Tutorial 4
Ordinary Differential Equations

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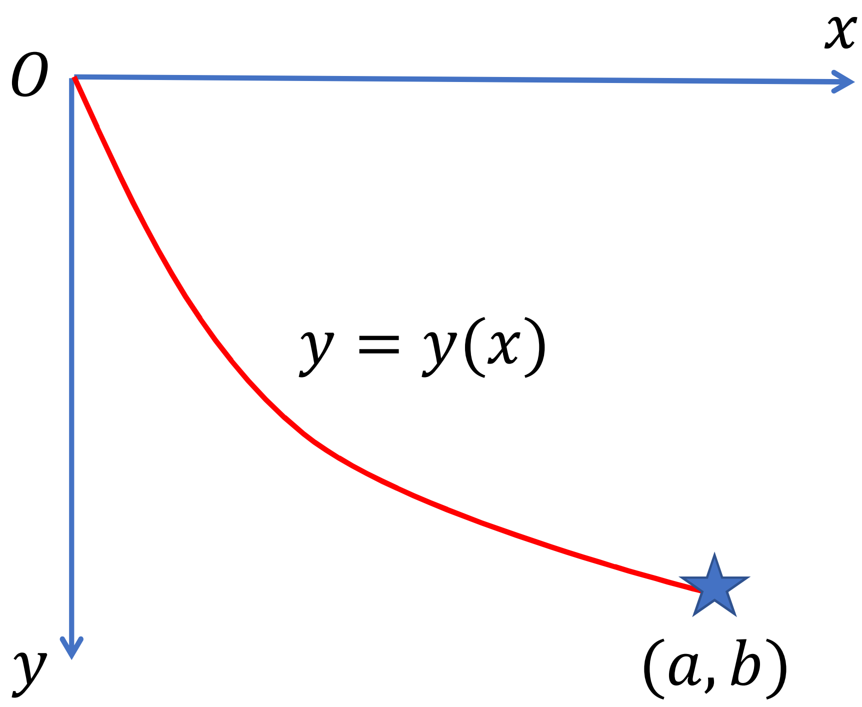
Travelling time along a path

What is the travelling time of an object sliding down along a frictionless path?
By the conservation of energy, we have

$$E = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$
$$= \frac{1}{2}m\dot{x}^2\left(1 + y'^2\right) - mgy = 0$$
$$\rightarrow \dot{x} = \sqrt{\frac{2gy}{1 + y'^2}}$$

Travelling time T,

$$T = \int dt = \int \frac{dt}{dx}dx = \int \frac{dx}{\dot{x}}$$
$$T = \int \sqrt{\frac{1 + y'^2}{2gy}}dx$$



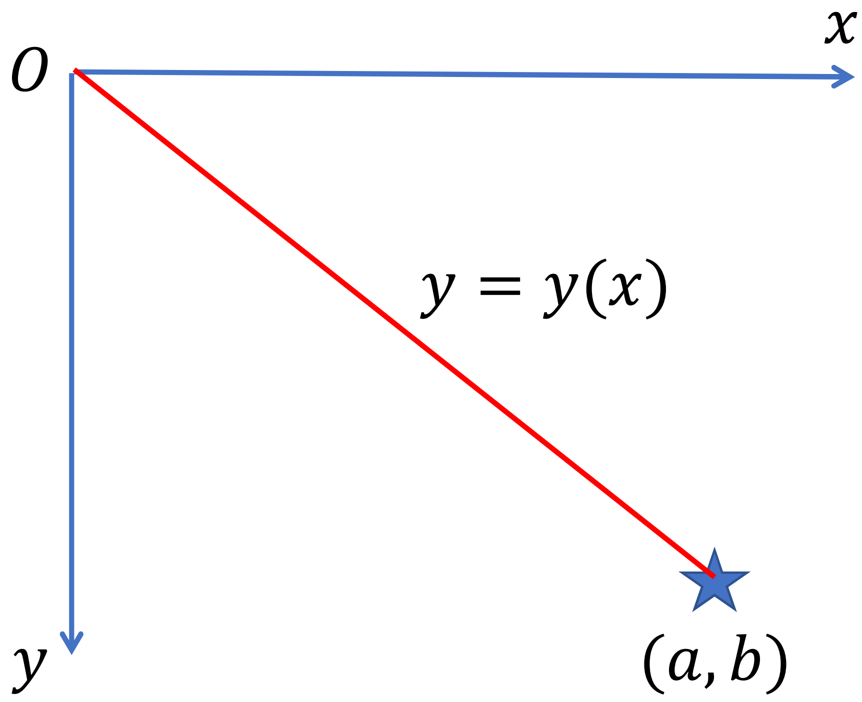
Example: if $y(x) = \frac{b}{a}\, x$

$$T = \int \sqrt{\frac{1+y'^2}{2gy}}dx$$

$$= \int_0^a \sqrt{\frac{1+(b/a)^2}{(2gb/a)x}}dx$$

$$= \sqrt{\frac{a^2+b^2}{2gab}} \int_0^a \frac{1}{\sqrt{x}}dx$$

$$= \sqrt{\frac{2(a^2+b^2)}{gb}}$$



Check: along a vertical path (i.e. a=0), we have $T = \sqrt{\frac{2b}{g}}$

which is what we have expected for a vertical constant acceleration!

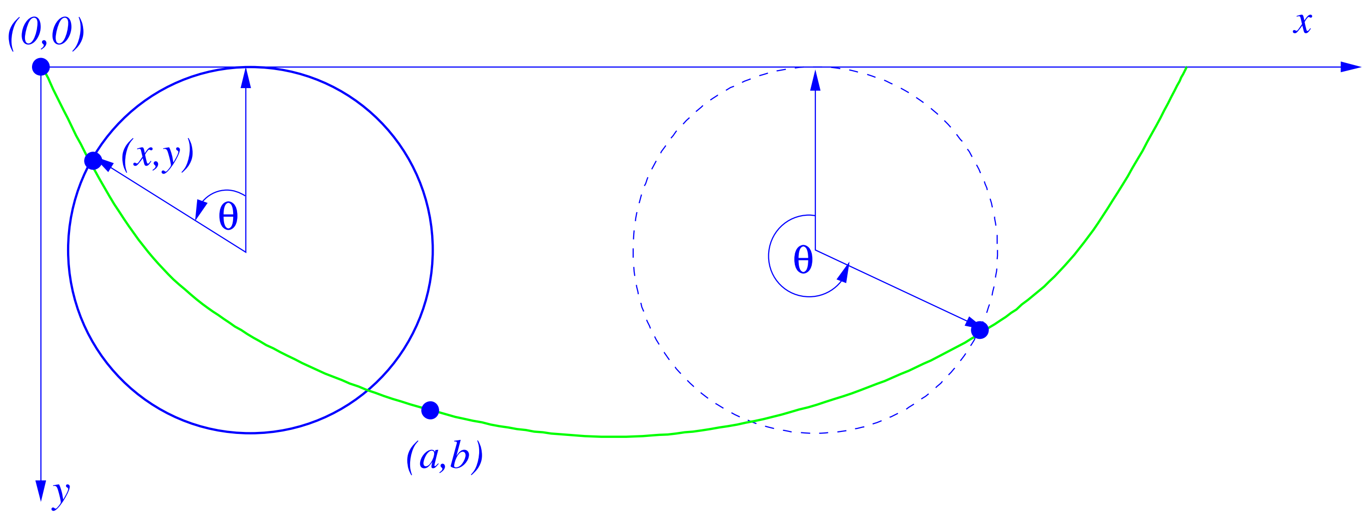
The traveling time is

$$T = \int_0^a \sqrt{\frac{1 + y'^2}{2gy}}\, dx$$

different y(x) will give different travelling time T

Question: Which path take the shortest time for travelling?

Calculus of variation



Moment of inertia

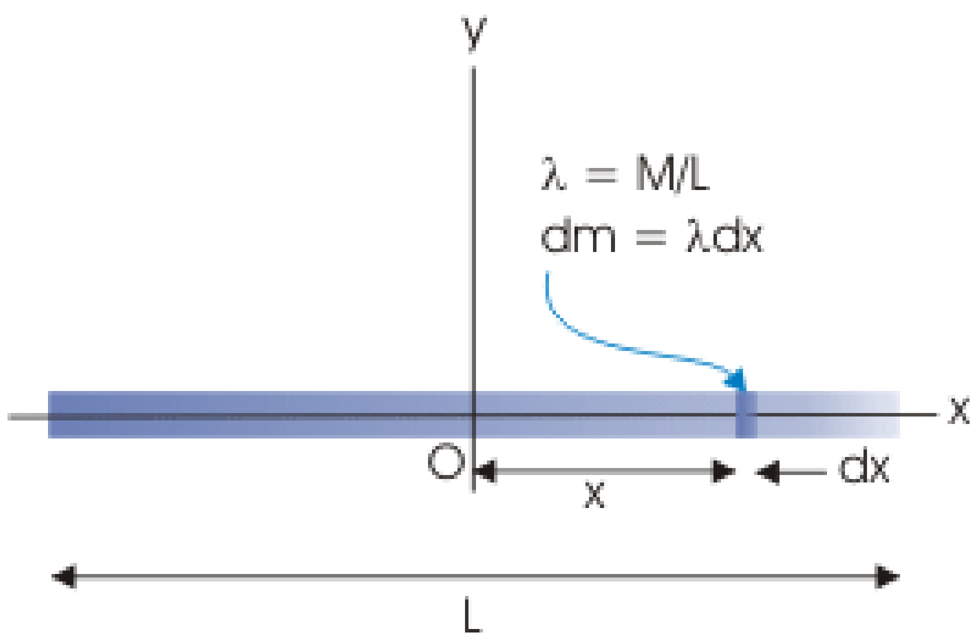
$$I = \sum m_i r_i^2 \longrightarrow \int r^2\, dm = \begin{cases} \int \rho r^2 dV & (3D) \\ \int \sigma r^2 dA & (2D) \\ \int \lambda r^2 dx & (1D) \end{cases}$$

Example: Moment of inertia of a rod rotating at CM

$$I = \int_{-L/2}^{L/2} \lambda x^2 dx$$

$$= \frac{M}{L}\frac{L^3}{12}$$

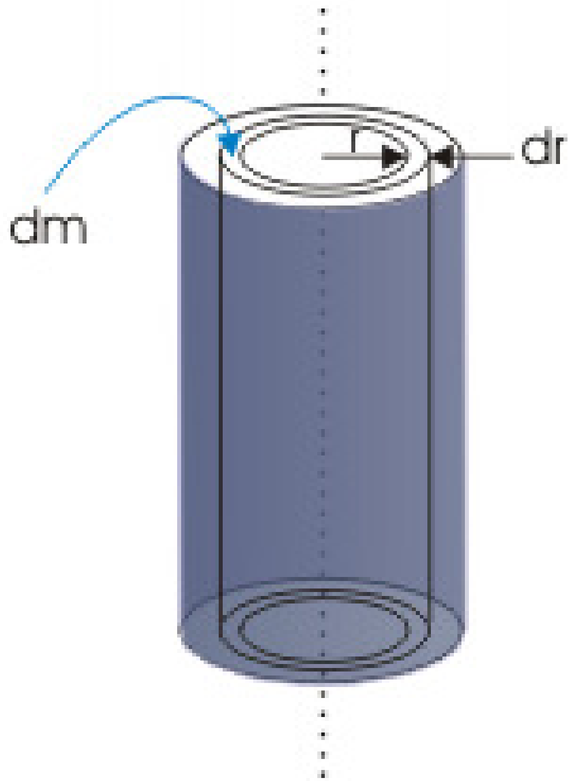
$$= \frac{ML^2}{12}$$



Exercise: Moment of inertia of a cone rotating along symmetric axis

$$I = \rho \int r^2 dV$$

$$\rho = \frac{dm}{dV} = \frac{M}{V} = \frac{M}{\pi R^2 H}$$



2. Differential equation

Many natural laws in physics and engineering are formulated by equations involving derivatives or differentials of physical quantities – **Differential equation (DE)**.

Example:

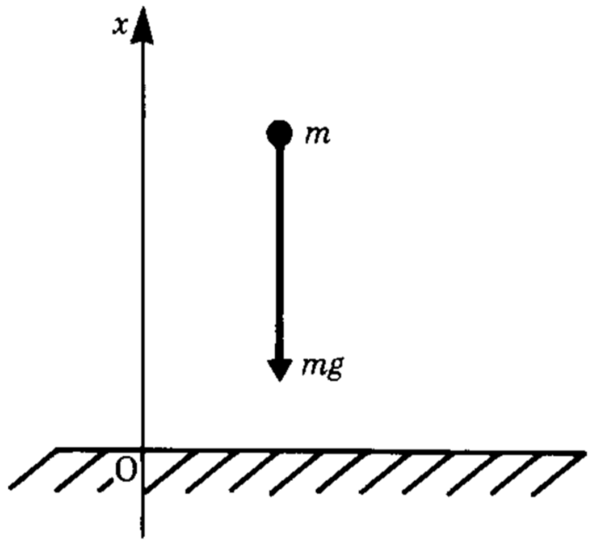
$$F = ma = -mg$$

$$\frac{d^2x}{dt^2} = -g$$

We are looking for the position of a body at an instant t x(t) which satisfies the DE.
We can show the general solution is given by

$$x(t) = -\frac{1}{2}gt^2 + C_1 t + C_2$$

where C_1 and C_2 are arbitrary (any) constants



$$x(t) = -\frac{1}{2}gt^2 + C_1 t + C_2$$

However, if we know the body is at the initial position x_0 with initial velocity v_0 at time t=0, we can determine the constants uniquely.

$$x(t) = -\frac{1}{2}gt^2 + v_0 t + x_0$$

We say that the solution is uniquely specified if two initial conditions (u_0 and v_0) are given.

Goals:

1. write down the differential equation for a specific problem
2. solve the differential equation

Order of a Differential Equation

Order of a DE is defined by the highest derivative contained in the equation. Thus an n-th order DE contains an n-th derivative.

Example:

$$y' + ax = 0\,, \quad \text{which is of the first order}$$
$$y'' + 7y = 0\,, \quad \text{which is of the second order.}$$

Linear Differential Equation

If the function y and its derivatives (y’, y’’, y’’’,….) in a DE are all to the first power and if no products likes yy’, y’y’’’,…. etc occur, then the DE is linear

Examples are

$$y'' + 7y + \sin x = 0 \quad \text{and} \quad 5y' = xy \quad \text{which are linear DEs.}$$

$$y'' + y^2 = 0 \quad \text{and} \quad (y'')^2 = x^2 y \quad \text{which are non-linear DEs.}$$

Linear DE with constant coefficients

$$a_2 y'' + a_1 y' + a_0 y = f(x)$$

where $a_2 \neq 0$ and a_2, a_1 and a_0 are arbitrary real constants, is called a second-order linear differential equation with constant coefficients since all a_j are constants

If $f(x) = 0$, the DE is called a **homogeneous** DE, eg,

$$my'' + \omega y' + ky = 0$$

If $f(x) \neq 0$, the DE is called a **inhomogeneous** DE, eg,

$$my'' + \omega y' + ky = \sin \omega x$$

Example:

Consider the 1st order linear inhomogeneous DE

$$\frac{dy}{dx} = f(x)$$

$$\rightarrow y(x) = \int f(x)dx + C$$

where C is an arbitrary constant.

This implies that the solution of the DE is **NOT uniquely determined.**

This solution is referred to as the general solution of the DE before the constants are specified.

Lemma

The general solution of a first-order DE contains exactly one undetermined constant. The general solution of a second-order DE contains exactly two constants, which can be chosen independently of each other.

Example: The linear inhomogeneous 2nd DE:

$$F = ma = -mg$$

$$\frac{d^2x}{dt^2} = -g$$

has the general solution:

$$x(t) = -\frac{1}{2}gt^2 + C_1 t + C_2$$

contains two independent constants.
Of course, **the constants can be uniquely determined if initial conditions are given**.

First order Differential equation

Separation of variables

If the DE can be rewritten in the following form

$$p(y)y' + q(x) = 0$$
$$p(y)dy = -q(x)dx$$

The solution of such an equation is obtained by simple integration

$$\int p(y)dy = -\int q(x)dx + C$$

Example

Example The variables in the following equation can be separated:

$$y'x^3 = 2y^2$$

Dividing by x^3y^2, we obtain

$$\frac{1}{y^2}y' = \frac{2}{x^3}, \quad \text{i.e.} \quad \frac{1}{y^2}\,\mathrm{d}y = \frac{2}{x^3}\,\mathrm{d}x$$

This is an equation of the type required with $p(y) = 1/y^2$ and $q(x) = -2/x^3$.
Now, straightforward integration gives

$$\frac{1}{y} = \frac{1}{x^2} + C$$

and hence

$$y = \frac{x^2}{Cx^2 + 1}$$

Integrating factor

We first consider homogeneous linear first-order DE

$$p(x)y' + q(x)y = 0$$

There is a straightforward and systematic way to solve it – integration factor

$$p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = 0$$

$$\frac{\mathrm{d}y}{y} = -\frac{q(x)}{p(x)}\,\mathrm{d}x$$

$$\int \frac{\mathrm{d}y}{y} = \ln|y| = -\int \frac{q(x)}{p(x)}\,\mathrm{d}x + C_1$$

Hence

$$y = C\mathrm{e}^{-\int \frac{q(x)}{p(x)}\,\mathrm{d}x}$$

$e^{\int (q/p)dx} = I(x)$ is called the integrating factor

Example

$$y' + \frac{y}{x} = 0$$

Its solution is

$$\begin{aligned}
\frac{\mathrm{d}y}{y} &= -\frac{\mathrm{d}x}{x} \\
\ln|y| &= -\ln|x| + C_1 \\
y &= \frac{C}{x}
\end{aligned}$$

Exercise: Solve the homogeneous first-order linear DE

$$xy' + (1+x)y = 0$$

Next, we consider inhomogeneous linear first-order DE

$$p(x)y' + q(x)y = f(x)$$

We know that $y_h(x) = Ce^{-\int \frac{q(x)}{p(x)}dx}$ is the solution of the homogeneous DE

Trick: We guess the solution of inhomogeneous DE is of the form,

$$y_(x) = v(x)e^{-\int \frac{q(x)}{p(x)}dx}$$

for some function $v(x)$.

Guess: $$y_(x) = v(x)e^{-\int \frac{q(x)}{p(x)}dx} = \frac{v(x)}{I(x)}$$

Compute

$$y' : \quad y' = \frac{v'(x)}{I(x)} - \frac{q(x)}{p(x)}\frac{v(x)}{I(x)} = \frac{1}{I(x)}\left(v'(x) - \frac{q(x)}{p(x)}v(x)\right)$$

Inserting this into the original equation gives

$$\frac{1}{I(x)}p(x)\omega'(x) = f(x)$$

This equation allows us to compute $v(x)$. Thus

$$v(x) = \int \omega'(x)\,\mathrm{d}x = \int I(x)\frac{f(x)}{p(x)}\,\mathrm{d}x$$

The solution of the equation $p(x)y' + q(x)y = f(x)$ reads

$$\boxed{y(x) = \frac{1}{I(x)}\int I(x)\frac{f(x)}{p(x)}\,\mathrm{d}x}$$

Example

Solve $$y' + \frac{y}{x} = 4x^2$$

We know $y(x) = \frac{C}{x}$ is the solution of the homogeneous DE

Hence we guess the solution of the inhomogeneous DE is of the form

$$y = \frac{\omega(x)}{x}, \quad y' = \frac{\omega'(x)}{x} - \frac{\omega(x)}{x^2}$$

Inserting into the original equation gives

$$\text{LHS} = y' + \frac{y}{x} = \frac{\omega'(x)}{x} - \frac{\omega(x)}{x^2} + \frac{\omega(x)}{x^2} = \frac{\omega'(x)}{x}$$

$$\text{RHS} = 4x^2$$

Thus $$\frac{\omega'(x)}{x} = 4x^2$$

and $$\omega(x) = \int 4x^3 \mathrm{d}x = x^4 + C$$

The general solution of the given equation reads

$$y(x) = x^3 + \frac{C}{x}$$

Exercise: Solve the inhomogeneous first-order linear DE

(b) $y' = \dfrac{y}{x} + x$

(d) $xy' + (1+x)y = xe^{-x}$

Second order Differential equation

Homogeneous second-order DE

$$a_2y'' + a_1y' + a_0y = 0$$

Table 10.1

Systematic procedure for the solution of the homogeneous second-order DE	Example
Let the equation be $a_2y'' + a_1y' + a_0y = 0$	$y'' + 3y' + 2y = 0$
Let $y = e^{rx}$ be a solution of the DE. Substituting for	
$y' = re^{rx}$	$y = e^{rx}, \quad y' = \frac{dy}{dx} = re^{rx}$
and $y'' = r^2e^{rx}$	$y'' = \frac{d^2y}{dx^2} = r^2e^{rx}$
gives $a_2r^2e^{rx} + a_1re^{rx} + a_0e^{rx} = 0$	
We can factorise e^{rx}: $e^{rx}(a_2r^2 + a_1r + a_0) = 0$	$e^{rx}(r^2 + 3r + 2) = 0$
Since $e^{rx} \neq 0$, the expression in the bracket must be zero: $a_2r^2 + a_1r + a_0 = 0$	$r^2 + 3r + 2 = 0$
This is a quadratic in r. It is called the *auxiliary equation* of the DE. Its roots are $r_{1,2} = \frac{-a_1 \pm \sqrt{a_1{}^2 - 4a_2a_0}}{2a_2}$	$r_1 = -1, \quad r_2 = -2$
Provided that r_1 and r_2 are different, the general solution of the DE is $y = C_1e^{r_1x} + C_2e^{r_2x}$	$y = C_1e^{-x} + C_2e^{-2x}$

Example

Example Solve $y'' + 4y' + 13y = 0$.

The auxiliary equation is $r^2 + 4r + 13 = 0$,
whose roots are $r_1 = -2 + 3j$ and $r_2 = -2 - 3j$.

$$y = C_1 e^{(-2+3i)x} + C_2 e^{(-2-3i)x}$$

$$y = e^{-2x}\left(C_1 e^{-3ix} + C_2 e^{3ix}\right)$$

$$= e^{-2x}\left(A \sin(3x) + B \cos 3x\right)$$

Summary

Summary The solution of the homogeneous second-order DE

$$a_2 y'' + a_1 y' + a_0 y = 0$$

The auxiliary equation is $a_2 r^2 + a_1 r + a_0 = 0$.

Calculate the roots r_1 and r_2 of the auxiliary equation:

$$r_{1,2} = \frac{-a_1 \pm \sqrt{a_1{}^2 - 4a_2 a_0}}{2a_2}$$

Obtain the general solution according to the following three possible cases.

Case 1

If $r_1 \neq r_2$ are real and unequal roots

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} \quad (10.2)$$

Case 2

If $r_1 = r_2$ are equal roots

$$y = e^{r_1 x}(C_1 + C_2 x) \quad (10.3)$$

Case 3

If r_1 and r_2 are complex roots with

$$\begin{aligned} r_1 &= a + jb \qquad \text{and} \qquad r_2 = a - jb \\ y &= e^{ax}(C_1 \cos bx + C_2 \sin bx) \end{aligned} \quad (10.4)$$

Exercise Consider the vertical motion of a particle in the presence of air resistance. The air-resistance (drag force) is given by

$$F_{\text{air}} = -kv = -k\frac{dx}{dt}$$

By taking downward as positive, find the trajectory of the particle where it is at the origin (i.e. $x(0) = 0$) with zero velocity ($v(0) = 0$) initially .

Example: Free undamped oscillator

By Newton's second law of motion,

$$m\ddot{x}(t) = -kx(t)$$

$$\ddot{x} + \omega_n^2 x = 0\,, \quad \omega_n^2 = \frac{k}{m}$$

$$\omega_n = \text{natural frequency}$$

This is a linear second-order DE. The auxiliary equation is

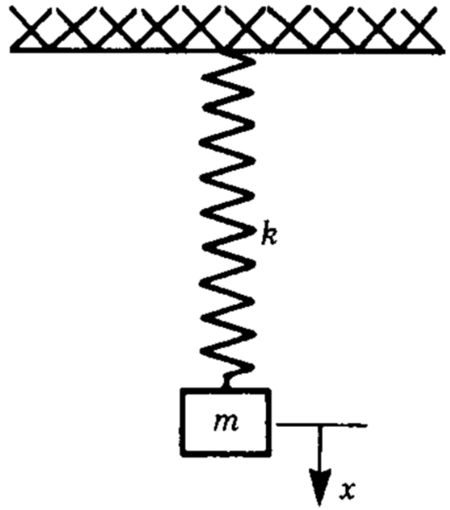
$$r^2 + \omega_n^2 = 0$$

The roots are $r_1 = \mathrm{j}\omega_n$ and $r_2 = -\mathrm{j}\omega_n$.

The general solution is (cf. Sect. 10.3.1, Case 3)

$$x = C_1 \cos \omega_n t + C_2 \sin \omega_n t$$

We need two boundary conditions to determine the values of C_1 and C_2.



For example, the boundary conditions of an oscillation are

$$x = 0 \quad \text{at} \quad t = 0 \quad (\text{position at the instant } t = 0)$$

$$\dot{x} = \omega_0 \quad \text{at} \quad t = 0 \quad (\text{velocity at the instant } t = 0)$$

Substituting the first condition in the DE above gives

$$0 = C_1 \cos 0 + C_2 \sin 0$$

$$\text{Hence} \quad C_1 = 0$$

Substituting the second boundary condition gives

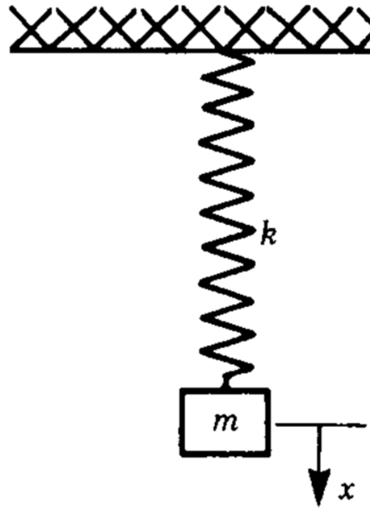
$$\dot{x} = \omega_0 = -\omega_n C_1 \sin 0 + \omega_n C_2 \cos 0 = \omega_n C_2$$

Hence $C_2 = v_0/\omega_n$

The particular solution is

$$x = \frac{\omega_0}{\omega_n} \sin \omega_n t$$

Example: Damped oscillator



The friction or damping force is given in some cases by

$$F = -c\dot{x}$$

where c is a friction or damping coefficient, $\dot{x}$ is the velocity and the minus sign indicates that the force acts in a direction opposite to the motion. By Newton's second law, the equation of motion for our spring-mass system becomes

$$m\ddot{x} + c\dot{x} + kx = 0$$

This is the DE of motion for free oscillations or vibrations, meaning that there are no external forces acting on the system.

The auxiliary equation is

$$mr^2 + cr + k = 0$$

whose roots are

$$r_{1,2} = \frac{-c}{2m} \pm \frac{\sqrt{c^2 - 4mk}}{2m} = -a \pm b$$

There are three cases to consider:

$$c^2 - 4mk > 0\,, \quad c^2 - 4mk < 0\,, \quad c^2 - 4mk = 0$$

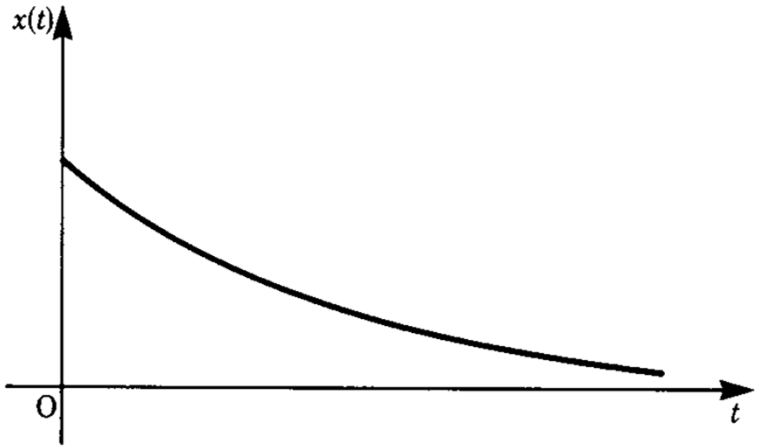
1. Over-damping

Case 1: $c^2 - 4mk > 0$.

This means that the roots are real and unequal. In this case the general solution is

$$x = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$
$$= e^{-at}\left[C_1 e^{bt} + C_2 e^{-bt}\right]$$

This corresponds to an *over-damped system*, and its response from a given initial displacement is shown in Fig. 10.5. No oscillations are present. The system will return to the equilibrium position slowly.



2. Critical damping

Case 2: $c^2 - 4mk = 0$.

The roots are equal, i.e. $r_1 = r_2 = -a$. The general solution is

$$x = (C_1 + C_2 t)e^{-at}$$

The system will return to the equilibrium position more quickly than the system in Case 1 but again there will be no oscillations. It is referred to as *critical* or *aperiodic* and the damping is called *critical damping*. Its response from a given initial displacement and initial velocity is shown in Fig. 10.6.

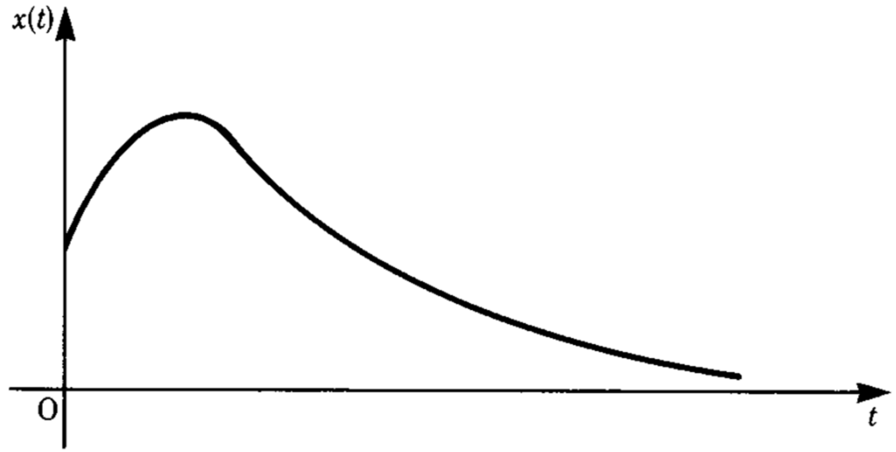


Fig. 10.6

3. Under-damping

Case 3: $c^2 - 4mk < 0$.

The roots in this case are complex conjugate, i.e. $r_1 = -a + jb, r_2 = -a - jb$, with $a > 0$. The general solution is

$$x = \mathrm{e}^{-at}\left[C_1\mathrm{e}^{\mathrm{j}bt} + C_2\mathrm{e}^{-\mathrm{j}bt}\right]$$

or

$$\begin{aligned} x &= \mathrm{e}^{-at}[C_1(\cos bt + \mathrm{j}\sin bt) + C_2(\cos bt - \mathrm{j}\sin bt)] \\ &= \mathrm{e}^{-at}(A\cos bt + B\sin bt) \end{aligned}$$

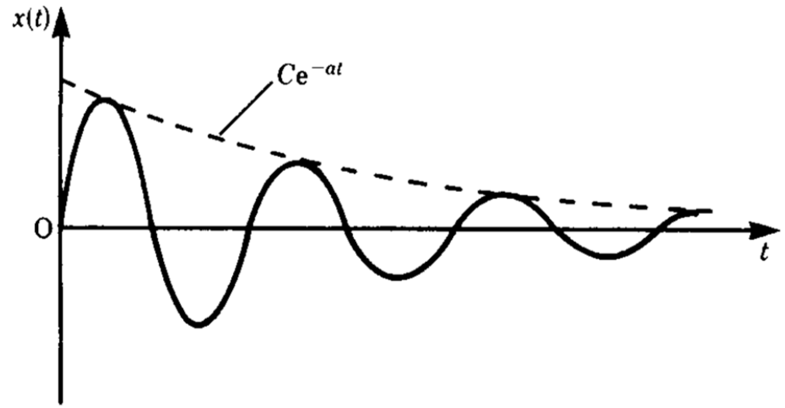
where $A = C_1 + C_2$ and $B = \mathrm{j}(C_1 - C_2)$ and A and B are arbitrary.

We should point out that although C_1 and C_2 may be complex, A and B are not necessarily complex. As we are dealing with a real physical problem, the solution must be real, hence A and B must be real, which means that C_1 and C_2 must be complex conjugate numbers.

The displacement x may be put in another form thus:

$$x = C\mathrm{e}^{-at}\cos(bt - \alpha)$$

An examination of this function shows that the system will oscillate, but the oscillations will die out due to the exponential factor. Its response from a given initial displacement and velocity is shown in Fig. 10.7. It is a *damped* oscillation.



Non-homogeneous 2^{nd} ODE

Lemma 10.2 *Consider the non-homogeneous DE*

$$a_2 y'' + a_1 y' + a_0 y = f(x)$$

Let y_c *be the general solution of the homogeneous equation*

$$a_2 y'' + a_1 y' + a_0 y = 0$$

y_c *is also called the complementary function.*

Let y_p *be a particular solution of the non-homogeneous DE*

$$a_2 y'' + a_1 y' + a_0 y = f(x)$$

Then the general solution of the DE is given by

$$\boxed{y = y_c + y_p}$$

Proof We will first show that $y = y_c + y_p$ is a solution of the DE.

According to the assumptions we have made for the homogeneous DE,

$$a_2 y_c'' + a_1 y_c' + a_0 y_c = 0 \quad [1]$$

For the non-homogeneous DE we have

$$a_2 y_p'' + a_1 y_p' + a_0 y_p = f(x) \quad [2]$$

Substituting $y = y_c + y_p$ in the non-homogeneous equation gives

$$a_2 (y_c + y_p)'' + a_1 (y_c + y_p)' + a_0 (y_c + y_p) = f(x)$$

Rearranging gives

$$\left(a_2 y_c'' + a_1 y_c' + a_0 y_c\right) + \left(a_2 y_p'' + a_1 y_p' + a_0 y_p\right) = f(x)$$

Q: How to find the particular solution?

A: There are systematic ways to find it. But in many cases, we can **guess** the particular solution.

Example

Example Find a particular integral of the DE

$$y'' - 3y' + 2y = 3 - 2x^2$$

Since the RHS is a quadratic, we assume

$$y_p = A + Bx + Cx^2$$

Hence ${y_p}' = B + 2Cx$ and ${y_p}'' = 2C$

Substituting in the DE gives

$$2C - 3(B + 2Cx) + 2(A + Bx + Cx^2) = 3 - 2x^2$$

Comparing coefficients we find

for x^2, $\quad 2C = -2\,, \quad C = -1$

for x, $\quad -6C + 2B = 0\,, \quad B = -3$

constant terms, $\quad 2C - 3B + 2A = 3\,, \quad A = -2$

A particular integral is

$$y_p = -2 - 3x - x^2$$

Example

Example Find a particular integral of

$$y'' - 4y' + 3y = 5\mathrm{e}^{-3x}$$

The roots of the auxiliary equation are 3 and 1. Thus e^{-3x} is not a term of the complementary function; hence we assume

$$\begin{aligned} y_\mathrm{p} &= A\mathrm{e}^{-3x} \\ y_\mathrm{p}' &= -3A\mathrm{e}^{-3x} \\ y_\mathrm{p}'' &= 9A\mathrm{e}^{-3x} \end{aligned}$$

Substituting in the DE gives

$$[9 - 4(-3) + 3]A\mathrm{e}^{-3x} = 5\mathrm{e}^{-3x}$$

so that
$$A = \frac{5}{24}$$

A particular integral is

$$y_\mathrm{p} = \frac{5}{24}\mathrm{e}^{-3x}$$

The complete solution is

$$y = C_1\mathrm{e}^{3x} + C_2\mathrm{e}^{x} + \frac{5}{24}\mathrm{e}^{-3x}$$

Example

Example Suppose that the RHS of the previous example was $5e^x$. As e^x is a term of the complementary function, we assume

$$y_p = Axe^x$$
$$y_p{}' = A\,xe^x + Ae^x = A(xe^x + e^x)$$
$$y_p{}'' = A\,xe^x + Ae^x + Ae^x = A(xe^x + 2e^x)$$

Substituting in the DE, we have

$$(x + 2 - 4x - 4 + 3x)Ae^x = 5e^x \qquad \text{or} \qquad -2A = 5$$

Hence $$A = -\frac{5}{2}$$

A particular integral is

$$y_p = -\frac{5}{2}xe^x$$

The complete solution is

$$y = C_1e^{3x} + C_2e^x - \frac{5}{2}xe^x$$

Exercise

Solve the non-homogeneous DE

$$y'' + y' + y = 2x + 3$$

subject to the initial conditions

$$y(0) = y'(0) = 0$$

Example: Forced oscillator

$$F_{\text{ext}}(t) + F_{\text{air}} + F_{\text{elastic}} = m\ddot{x}$$
$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t$$

We consider the solution of the form:

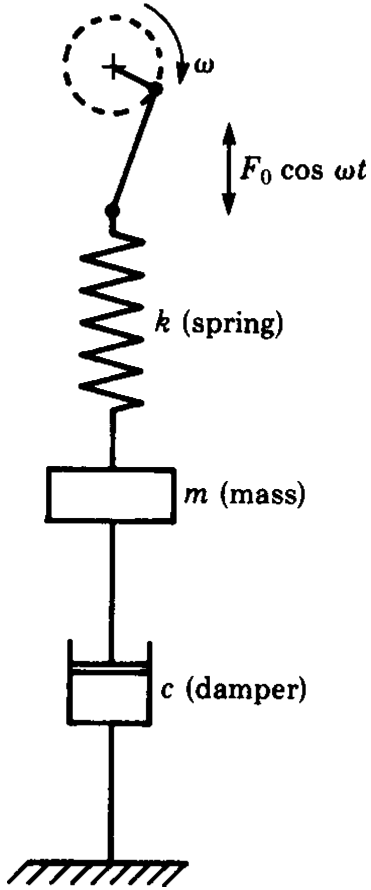
$$x(t) = x_{\text{h}}(t) + x_{\text{p}}(t)$$

General solution

$$m\ddot{x}_h + c\dot{x}_h + kx_h = 0$$

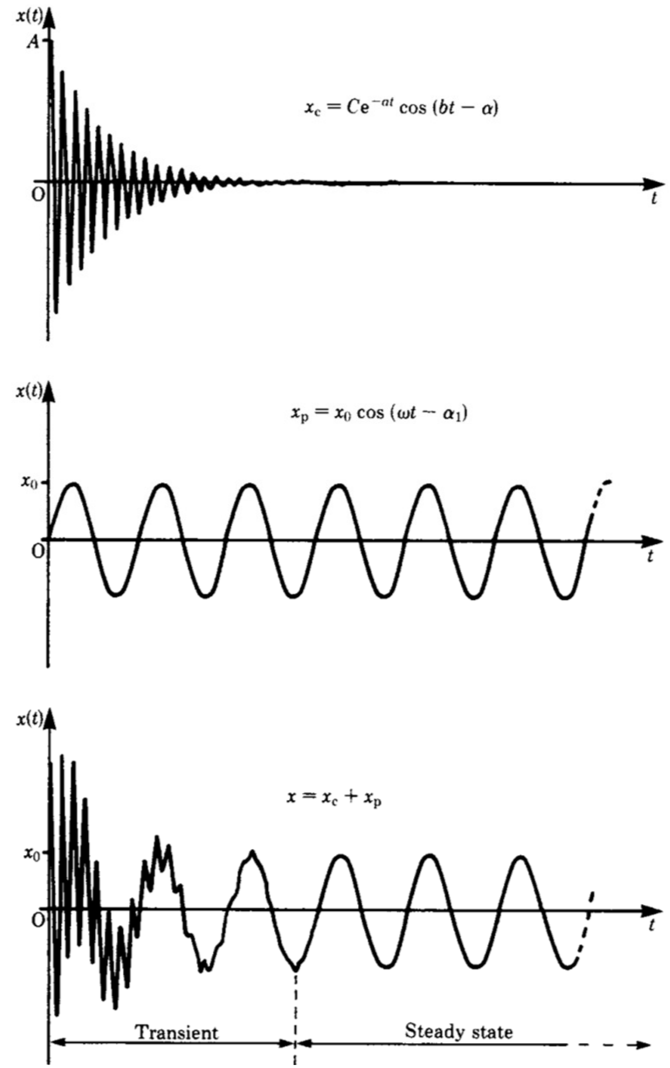
particular solution

$$m\ddot{x}_p + c\dot{x}_p + kx_p = F_0 \cos \omega t$$



Trial solution: $x_p = x_0 \cos(\omega t - \alpha_1)$

oscillates with the same frequency as the external force



Substituting in the DE and comparing coefficients, we find

$$x_0 = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}$$

Check!!

and $$\tan \alpha_1 = \frac{\omega c}{k - m\omega^2}$$

The general solution is

$$x = x_c + x_p, \quad x_c = \text{complementary function}$$

i.e. $$x = x_c + \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \cos(\omega t - \alpha_1)$$

If there is damping, the complementary function die out after some time and the motion is given by

$$x = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \cos(\omega t - \alpha_1)$$

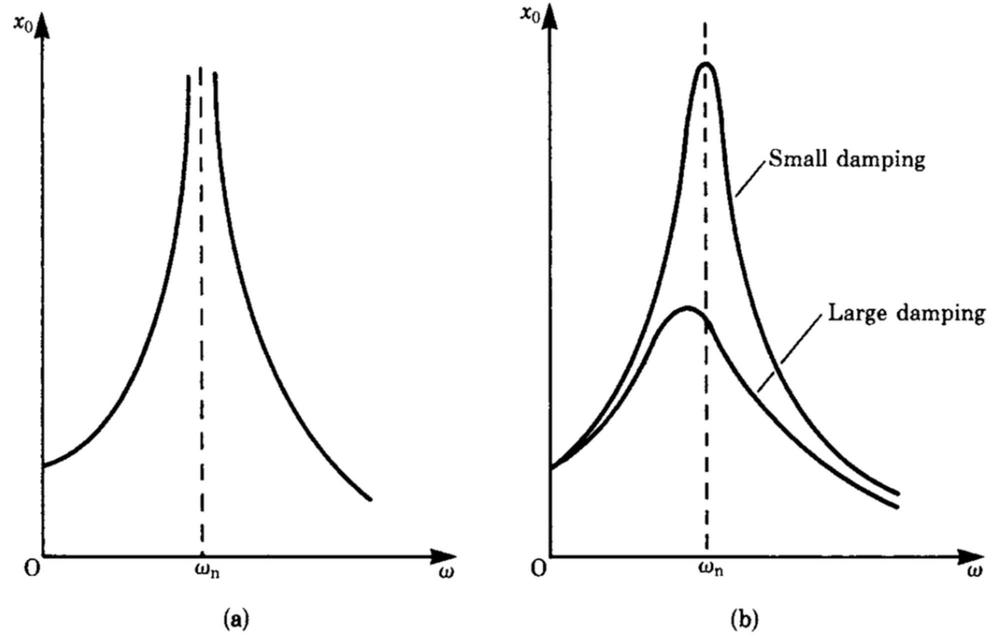
The amplitude is

$$x_0 = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}$$

and peaks at

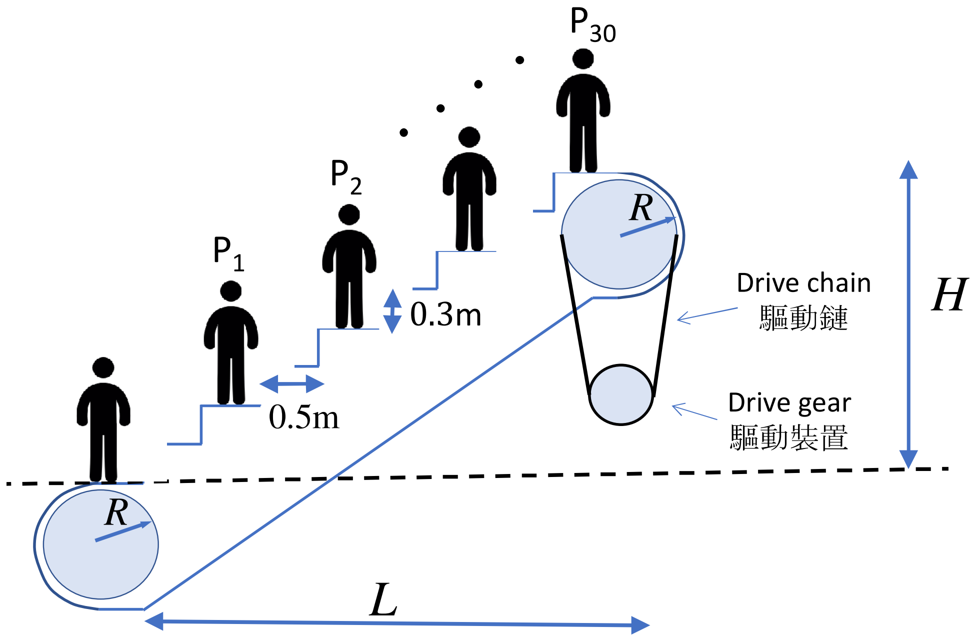
$$\omega = \omega_d = \sqrt{\left(\frac{k}{m}\right)^2 - \frac{c^2}{2m^2}}$$

which is the damped natural frequency of the system.



Escalator Breakdown (HKPO 2017)

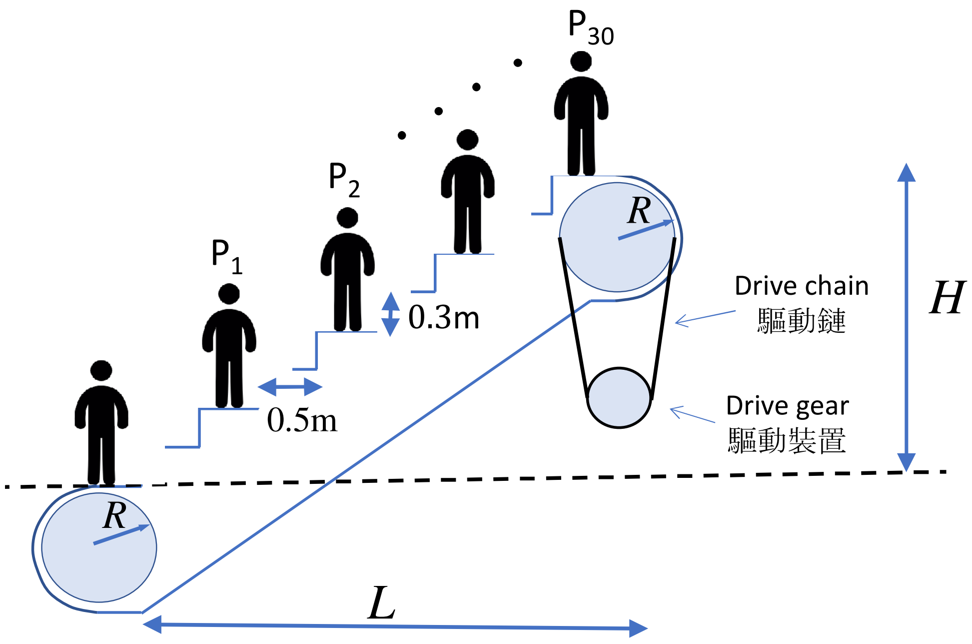
An escalator has a horizontal length $L = 30$ m and a height $H = 18$ m. It transports passengers upward through its length in 90 s. Each step on the escalator is 0.5 m deep and 0.3 m high. There are 60 steps on the escalator.



(a) Assume that there is one passenger with mass $m = 70$ kg standing on alternating steps of the escalator (i.e. there are 30 passengers on the escalator at any time). What is the minimum power of the drive gear to keep the escalator moving in the steady speed?

The vertical velocity = v_y = 18m/90s
Number of passengers = n = 30
The power required = the power to transport one passenger × (30) = $nmgv_y = 4116$ W

(B) Suppose the drive chain is suddenly broken and all the braking devices malfunction. The escalator reverses direction, and sends passengers careening downward with an acceleration. Eventually, all passengers will hit on the ground.



Assume both wheels have mass $M = 7{,}000$ kg, radius $R = 1$ m, and there is one passenger on alternating steps. Initially, the passengers are standing as shown in the figure when the drive chain is broken. What is the velocity v_1 of the passengers when the first passenger (P_1) hits on the ground?

(Hint: The rotational kinetic energy of a wheel is $\frac{1}{4}MR^2\omega^2$ where ω is the angular velocity of the wheel.)

The vertical velocity $v_y = 18\text{m}/90\text{s}$
The horizontal velocity $v_x = 30\text{m}/90\text{s}$
The velocity $v_0 = \sqrt{v_x^2 + v_y^2} = 0.39\ \text{m s}^{-1}$

Initially, the passengers are moving upward with an initial velocity $v_0 = 0.4\ \text{m s}^{-1}$ and the wheels rotate with an angular velocity $\omega_0 = v_0/R = 0.4$ rad s^{-1}.

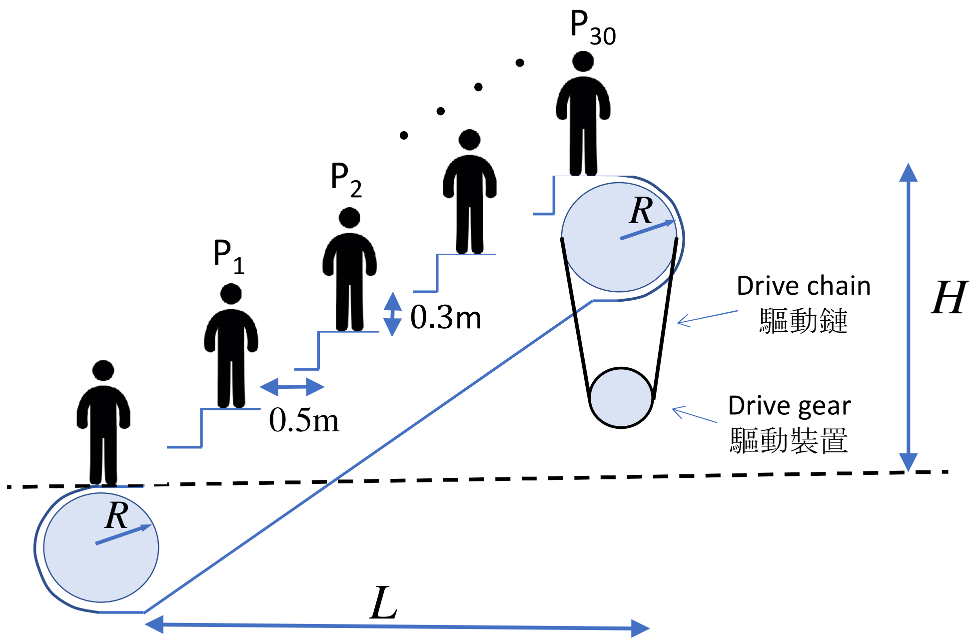
The initial energy is $E_0 = 2 \times \frac{1}{4} MR^2 \left(\frac{v_0}{R}\right)^2 + 30 \times \frac{1}{2} m v_0^2$ =692 J

When the first passenger hits on the ground, the total energy becomes

$$E_1 = 2 \times \frac{1}{4} MR^2 \left(\frac{v_1}{R}\right)^2 + 30 \times \frac{1}{2} m v_1^2 - 30mg(0.6) = E_0$$

By the conservation of energy, we have $v_1 = 1.693\ \text{m s}^{-1}$.

(c) What is the velocity v_2 of the second passenger (P_2) before he/she hits the ground? Assume that the first passenger has left the location.



The total energy after the first person hits on the ground

$$E_2 = 2 \times \frac{1}{4} MR^2 \left(\frac{v_1}{R}\right)^2 + 29 \times \frac{1}{2} m v_1^2 = 12941\,J$$

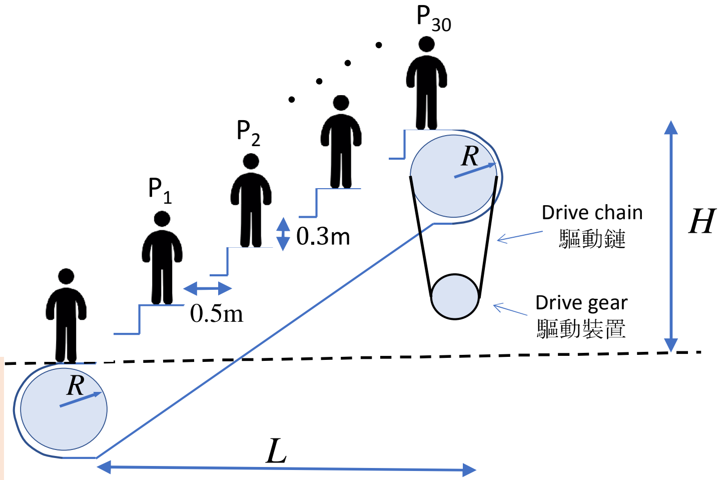
When the second person hits on the ground, the total energy becomes

$$E_3 = 2 \times \frac{1}{4} MR^2 \left(\frac{v_2}{R}\right)^2 + 29 \times \frac{1}{2} m v_2^2 - 29mg(0.6)$$

Hence, we have $v_2 = 2.347 \text{ m s}^{-1}$.

(d) What is the velocity of the last passenger (P_{30}) before he/she hits the ground? Assume that all passengers hitting the ground earlier have left the location.

(Hint: A useful approximation is $\frac{1}{A+1}+\frac{1}{A+2}+\cdots+\frac{1}{A+N}\approx \ln\left(1+\frac{N}{A+1/2}\right)$.)



In general, we have the recursive formula

$$v_1^2 = v_0^2 + \frac{2(n)mgh}{M+m(n)} = v_0^2 + 2gh\left(1-\frac{M/m}{M/m+30}\right)$$

$$v_2^2 = v_1^2 + \frac{2(n-1)mgh}{M+m(n-1)} = v_1^2 + 2gh\left(1-\frac{M/m}{M/m+29}\right)$$

$$\cdots\cdots\cdots$$

$$v_{30}^2 = v_{29}^2 + \frac{2(n-29)mgh}{M+m(n-29)} = v_{29}^2 + 2gh\left(1-\frac{M/m}{M/m+1}\right)$$

Adding the equations,

$$v_{30}^2 = v_0^2 + 2gh\left[30-\frac{M}{m}\left(\frac{1}{\frac{M}{m}+30}+\cdots\frac{1}{\frac{M}{m}+1}\right)\right]$$

$$\approx v_0^2 + 2gh\left[30-100\ln\left(1+\frac{30}{100.5}\right)\right]$$

$$v_{30}^2 = v_0^2 + 2gh(3.878)$$

$$v_{30} = 6.76\ \mathrm{m\ s^{-1}}$$

In contrast to the free fall, $mg(18) = 0.5mv^2 \to v = 18.78\ \mathrm{m\ s^{-1}}$, the velocity is lower.