Tutorial 3
Series expansion and Complex numbers

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0. Work done along an arbitrary path

Work-Energy theorem

$$W_{tot} = \int_{P_1}^{P_2} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{l}} = \int_{P_1}^{P_2} mvdv = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

Question:

How do we evaluate the line integral $\int_{P_1}^{P_2} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{l}}$ along a specific path?

Suppose we can parameterize the trajectory $\vec{r}(t) = (x(t), y(t), z(t))$
And the force is given by $\vec{F}(\vec{r}) = \left(F_x(\vec{r}), F_y(\vec{r}), F_z(\vec{r})\right)$

$$U = \int_{P_1}^{P_2} \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \mathrm{d}\boldsymbol{r}$$
$$= \int_{t_1}^{t_2} \boldsymbol{F}(x(t), y(t), z(t)) \cdot \mathrm{d}\boldsymbol{r}$$
$$= \int_{t_1}^{t_2} \left(F_x(\boldsymbol{r}) \cdot \boldsymbol{i} + F_y(\boldsymbol{r}) \cdot \boldsymbol{j} + F_z(\boldsymbol{r}) \cdot \boldsymbol{k}\right) \left(\frac{\mathrm{d}x}{\mathrm{d}t} \mathrm{d}t \cdot \boldsymbol{i} + \frac{\mathrm{d}y}{\mathrm{d}t} \mathrm{d}t \cdot \boldsymbol{j} + \frac{\mathrm{d}z}{\mathrm{d}t} \mathrm{d}t \cdot \boldsymbol{k}\right)$$

Hence the formula for the work U reads

$$U = \int_{t_1}^{t_2} \left(F_x(\boldsymbol{r}) \frac{\mathrm{d}x}{\mathrm{d}t} \mathrm{d}t + F_y(\boldsymbol{r}) \frac{\mathrm{d}y}{\mathrm{d}t} \mathrm{d}t + F_z(\boldsymbol{r}) \frac{\mathrm{d}z}{\mathrm{d}t} \mathrm{d}t\right)$$

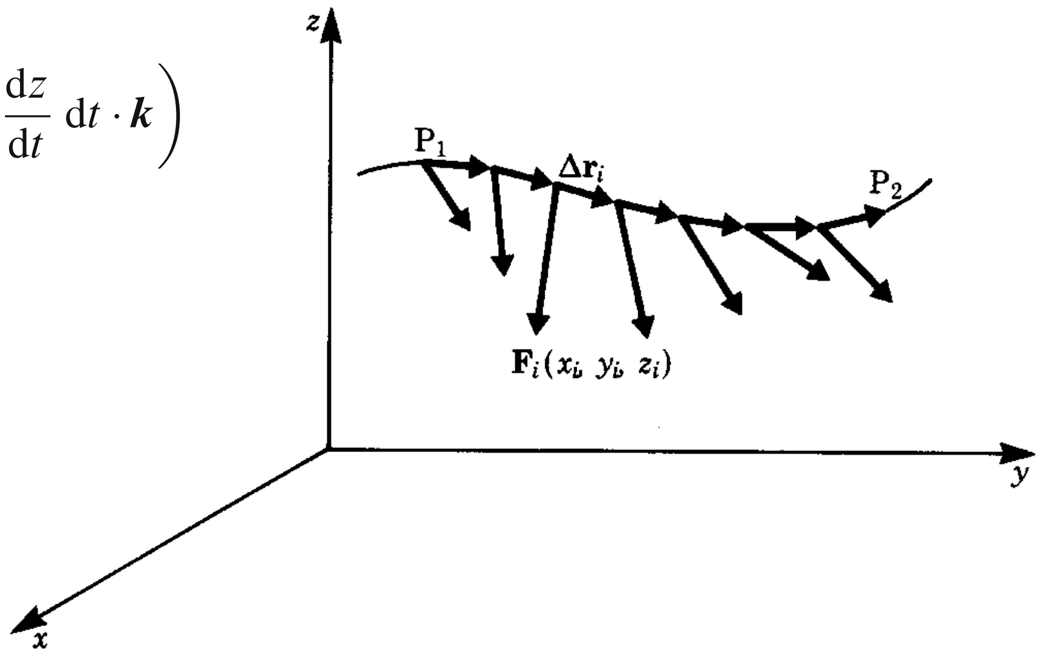


Fig. 6.27

Example

Consider the fairground Ferris wheel. We want to find the work done during the ascent of the Ferris wheel (mass m). The path is the semicircle from P_1 to P_2.

$$\vec{F} = (0, -mg)$$
$$\vec{r}(\phi) = (R\sin\phi, -R\cos\phi)$$
$$d\vec{r} = (R\cos\phi, R\sin\phi)d\phi$$

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$$
$$= \int_0^{\pi} -mgR\sin\phi d\phi = 2mgR$$

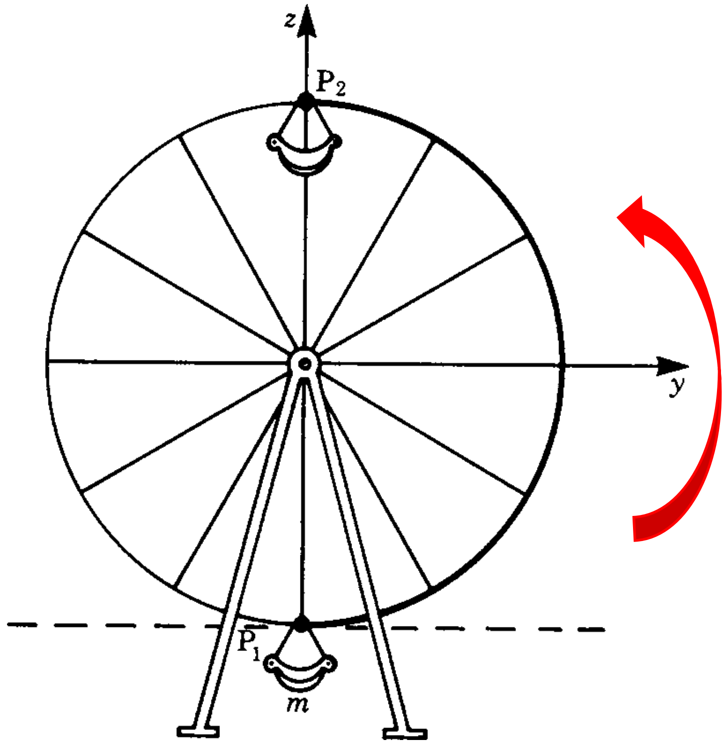


Fig. 6.28

Exercise:

16. A force in a conservative field is given by

$$\boldsymbol{F} = (x, y, z)\,\mathrm{N}$$

A body moves from the origin of the coordinate system to the point

$$P = (5, 0, 0)$$

Calculate the work done.

17. Given the force

$$\boldsymbol{F} = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right)$$

Evaluate the line integral along a semicircle around the origin of the coordinate system with radius R. Can you give the answer without computing?

18. Given a force $\boldsymbol{F} = (0, -z, y)$. calculate the line integral along the curve

$$\boldsymbol{r}(t) = \left(\sqrt{2}\cos t, \quad \cos 2t, \quad \frac{2t}{\omega} \right)$$

from $t = 0$ to $t = \frac{\omega}{2}$.

1. Taylor series and power series

From the sum of a geometric series, we know

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

when -1<x<1.

We now consider this result from a different point of view.
Assume x is small (i.e. |x|<<1), we approximate a function f(x) by a series,

$$f(x) = \frac{1}{1-x} \approx 1 + x + x^2 + x^3 + \cdots$$

Now, we investigate functions f(x) which can be expressed as infinite power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

(In particular, when x is small, we may truncate the series up to the 2 second or third terms and get a good approximation)

$$f(x) \approx a_0 + a_1 x + a_2 x^2$$

Application in physics:

1. Evaluation
2. Approximation
3. Term-by-term integration

Taylor’s series

Suppose a function f(x) can be expressed as

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

with some coefficients a_0, a_1, a_2.......

We want to determine the values of these coefficients

Trick

1. We notice that

$$a_0 = f(0)$$

2. Next, we take the derivative on both sides

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 \cdots$$

And substitute x=0 in the equation

$$a_1 = f'(0)$$

3. We take another derivative on both sides

$$f''(x) = 2a_2 + 6a_3x + \cdots$$

And substitute x=0 in the equation

$$a_2 = \frac{1}{2}f^{(2)}(0)$$

4. Similarly, we take n derivatives on both sides

$$f^{(n)}(x) = n!a_n + (n+1)!a_{n+1}x + \cdots$$

And substitute x=0 in the equation

$$a_n = \frac{1}{n!}f^{(n)}(0)$$

Taylor's (Maclaurin's) series

The expansion of a function f(x) expressed in a power series is given by

$$f(x) = f(0) + \frac{1}{1!}f'(0)x + \frac{1}{2!}f^{(2)}(0)x^2 + \frac{1}{3!}f^{(3)}(0)x^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

In general, we can generalize the argument and obtain the general Taylor's series

$$f(x) = f(x_0) + \frac{1}{1!}f'(x_0)(x-x_0) + \frac{1}{2!}f^{(2)}(x_0)(x-x_0)^2 + \frac{1}{3!}f^{(3)}(x_0)(x-x_0)^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

We say we expand the function by a Taylor's series with respect to the point x_0

Example:

Expansion of the Sine Function $f(x) = \sin x$

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \end{aligned}$$

Substituting in (8.2) gives

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1} \end{aligned}$$

Example:

Expansion of the Binomial Series $f(x) = (a+x)^x$

$$f(x) = (a+x)^n \qquad f(0) = a^n$$
$$f'(x) = n(a+x)^{n-1} \qquad f'(0) = na^{n-1}$$
$$f''(x) = n(n-1)(a+x)^{n-2} \qquad f''(0) = n(n-1)a^{n-2}$$
$$\vdots \qquad \vdots$$
$$f^k(x) = n(n-1)\cdots(n-k+1)(a+x)^{n-k} \qquad f^k(0) = n(n-1)\cdots(n-k+1)a^{n-k}$$

Note that n need not be an integer. Thus the expansion is valid for, e.g. $n = 1/2$. Substituting in (8.2) gives

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)\cdots(n-k+1)}{k!}a^{n-k}x^k + \cdots$$

A useful version of this series is when $a = 1$. We then have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \qquad (8.5)$$

Example:

Expansion of the Function $f(x) = \frac{1}{1-x}$

We know the result already because this is the sum of a geometric series.

$$
\begin{aligned}
f(x) &= \frac{1}{1-x} & f(0) &= 1 \\
f'(x) &= \frac{1}{(1-x)^2} & f'(0) &= 1 \\
f''(x) &= \frac{1\times 2}{(1-x)^3} & f''(0) &= 2! \\
f'''(x) &= \frac{1\times 2\times 3}{(1-x)^4} & f'''(0) &= 3! \\
&\vdots & &\vdots \\
f^n(x) &= \frac{n!}{(1-x)^{n+1}} & f^n(0) &= n!
\end{aligned}
$$

Substituting in (8.2) gives the familiar result

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n \qquad (|x| < 1) \qquad (8.6)$$

(Absolute value of $x < 1$)

Example:

Example Expand the cosine $f(x) = \cos x$ about the point $x_0 = \omega/3$ or (60°).

Differentiating gives

$$f'(x) = -\sin x\,, \quad f''(x) = -\cos x\,, \quad f'''(x) = \sin x$$

$$f'\left(\frac{\omega}{3}\right) = -\frac{\sqrt{3}}{2}\,,\; f''\left(\frac{\omega}{3}\right) = -\frac{1}{2}\,, \quad f'''\left(\frac{\omega}{3}\right) = \frac{\sqrt{3}}{2}$$

and so on.

Substituting in (8.9) gives

$$\cos x = \frac{1}{2} - \left(x - \frac{\omega}{3}\right)\frac{\sqrt{3}}{2} - \left(x - \frac{\omega}{3}\right)^2 \frac{1}{4} + \left(x - \frac{\omega}{3}\right)^3 \frac{\sqrt{3}}{12} + \cdots$$

Suppose we wish to calculate the value of cosine 61° without using tables. Then

$$\cos 61^\circ = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{\omega}{180}\right) - \frac{1}{4}\left(\frac{\omega}{180}\right)^2 + \frac{\sqrt{3}}{12}\left(\frac{\omega}{180}\right)^3 + \cdots$$

If we use two terms only

$$\cos 61^\circ \approx \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{\omega}{180}\right) = 0.5000 - 0.01511 = 0.48489$$

Exercise:

$$f(x) = \sqrt{1 - x}$$

$$f(x) = \tan x$$

with respect to $x = 0$ and $x = \frac{\pi}{4}$ respectively

And approximate $\tan 44°$

Application

Example Detour problem. Figure 8.4 shows two possible paths that can be taken when travelling a distance S from A to B, a direct one and an indirect one via C. The problem is to find how much longer is the detour via C than the direct path?

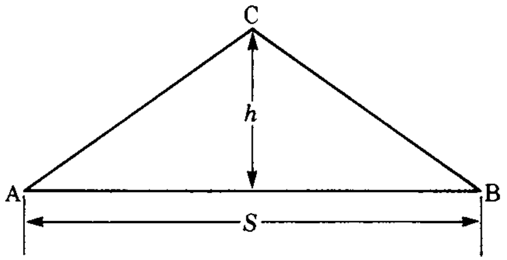


Fig. 8.4

Let u be the detour. If h is the height of an assumed equilateral triangle, then, by Pythagoras' theorem, we have

$$u = 2\left(\sqrt{\left(\frac{S}{2}\right)^2 + h^2} - \frac{S}{2}\right)$$

$$u = S\left(\sqrt{1 + \left(\frac{2h}{S}\right)^2} - 1\right)$$

To investigate the behaviour of u as a function of h, it is much simpler to express it by an approximate polynomial. Using the binomial expansion, we have

$$\frac{1+u}{S} = f(h) = \left(1 + \left(\frac{2h}{S}\right)^2\right)^{1/2} = 1 + \frac{1}{2}\left(\frac{2h}{S}\right)^2 - \frac{\frac{1}{2}\left(1-\frac{1}{2}\right)}{2!}\left(\frac{2h}{S}\right)^4 + \cdots$$

Provided that $h < S$, we can use a first-degree approximation by taking the first two terms of the series:

$$f(h) \approx 1 + \frac{1}{2}\left(\frac{2h}{S}\right)^2$$

Substituting in the equation for u gives

$$u = S\left(1 + \frac{1}{2}\left(\frac{2h}{S}\right)^2 - 1\right) = \frac{2h^2}{S}$$

As an example, let $S = 100\,\text{km}$. The function is shown in Fig. 8.5. An examination of the graph shows, e.g. that when $h = 5\,\text{km}$, the detour is only 0.5 km.

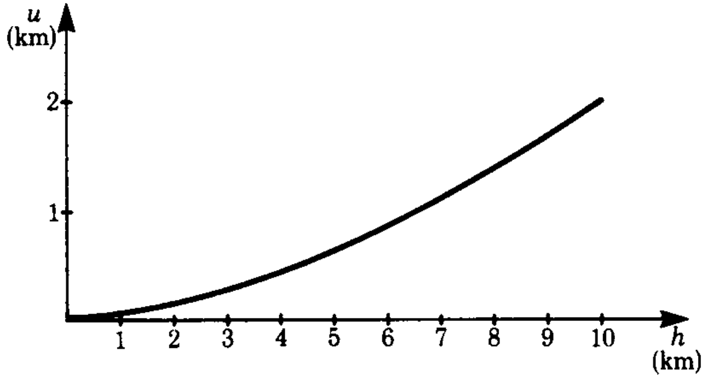


Fig. 8.5

Application

Example Obtain a closer approximation for one of the roots of the equation

$$x^4 - 1.5x^3 + 3.7x - 21.554 = 0$$

A rough estimate gave $x = 2.4$.

Let x be a rough approximation for the root of an equation found by trial and error. If the true solution is $x + h$, then, by Taylor's theorem, we have

$$0 = f(x+h) \approx f(x) + h \cdot f'(x)$$

Solving for h gives

$$h \approx -\frac{f(x)}{f'(x)}; \quad \text{hence} \quad x - \frac{f(x)}{f'(x)} \quad \text{is a better approximation.}$$

This is also known as the Newton-Raphson approximation formula (see Chap. 17). Returning to the example, we find

$$f'(x) = 4x^3 - 4.5x^2 + 3.7 \quad \text{and} \quad f'(2.4) = 33.076$$

$$\text{Also} \quad f(2.4) = -0.2324$$

It follows that $h = 0.2324/33.076 = 0.007$.

A more accurate approximation is $x = 2.4 + 0.007 = 2.407$.

Application

Example Evaluate

$$\int_0^{0.4} \sqrt{\frac{4-x^2}{4+4x^3}}\,\mathrm{d}x$$

First we express the integrand as a product, i.e.

$$\int \sqrt{\frac{4-x^2}{4+4x^3}}\,\mathrm{d}x = \int \left(1-\left(\frac{x}{2}\right)^2\right)^{1/2} (1+x^3)^{-1/2}\,\mathrm{d}x$$

The binomial series converges for $|x| < 1$. The condition is satisfied in the case of our two functions. The expansions are

$$\left(1-\left(\frac{x}{2}\right)^2\right)^{1/2} = 1-\frac{1}{8}x^2-\frac{1}{128}x^4-\frac{1}{1\,024}x^6-\cdots$$

$$(1+x^3)^{-1/2} = 1-\frac{x^3}{2}+\frac{3}{8}x^6-+\cdots$$

Multiplying the two series gives, for the integrand I

$$I = 1-\frac{1}{8}x^2-\frac{1}{2}x^3-\frac{1}{128}x^4+\frac{1}{16}x^5+\frac{383}{1\,024}x^6+\cdots$$

Integrating term by term we find

$$\int_0^{0.4} I\,\mathrm{d}x = \left[x-\frac{1}{24}x^3-\frac{1}{8}x^4-\frac{1}{640}x^5+\frac{1}{96}x^6+\frac{383}{7\,168}x^7+\cdots\right]_0^{0.4}$$

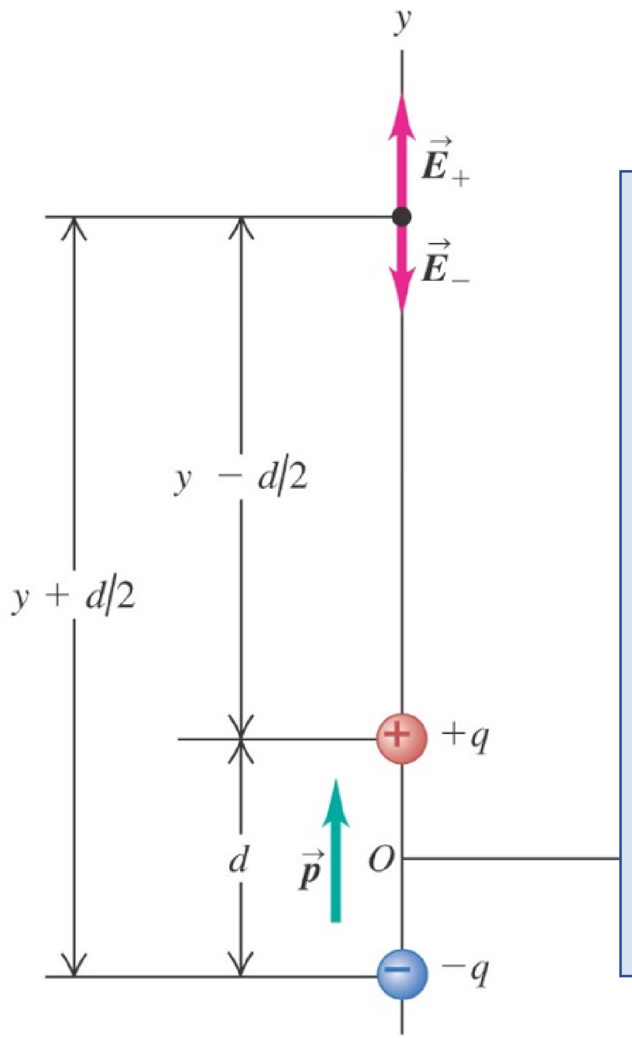
$$\int_0^{0.4} I\,\mathrm{d}x \approx 0.4-0.00267-0.00320-0.00002+0.00004+0.00009$$

$$\approx 0.3942$$

Exercise:

Field of an electric dipole Example 21.14

An **electric dipole** is a pair of equal and opposite charges $+q$ and $-q$ at a fixed distance d apart. Its **electric dipole moment** $\vec{p}$ is defined as $p = qd$ and points from $-q$ to $+q$



$$E_y = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{(y-d/2)^2} + \frac{-q}{(y+d/2)^2}\right]$$

Exercise:

12. Solve the following integrals using a series expansion:

(a) $\int_0^{0.58} \sqrt{1+x^2}\,\mathrm{d}x$

(b) $\int_0^{x} \frac{\sin t}{t}\,\mathrm{d}t$ (Integral (b) cannot be evaluated by any other method.)

Find the solution of $\sin x = x$ using the Taylor's series

2. Euler number e and natural logarithm

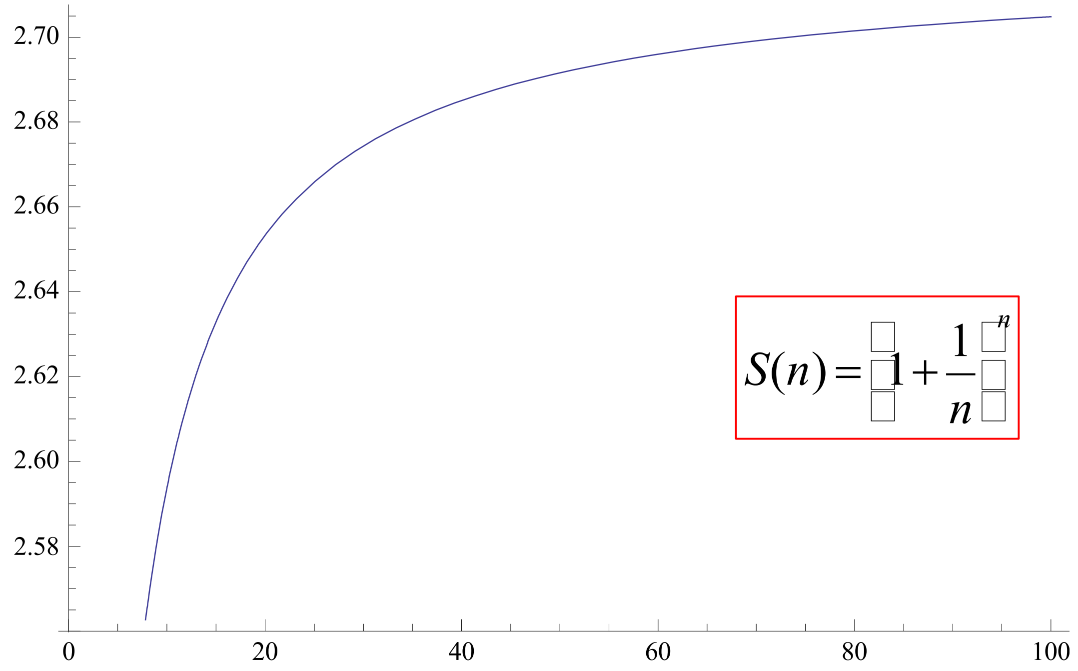
Mathematically, there are many different ways to define the Euler number e

$$e = \sum_{n=1}^{\infty} \frac{1}{n!} = \frac{1}{1} + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2 \cdot 1} + \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} + \ldots\ldots = 2.71828\ldots..$$

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n}$$

$$e = \lim_{x \to 0} (1 + x)^{\frac{1}{x}}$$

$$e = \lim_{n \to \infty} \frac{n}{\sqrt[n]{n!}}$$



e in Calculus I

$$\frac{d}{dx}a^x = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = a^x\left(\lim_{h\to 0}\frac{a^h-1}{h}\right)$$

$$\lim_{h\to 0}\frac{a^h-1}{h} = 1 \to a = \lim_{h\to 0}(1+h)^{1/h} \to a = e$$

$$\frac{d}{dx}e^x = e^x$$

e in Calculus II

$$\frac{d}{dx}\log_a x = \lim_{h\to 0}\frac{\log_a(x+h)-\log_a x}{h} = \frac{1}{x}\left(\lim_{h\to 0}\frac{1}{h}\log_a(1+h)\right)$$

$$\lim_{h\to 0}\frac{1}{h}\log_a(1+h) = 1 \Rightarrow a = \lim_{h\to 0}(1+h)^{\frac{1}{h}} = e$$

$$\frac{d}{dx}\log_e x = \frac{1}{x}$$

Notation: $\log_e = \ln$ is called natural logarithm

Exercise: Take the derivative of the following functions

$$e^{-x^2}$$

$$10^x$$

$$\ln(\sqrt{1-x^2})$$

$$\log_a x$$

Find the Taylor’s series expansion of e^x for small x

Find the Taylor’s series expansion of $\ln x$ when x closes to 1

3. Complex number

What is $\sqrt{-1}$?

Definition

$$\mathrm{i} = \mathrm{j} = \sqrt{-1}$$

Is called the imaginary number such that

$$i^2 = \sqrt{-1} \times \sqrt{-1} = -1$$

For other number, for example, $\sqrt{-5} = \sqrt{5} \times \sqrt{-1} = \sqrt{5}i$

Real number: $0, 1, 0.3, \pi, e, \sqrt{2}, \ldots\ldots$

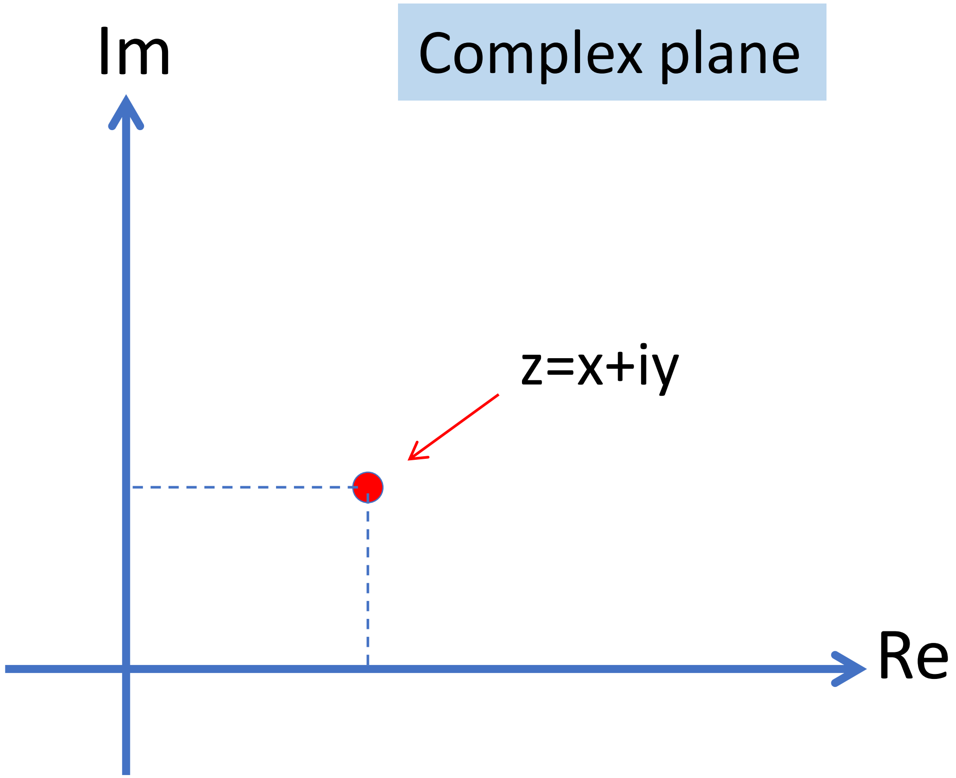
Imaginary number: $i, 2i, ei, \pi i, \ldots\ldots$

We define the symbol i such that $i \cdot i = i^2 = -1$

Complex number $z = x + iy$ (x and y are two arbitary real numbers)

Real part: $Re(z) = x$

Imaginary part: $Im(z) = y$



Example:

Two complex number:

z_1=1+i

z_2=2+3i

Then

$$z_1 + z_2 = 3 + 4i$$

$$2z_1 = 2 + 2i$$

$$iz_2 = 2i + 3i^2 = -3 + 2i$$

$$z_1 \cdot z_2 = (1+i)(2+3i) = 2 + 3i + 2i + 3i^2 = -1 + 5i$$

$$\frac{z_2}{z_1} = \frac{2+3i}{1+i} = \frac{(2+3i)(1-i)}{(1+i)(1-i)} = \frac{2-2i+3i-3i^2}{2} = \frac{5+i}{2}$$

Notation:

z=x+iy

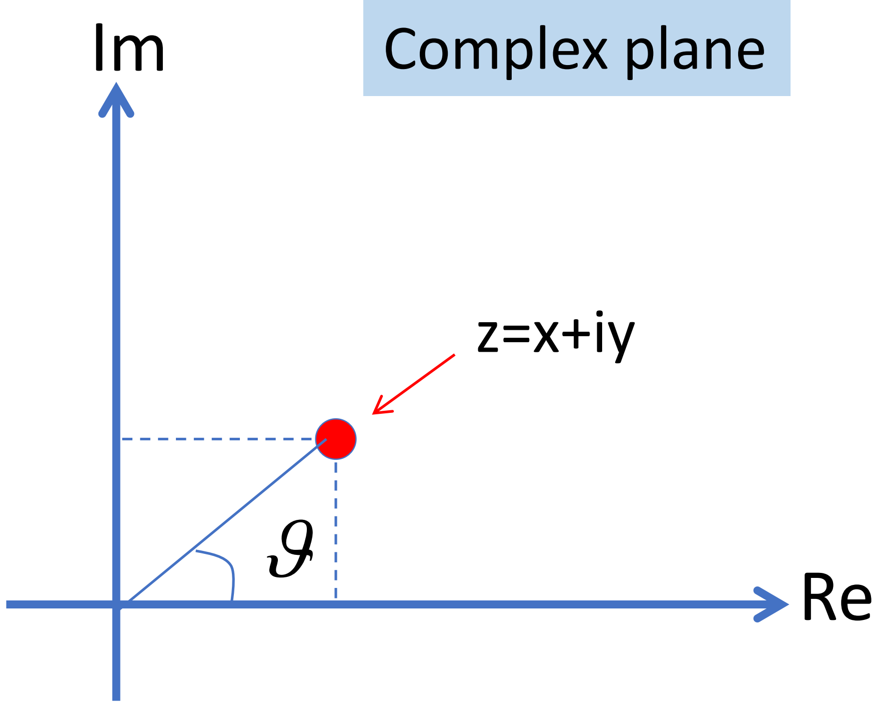
Complex conjugate $\overline{z}$=x-iy

absolute value (or modulus or magnitude) of a complex number: r=$|z|$ □ $\sqrt{x^2+y^2}$

argument of z: ϑ=arg(z) □ arctan($\frac{y}{x}$)

$$z = r(\cos\vartheta + i\sin\vartheta)$$

$$r^2 = x^2 + y^2 = z\overline{z}$$



Euler's formula

For any real number x, $e^{ix} = \cos x + i \sin x$

In particular, $e^{i\pi} + 1 = 0$

One of the most beautiful identity in mathematics. Relating 5 fundamental numbers (e,i,π,0,1) in a single formula.

Example:

$$e^{i(a+b)} = \cos(a+b) + i\sin(a+b)$$

$$e^{i(a+b)} = e^{ia}e^{ib} = (\cos a + i\sin a)(\cos b + i\sin b)$$

$$= (\cos a\cos b - \sin a\sin b) + i(\cos a\sin b + \sin a\cos b)$$

□ $\cos(a+b) = \cos a\cos b - \sin a\sin b$

□ $\sin(a+b) = \cos a\sin b + \sin a\cos b$

$$\frac{d}{dx}e^{ix} = ie^{ix} = -\sin x + i\cos x$$

$$\frac{d}{dx}(\cos x + i\sin x) = -\sin x + i\cos x$$

$$\frac{d^2}{dx^2}e^{icx} = -c^2 e^{ix}$$

Root of a complex number

De Moivre's theorem holds true for *positive, negative* and *fractional powers*. We can, therefore, use this fact to determine all the distinct roots of any number.

Since $x + \mathrm{j}y = r(\cos\alpha + \mathrm{j}\sin\alpha)$, then, by De Moivre's theorem, it follows that

$$\sqrt[n]{x+\mathrm{j}y} = \sqrt[n]{r}\left(\cos\frac{\alpha}{n} + \mathrm{j}\sin\frac{\alpha}{n}\right)$$

However, using this equation, we obtain one root only. In order to obtain all the roots we must consider the fact that the cosine and sine functions are periodic functions of period 2ω radians or $360°$. Thus we can write

$$\begin{aligned}(\cos\alpha + \mathrm{j}\sin\alpha)^n &= [\cos(\alpha + 2\omega k) + \mathrm{j}\sin(\alpha + 2\omega k)]^n \\ &= \cos(n\alpha + 2\omega nk) + \mathrm{j}\sin(n\alpha + 2\omega nk)\end{aligned}$$

$$\text{where} \quad k = 0, \pm 1, \pm 2, \pm 3, \cdots$$

When raising a complex number to an integral power there is no ambiguity: the result is independent of periodicity. But extracting the roots of a complex number

Example: Calculate all the roots of $z^4 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

Since $z^4 = \cos(\frac{2\pi}{3} + 2\pi k) + i(\sin\frac{2\pi}{3} + 2\pi k)$ for any integer k, we have

$$k = 0\,, \quad z_1 = \cos\frac{\omega}{6} + \mathrm{j}\sin\frac{\omega}{6} = \frac{1}{2}\sqrt{3} + \frac{1}{2}\mathrm{j}$$

$$k = 1\,, \quad z_2 = \cos\frac{4\omega}{6} + \mathrm{j}\sin\frac{4\omega}{6} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{j}$$

$$k = 2\,, \quad z_3 = \cos\frac{7\omega}{6} + \mathrm{j}\sin\frac{7\omega}{6} = -\frac{\sqrt{3}}{2} - \frac{1}{2}\mathrm{j}$$

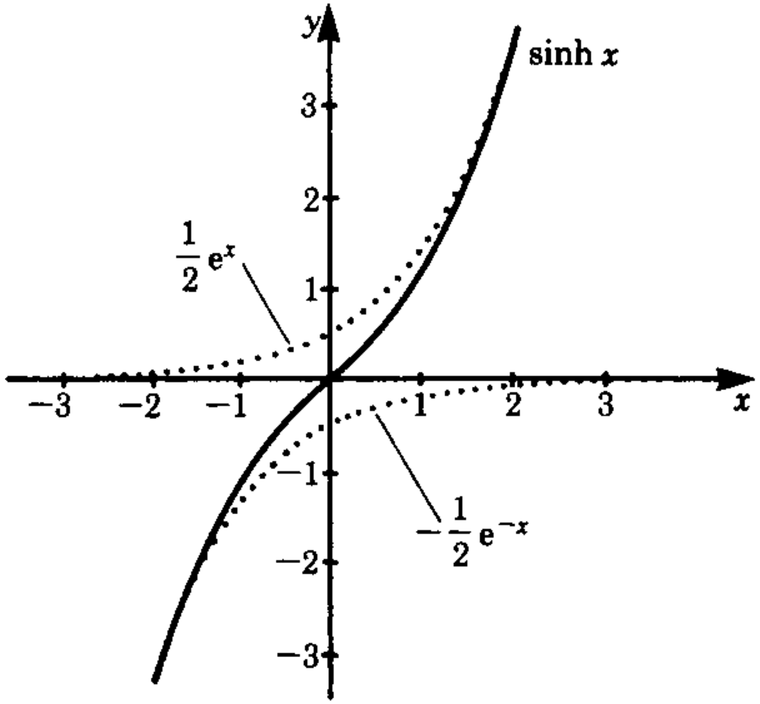
$$k = 3\,, \quad z_4 = \cos\frac{10\omega}{6} + \mathrm{j}\sin\frac{10\omega}{6} = \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{j}$$

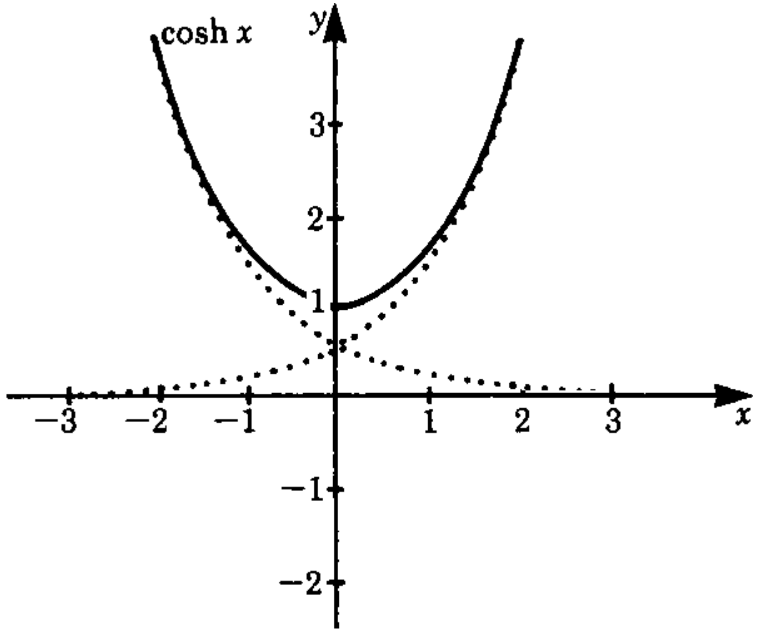
Hyperbolic function:

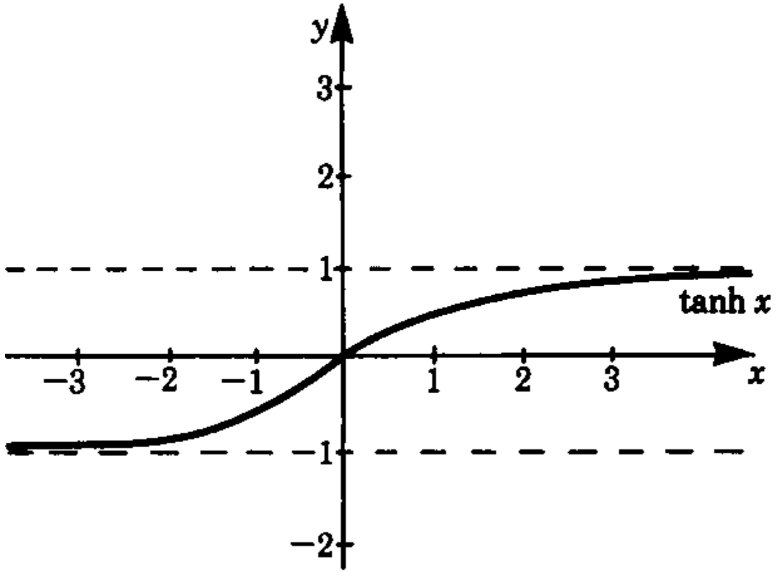
$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$







Exercise:

1. Compute i^8 and $(-i)^3$
2. Evaluate $\sqrt{a-b} \times \sqrt{b-a}$
3. Evaluate the roots of the quadratic equation $x^2 + x + 1 = 0$

Compute $z_1 z_2$:

(a) $z_1 = 2(\cos 15^\circ + \mathrm{j}\sin 15^\circ)$
$z_2 = 3(\cos 45^\circ + \mathrm{j}\sin 45^\circ)$

Calculate z_1/z_2:

(a) $z_1 = \cos 70^\circ + \mathrm{j}\sin 70^\circ$
$z_2 = \cos 25^\circ + \mathrm{j}\sin 25^\circ$

Given $z = \frac{1}{2}\mathrm{e}^{\mathrm{j}\omega/4}$, calculate z^3.

Given $z = 32\mathrm{e}^{\mathrm{j}10\omega}$, calculate $z^{1/5}$.

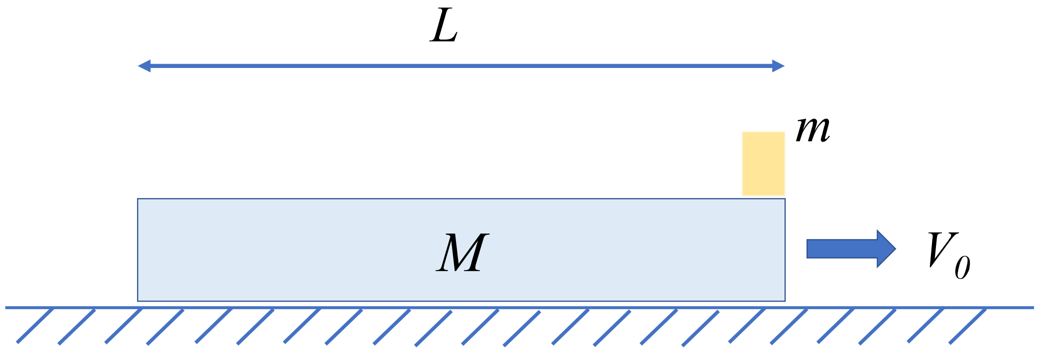
$$\frac{d\sinh x}{dx} = ?$$

$$\frac{d\cosh x}{dx} = ?$$

$$\frac{d\tanh x}{dx} = ?$$

Exercise: (HKPhO2017)

4. A block of mass M and length L is sliding on the frictionless table and moves with constant velocity V_0 to the right. Suddenly, a small mass m is put on the right end of the block. The mass m slides relative to the block and fall on the left end of the block. Let the coefficient of kinetic friction between the block M and mass m be μ.
質量為 M ,長為 L 的木板，在光滑水平面上以速度向右作勻速正線運動。突然，在長木板右端放上一個質量為 m 的小物塊。小物塊產生相對滑動，並從木板左端滑出。設 M 和 m 之間的動摩擦系數為μ。



(a) What is the loss of the total mechanical energy during the process?
求此過程中損失的總機械能。

(b) What is the final velocity of the mass m and the total travelled time of the mass m before it falls on the left end?
求小物塊脫離木板時的速度和小物塊在本板上運動的時間。

(c) What is the minimum value of V_0?
求 V_0 的最小值。

The friction acts on the mass m (-ve direction)

$$f = -\mu mg = ma$$

Let V and v be the velocities of the block M and the mass m when the mass m leaves the block.

$$v = \mu gt$$

By the conservation of momentum

$$MV_0 = mv + MV$$

and the work-energy theorem,

$$-fL = \left(\frac{1}{2}MV^2 + \frac{1}{2}mv^2\right) - \frac{1}{2}MV_0^2$$

(a) The loss of total mechanical energy $\Delta E = fL = \mu mgL$

(b) By elimination, we have

$$(Mm + m^2)v^2 - 2mMV_0v + 2\mu mMgL = 0$$

$$v = \frac{1}{M+m}\left(MV_0 \pm \sqrt{M^2V_0^2 - 2\mu gML(M+m)}\right)$$

Since the velocity of the mass m must less than V, we have $v \leq \frac{1}{M+m}V_0$ and hence

$$v = \frac{1}{M+m}\left(MV_0 - \sqrt{M^2V_0^2 - 2\mu gML(M+m)}\right)$$

and $t = \frac{v}{\mu g} = \frac{1}{\mu g(M+m)}\left(MV_0 - \sqrt{M^2V_0^2 - 2\mu gML(M+m)}\right)$

(c) For v to be real, we much have $M^2V_0^2 - 2\mu gML(M+m) \geq 0$

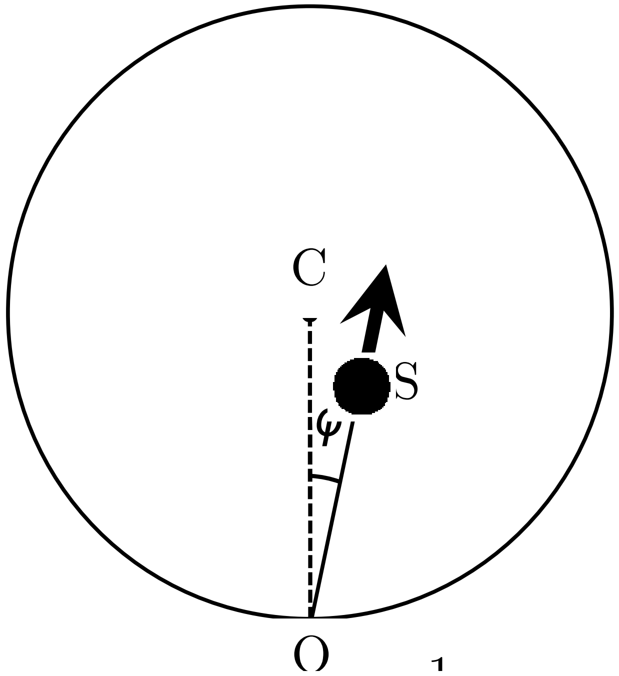
$$V_0 \geq \sqrt{\frac{2\mu gL(M+m)}{M}}$$

2. A Round Snooker Table

A smooth snooker ball S is struck from point O of a specially-designed circular snooker table. The ball then moves off horizontally in a direction making an angle ϕ with the radius CO. The ball makes m impacts with the smooth vertical wall of the table before returning to point O. If $m = 1$, $\phi = 0$. The coefficient of restitution between the ball and the wall is e.

(a) If $m = 2$, find ϕ in terms of positive exponents of e.

(b) If $m = 3$, find ϕ in terms of positive exponents of e.



$$\text{Coefficient of restitution} = \frac{\text{Relative velocity after collision}}{\text{Relative velocity before collision}}$$

e = 1 for elastic collision
e = 0 for perfectly inelastic

(a) $m = 2$:

From conservation of linear momentum, 1st impact: $v_1 \sin\phi_1 = v \sin\phi$.

From Newton's law of impact, 1st impact: $v_1 \cos\phi_1 = ev\cos\phi$.

Thus $\tan\phi_1 = e^{-1}\tan\phi$.

Similarly for 2nd impact: $\tan\phi_2 = e^{-1}\tan\phi_1 = e^{-2}\tan\phi$.

From geometry,

$$2(\phi + \phi_1 + \phi_2) = \pi,$$

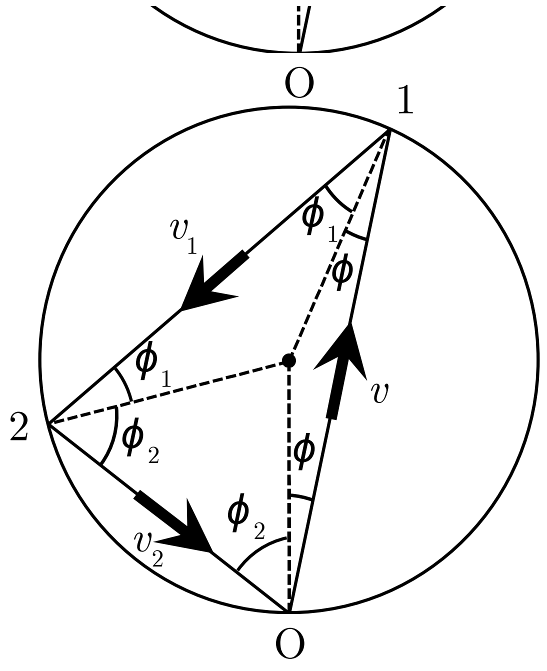
$$\tan(\phi_1 + \phi_2) = \tan(\frac{\pi}{2} - \phi)$$

$$\frac{\tan\phi_1 + \tan\phi_2}{1 - \tan\phi_1 \tan\phi_2} = \frac{1}{\tan\phi}$$

$$\frac{e^{-1}\tan\phi + e^{-2}\tan\phi}{1 - e^{-3}\tan^2\phi} = \frac{1}{\tan\phi}$$

$$\tan^2\phi = \frac{1}{e^{-1} + e^{-2} + e^{-3}} = \frac{e^3}{1 + e + e^2}$$

$$\phi = \tan^{-1}\frac{1}{\sqrt{e^{-1} + e^{-2} + e^{-3}}} = \tan^{-1}\sqrt{\frac{e^3}{1 + e + e^2}}.$$



(b) m=3

As in (a), from conservation of linear momentum and Newton's law of impact, 3rd impact: $\tan\phi_3 = e^{3}\tan\phi$.

From geometry,

$$2(\phi+\phi_1+\phi_2+\phi_3) = 2\pi,$$

$$\tan(\phi+\phi_1) = \tan\left[\pi-(\phi_2+\phi_3)\right] = -\tan(\phi_2+\phi_3)$$

$$\frac{\tan\phi+\tan\phi_1}{1-\tan\phi\tan\phi_1} = -\frac{\tan\phi_2+\tan\phi_3}{1-\tan\phi_2\tan\phi_3}$$

$$\frac{\tan\phi+e^{-1}\tan\phi}{1-e^{-1}\tan^2\phi} = -\frac{e^{-2}\tan\phi+e^{-3}\tan\phi}{1-e^{-5}\tan^2\phi}$$

$$\Rightarrow \tan^2\phi = \frac{1+e^{-1}+e^{-2}+e^{-3}}{e^{-3}+e^{-4}+e^{-5}+e^{-6}} = e^3$$

$$\Rightarrow \phi = \underline{\underline{\tan^{-1}e^{3/2}}}.$$