Partial differentiation

Table 12.2

Partial derivative	Rule	Example: $f(x,y,z) = 2x^3y + z^2$	
Partial derivative with respect to x: $\frac{\partial}{\partial x}$	Treat all variables as constants except for x	$\frac{\partial f}{\partial x} = 6x^2y$	(12.1a)
Partial derivative with respect to y: $\frac{\partial}{\partial y}$	Treat all variables as constants except for y	$\frac{\partial f}{\partial y} = 2x^3$	(12.1b)
Partial derivative with respect to z: $\frac{\partial}{\partial z}$	Treat all variables as constants except for z	$\frac{\partial f}{\partial z} = 2z$	(12.1c)

Wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

Example: Show that the following functions are solutions of the wave equation.

$$y(x,t) = \sin(kx - \omega t)$$

$$y(x,t) = e^{i(kx-\omega t)}$$

$$y(x,t) = (x - vt)^n$$

$$y(x,t) = (x + vt)^n$$

$$y(x,t) = \frac{1}{A + x - vt}$$

In fact, we can show that any functions with the form,

$$y(x,t) = f(x+vt) + g(x-vt)$$

for any differentiable functions f(u) and g(u) are solutions of the wave equation.

$$y(x, t) = A\cos\left[\omega\left(\frac{x}{v} - t\right)\right] = A\cos\left[2\pi f\left(\frac{x}{v} - t\right)\right] \quad \text{(sinusoidal wave moving in } +x\text{-direction)} \tag{15.3}$$

15.12 •• CALC Speed of Propagation vs. Particle Speed. (a) Show that Eq. (15.3) may be written as

$$y(x, t) = A\cos\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

(b) Use $y(x, t)$ to find an expression for the transverse velocity v_y of a particle in the string on which the wave travels. (c) Find the maximum speed of a particle of the string. Under what circumstances is this equal to the propagation speed v? Less than v? Greater than v?

15.12. **IDENTIFY:** $v_y = \frac{\partial y}{\partial t}$. $v = f\lambda = \lambda/T$.

SET UP: $\frac{\partial}{\partial t} A\cos\left(\frac{2\pi}{\lambda}(x - vt)\right) = +A\left(\frac{2\pi v}{\lambda}\right)\sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$

EXECUTE: **(a)** $A\cos 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right) = +A\cos\frac{2\pi}{\lambda}\left(x - \frac{\lambda}{T}t\right) = +A\cos\frac{2\pi}{\lambda}(x - vt)$ where $\frac{\lambda}{T} = \lambda f = v$ has been used.

(b) $v_y = \frac{\partial y}{\partial t} = \frac{2\pi v}{\lambda} A\sin\frac{2\pi}{\lambda}(x - vt)$.

(c) The speed is the greatest when the sine is 1, and that speed is $2\pi vA/\lambda$. This will be equal to v if $A = \lambda/2\pi$, less than v if $A < \lambda/2\pi$ and greater than v if $A > \lambda/2\pi$.

EVALUATE: The propagation speed applies to all points on the string. The transverse speed of a particle of the string depends on both x and t.

Example 3.2.

(1) Find the amplitude, frequency, wavelength, and speed of propagation of the wave described by the equation

$$y = 0.2 \cos [\pi(5t - 2x)].$$

Here, the units of length and time are taken to be meter and second, respectively.

(2) When a sinusoidal wave of amplitude 0.1 m and frequency 2 Hz travels at a speed of 2 m/s in the $-x$ direction, derive the expression for the displacement y at position x at time t by using an integer, n. Here, we assume that the displacement at the origin ($x = 0$) at time $t = 0$ is zero, i.e., $y = 0$.

Solution

(1) From Eq. (3.10), the amplitude is A = <u>0.2 m</u>, the period is T = <u>0.4 s</u>, and the wavelength is λ = <u>1.0 m</u>. From these values, the frequency, $f = 1/T$ = <u>2.5 Hz</u>, and the propagation speed of the wave, $V = f\lambda$ = <u>2.5 m/s</u>.

(2) Using A = 0.1 m, T = 0.5 s and $\lambda = VT$ = 1 m (because V = 2 m/s), we have

$$y(x,t) = \underline{0.1\cos\left\{4\pi\left(t+\frac{x}{2}\right)+\left(n+\frac{1}{2}\right)\pi\right\}}$$

$$= \underline{-0.1\sin\left\{4\pi\left(t+\frac{x}{2}+\frac{n\pi}{4}\right)\right\}}.$$

■

15.23 • A horizontal wire is stretched with a tension of 94.0 N, and the speed of transverse waves for the wire is 492 m/s. What must the amplitude of a traveling wave of frequency 69.0 Hz be in order for the average power carried by the wave to be 0.365 W?

15.24 •• A light wire is tightly stretched with tension F. Transverse traveling waves of amplitude A and wavelength λ_1 carry average power $P_{av,1} = 0.400$ W. If the wavelength of the waves is doubled, so $\lambda_2 = 2\lambda_1$, while the tension F and amplitude A are not altered, what then is the average power $P_{av,2}$ carried by the waves?

15.23. IDENTIFY: The average power carried by the wave depends on the mass density of the wire and the tension in it, as well as on the square of both the frequency and amplitude of the wave (the target variable).

SET UP: $P_{\text{av}} = \frac{1}{2}\sqrt{\mu F}\,\omega^2 A^2$, $v = \sqrt{\frac{F}{\mu}}$.

EXECUTE: Solving $P_{\text{av}} = \frac{1}{2}\sqrt{\mu F}\,\omega^2 A^2$ for A gives $A = \left(\frac{2P_{\text{av}}}{\omega^2\sqrt{\mu F}}\right)^{1/2}$. $P_{\text{av}} = 0.365\text{ W}$.

$\omega = 2\pi f = 2\pi(69.0\text{ Hz}) = 433.5\text{ rad/s}$. The tension is $F = 94.0\text{ N}$ and $v = \sqrt{\frac{F}{\mu}}$ so

$$\mu = \frac{F}{v^2} = \frac{94.0\text{ N}}{(492\text{ m/s})^2} = 3.883\times10^{-4}\text{ kg/m}.$$

$$A = \left(\frac{2(0.365\text{ W})}{(433.5\text{ rad/s})^2\sqrt{(3.883\times10^{-4}\text{ kg/m})(94.0\text{ N})}}\right)^{1/2} = 4.51\times10^{-3}\text{ m} = 4.51\text{ mm}$$

EVALUATE: Vibrations of strings and wires normally have small amplitudes, which this wave does.

15.24. IDENTIFY: The average power (the target variable) is proportional to the square of the frequency of the wave and therefore it is inversely proportional to the square of the wavelength.

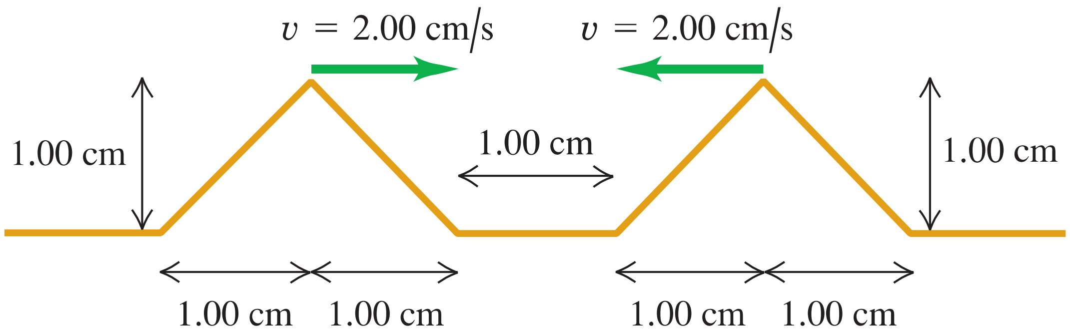
SET UP: $P_{\text{av}} = \frac{1}{2}\sqrt{\mu F}\,\omega^2 A^2$ where $\omega = 2\pi f$. The wave speed is $v = \sqrt{\frac{F}{\mu}}$.

EXECUTE: $\omega = 2\pi f = 2\pi\frac{v}{\lambda} = \frac{2\pi}{\lambda}\sqrt{\frac{F}{\mu}}$ so $P_{\text{av}} = \frac{1}{2}\sqrt{\mu F}\,\frac{4\pi^2}{\lambda^2}\left(\frac{F}{\mu}\right)A^2$. This shows that P_{av} is proportional to $\frac{1}{\lambda^2}$. Therefore $P_{\text{av},1}\lambda_1^2 = P_{\text{av},2}\lambda_2^2$ and $P_{\text{av},2} = P_{\text{av},1}\left(\frac{\lambda_1}{\lambda_2}\right)^2 = (0.400\text{ W})\left(\frac{\lambda_1}{2\lambda_1}\right)^2 = 0.100\text{ W}.$

EVALUATE: The wavelength is increased by a factor of 2, so the power is decreased by a factor of $2^2 = 4$.

15.32 • **Interference of Triangular Pulses.** Two triangular wave pulses are traveling toward each other on a stretched string as shown in Fig. E15.32. Each pulse is identical to the other and travels at 2.00 cm/s. The leading edges of the pulses are 1.00 cm apart at $t = 0$. Sketch the shape of the string at $t = 0.250$ s, $t = 0.500$ s, $t = 0.750$ s, $t = 1.000$ s, and $t = 1.250$ s.

Figure **E15.32**



15.32. **IDENTIFY:** Apply the principle of superposition.
SET UP: The net displacement is the algebraic sum of the displacements due to each pulse.
EXECUTE: The shape of the string at each specified time is shown in Figure 15.32.
EVALUATE: The pulses interfere when they overlap but resume their original shape after they have completely passed through each other.

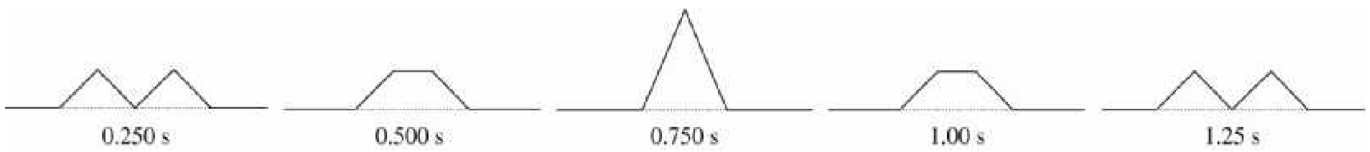


Figure 15.32

15.38 • **CALC** **Wave Equation and Standing Waves.** (a) Prove by direct substitution that $y(x, t) = (A_{SW} \sin kx) \sin \omega t$ is a solution of the wave equation, Eq. (15.12), for $v = \omega/k$. (b) Explain why the relationship $v = \omega/k$ for *traveling* waves also applies to *standing* waves.

15.39 • **CALC** Let $y_1(x, t) = A \cos(k_1 x - \omega_1 t)$ and $y_2(x, t) = A \cos(k_2 x - \omega_2 t)$ be two solutions to the wave equation, Eq. (15.12), for the same v. Show that $y(x, t) = y_1(x, t) + y_2(x, t)$ is also a solution to the wave equation.

15.38. **IDENTIFY:** Evaluate $\partial^2 y/\partial x^2$ and $\partial^2 y/\partial t^2$ and see if Eq. (15.12) is satisfied for $v = \omega/k$.

SET UP: $\frac{\partial}{\partial x}\sin kx = k\cos kx$. $\frac{\partial}{\partial x}\cos kx = -k\sin kx$. $\frac{\partial}{\partial t}\sin\omega t = \omega\cos\omega t$. $\frac{\partial}{\partial t}\cos\omega t = -\omega\sin\omega t$

EXECUTE: **(a)** $\frac{\partial^2 y}{\partial x^2} = -k^2[A_{\text{sw}}\sin\omega t]\sin kx$, $\frac{\partial^2 y}{\partial t^2} = -\omega^2[A_{\text{sw}}\sin\omega t]\sin kx$, so for $y(x,t)$ to be a solution of Eq. (15.12), $-k^2 = \frac{-\omega^2}{v^2}$, and $v = \frac{\omega}{k}$.

(b) A standing wave is built up by the superposition of traveling waves, to which the relationship $v = \lambda/k$ applies.

EVALUATE: $y(x,t) = (A_{\text{SW}}\sin kx)\sin\omega t$ is a solution of the wave equation because it is a sum of solutions to the wave equation.

15.39. **IDENTIFY:** Evaluate $\partial^2 y/\partial x^2$ and $\partial^2 y/\partial t^2$ and show that Eq. (15.12) is satisfied.

SET UP: $\frac{\partial}{\partial x}(y_1 + y_2) = \frac{\partial y_1}{\partial x} + \frac{\partial y_2}{\partial x}$ and $\frac{\partial}{\partial t}(y_1 + y_2) = \frac{\partial y_1}{\partial t} + \frac{\partial y_2}{\partial t}$

EXECUTE: $\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y_1}{\partial x^2} + \frac{\partial^2 y_2}{\partial x^2}$ and $\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y_1}{\partial t^2} + \frac{\partial^2 y_2}{\partial t^2}$. The functions y_1 and y_2 are given as being solutions to the wave equation, so

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y_1}{\partial x^2} + \frac{\partial^2 y_2}{\partial x^2} = \left(\frac{1}{v^2}\right)\frac{\partial^2 y_1}{\partial t^2} + \left(\frac{1}{v^2}\right)\frac{\partial^2 y_2}{\partial t^2} = \left(\frac{1}{v^2}\right)\left[\frac{\partial^2 y_1}{\partial t^2} + \frac{\partial^2 y_2}{\partial t^2}\right] = \left(\frac{1}{v^2}\right)\frac{\partial^2 y}{\partial t^2}$$

and so $y = y_1 + y_2$ is a solution of Eq. (15.12).

EVALUATE: The wave equation is a linear equation, as it is linear in the derivatives, and differentiation is a linear operation.

15.59 ••• **CP** The lower end of a uniform bar of mass 45.0 kg is attached to a wall by a frictionless hinge. The bar is held by a horizontal wire attached at its upper end so that the bar makes an angle of 30.0° with the wall. The wire has length 0.330 m and mass 0.0920 kg. What is the frequency of the fundamental standing wave for transverse waves on the wire?

15.59. **IDENTIFY:** The frequency of the fundamental (the target variable) depends on the tension in the wire. The bar is in rotational equilibrium so the torques on it must balance.

SET UP: $v = \sqrt{\dfrac{F}{\mu}}$ and $f = \dfrac{v}{\lambda}$. $\Sigma\tau_z = 0$.

EXECUTE: $\lambda = 2L = 0.660\text{ m}$. The tension F in the wire is found by applying the rotational equilibrium methods of Chapter 11. Let l be the length of the bar. Then $\Sigma\tau_z = 0$ with the axis at the hinge gives

$$Fl\cos 30° = \frac{1}{2}lmg\sin 30°. \quad F = \frac{mg\tan 30°}{2} = \frac{(45.0\text{ kg})(9.80\text{ m/s}^2)\tan 30°}{2} = 127.3\text{ N}.$$

$$v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{127.3\text{ N}}{(0.0920\text{ kg}/0.330\text{ m})}} = 21.37\text{ m/s}. \quad f = \frac{v}{\lambda} = \frac{21.37\text{ m/s}}{0.660\text{ m}} = 32.4\text{ Hz}$$

EVALUATE: This is an audible frequency for humans.

15.60 ••• **CP** You are exploring a newly discovered planet. The radius of the planet is 7.20×10^7 m. You suspend a lead weight from the lower end of a light string that is 4.00 m long and has mass 0.0280 kg. You measure that it takes 0.0600 s for a transverse pulse to travel from the lower end to the upper end of the string. On earth, for the same string and lead weight, it takes 0.0390 s for a transverse pulse to travel the length of the string. The weight of the string is small enough that its effect on the tension in the string can be neglected. Assuming that the mass of the planet is distributed with spherical symmetry, what is its mass?

15.60. **IDENTIFY:** The mass of the planet (the target variable) determines g at its surface, which in turn determines the weight of the lead object hanging from the string. The weight is the tension in the string, which determines the speed of a wave pulse on that string.

SET UP: At the surface of the planet $g = G\dfrac{m_{\rm p}}{R_{\rm p}^2}$. The pulse speed is $v = \sqrt{\dfrac{F}{\mu}}$.

EXECUTE: On earth, $v = \dfrac{4.00\text{ m}}{0.0390\text{ s}} = 1.0256\times10^2\text{ m/s}$. $\mu = \dfrac{0.0280\text{ kg}}{4.00\text{ m}} = 7.00\times10^{-3}\text{ kg/m}^3$. $F = Mg$, so

$v = \sqrt{\dfrac{Mg}{\mu}}$ and the mass of the lead weight is

$$M = \left(\frac{\mu}{g}\right)v^2 = \left(\frac{7.00\times10^{-3}\text{ kg/m}}{9.8\text{ m/s}^2}\right)(1.0256\times10^2\text{ m/s})^2 = 7.513\text{ kg}.$$ On the planet,

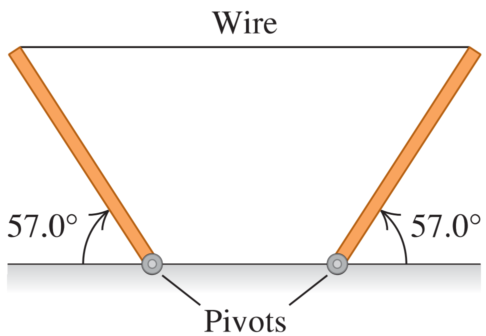
$$v = \frac{4.00\text{ m}}{0.0600\text{ s}} = 66.67\text{ m/s}.$$ Therefore $$g = \left(\frac{\mu}{M}\right)v^2 = \left(\frac{7.00\times10^{-3}\text{ kg/m}}{7.513\text{ kg}}\right)(66.67\text{ m/s})^2 = 4.141\text{ m/s}^2.$$

$$g = G\frac{m_{\rm p}}{R_{\rm p}^2} \text{ and } m_p = \frac{gR_{\rm p}^2}{G} = \frac{(4.141\text{ m/s}^2)(7.20\times10^7\text{ m})^2}{6.6742\times10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2} = 3.22\times10^{26}\text{ kg}.$$

EVALUATE: This mass is about 50 times that of Earth, but its radius is about 10 times that of Earth, so the result is reasonable.

15.62 ••• **CP** A 5.00-m, 0.732-kg wire is used to support two uniform 235-N posts of equal length (Fig. P15.62). Assume that the wire is essentially horizontal and that the speed of sound is 344 m/s. A strong wind is blowing, causing the wire to vibrate in its 5th overtone. What are the frequency and wavelength of the sound this wire produces?

Figure **P15.62**



15.62. **IDENTIFY:** Apply $\Sigma\tau_z = 0$ to one post and calculate the tension in the wire. $v = \sqrt{F/\mu}$ for waves on the wire. $v = f\lambda$. The standing wave on the wire and the sound it produces have the same frequency. For standing waves on the wire, $\lambda_n = \frac{2L}{n}$.

SET UP: For the 5th overtone, $n = 6$. The wire has $\mu = m/L = (0.732\text{ kg})/(5.00\text{ m}) = 0.146\text{ kg/m}$. The free-body diagram for one of the posts is given in Figure 15.62. Forces at the pivot aren't shown. We take the rotation axis to be at the pivot, so forces at the pivot produce no torque.

EXECUTE: $\Sigma\tau_z = 0$ gives $w\left(\frac{L}{2}\cos 57.0°\right) - T(L\sin 57.0°) = 0$. $T = \frac{w}{2\tan 57.0°} = \frac{235\text{ N}}{2\tan 57.0°} = 76.3\text{ N}$. For waves on the wire, $v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{76.3\text{ N}}{0.146\text{ kg/m}}} = 22.9\text{ m/s}$. For the 5th overtone standing wave on the wire, $\lambda = \frac{2L}{6} = \frac{2(5.00\text{ m})}{6} = 1.67\text{ m}$. $f = \frac{v}{\lambda} = \frac{22.9\text{ m/s}}{1.67\text{ m}} = 13.7\text{ Hz}$. The sound waves have frequency 13.7 Hz and wavelength $\lambda = \frac{344\text{ m/s}}{13.7\text{ Hz}} = 25.0\text{ m}$.

EVALUATE: The frequency of the sound wave is just below the lower limit of audible frequencies. The wavelength of the standing wave on the wire is much less than the wavelength of the sound waves, because the speed of the waves on the wire is much less than the speed of sound in air.

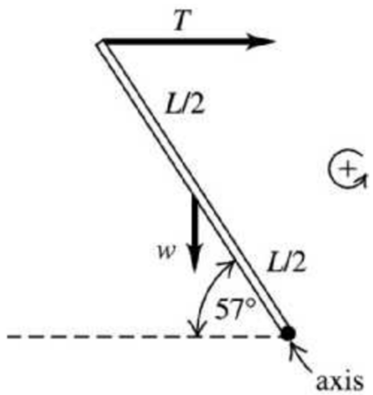


Figure 15.62

15.74 ••• **CALC** A guitar string is vibrating in its fundamental mode, with nodes at each end. The length of the segment of the string that is free to vibrate is 0.386 m. The maximum transverse acceleration of a point at the middle of the segment is $8.40 \times 10^3\ \mathrm{m/s^2}$ and the maximum transverse velocity is 3.80 m/s. (a) What is the amplitude of this standing wave? (b) What is the wave speed for the transverse traveling waves on this string?

15.74. **IDENTIFY:** The displacement of the string at any point is $y(x,t) = (A_{SW} \sin kx)\sin \omega t$. For the fundamental mode $\lambda = 2L$, so at the midpoint of the string $\sin kx = \sin(2\pi/\lambda)(L/2) = 1$, and $y = A_{SW} \sin \omega t$. The transverse velocity is $v_y = \partial y/\partial t$ and the transverse acceleration is $a_y = \partial v_y/\partial t$.

SET UP: Taking derivatives gives $v_y = \dfrac{\partial y}{\partial t} = \omega A_{SW} \cos \omega t$, with maximum value $v_{y,\,\max} = \omega A_{SW}$, and

$a_y = \dfrac{\partial v_y}{\partial t} = -\omega^2 A_{SW} \sin \omega t$, with maximum value $a_{y,\,\max} = \omega^2 A_{SW}$.

EXECUTE: $\omega = a_{y,\,\max}/v_{y,\,\max} = (8.40\times10^3 \text{ m/s}^2)/(3.80 \text{ m/s}) = 2.21\times10^3 \text{ rad/s}$, and then

$A_{SW} = v_{y,\,\max}/\omega = (3.80 \text{ m/s})/(2.21\times10^3 \text{ rad/s}) = 1.72\times10^{-3} \text{ m}$.

(b) $v = \lambda f = (2L)(\omega/2\pi) = L\omega/\pi = (0.386 \text{ m})(2.21\times10^3 \text{ rad/s})/\pi = 272 \text{ m/s}$.

EVALUATE: The maximum transverse velocity and acceleration will have different (smaller) values at other points on the string.

16.39 •• Tuning a Violin. A violinist is tuning her instrument to concert A (440 Hz). She plays the note while listening to an electronically generated tone of exactly that frequency and hears a beat of frequency 3 Hz, which increases to 4 Hz when she tightens her violin string slightly. (a) What was the frequency of the note played by her violin when she heard the 3-Hz beat? (b) To get her violin perfectly tuned to concert A, should she tighten or loosen her string from what it was when she heard the 3-Hz beat?

16.39. **IDENTIFY:** The beat is due to a difference in the frequencies of the two sounds.

SET UP: $f_{\text{beat}} = f_1 - f_2$. Tightening the string increases the wave speed for transverse waves on the string and this in turn increases the frequency.

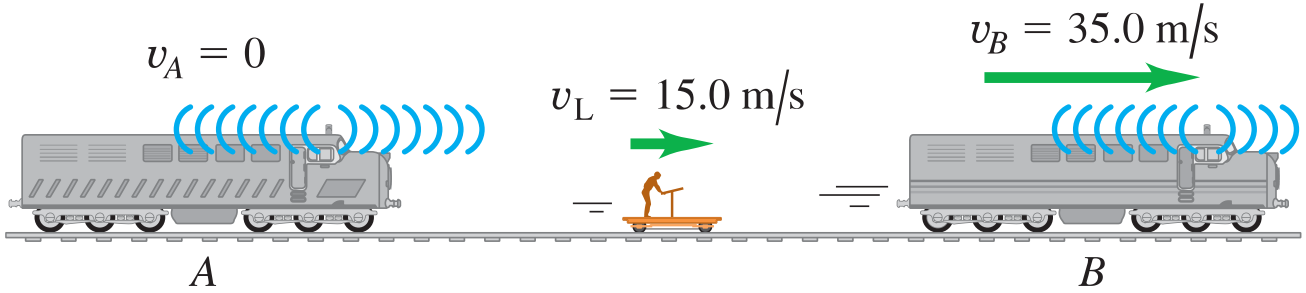
EXECUTE: **(a)** If the beat frequency increases when she raises her frequency by tightening the string, it must be that her frequency is 433 Hz, 3 Hz above concert *A*.

(b) She needs to lower her frequency by loosening her string.

EVALUATE: The beat would only be audible if the two sounds are quite close in frequency. A musician with a good sense of pitch can come very close to the correct frequency just from hearing the tone.

16.45 • Two train whistles, A and B, each have a frequency of 392 Hz. A is stationary and B is moving toward the right (away from A) at a speed of 35.0 m/s. A listener is between the two whistles and is moving toward the right with a speed of 15.0 m/s (Fig. E16.45). No wind is blowing. (a) What is the frequency from A as heard by the listener? (b) What is the frequency from B as heard by the listener? (c) What is the beat frequency detected by the listener?

Figure **E16.45**



16.45. **IDENTIFY:** Apply the Doppler shift equation $f_L = \left(\frac{v + v_L}{v + v_S}\right) f_S.$

SET UP: The positive direction is from listener to source. $f_S = 392$ Hz.

(a) $v_S = 0.$ $v_L = -15.0$ m/s. $f_L = \left(\frac{v + v_L}{v + v_S}\right) f_S = \left(\frac{344 \text{ m/s} - 15.0 \text{ m/s}}{344 \text{ m/s}}\right)(392 \text{ Hz}) = 375 \text{ Hz}$

(b) $v_S = +35.0$ m/s. $v_L = +15.0$ m/s. $f_L = \left(\frac{v + v_L}{v + v_S}\right) f_S = \left(\frac{344 \text{ m/s} + 15.0 \text{ m/s}}{344 \text{ m/s} + 35.0 \text{ m/s}}\right)(392 \text{ Hz}) = 371 \text{ Hz}$

(c) $f_{beat} = f_1 - f_2 = 4$ Hz

EVALUATE: The distance between whistle *A* and the listener is increasing, and for whistle *A* $f_L < f_S$. The distance between whistle *B* and the listener is also increasing, and for whistle *B* $f_L < f_S$.

16.56 • The shock-wave cone created by the space shuttle at one instant during its reentry into the atmosphere makes an angle of 58.0° with its direction of motion. The speed of sound at this altitude is $331\ \mathrm{m/s}$. (a) What is the Mach number of the shuttle at this instant, and (b) how fast (in m/s and in mi/h) is it traveling relative to the atmosphere? (c) What would be its Mach number and the angle of its shock-wave cone if it flew at the same speed but at low altitude where the speed of sound is $344\ \mathrm{m/s}$?

16.56. **IDENTIFY:** Apply Eq. (16.31).

SET UP: The Mach number is the value of v_S/v, where v_S is the speed of the shuttle and v is the speed of sound at the altitude of the shuttle.

EXECUTE: **(a)** $\frac{v}{v_S} = \sin\alpha = \sin 58.0° = 0.848$. The Mach number is $\frac{v_S}{v} = \frac{1}{0.848} = 1.18$.

(b) $v_S = \frac{v}{\sin\alpha} = \frac{331 \text{ m/s}}{\sin 58.0°} = 390 \text{ m/s}$

(c) $\frac{v_S}{v} = \frac{390 \text{ m/s}}{344 \text{ m/s}} = 1.13$. The Mach number would be 1.13. $\sin\alpha = \frac{v}{v_S} = \frac{344 \text{ m/s}}{390 \text{ m/s}}$ and $\alpha = 61.9°$.

EVALUATE: The smaller the Mach number, the larger the angle of the shock-wave cone.

16.64 ••• **CP** **A New Musical Instrument.** You have designed a new musical instrument of very simple construction. Your design consists of a metal tube with length L and diameter $L/10$. You have stretched a string of mass per unit length μ across the open end of the tube. The other end of the tube is closed. To produce the musical effect you're looking for, you want the frequency of the third-harmonic standing wave on the string to be the same as the fundamental frequency for sound waves in the air column in the tube. The speed of sound waves in this air column is v_S. (a) What must be the tension of the string to produce the desired effect? (b) What happens to the sound produced by the instrument if the tension is changed to twice the value calculated in part (a)? (c) For the tension calculated in part (a), what other harmonics of the string, if any, are in resonance with standing waves in the air column?

16.64. IDENTIFY: The harmonics of the string are $f_n = nf_1 = n\left(\frac{v}{2l}\right)$, where l is the length of the string. The tube is a stopped pipe and its standing wave frequencies are given by Eq. (16.22). For the string, $v = \sqrt{F/\mu}$, where F is the tension in the string.

SET UP: The length of the string is $d = L/10$, so its third harmonic has frequency $f_3^{\text{string}} = 3\frac{1}{2d}\sqrt{F/\mu}$.

The stopped pipe has length L, so its first harmonic has frequency $f_1^{\text{pipe}} = \frac{v_s}{4L}$.

EXECUTE: **(a)** Equating f_1^{string} and f_1^{pipe} and using $d = L/10$ gives $F = \frac{1}{3600}\mu v_s^2$.

(b) If the tension is doubled, all the frequencies of the string will increase by a factor of $\sqrt{2}$. In particular, the third harmonic of the string will no longer be in resonance with the first harmonic of the pipe because the frequencies will no longer match, so the sound produced by the instrument will be diminished.
(c) The string will be in resonance with a standing wave in the pipe when their frequencies are equal. Using $f_1^{\text{pipe}} = 3f_1^{\text{string}}$, the frequencies of the pipe are $nf_1^{\text{pipe}} = 3nf_1^{\text{string}}$ (where $n = 1, 3, 5, \ldots$). Setting this equal to the frequencies of the string $n'f_1^{\text{string}}$, the harmonics of the string are $n' = 3n = 3, 9, 15, \ldots$ The nth harmonic of the pipe is in resonance with the $3n$th harmonic of the string.
EVALUATE: Each standing wave for the air column is in resonance with a standing wave on the string. But the reverse is not true; not all standing waves of the string are in resonance with a harmonic of the pipe.

16.82 •• On a clear day you see a jet plane flying overhead. From the apparent size of the plane, you determine that it is flying at a constant altitude h. You hear the sonic boom at time T after the plane passes directly overhead. Show that if the speed of sound v is the same at all altitudes, the speed of the plane is

$$v_S = \frac{hv}{\sqrt{h^2 - v^2T^2}}$$

(*Hint:* Trigonometric identities will be useful.)

16.82. **IDENTIFY** and **SET UP:** Use Figure (16.37) to relate α and T.
Use this in Eq. (16.31) to eliminate $\sin\alpha$.

EXECUTE: Eq. (16.31): $\sin\alpha = v/v_S$ From Figure 16.37 $\tan\alpha = h/v_S T$. And $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{\sin\alpha}{\sqrt{1-\sin^2\alpha}}$.

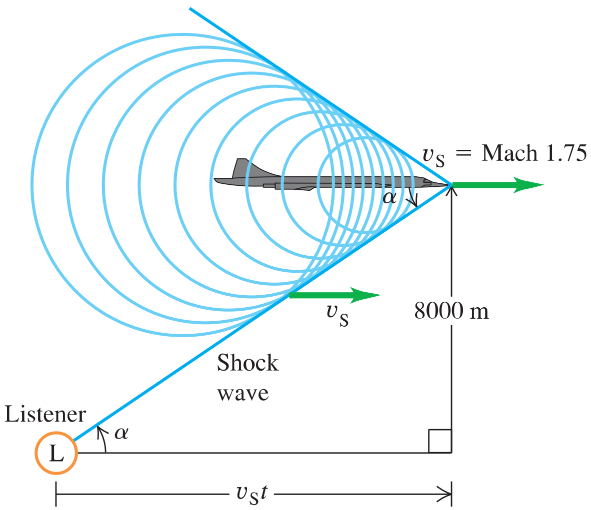
Combining these equations we get $\dfrac{h}{v_S T} = \dfrac{v/v_S}{\sqrt{1-(v/v_S)^2}}$ and $\dfrac{h}{T} = \dfrac{v}{\sqrt{1-(v/v_S)^2}}$.

$$1-(v/v_S)^2 = \frac{v^2T^2}{h^2} \quad \text{and} \quad v_S^2 = \frac{v^2}{1-v^2T^2/h^2}$$

$$v_S = \frac{hv}{\sqrt{h^2 - v^2T^2}}$$ as was to be shown.

EVALUATE: For a given h, the faster the speed v_S of the plane, the greater is the delay time T. The maximum delay time is h/v, and T approaches this value as $v_S \to \infty$. $T \to 0$ as $v \to v_S$.

16.37 You hear a sonic boom when the shock wave reaches yo at L (*not* just when the plane breaks the sound barrier). A listener the right of L has not yet heard the sonic boom but will shortly; a listener to the left of L has already heard the sonic boom.



Example 3.5. Let us discuss a sinusoidal wave moving in the $+x$ direction.

$$y_1(x,t) = A \sin \frac{2\pi}{T}\left(t - \frac{x}{v}\right).$$

A wall located at $x = L$ reflects this wave. Answer the following questions for each of the following two cases: (a) The wall is a fixed end. (b) The wall is a free end.

(1) Find the expression for the reflected wave.
(2) Find the expression for the resultant wave produced by the incident wave and the reflected wave.

Note that a fixed end is an end where the amplitude of the resultant wave of the incident and reflected waves vanishes at all times, whereas a free end is an end where the displacement of the reflected wave is equal to that of the incident wave.

Solution

(1) The reflected wave moves in the $-x$ direction, and has the same amplitude, period, and velocity as the incident wave. Therefore, the reflected wave can be written as

$$y_2(x,t) = A\sin\left\{\frac{2\pi}{T}\left(t+\frac{x}{v}+\beta\right)\right\},$$

where β is a constant that is to be determined by the condition that the wave should satisfy at the end $x = L$ (a boundary condition).

(a) In the case where $x = L$ is a fixed end, the displacement y must always satisfy the condition $y(L,t) = y_1(L,t) + y_2(L,t) = 0$. Therefore,

$$\sin\frac{2\pi}{T}\left(t-\frac{L}{v}\right) + \sin\left\{\frac{2\pi}{T}\left(t+\frac{L}{v}+\beta\right)\right\} = 0.$$

From this equation, β is determined to be $\beta = \frac{T}{2} - \frac{2L}{v}$, and we obtain

$$y_2(x,t) = A\sin\left\{\frac{2\pi}{T}\left(t+\frac{x-2L}{v}\right)+\pi\right\}$$
$$= \underline{-A\sin\left\{\frac{2\pi}{T}\left(t+\frac{x-2L}{v}\right)\right\}}.$$

(b) On the other hand, in the case where $x = L$ is a free end, the displacement y must satisfy the condition. $y_1(L,t) = y_2(L,t)$. Therefore,

$$\sin\frac{2\pi}{T}\left(t-\frac{L}{v}\right) = \sin\left\{\frac{2\pi}{T}\left(t+\frac{L}{v}+\beta\right)\right\}.$$

From this equation, β is determined to be $\beta = -\frac{2L}{v}$, and we obtain

$$y_2(x,t) = \underline{A\sin\left\{\frac{2\pi}{T}\left(t+\frac{x-2L}{v}\right)\right\}}.$$

(2) The resultant wave is $y(x,t) = y_1(x,t) + y_2(x,t)$. Then, we have

(a) $y(x,t) = \underline{2A\sin\left(\frac{2\pi}{T}\cdot\frac{L-x}{v}\right)\cdot\cos\left\{\frac{2\pi}{T}\left(t-\frac{L}{v}\right)\right\}}$

(b) $y(x,t) = \underline{2A\cos\left(\frac{2\pi}{T}\cdot\frac{L-x}{v}\right)\cdot\sin\left\{\frac{2\pi}{T}\left(t-\frac{L}{v}\right)\right\}}.$

We can see that these resultant waves are standing waves, oscillating independently in time and in space. ■