Coupled oscillators

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Example:

Two Coupled Harmonic Oscillators

Consider a system of two objects of mass M. The two objects are attached to two springs with spring constants κ (see Figure 1). The interaction force between the masses is represented by a third spring with spring constant κ_{12}, which connects the two masses.

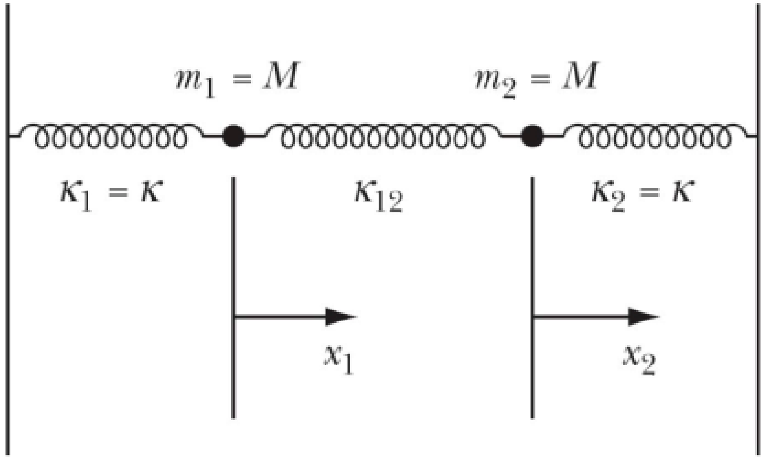


Figure 1. Two coupled harmonic oscillators.

We will assume that when the masses are in their equilibrium position, the springs are also in their equilibrium positions. The force on the left mass is equal to

$$F_1 = -\kappa x_1 + \kappa_{12}\left(x_2 - x_1\right) = -\left(\kappa + \kappa_{12}\right)x_1 + \kappa_{12}x_2 = M\ddot{x}_1$$

The force on the right mass is equal to

$$F_2 = -\kappa x_2 + \kappa_{12}\left(x_1 - x_2\right) = -\left(\kappa + \kappa_{12}\right)x_2 + \kappa_{12}x_1 = M\ddot{x}_2$$

The equation of motion are:

$$M\ddot{x}_1 + (\kappa + \kappa_{12})x_1 - \kappa_{12}x_2 = 0$$

$$M\ddot{x}_2 + (\kappa + \kappa_{12})x_2 - \kappa_{12}x_1 = 0$$

Coupled 2nd Differential equations!!

Since it is reasonable to assume that the resulting motion has an oscillatory behavior, we consider following trial functions:

$$\boxed{\begin{aligned} x_1(t) &= B_1 e^{i\omega t} \\ x_2(t) &= B_2 e^{i\omega t} \end{aligned}}$$

Substituting these trial functions into the equations of motion we obtain the following conditions:

$$(\kappa + \kappa_{12} - M\omega^2)B_1 - \kappa_{12}B_2 = 0$$

$$-\kappa_{12}B_1 + (\kappa + \kappa_{12} - M\omega^2)B_2 = 0$$

Consider two simultaneous equations

$$Ax + By = 0$$
$$Cx + Dy = 0$$

where A,B,C,D are constants. We want to find the solutions (x,y) which satisfy two equations simultaneous.
From eqtn. 1 and 2, we have

$$x = -\frac{B}{A}y$$
$$x = -\frac{D}{C}y$$

For the solution which satisfy two equations, it requires

$$\frac{B}{A} = \frac{D}{C} \rightarrow \boxed{AD - BC = 0}$$

It means that if $AD - BC \neq 0$, we can't find a solution which satisfies two equations simultaneous (except x=y=0).

$$\boxed{\text{Determinant} = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = AD - BC}$$

These equations only will have a non-trivial solution if

$$\begin{vmatrix} \kappa + \kappa_{12} - M\omega^2 & -\kappa_{12} \\ -\kappa_{12} & \kappa + \kappa_{12} - M\omega^2 \end{vmatrix} = 0$$

Note: the trivial solution is $B_1 = B_2 = 0$. The requirement for a non-trivial solution requires that the angular frequency of the system is equal to one of the following two characteristic frequencies (the so called eigen frequencies):

$$\omega_1 = \pm\sqrt{\frac{\kappa + 2\kappa_{12}}{M}}$$

$$\omega_2 = \pm\sqrt{\frac{\kappa}{M}}$$

For each of these frequencies, we can now determine the amplitudes B_1 and B_2. Let us first consider the eigen frequency ω_1. For this frequency we obtain the following relations between B_1 and B_2:

$$\left(\kappa+\kappa_{12}-\left(\kappa+2\kappa_{12}\right)\right)B_1-\kappa_{12}B_2=-\kappa_{12}B_1-\kappa_{12}B_2=-\kappa_{12}\left(B_1+B_2\right)=0$$

or $B_1 = -B_2$. For the eigen frequency ω_2 we obtain the following relations between B_1 and B_2:

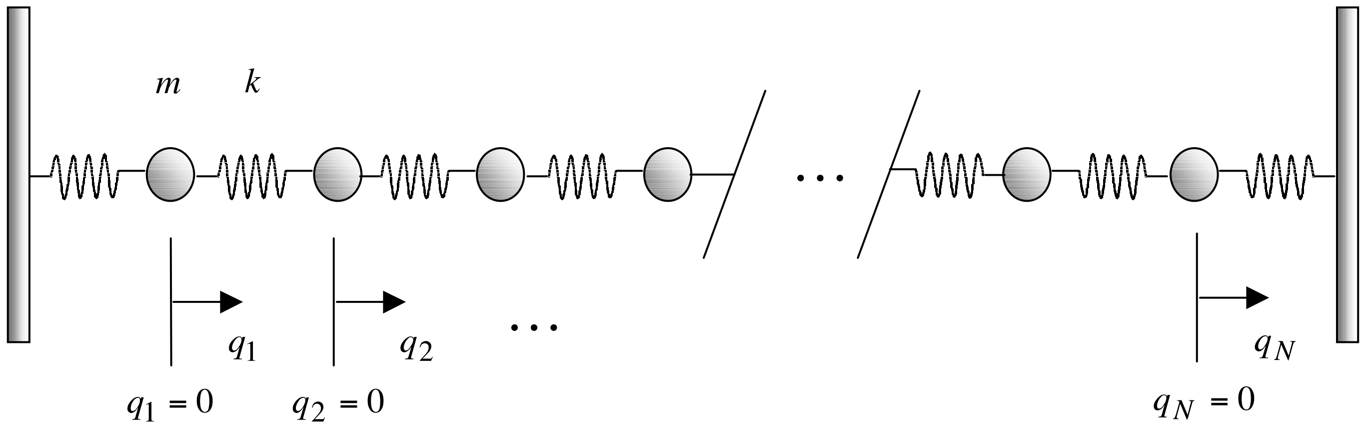
$$\left(\kappa+\kappa_{12}-\kappa\right)B_1-\kappa_{12}B_2=\kappa_{12}B_1-\kappa_{12}B_2=\kappa_{12}\left(B_1-B_2\right)=0$$

or $B_1 = B_2$. The most general solution of the coupled harmonic oscillator problem is thus

$$\boxed{\begin{aligned} x_1(t) &= B_1^{+}e^{+i\omega_1 t}+B_1^{-}e^{-i\omega_1 t}+B_2^{+}e^{+i\omega_2 t}+B_2^{-}e^{-i\omega_2 t} \\ x_2(t) &= -B_1^{+}e^{+i\omega_1 t}-B_1^{-}e^{-i\omega_1 t}+B_2^{+}e^{+i\omega_2 t}+B_2^{-}e^{-i\omega_2 t} \end{aligned}}$$

How to specify the values of 4 constants?

N-coupled oscillators



All the masses and the elastic spring are identical

Equation of motion (EOM): $(\omega^2 = k/m)$

$$\frac{d^2 q_j}{dt^2} - \omega^2 (q_{j+1} - 2q_j + q_{j-1}) = 0, \quad j = 1, 2, \ldots, N$$

$$q_0 = q_{N+1} = 0$$

N coupled 2nd linear homogeneous differential equations

1 Guess: $q_j(t) = a\sin(j\phi)e^{i\Omega t}$ $\quad a, \phi, \Omega$ are 3 undetermined constants

EOM implies:

$$-\Omega^2 a\sin(j\phi) = \omega^2\Big\{a\sin[(j-1)\phi] - 2a\sin[j\phi] + a\sin[(j+1)\phi]\Big\}.$$

Simplify:

$$\Omega^2\sin(j\phi) = 2\omega^2[1-\cos(\phi)]\sin(j\phi). \qquad \text{OR} \qquad \boxed{\Omega = 2\omega|\sin(\phi/2)|.}$$

boundary conditions

2 $q_0(t) = q_{N+1}(t) = 0 \rightarrow \sin((N+1)\phi) = 0 \rightarrow (N+1)\phi = n\pi$ $\quad$ for any integer n

3 We can obtain N different values of Ω_n

$$\boxed{\Omega_n = 2\omega\left|\sin\left(\frac{n\pi}{2N+2}\right)\right|, \quad n = 1, 2, \ldots, N.}$$

Normal mode

The general solution of the oscillators are:

$$q_j(t) = \sum_{n=1}^{N} a_n \overbrace{\sin\left(\frac{n\pi j}{N+1}\right)} \cos(\Omega_n t + \alpha_n), \quad j = 1, 2, \ldots, N$$

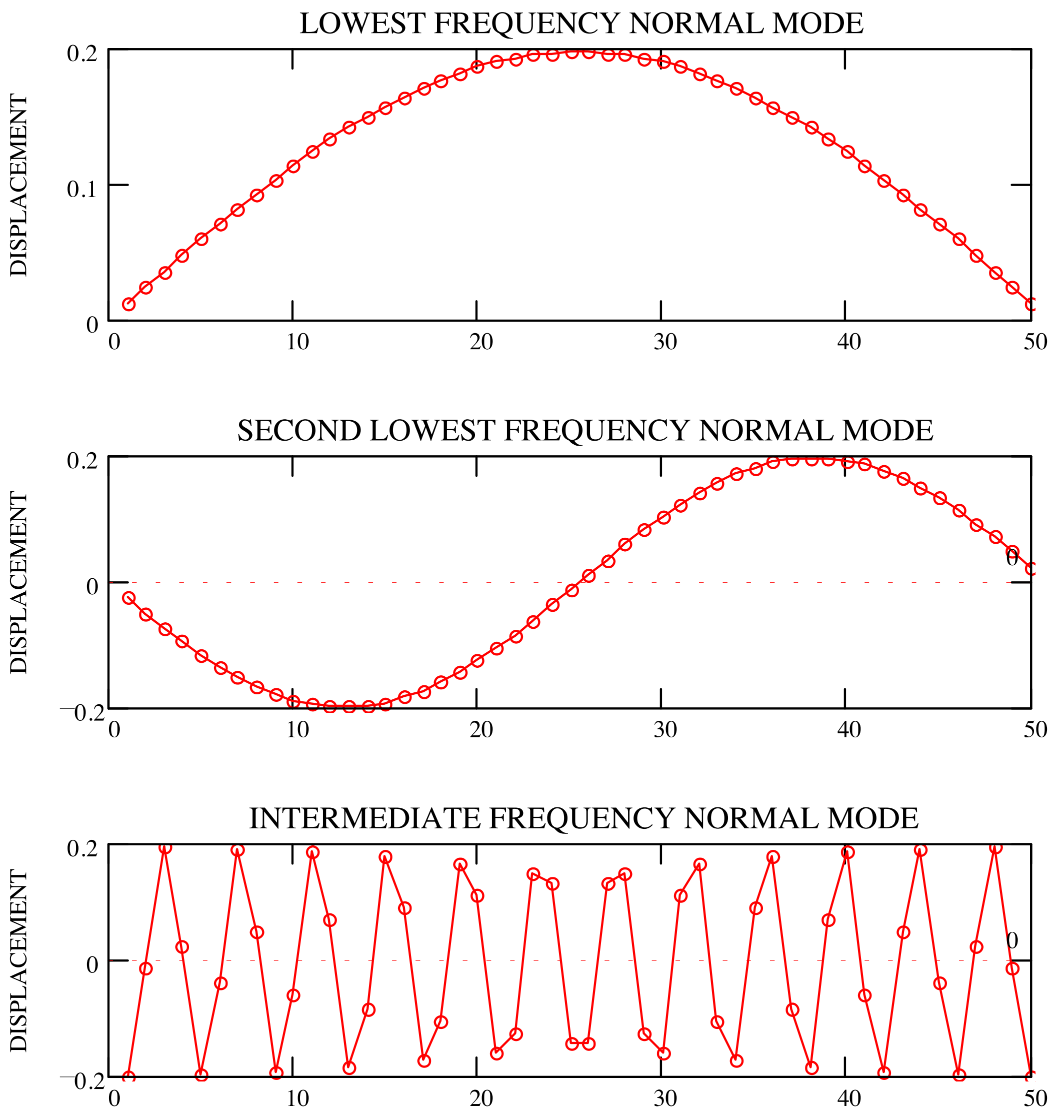
where

$$\Omega_n = 2\omega \left| \sin\left(\frac{n\pi}{2(N+1)}\right) \right|$$

There are 2N undetermined constants $\{a_n, \alpha_n\}$!!

Fit by the 2N initial conditions (position and velocity) of N oscillators!!

Normal modes for 50 coupled oscillators (N=50)



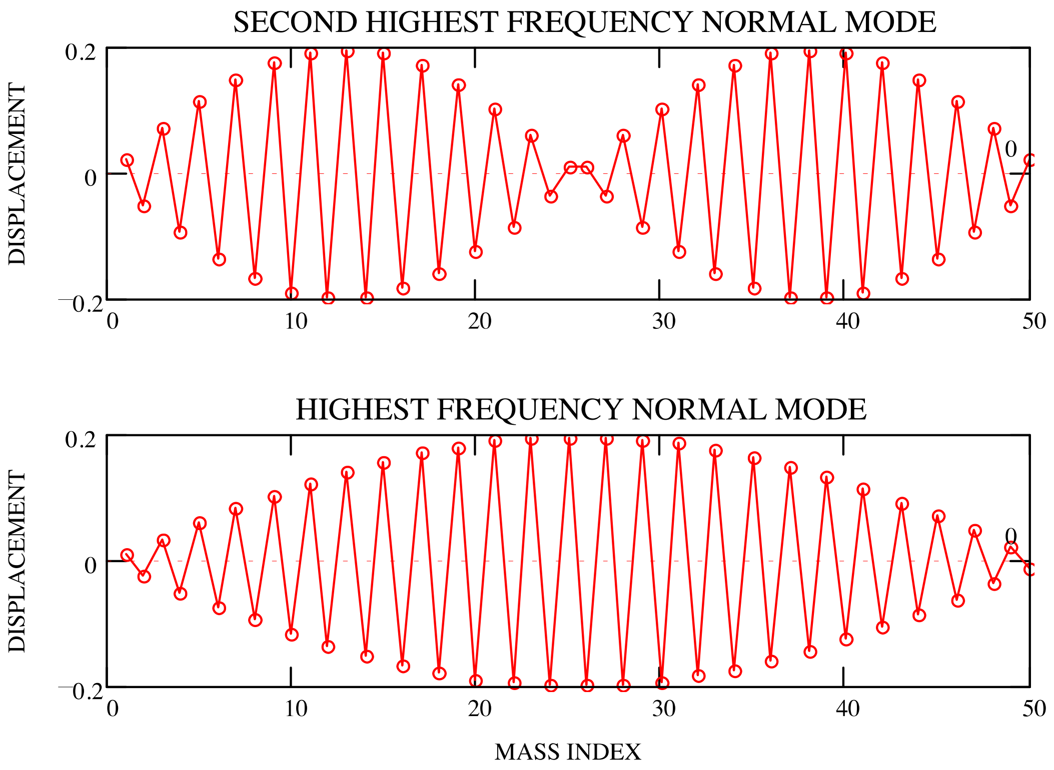


Figure 8. Selected normal modes for an $N = 50$ linear chain of coupled oscillators.

Some exercises in mechanics

Problem 1
A sprinter running a 100 meter race starts at rest, accelerates at constant acceleration with magnitude **A** for 2 seconds, and then runs at constant speed until the end.

a) Find the position (relative to the start position) and speed of the runner at the end of the 2 second in terms of **A.**

b) Assume that the runner takes a total of 10 seconds to run the 100 meters. Find the value of the acceleration **A**.

Problem 1

a)

$$x = \frac{1}{2}At^2 = \frac{1}{2}A(2)^2 = 2A$$

$$V_x = At = 2A$$

b)

8 seconds to run $100 - 2Am$ at speed $2A$.

$$8(2A) = 100 - 2A \quad 18A = 100 \quad A = \frac{100}{18}$$

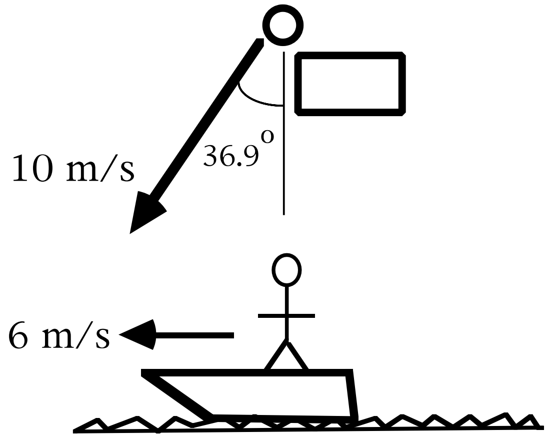
$$x = 2A + 2At$$

$$x = 100, \text{ at } t = 8$$

$$100 = 2A + 2A(8)$$

$$A = \frac{100}{18}$$

Problem 3



A rock is thrown downward from a bridge at an initial speed of 10 m/s and an angle of 36.9^{0} from the vertical as shown. At the same instant a boat is passing under the bridge traveling 6 m/s in the direction shown. See note on formula sheet about the values of trigonometric functions for this angle.

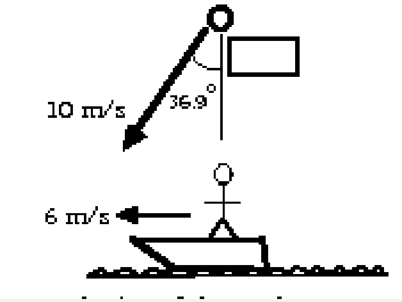
a) Find the vertical and horizontal components of the initial velocity of the rock as seen by a person on the bridge. Clearly indicate on your drawing your choice of axes.

b) Find the vertical and horizontal components of the initial velocity of the rock as seen by the person on the boat. Clearly indicate on your drawing your choice of axes.

c) Draw a clear vector diagram showing how to relate the velocity the rock appears to be moving as seen from the bridge, the velocity the rock appears to be moving as seen by the person in the boat, and the velocity of the boat with respect to the bridge.

Problem 3

a)

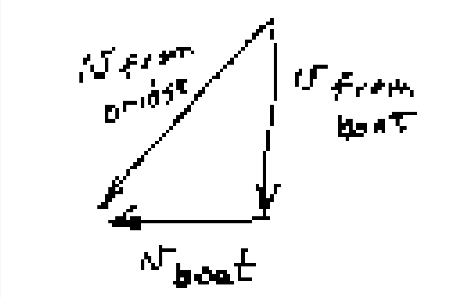


$$v_x = 10sin(36.9°) = 6 \quad m/s$$
$$v_y = -10cos(36.9°) = -8 \quad m/s$$

b)

$$v_x = 6 - 6 = 0m/s$$
$$v_y = -8 - 0 = -8m/s$$

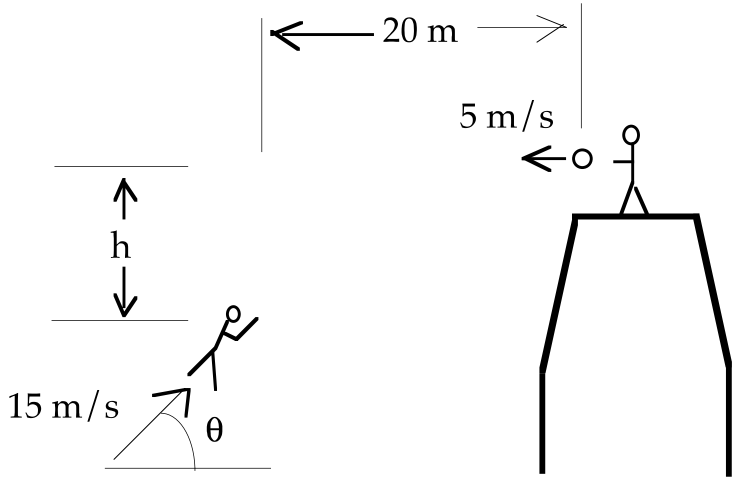
c)



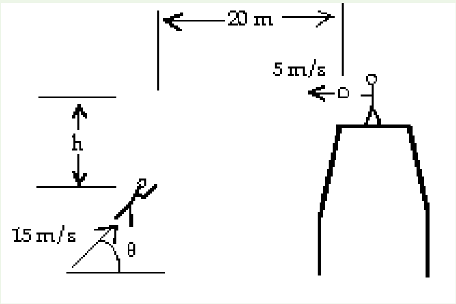
Problem 11

A circus acrobat is launched by a catapult at a speed of 15 $^{m}/_{s}$ at an angle of $\theta = 40°$ above the horizontal as shown. At a distance of 20 *m* away, her partner is standing on a platform at a height of *h* meters. At the instant that the acrobat is launched, her partner throws a basketball towards her horizontally at a speed of 5 $^{m}/_{s}$. Ignore air resistance in solving this problem.

a) Write equations for the horizontal and vertical positions as functions of time for both the acrobat and the basketball. Be consistent in your choice of origin.
b) When will the performer and basketball be at the same horizontal position?
c) Find the value of *h* for which the acrobat will catch the ball. Assume that she and the ball must be at the same height for her to catch it.
d) Find the magnitude of the velocity of the ball relative to the acrobat at the instant that she catches it.



Problem 11



a)

Acrobat	Basketball
$x = 0 + 15cos(\theta)t + 0$	$x = 20 - 5t$
$x = 11.5t$	$y = h - \frac{1}{2}gt^2 = h - 4.9t^2$
$y = 0 + 15sin(\theta)t - \frac{1}{2}gt^2$	
$y = 9.6t - 4.9t^2$	

b) $x_1 = x_2$ at $11.5t = 20 - 5t \Rightarrow t = 1.21s.$

c) Acrobat: $y = 4.44m.$

Basketball: $y = h - 4.9t^2 = h - 7.20 = 4.44m.$

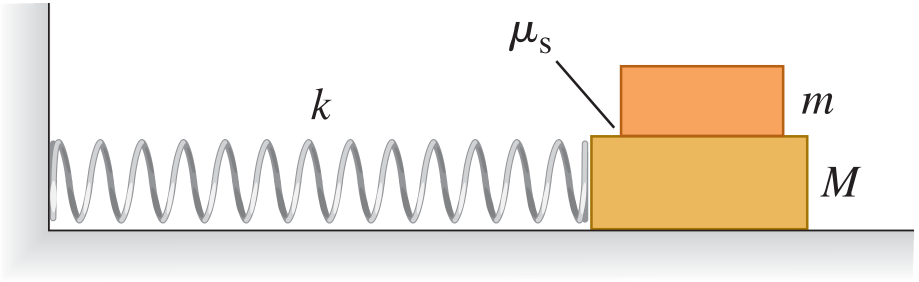
$h = 11.6m.$

d)

Acrobat	Basketball	Relative
$v_x = 11.5m/s$	$v_x = -5m/s$	$v_x = -5(-11.5) = -16.5m/s$
$v_y = -2.26m/s$	$v_y = -11.9m/s$	$v_y = -11.9 - (-2.26) = -9.6m/s$
		Mag $= 19.1m/s$

14.72 •• **CP** A block with mass M rests on a frictionless surface and is connected to a horizontal spring of force constant k. The other end of the spring is attached to a wall (Fig. P14.72). A second block with mass m rests on top of the first block. The coefficient of static friction between the blocks is μ_s. Find the *maximum* amplitude of oscillation such that the top block will not slip on the bottom block.

Figure **P14.72**



14.72. **IDENTIFY:** In SHM, $a_{\max} = \dfrac{k}{m_{\text{tot}}} A.$ Apply $\sum \vec{F} = m\vec{a}$ to the top block.

SET UP: The maximum acceleration of the lower block can't exceed the maximum acceleration that can be given to the other block by the friction force.

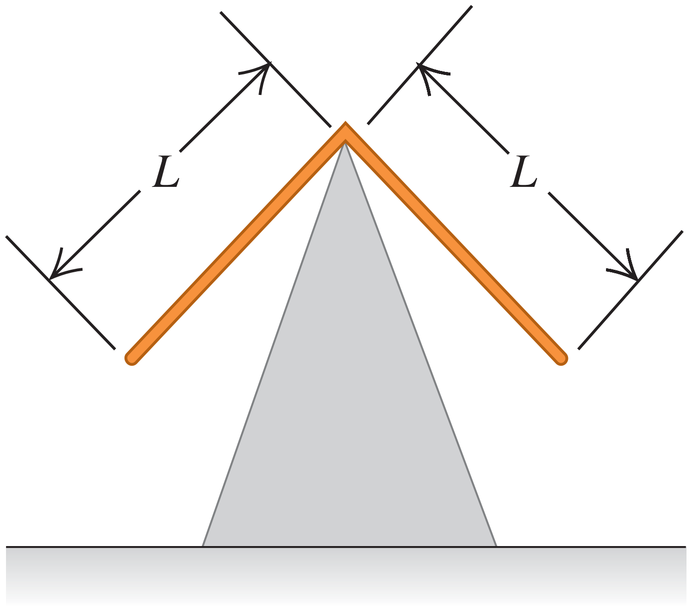
EXECUTE: For block m, the maximum friction force is $f_{\text{s}} = \mu_{\text{s}} n = \mu_{\text{s}} mg.$ $\sum F_x = ma_x$ gives $\mu_{\text{s}} mg = ma$ and $a = \mu_{\text{s}} g.$ Then treat both blocks together and consider their simple harmonic motion.

$a_{\max} = \left(\dfrac{k}{M+m}\right) A.$ Set $a_{\max} = a$ and solve for A: $\mu_{\text{s}} g = \left(\dfrac{k}{M+m}\right) A$ and $A = \dfrac{\mu_{\text{s}} g (M+m)}{k}.$

EVALUATE: If A is larger than this the spring gives the block with mass M a larger acceleration than friction can give the other block, and the first block accelerates out from underneath the other block.

14.99 ••• Two identical thin rods, each with mass m and length L, are joined at right angles to form an L-shaped object. This object is balanced on top of a sharp edge (Fig. P14.99). If the L-shaped object is deflected slightly, it oscillates. Find the frequency of oscillation.

Figure **P14.99**



14.99. **IDENTIFY**: The object oscillates as a physical pendulum, with $f = \frac{1}{2\pi}\sqrt{\frac{mgd}{I}}$, where m is the total mass of the object.

SET UP: The moment of inertia about the pivot is $2(1/3)ML^2 = (2/3)\,ML^2$, and the center of gravity when balanced is a distance $d = L/(2\sqrt{2})$ below the pivot.

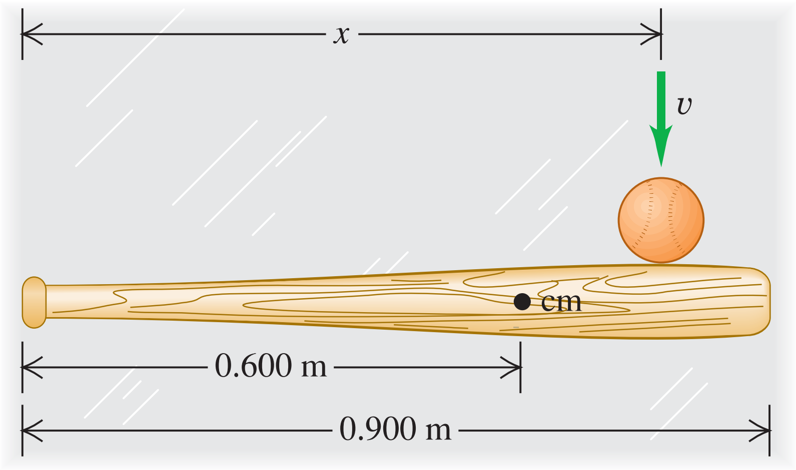
EXECUTE: The frequency is $f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{6g}{4\sqrt{2}L}} = \frac{1}{4\pi}\sqrt{\frac{6g}{\sqrt{2}L}}$.

EVALUATE: If $f_{sp} = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$ is the frequency for a simple pendulum of length L,

$$f = \frac{1}{2}\sqrt{\frac{6}{\sqrt{2}}}f_{sp} = 1.03 f_{sp}.$$

10.99 •• **Center of Percussion.** A baseball bat rests on a frictionless, horizontal surface. The bat has a length of 0.900 m, a mass of 0.800 kg, and its center of mass is 0.600 m from the handle end of the bat (Fig. P10.99). The moment of inertia of the bat about its center of mass is $0.0530\ \text{kg} \cdot \text{m}^2$. The bat is struck by a baseball traveling perpendicular to the bat. The impact applies an impulse $J = \int_{t_1}^{t_2} F\,dt$ at a point a distance x from the handle end of the bat. What must x be so that the handle end of the bat remains at rest as the bat begins to move? [*Hint:* Consider the motion of the center of mass and the rotation about the center of mass. Find x so that these two motions combine to give $v = 0$ for the end of the bat just after the collision. Also, note that integration of Eq. (10.29) gives $\Delta L = \int_{t_1}^{t_2} (\sum \tau)\,dt$ (see Problem 10.92).] The point on the bat you have located is called the *center of percussion.* Hitting a pitched ball at the center of percussion of the bat minimizes the "sting" the batter experiences on the hands.

Figure **P10.99**



10.99. **IDENTIFY:** Follow the method outlined in the hint.

SET UP: $J = m\Delta v_{cm}$. $\Delta L = J(x - x_{cm})$.

EXECUTE: The velocity of the center of mass will change by $\Delta v_{cm} = J/m$ and the angular velocity will change by $\Delta\omega = \dfrac{J(x - x_{cm})}{I}$. The change in velocity of the end of the bat will then be $\Delta v_{end} = \Delta v_{cm} - \Delta\omega x_{cm} = \dfrac{J}{m} - \dfrac{J(x - x_{cm})x_{cm}}{I}$. Setting $\Delta v_{end} = 0$ allows cancellation of J and gives $I = (x - x_{cm})x_{cm}m$, which when solved for x is $x = \dfrac{I}{x_{cm}m} + x_{cm} = \dfrac{(5.30\times10^{-2}\,\text{kg}\cdot\text{m}^2)}{(0.600\text{ m})(0.800\text{ kg})} + (0.600\text{ m}) = 0.710\text{ m}$.

EVALUATE: The center of percussion is farther from the handle than the center of mass.

10.100 ••• A uniform ball of radius R rolls without slipping between two rails such that the horizontal distance is d between the two contact points of the rails to the ball. (a) In a sketch, show that at any instant $v_{cm} = \omega\sqrt{R^2 - d^2/4}$. Discuss this expression in the limits $d = 0$ and $d = 2R$. (b) For a uniform ball starting from rest and descending a vertical distance h while rolling without slipping down a ramp, $v_{cm} = \sqrt{10gh/7}$. Replacing the ramp with the two rails, show that

$$v_{cm} = \sqrt{\frac{10gh}{5 + 2/(1 - d^2/4R^2)}}$$

In each case, the work done by friction has been ignored. (c) Which speed in part (b) is smaller? Why? Answer in terms of how the loss of potential energy is shared between the gain in translational and rotational kinetic energies. (d) For which value of the ratio d/R do the two expressions for the speed in part (b) differ by 5.0%? By 0.50%?

10.100. **IDENTIFY:** Apply conservation of energy to the motion of the ball.

SET UP: In relating $\frac{1}{2}mv_{\text{cm}}^2$ and $\frac{1}{2}I\omega^2$, instead of $v_{\text{cm}} = R\omega$ use the relation derived in part (a). $I = \frac{2}{5}mR^2$.

EXECUTE: **(a)** Consider the sketch in Figure 10.100.

The distance from the center of the ball to the midpoint of the line joining the points where the ball is in contact with the rails is $\sqrt{R^2 - (d/2)^2}$, so $v_{\text{cm}} = \omega\sqrt{R^2 - d^2/4}$. When $d = 0$, this reduces to $v_{\text{cm}} = \omega R$, the same as rolling on a flat surface. When $d = 2R$, the rolling radius approaches zero, and $v_{\text{cm}} \to 0$ for any ω.

(b) $K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}\left[mv_{\text{cm}}^2 + (2/5)mR^2\left(\frac{v_{\text{cm}}}{\sqrt{R^2 - (d^2/4)}}\right)^2\right] = \frac{mv_{\text{cm}}^2}{10}\left[5 + \frac{2}{(1 - d^2/4R^2)}\right]$

Setting this equal to mgh and solving for v_{cm} gives the desired result.

(c) The denominator in the square root in the expression for v_{cm} is larger than for the case $d = 0$, so v_{cm} is smaller. For a given speed, ω is larger than in the $d = 0$ case, so a larger fraction of the kinetic energy is rotational, and the translational kinetic energy, and hence v_{cm}, is smaller.

(d) Setting the expression in part (b) equal to 0.95 of that of the $d = 0$ case and solving for the ratio d/R gives $d/R = 1.05$. Setting the ratio equal to 0.995 gives $d/R = 0.37$.

EVALUATE: If we set $d = 0$ in the expression in part (b), $v_{\text{cm}} = \sqrt{\frac{10gh}{7}}$, the same as for a sphere rolling down a ramp. When $d \to 2R$, the expression gives $v_{\text{cm}} = 0$, as it should.

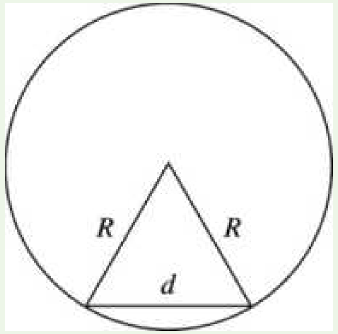


Figure 10.100

9.98 •• CALC Neutron Stars and Supernova Remnants. The Crab Nebula is a cloud of glowing gas about 10 light-years across, located about 6500 light-years from the earth (Fig. P9.98). It is the remnant of a star that underwent a *supernova explosion,* seen on earth in 1054 A.D. Energy is released by the Crab Nebula at a rate of about 5×10^{31} W, about 10^5 times the rate at which the sun radiates energy. The Crab Nebula obtains its energy from the rotational kinetic energy of a rapidly spinning *neutron star* at its center. This object rotates once every 0.0331 s, and this period is increasing by 4.22×10^{-13} s for each second of time that elapses. (a) If the rate at which energy is lost by the neutron star is equal to the rate at which energy is released by the nebula, find the moment of inertia of the neutron star. (b) Theories of supernovae predict that the neutron star in the Crab Nebula has a mass about 1.4 times that of the sun. Modeling the neutron star as a solid uniform sphere, calculate its radius in kilometers. (c) What is the linear speed of a point on the equator of the neutron star? Compare to the speed of light. (d) Assume that the neutron star is uniform and calculate its density. Compare to the density of ordinary rock (3000 kg/m^3) and to the density of an atomic nucleus (about 10^{17} kg/m^3). Justify the statement that a neutron star is essentially a large atomic nucleus.

Figure **P9.98**

9.98. **IDENTIFY:** Write K in terms of the period T and take derivatives of both sides of this equation to relate dK/dt to dT/dt.

SET UP: $\omega = \dfrac{2\pi}{T}$ and $K = \frac{1}{2}I\omega^2$. The speed of light is $c = 3.00\times10^8$ m/s.

EXECUTE: **(a)** $K = \dfrac{2\pi^2 I}{T^2}$. $\dfrac{dK}{dt} = -\dfrac{4\pi^2 I}{T^3}\dfrac{dT}{dt}$. The rate of energy loss is $\dfrac{4\pi^2 I}{T^3}\dfrac{dT}{dt}$. Solving for the moment of inertia I in terms of the power P,

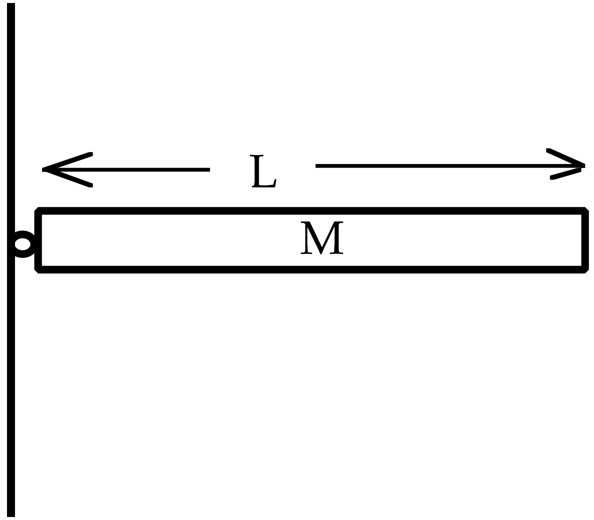
$$I = \frac{PT^3}{4\pi^2}\frac{1}{dT/dt} = \frac{(5\times10^{31}\ \text{W})(0.0331\ \text{s})^3}{4\pi^2}\frac{1\ \text{s}}{4.22\times10^{-13}\ \text{s}} = 1.09\times10^{38}\ \text{kg}\cdot\text{m}^2$$

(b) $R = \sqrt{\dfrac{5I}{2M}} = \sqrt{\dfrac{5(1.08\times10^{38}\ \text{kg}\cdot\text{m}^2)}{2(1.4)(1.99\times10^{30}\ \text{kg})}} = 9.9\times10^3$ m, about 10 km.

(c) $v = \dfrac{2\pi R}{T} = \dfrac{2\pi(9.9\times10^3\,\text{m})}{(0.0331\ \text{s})} = 1.9\times10^6\ \text{m/s} = 6.3\times10^{-3}c.$

Problem 8

A uniform bar of mass **M** and length **L** is attached to a wall by a frictionless hinge (i.e. there are no torques exerted by the hinge). The bar is released from rest in a horizontal position as shown. The bar has moments of inertia about its center and about its end of $I_{cm} = {ML^2}/{12}$ and $I_{end} = {ML^2}/{3}$, respectively.



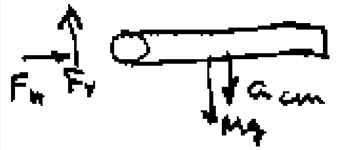
a) What is the angular acceleration of the bar at the instant that it is released?
b) At the instant after the bar is released, find the magnitude and direction of the force exerted on the bar by the hinge.
c) Use Work/Energy concepts to find the angular velocity of the bar when it has swung down to a vertical position and is just about the hit the wall.

Problem 8

a) $\tau = I\alpha,$ take torques about hinge.

$(Mg)(\frac{L}{2})(\sin(90°)) = (\frac{ML^2}{3})(\alpha) \Rightarrow \alpha = \frac{\frac{gL}{2}}{\frac{L^2}{3}} = \frac{3g}{2L},\ \boxed{\alpha = \frac{3g}{2L}}$

b)



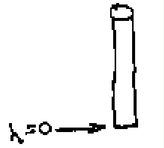
$F = Ma_{cm},\ a_{cm} = \alpha(\frac{L}{2})$, downward.

All forces and acceleration are vertical $\Rightarrow \boxed{F_H = 0}$.

$F_V - Mg = -Ma = -M\alpha(\frac{L}{2}) = -M(\frac{3g}{2L})(\frac{L}{2}) = \frac{-3Mg}{4}$

$F_V = Mg - \frac{3Mg}{4} \Rightarrow \boxed{F_V = \frac{Mg}{4}},\ \boxed{F_{TOT} = \frac{Mg}{4},\ up}.$

c) Used fixed pivot:



$KE_I = 0,\ PE_I = MgL,\ KE_F = \frac{1}{2}I_{end}\omega^2,\ PE_F = Mg(\frac{L}{2}),$ Work $= 0.$

$\frac{1}{2}(\frac{ML^2}{3})\omega^2 + Mg\frac{L}{2} = MgL,\ \frac{ML^2\omega^2}{6} = \frac{MgL}{2},\ \boxed{\omega^2 = \frac{3g}{L} \text{ or } \omega = \sqrt{\frac{3g}{L}}}.$

Used center of mass:

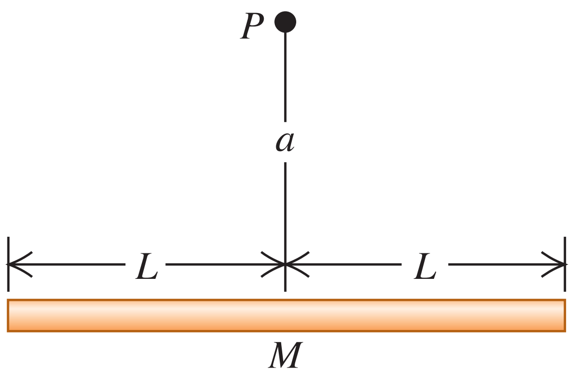
$KE_I = 0,\ KE_F = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2,\ v_{CM} = \omega(\frac{L}{2})$

$KE_F = \frac{1}{2}M(\frac{L}{2})^2\omega^2 + \frac{1}{2}(\frac{ML^2}{12})\omega^2 = ML^2\omega^2(\frac{1}{8} + \frac{1}{24}) = ML^2\omega^2(\frac{3}{24} + \frac{1}{24})$

$= ML^2\omega^2(\frac{1}{6}) \Rightarrow$ Same answer.

13.90 ••• **CALC** Mass M is distributed uniformly along a line of length $2L$. A particle with mass m is at a point that is a distance a above the center of the line on its perpendicular bisector (point P in Fig. P13.90). For the gravitational force that the line exerts on the particle, calculate the components perpendicular and parallel to the line. Does your result reduce to the correct expression as a becomes very large?

Figure **P13.90**



13.90. **IDENTIFY:** Divide the rod into infinitesimal segments. Calculate the force each segment exerts on m and integrate over the rod to find the total force.

SET UP: From symmetry, the component of the gravitational force parallel to the rod is zero. To find the perpendicular component, divide the rod into segments of length dx and mass $dm = dx\dfrac{M}{2L}$, positioned at a distance x from the center of the rod.

EXECUTE: The magnitude of the gravitational force from each segment is

$dF = \dfrac{Gm\,dM}{x^2+a^2} = \dfrac{GmM}{2L}\dfrac{dx}{x^2+a^2}$. The component of dF perpendicular to the rod is $dF\dfrac{a}{\sqrt{x^2+a^2}}$ and so the net gravitational force is $F = \displaystyle\int_{-L}^{L} dF = \dfrac{GmMa}{2L}\int_{-L}^{L}\dfrac{dx}{(x^2+a^2)^{3/2}}$.

The integral can be found in a table, or found by making the substitution $x = a\tan\theta$. Then,

$dx = a\sec^2\theta\,d\theta, (x^2+a^2) = a^2\sec^2\theta$, and so

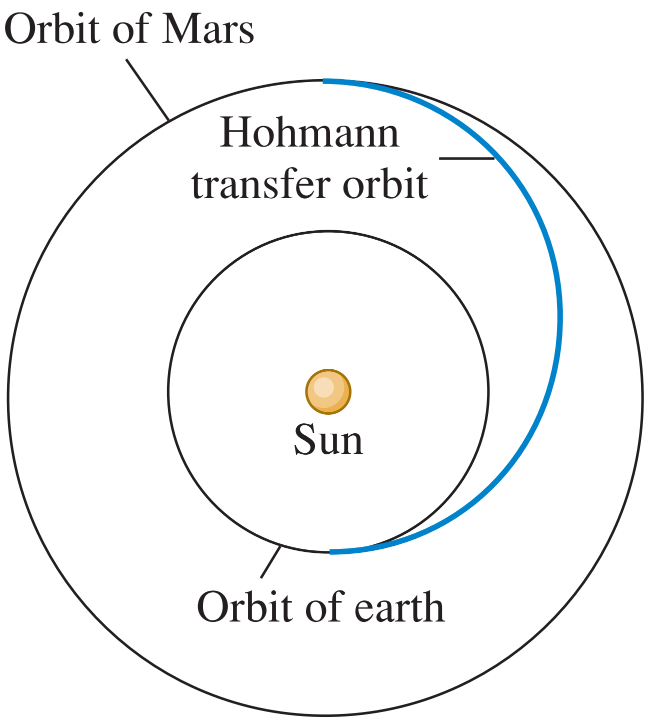
$$\int\frac{dx}{(x^2+a^2)^{3/2}} = \int\frac{a\sec^2\theta\,d\theta}{a^3\sec^3\theta} = \frac{1}{a^2}\int\cos\theta\,d\theta = \frac{1}{a^2}\sin\theta = \frac{x}{a^2\sqrt{x^2+a^2}},$$

and the definite integral is $F = \dfrac{GmM}{a\sqrt{a^2+L^2}}$.

EVALUATE: When $a \gg L$, the term in the square root approaches a^2 and $F \to \dfrac{GmM}{a^2}$, as expected.

13.87 ••• **Interplanetary Navigation.** The most efficient way to send a spacecraft from the earth to another planet is by using a *Hohmann transfer orbit* (Fig. P13.87). If the orbits of the departure and destination planets are circular, the Hohmann transfer orbit is an elliptical orbit whose perihelion and aphelion are tangent to the orbits of the two planets. The rockets are fired briefly at the departure planet to put the spacecraft into the transfer orbit; the spacecraft then coasts until it reaches the destination planet. The rockets are then fired again to put the spacecraft into the same orbit about the sun as the destination planet. (a) For a flight from earth to Mars, in what direction must the rockets be fired at the earth and at Mars: in the direction of motion, or opposite the direction of motion? What about for a flight from Mars to the earth? (b) How long does a one-way trip from the the earth to Mars take, between the firings of the rockets? (c) To reach Mars from the earth, the launch must be timed so that Mars will be at the right spot when the spacecraft reaches Mars's orbit around the sun. At launch, what must the angle between a sun–Mars line and a sun–earth line be? Use data from Appendix F.

Figure **P13.87**



13.87. IDENTIFY: Apply Eq. (13.19) to the transfer orbit.

SET UP: The orbit radius for earth is $r_E = 1.50\times10^{11}$ m and for Mars it is $r_M = 2.28\times10^{11}$ m. From Figure 13.18 in the textbook, $a = \frac{1}{2}(r_E + r_M)$.

EXECUTE: **(a)** To get from the circular orbit of the earth to the transfer orbit, the spacecraft's energy must increase, and the rockets are fired in the direction opposite that of the motion, that is, in the direction that increases the speed. Once at the orbit of Mars, the energy needs to be increased again, and so the rockets need to be fired in the direction opposite that of the motion. From Figure 13.18 in the textbook, the semimajor axis of the transfer orbit is the arithmetic average of the orbit radii of the earth and Mars, and so from Eq. (13.13), the energy of the spacecraft while in the transfer orbit is intermediate between the energies of the circular orbits. Returning from Mars to the earth, the procedure is reversed, and the rockets are fired against the direction of motion.

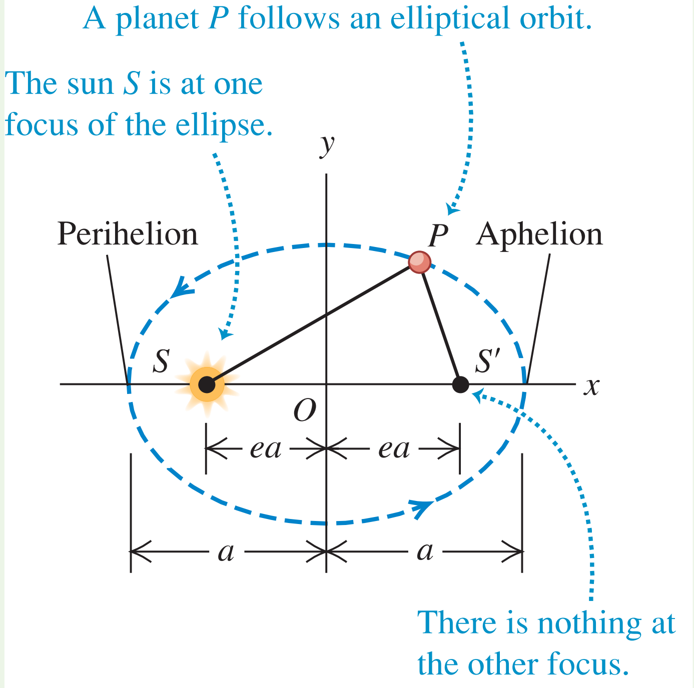
(b) The time will be half the period as given in Eq. (13.17), with the semimajor axis equal to $a = \frac{1}{2}(r_E + r_M) = 1.89\times10^{11}\ \text{m}$ so

$$t = \frac{T}{2} = \frac{\pi\,(1.89\times10^{11}\ \text{m})^{3/2}}{\sqrt{(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(1.99\times10^{30}\ \text{kg})}} = 2.24\times10^{7}\ \text{s} = 259\ \text{days},$$ which is more than $8\frac{1}{2}$ months.

(c) During this time, Mars will pass through an angle of $(360°)\dfrac{(2.24\times10^{7}\ \text{s})}{(687\ \text{d})(86{,}400\ \text{s/d})} = 135.9°$, and the spacecraft passes through an angle of $180°$, so the angle between the earth-sun line and the Mars-sun line must be $44.1°$.

EVALUATE: The period T for the transfer orbit is 526 days, the average of the orbital periods for earth and Mars.

13.18 Geometry of an ellipse. The sum of the distances SP and $S'P$ is the same for every point on the curve. The sizes of the sun (S) and planet (P) are exaggerated for clarity.



13.88 ••• CP Tidal Forces near a Black Hole. An astronaut inside a spacecraft, which protects her from harmful radiation, is orbiting a black hole at a distance of 120 km from its center. The black hole is 5.00 times the mass of the sun and has a Schwarzschild radius of 15.0 km. The astronaut is positioned inside the spaceship such that one of her 0.030-kg ears is 6.0 cm farther from the black hole than the center of mass of the spacecraft and the other ear is 6.0 cm closer. (a) What is the tension between her ears? Would the astronaut find it difficult to keep from being torn apart by the gravitational forces? (Since her whole body orbits with the same angular velocity, one ear is moving too slowly for the radius of its orbit and the other is moving too fast. Hence her head must exert forces on her ears to keep them in their orbits.) (b) Is the center of gravity of her head at the same point as the center of mass? Explain.

13.88. **IDENTIFY:** Apply $\Sigma\vec{F} = m\vec{a}$ to each ear.

SET UP: Denote the orbit radius as r and the distance from this radius to either ear as δ. Each ear, of mass m, can be modeled as subject to two forces, the gravitational force from the black hole and the tension force (actually the force from the body tissues), denoted by F.

EXECUTE: The force equation for either ear is $\dfrac{GMm}{(r+\delta)^2} - F = m\omega^2(r+\delta)$, where δ can be of either sign.

Replace the product $m\omega^2$ with the value for $\delta = 0$, $m\omega^2 = GMm/r^3$, and solve for F:

$$F = (GMm)\left[\frac{r+\delta}{r^3} - \frac{1}{(r+\delta)^2}\right] = \frac{GMm}{r^3}\left[r+\delta - r(1+(\delta/r)^{-2}\right].$$

Using the binomial theorem to expand the term in square brackets in powers of δ/r,

$$F \approx \frac{GMm}{r^3}[r+\delta - r(1-2(\delta/r))] = \frac{GMm}{r^3}(3\delta) = 2.1\text{ kN}.$$

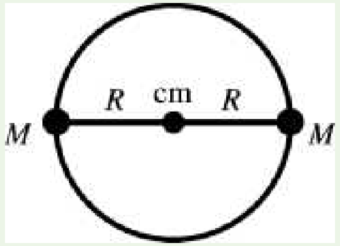
This tension is much larger than that which could be sustained by human tissue, and the astronaut is in trouble.

13.71 • Binary Star—Equal Masses. Two identical stars with mass M orbit around their center of mass. Each orbit is circular and has radius R, so that the two stars are always on opposite sides of the circle. (a) Find the gravitational force of one star on the other. (b) Find the orbital speed of each star and the period of the orbit. (c) How much energy would be required to separate the two stars to infinity?

13.72 •• CP Binary Star—Different Masses. Two stars, with masses M_1 and M_2, are in circular orbits around their center of mass. The star with mass M_1 has an orbit of radius R_1; the star with mass M_2 has an orbit of radius R_2. (a) Show that the ratio of the orbital radii of the two stars equals the reciprocal of the ratio of their masses—that is, $R_1/R_2 = M_2/M_1$. (b) Explain why the two stars have the same orbital period, and show that the period T is given by $T = 2\pi(R_1 + R_2)^{3/2}/\sqrt{G(M_1 + M_2)}$. (c) The two stars in a certain binary star system move in circular orbits. The first star, Alpha, has an orbital speed of 36.0 km/s. The second star, Beta, has an orbital speed of 12.0 km/s. The orbital period is 137 d. What are the masses of each of the two stars? (d) One of the best candidates for a black hole is found in the binary system called A0620-0090. The two objects in the binary system are an orange star, V616 Monocerotis, and a compact object believed to be a black hole (see Fig. 13.27). The orbital period of A0620-0090 is 7.75 hours, the mass of V616 Monocerotis is estimated to be 0.67 times the mass of the sun, and the mass of the black hole is estimated to be 3.8 times the mass of the sun. Assuming that the orbits are circular, find the radius of each object's orbit and the orbital speed of each object. Compare these answers to the orbital radius and orbital speed of the earth in its orbit around the sun.

13.71. **IDENTIFY:** Use Eq. (13.2) to calculate F_g. Apply Newton's second law to circular motion of each star to find the orbital speed and period. Apply the conservation of energy expression, Eq. (7.13), to calculate the energy input (work) required to separate the two stars to infinity.

(a) SET UP: The cm is midway between the two stars since they have equal masses. Let R be the orbit radius for each star, as sketched in Figure 13.71.



The two stars are separated by a distance $2R$, so

$$F_g = GM^2/(2R)^2 = GM^2/4R^2$$

Figure 13.71

(b) EXECUTE: $F_g = ma_{rad}$

$$GM^2/4R^2 = M(v^2/R) \text{ so } v = \sqrt{GM/4R}$$

And $T = 2\pi R/v = 2\pi R\sqrt{4R/GM} = 4\pi\sqrt{R^3/GM}$

(c) SET UP: Apply $K_1 + U_1 + W_{other} = K_2 + U_2$ to the system of the two stars. Separate to infinity implies $K_2 = 0$ and $U_2 = 0$.

EXECUTE: $K_1 = \frac{1}{2}Mv^2 + \frac{1}{2}Mv^2 = 2\left(\frac{1}{2}M\right)(GM/4R) = GM^2/4R$

$$U_1 = -GM^2/2R$$

Thus the energy required is $W_{other} = -(K_1 + U_1) = -(GM^2/4R - GM^2/2R) = GM^2/4R.$

EVALUATE: The closer the stars are and the greater their mass, the larger their orbital speed, the shorter their orbital period and the greater the energy required to separate them.

13.72. **IDENTIFY:** In the center of mass coordinate system, $r_{cm} = 0$. Apply $\vec{F} = m\vec{a}$ to each star, where F is the gravitational force of one star on the other and $a = a_{rad} = \dfrac{4\pi^2 R}{T^2}$.

SET UP: $v = \dfrac{2\pi R}{T}$ allows R to be calculated from v and T.

EXECUTE: **(a)** The radii R_1 and R_2 are measured with respect to the center of mass, and so $M_1R_1 = M_2R_2$, and $R_1/R_2 = M_2/M_1$.

(b) The forces on each star are equal in magnitude, so the product of the mass and the radial accelerations are equal: $\dfrac{4\pi^2 M_1R_1}{T_1^2} = \dfrac{4\pi^2 M_2R_2}{T_2^2}$. From the result of part (a), the numerators of these expressions are equal, and so the denominators are equal, and the periods are the same. To find the period in the symmetric form desired, there are many possible routes. An elegant method, using a bit of hindsight, is to use the above expressions to relate the periods to the force $F_g = \dfrac{GM_1M_2}{(R_1+R_2)^2}$, so that equivalent expressions for the period are $M_2T^2 = \dfrac{4\pi^2R_1(R_1+R_2)^2}{G}$ and $M_1T^2 = \dfrac{4\pi^2R_2(R_1+R_2)^2}{G}$. Adding the expressions gives $(M_1+M_2)T^2 = \dfrac{4\pi^2(R_1+R_2)^3}{G}$ or $T = \dfrac{2\pi(R_1+R_2)^{3/2}}{\sqrt{G(M_1+M_2)}}$.

(c) First we must find the radii of each orbit given the speed and period data. In a circular orbit, $v = \dfrac{2\pi R}{T}$, or $R = \dfrac{vT}{2\pi}$. Thus $R_\alpha = \dfrac{(36\times10^3 \text{ m/s})(137 \text{ d})(86{,}400 \text{ s/d})}{2\pi} = 6.78\times10^{10} \text{ m}$ and

$R_\beta = \dfrac{(12\times10^3 \text{ m/s})(137 \text{ d})(86{,}400 \text{ s/d})}{2\pi} = 2.26\times10^{10} \text{ m}$. Now find the sum of the masses.

$(M_\alpha + M_\beta) = \dfrac{4\pi^2(R_\alpha+R_\beta)^3}{T^2G}$. Inserting the values of T and the radii gives

$(M_\alpha + M_\beta) = \dfrac{4\pi^2(6.78\times10^{10}\text{m} + 2.26\times10^{10} \text{ m})^3}{[(137 \text{ d})(86{,}400 \text{ s/d})]^2(6.673\times10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2)} = 3.12\times10^{30} \text{ kg}$. Since

$M_\beta = M_\alpha R_\alpha/R_\beta = 3M_\alpha$, $4M_\alpha = 3.12\times10^{30}$ kg, or $M_\alpha = 7.80\times10^{29}$ kg, and $M_\beta = 2.34\times10^{30}$ kg.

(d) Let α refer to the star and β refer to the black hole. Use the relationships derived in parts (a) and (b):

$R_\beta = (M_\alpha/M_\beta)R_\alpha = (0.67/3.8)R_\alpha = (0.176)R_\alpha$, $R_\alpha + R_\beta = \sqrt[3]{\dfrac{(M_\alpha+M_\beta)T^2G}{4\pi^2}}$. For Monocerotis, inserting the values for M and T gives $R_\alpha = 1.9\times10^9$ m, $v_\alpha = 4.4\times10^2$ km/s and for the black hole $R_\beta = 34\times10^8$ m, $v_\beta = 77$ km/s.

EVALUATE: Since T is the same, v is smaller when R is smaller.