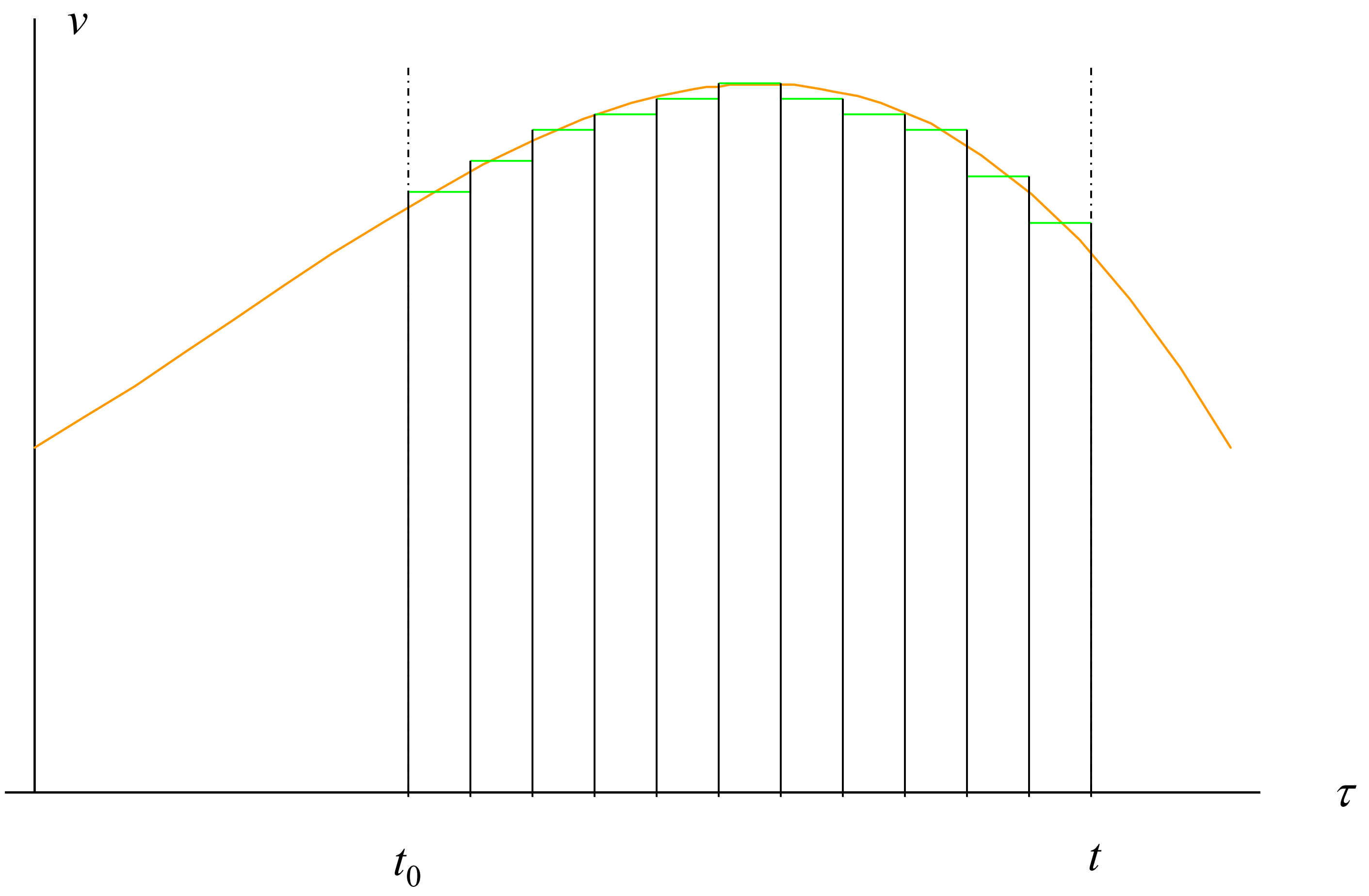
which is the area under the velocity-time graph



What if velocity varies with time?

We can cut the time interval into very small subintervals

Each time subinterval is so small that the velocity doesn't vary much and can be well approximated by a constant



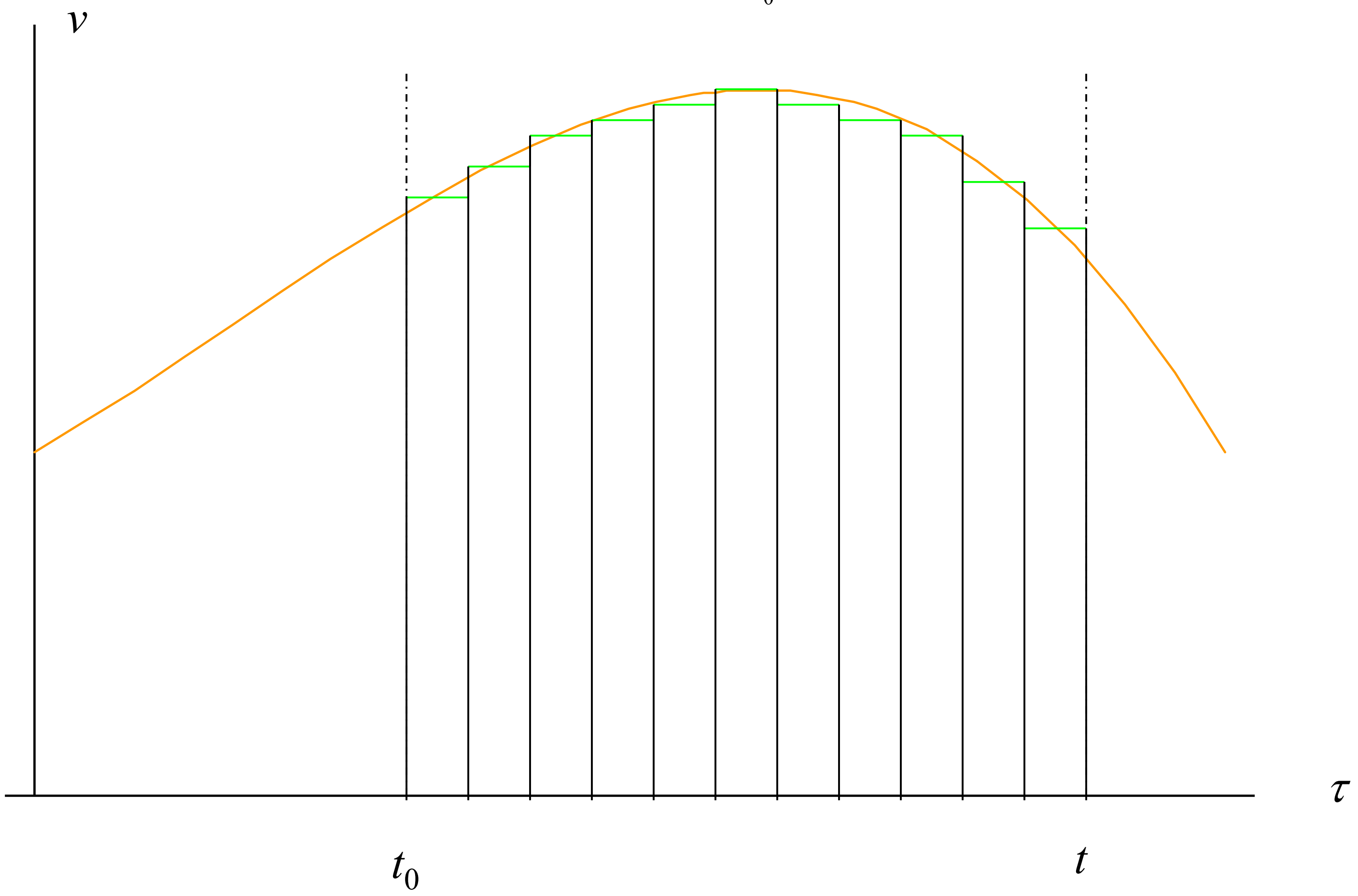
Constant v in each subinterval

→ area of each rectangle ~ Δs during that $\Delta\tau$

When each $\Delta\tau \to 0$

→ sum of areas of rectangles = the total change in s

$$s(t) - s(t_0) = \int_{t_0}^{t} v(\tau)\, d\tau$$



Area

Example Calculate the area enclosed between the straight line $y = 4x$ and the parabola $y = 2 + x^2$.

The required area is shown shaded. The curves cross each other at A and B, corresponding to $x = a$ and $x = b$, respectively. We need to calculate the values of a and b, our limits of integration. These are given by solving the equation

$$4x = 2 + x^2$$

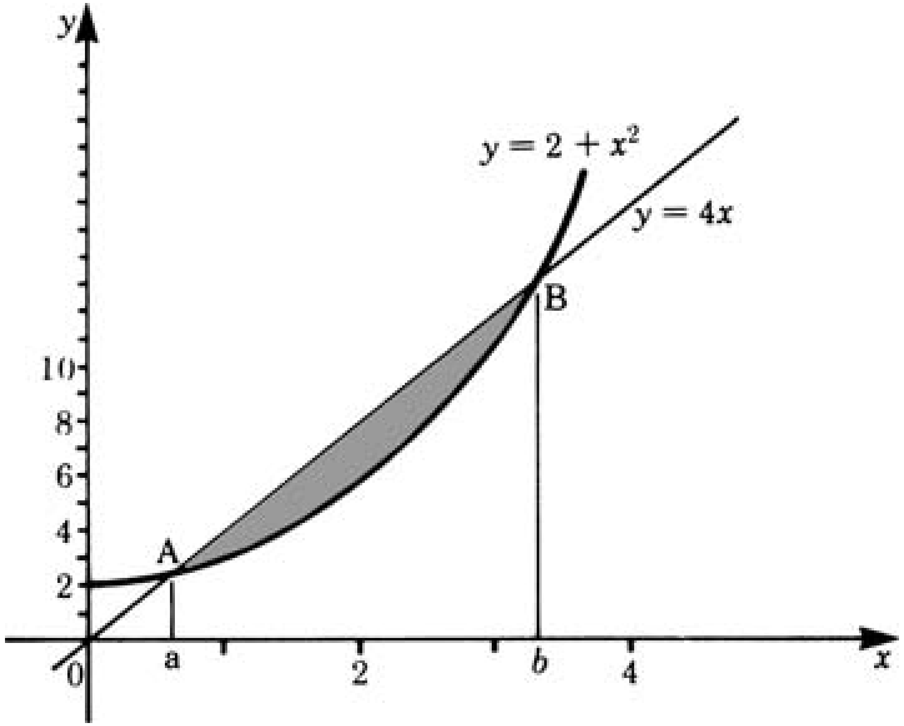
This is a quadratic equation whose roots are $x_1 = a = 0.59, x_2 = b = 3.41$ to 2 d.p.

Since the straight line between A and B is above the parabola, we have

$$f_2(x) = 4x \ , \ f_1(x) = 2 + x^2$$

Hence the area is given by

$$A = \int_{0.59}^{3.41} (4x - 2 - x^2)\,\mathrm{d}x = \left[2x^2 - 2x - \frac{1}{3}x^3\right]_{0.59}^{3.41} = 3.77 \text{ square units}$$



Area

Occasionally a curve is defined by parametric equation of the form

$$x = f(t) \quad \text{and} \quad y = g(t)$$

In this case, the areas are given by the following integrals:

$$A = \int_{x_1}^{x_2} y \, \mathrm{d}x = \int_{t_1}^{t_2} y \frac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t = \int_{t_1}^{t_2} g(t) \frac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t$$

The limits t_1 and t_2 are those values of t which correspond to x_1 and x_2.

Area

Example Suppose that the closed curve ABCD (Fig. 7.6) is an ellipse whose equation is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \quad (h,k,a \text{ and } b \text{ are constants})$$

What is the area of the ellipse?

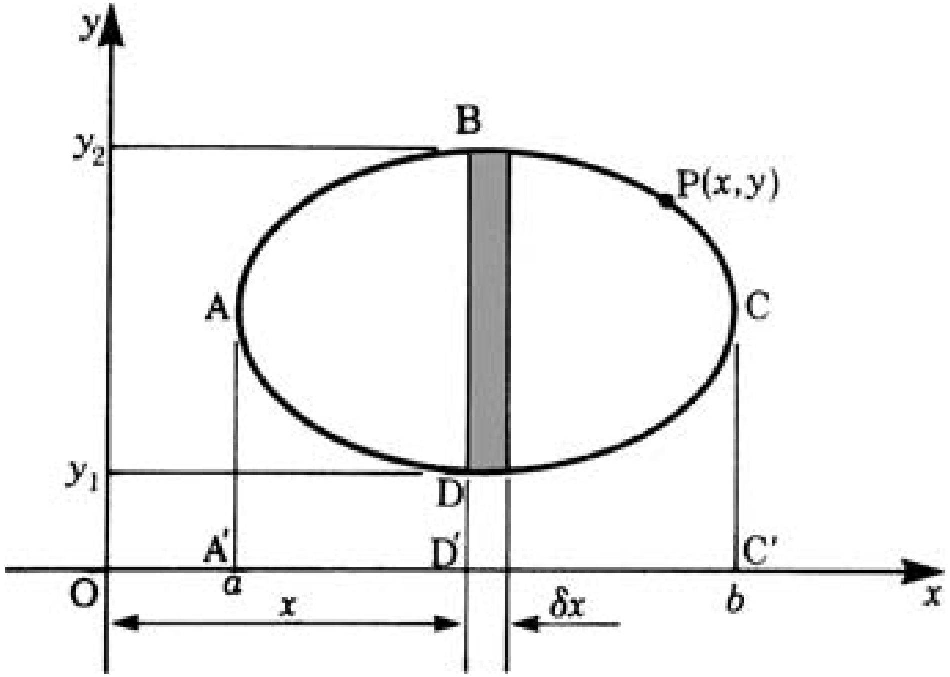
Let $x = h - a\cos t$ and $y = k + b\sin t$.

Then, as t varies from 0 to 2ω, a point $\mathrm{P}(x, y)$ goes round the curve in the direction ABCDA.

The area is

$$\int_0^{2\omega}(k+b\sin t)a\sin t\ \mathrm{d}t = ka\int_0^{2\omega}\sin t\ \mathrm{d}t + ab\int_0^{2\omega}\sin^2 t\ \mathrm{d}t = \omega ab$$

Note that the first integral $= ka \int_0^{2\omega} \sin t\ \mathrm{d}t = 0$.



Arc Length

Let δs = length of the arc BC, BD = δx and CD = δy, as shown by the small triangle BCD.

Then the arc BC is nearly equal to the chord BC, so we may write

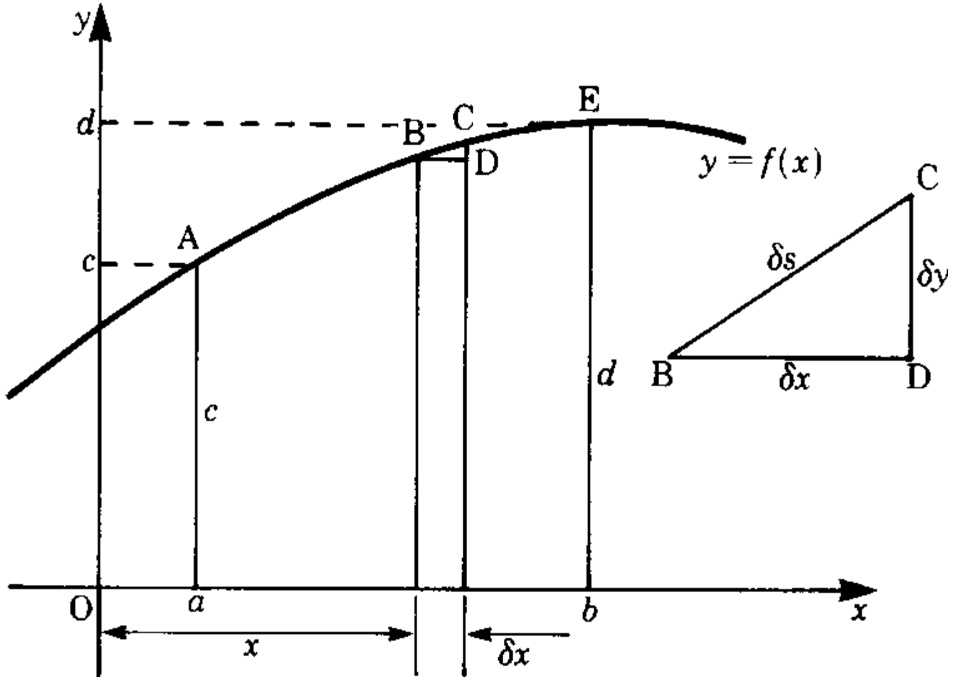
$$(\delta s)^2 \approx (\text{chord BC})^2 = (\delta x)^2 + (\delta y)^2$$

Therefore

$$\left(\frac{\delta s}{\delta x}\right)^2 \approx 1 + \left(\frac{\delta y}{\delta x}\right)^2 \quad \text{or} \quad \left(\frac{\delta s}{\delta y}\right)^2 \approx \left(\frac{\delta x}{\delta y}\right)^2 + 1$$

$$\frac{\delta s}{\delta x} \approx \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \quad \text{or} \quad \frac{\delta s}{\delta y} \approx \sqrt{1 + \left(\frac{\delta x}{\delta y}\right)^2}$$

Hence, as $\delta x \to 0, \delta s/\delta x \to \mathrm{d}s/\mathrm{d}x$ and $\delta y/\delta x \to \mathrm{d}y/\mathrm{d}x$ and
$\mathrm{d}s/\mathrm{d}x = \sqrt{1 + (\mathrm{d}y/\mathrm{d}x)^2}$ and $\mathrm{d}s/\mathrm{d}y = \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}$



The total length s of the curve from A to E, corresponding to $x = a$ and $x = b$, respectively, is

$$s = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x = \int_a^b (1 + y'^2)^{1/2}\, \mathrm{d}x \qquad (7.6)$$

The length is also given by

$$s = \int_c^d (1 + x'^2)^{1/2}\, \mathrm{d}y$$

Arc Length

Example Let us find the length of the circumference of a circle of radius R, which, of course, is well known to us.

The equation of a circle is

$$x^2 + y^2 = R^2$$

Differentiating implicitly with respect to x gives

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad \text{or} \quad yy' = -x$$

Hence $y' = -x/y = -x/\sqrt{R^2 - x^2}$ and $(1 + y'^2) = R^2/(R^2 - x^2)$

The length of the circumference

$$L = 4 \times \text{length of } \frac{1}{4} \text{ circumference} = 4R\int_0^R \frac{\mathrm{d}x}{\sqrt{R^2 - x^2}}$$

Note that to evaluate the integral we can substitute $x = R\sin\omega$.

Then $\mathrm{d}x = R\cos\omega\,\mathrm{d}\omega$, so that

$$L = 4R\int_0^{\omega/2} \frac{R\cos\omega\,\mathrm{d}\omega}{R\cos\omega} = 4R\int_0^{\omega/2} \mathrm{d}\omega = 2\varepsilon R$$

Surface area and volume of a solid of revolution

Consider the curve AB, defined by $y = f(x)$, and shown in Fig. 7.11 between $x = a$ and $x = b$.

Let us revolve the curve AB about the x-axis. Two figures are generated: (a) a surface and (b) a solid. If we consider a small strip of width δx and height y, then the small surface generated is given by $\delta A = 2\omega y \delta s$, where δs is the length of the curve corresponding to δx. The total surface will be the sum of all such elements, i.e. surface $\approx \Sigma 2\omega y \delta s$. If δx becomes smaller and smaller we have, in the limit,

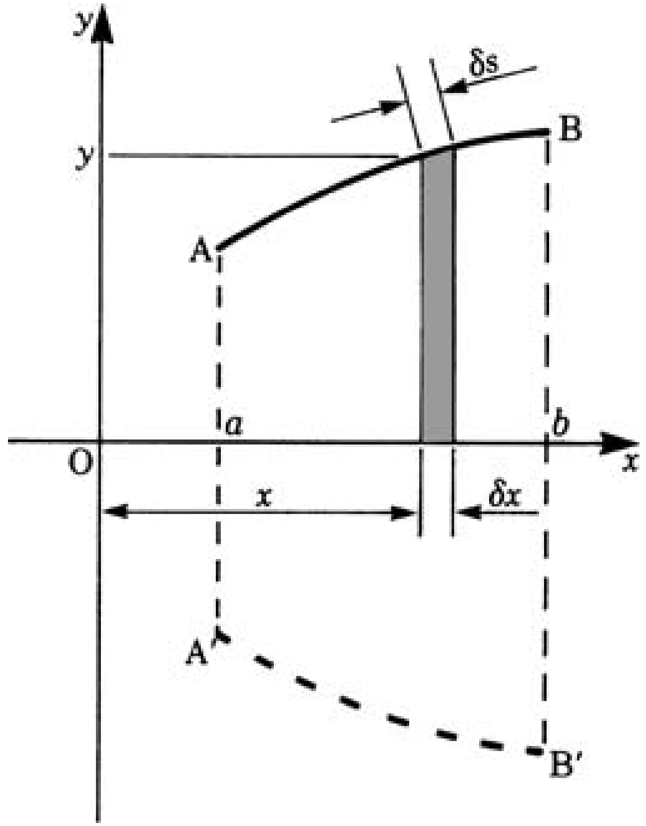
$$A = \int_a^b 2\omega y \, \mathrm{d}s = 2\omega \int_a^b y \left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{1/2} \mathrm{d}x \qquad (7.8)$$

Furthermore, as the strip is rotated, it generates a thin circular slice whose volume δV is approximately

$$\delta V = \omega y^2 \delta x$$

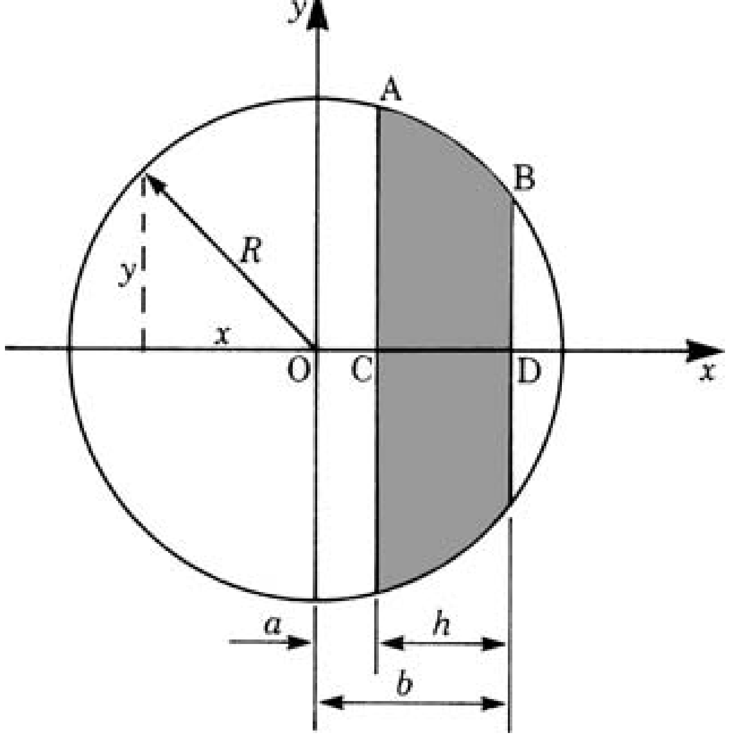
For the whole curve, as $\delta x \to 0$, the volume of the solid generated is

$$V = \omega \int_a^b y^2 \mathrm{d}x \qquad (7.9)$$



Example: volume and surface area of a sphere

The surface will be generated by rotating the arc AB and the volume by rotating the area ABCD about the x-axis.



From the figure, we have

$$y^2 = R^2 - x^2$$

Differentiating implicitly gives

$$yy' = -x$$

Thus

$$y'^2 = \frac{x^2}{y^2} \quad \text{and} \quad 1 + y'^2 = \frac{y^2 + x^2}{y^2} = \frac{R^2}{R^2 - x^2}$$

(a) The surface area is

$$A = 2\omega \int_a^b (R^2 - x^2)^{1/2} \frac{R}{(R^2 - x^2)^{1/2}} \, \mathrm{d}x = 2\omega R \int_a^b \mathrm{d}x = 2\omega R(b - a)$$

Hence $A = 2\omega Rh$

(b) The volume V is

$$V = \omega \int_a^b y^2 \mathrm{d}x = \omega \int_a^b (R^2 - x^2) \, \mathrm{d}x = \omega \left[R^2 x - \frac{x^3}{3} \right]_a^b$$

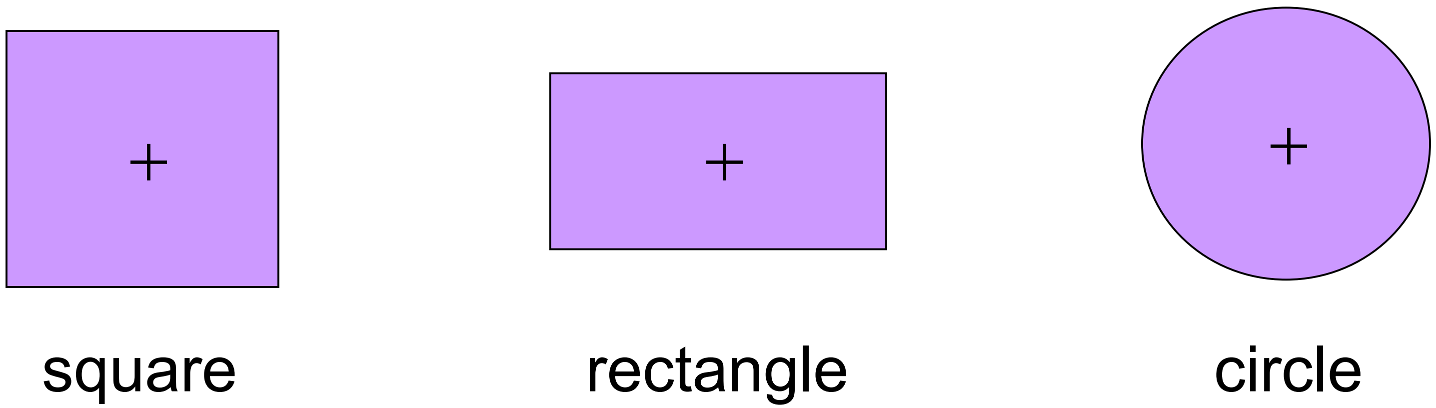
For the special case where $b = R$ and $a = 0$, we have

$$V = \frac{2}{3}\omega R^3$$

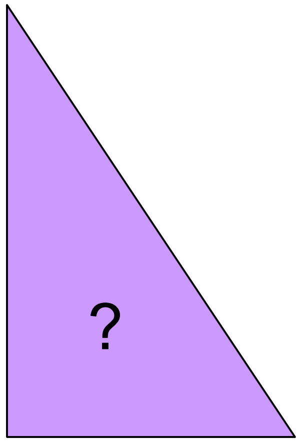
This is the volume of a half sphere or hemisphere. Hence the volume of a sphere is $V = \frac{4}{3}\omega R^3$.

“Center” of an Object – center of mass

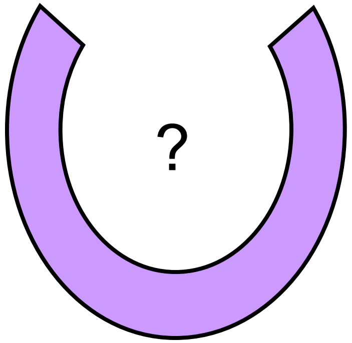
What are the “centers” of the following figures”?

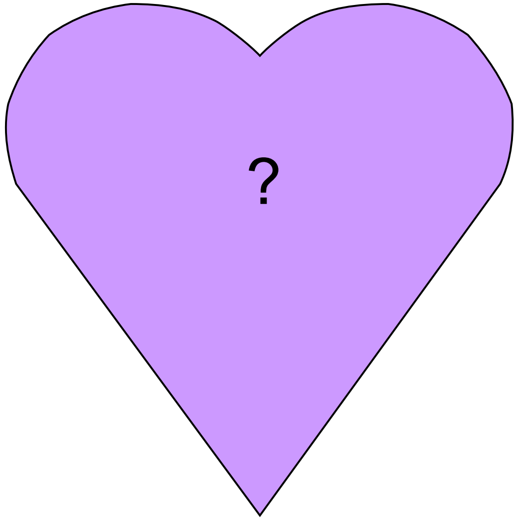


What about these?



right triangle





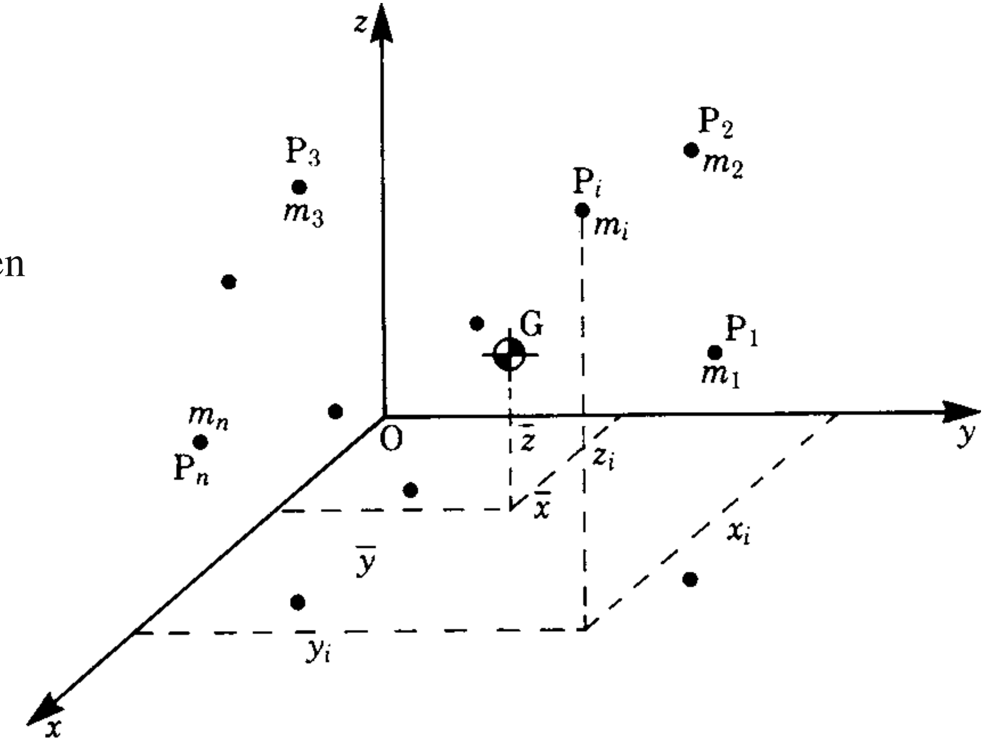
We need to define “center”!!

Center of mass

If M is the total mass of the particles, the position of the center of mass G is given by the following equations:

$$\bar{x} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i \ , \quad \bar{y} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i \ , \quad \bar{z} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i$$

where $$M = \sum_{i=1}^{n} m_i$$



When the particles form a solid body, the above summations become integrals. If δm is the mass of a typical particle in the body at distances x, y and z from the planes, then the center of mass of the body is given by

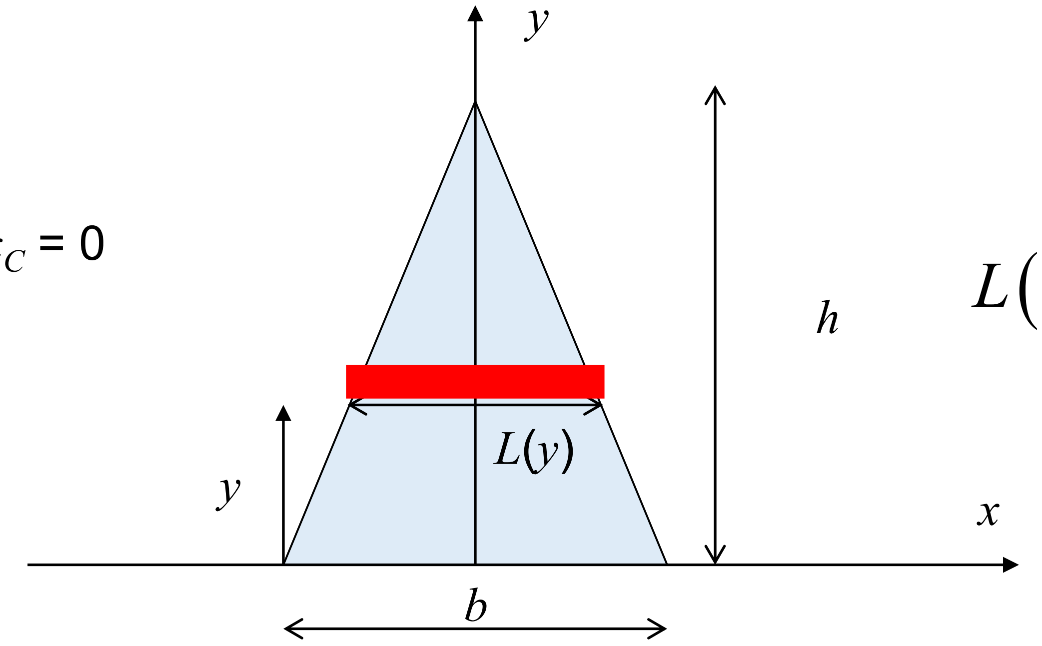
$$\bar{x} = \frac{\int x \,\mathrm{d}m}{\int \mathrm{d}m} \ , \quad \bar{y} = \frac{\int y \,\mathrm{d}m}{\int \mathrm{d}m} \ , \quad \bar{z} = \frac{\int z \,\mathrm{d}m}{\int \mathrm{d}m}$$

between appropriate limits.

$$\int \mathrm{d}m = M = \text{total mass of the body}$$

Example: Center of an isosceles triangle

By symmetry, $x_C = 0$



$$L(y) = b\frac{h-y}{h}$$

$$dm = mL(y)dy = \frac{mb}{h}(h-y)dy$$

$$M = \int dm = \int_0^h \frac{mb}{h}(h-y)dy = m\frac{bh}{2}$$

$$\begin{aligned}\tilde{y} &= \frac{1}{M}\int y dm \\ &= \frac{1}{M}\int_0^h \frac{mb}{h}(h-y)y dy \\ &= \frac{1}{M}\frac{mbh^2}{6} = \frac{h}{3}\end{aligned}$$

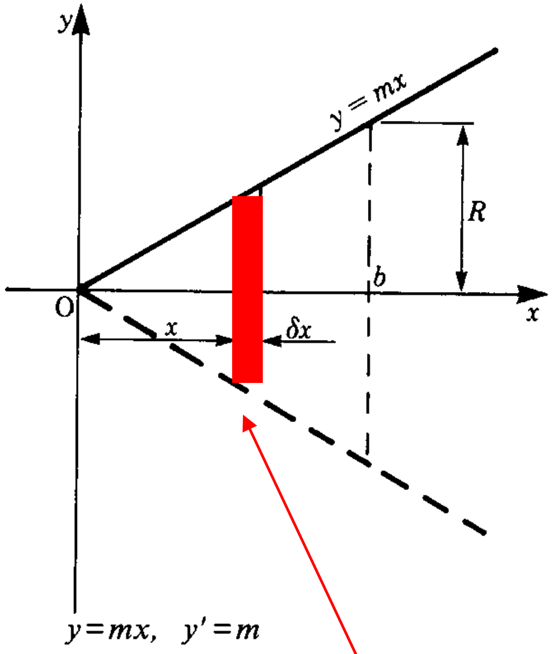
Example: Center of a cone

The equation of the straight line is $y = \frac{R}{b} \cdot x$.

The mass of the thin slice obtained by rotating the element δx about the x-axis is $m\varepsilon\, y^2 \delta x$, where m is the mass per unit volume. The total mass of the cone is

$$M = m\varepsilon \int_0^b y^2 \mathrm{d}x = m\varepsilon \frac{R^2}{b^2} \int_0^b x^2 \mathrm{d}x$$

$$= \frac{1}{3} m\varepsilon\, R^2 b$$



$$dm = m\pi y^2 dx$$

$$x dm = m\pi x y^2 dx = m\pi \frac{R^2}{b^2} x^3 dx$$

Hence, the center of mass located at

$$\tilde{x} = \frac{1}{M} \int_0^b x dm = \frac{1}{M} \int_0^b \left(m\pi \frac{R^2}{b^2} \right) x^3 dx = \frac{3}{4} b$$